MAP 2210 – Aplicações de Álgebra Linear

1º Semestre – 2020

Prof. Dr. Luis Carlos de Castro Santos

lsantos@ime.usp.br

Objetivos

Formação básica de álgebra linear aplicada a problemas numéricos. Resolução de problemas em microcomputadores usando linguagens e/ou software adequados fora do horário de aula.

MAP2210

MÁRIO BARONE JÚNIOR

ÁLGEBRA LINEAR

3ª edição - 1988

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São Paulo

Capítulo 2

ESPAÇOS VETORIAIS

2.2 – DEFINIÇÃO. Um *espaço vetorial* é um conjunto munido de uma operação de adição e de uma operação de multiplicação por escalar que verificam as oito propriedades A–1 a A–4 e M–1 a M–4 enunciadas no exercício 1.4.

1.4 – EXERCÍCIO. Indiquemos por E um qualquer dos conjuntos V^3, $\mathbf{R}^n$ ou $\mathcal{F}(I)$; se $u, v \in E$ e $\lambda \in \mathbf{R}$, $u + v$ e λu indicarão as operações adequadas. Verifique que valem as seguintes propriedades:

A–1) $\forall u, v, w \in E,$ $\quad (u + v) + w = u + (v + w);$

A–2) $\forall u, v \in E,$ $\quad u + v = v + u;$

A–3) $\exists 0 \in E$ tal que, $\forall u \in E,$ $\quad u + 0 = 0 + u;$

A–4) $\forall u \in E,$ $\exists (-u) \in E$ tal que $\quad u + (-u) = (-u) + u = 0;$

M–1) $\forall \alpha \in \mathbf{R},$ $\forall u, v \in E,$ $\quad \alpha(u + v) = \alpha u + \alpha v;$

M–2) $\forall \alpha, \beta \in \mathbf{R},$ $\forall u \in E,$ $\quad (\alpha + \beta)u = \alpha u + \beta u;$

M–3) $\forall \alpha, \beta \in \mathbf{R},$ $\forall u \in E,$ $\quad \alpha(\beta u) = (\alpha\beta)u;$

M–4) $\forall u \in E,$ $\quad 1u = u.$

2.4 – EXEMPLOS. **a)** De acordo com o exercício 1.4, os conjuntos V^3, $\mathbf{R}^n$ e $\mathcal{F}(I)$, com as operações usuais, são espaços vetoriais.

b) O conjunto $M_{p\times n}(\mathbf{R})$ das matrizes reais com p linhas e n colunas, com as operações usuais de adição de matrizes e de multiplicação de matriz por número real, é um espaço vetorial (verifique). Em particular temos o espaço das matrizes reais quadradas de ordem n: $M_n(\mathbf{R}) = M_{n\times n}(\mathbf{R})$.

2) Em $M_{p\times n}(\mathbf{R})$, o vetor nulo é a matriz que tem todos os elementos iguais a zero (matriz nula) e a oposta de u'a matriz A é a matriz que se obtém trocando o sinal de todos os elementos de A.

Capítulo 3

COMBINAÇÃO LINEAR – SUBESPAÇO

3.1 – DEFINIÇÃO. Seja $\{u_1, u_2, \ldots, u_q\}$ um subconjunto finito formado por q vetores de um espaço vetorial V, com $q \geq 1$.

Uma *combinação linear* dos vetores $u_1, \ldots, u_q$ é qualquer vetor de V que possa ser colocado na forma

$$\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_q u_q.$$

Os escalares (números reais) $\alpha_1, \ldots, \alpha_q$ são chamados coeficientes da combinação linear.

Subespaços

Consideremos o sistema linear homogêneo

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0 \\ \quad\vdots \qquad\qquad\qquad\qquad \vdots \\ a_{p1}x_1 + a_{p2}x_2 + \cdots + a_{pn}x_n = 0, \end{cases}$$

com p equações e n incógnitas.

Consideremos então o sistema $Ax = 0$ e seja S o conjunto das n-uplas do $\mathbf{R}^n$ que são soluções deste sistema homogêneo. Valem as seguintes propriedades:

1) O vetor nulo do $\mathbf{R}^n$ está em S pois $A \cdot 0 = 0$. (Todo sistema homogêneo tem pelo menos a solução trivial ou nula.)

2) Se as n-uplas u e v são soluções então, $A(u + v) = Au + Av = 0 + 0 = 0$, donde $u + v$ também é solução, ou seja

$$u, v \in S \Longrightarrow u + v \in S.$$

3) Se a n-upla u é solução e $\lambda \in \mathbf{R}$, então $A(\lambda u) = \lambda(Au) = \lambda 0 = 0$, donde λu também é solução, ou seja,

$$u \in S,\ \lambda \in \mathbf{R} \Longrightarrow \lambda u \in S.$$

3.5 – DEFINIÇÃO. Um subconjunto S de um espaço vetorial V é chamado um *subespaço vetorial* de V se verifica as seguintes condições:

S–1) O vetor nulo de V pertence a S.

S–2) Se os vetores u e v de V estão em S, então $u+v$ também pertence a S.

S–3) Se o vetor u de V está em S e $\lambda \in \mathbf{R}$ é um escalar qualquer, então λu também pertence a S.

3.6 – EXEMPLOS. **1)** Em qualquer espaço vetorial V, os exemplos mais simples de subespaços vetoriais são o próprio V e o subespaço $\{0\}$ (verifique).

2) Como acabamos de ver, o conjunto das soluções de um sistema linear homogêneo com n incógnitas é um subespaço vetorial do $\mathbf{R}^n$ e o conjunto das soluções de uma equação diferencial linear homogênea de segunda ordem com coeficientes constantes é um subespaço vetorial de $\mathcal{F}(\mathbf{R})$.

3.10 – PROPOSIÇÃO. *(Exercício.)* Se $A = \{u_1, u_2, \ldots, u_q\}$, com $q \geq 1$, é um subconjunto *finito* de um espaço vetorial V, então o subconjunto de V formado por *todas as combinações lineares* de $u_1, u_2, \ldots, u_q$ é um subespaço vetorial de V.

3.11 – DEFINIÇÃO. O subespaço construido na proposição anterior é chamado *subespaço gerado* pelos vetores $u_1, u_2, \ldots, u_q$ ou pelo conjunto A e é representado por $[u_1, u_2, \ldots, u_q]$ ou por $[A]$.

Capítulo 17

TRANSFORMAÇÕES LINEARES

Sabemos que um sistema linear com p equações e n incógnitas pode ser escrito na forma

$$Ax = b,$$

onde A é uma matriz $p \times n$, x é $n \times 1$ e b é $p \times 1$.

O que nos interessa no momento é perceber que, através da matriz A fica definida uma regra que a cada vetor $x \in \mathbf{R}^n$ associa um e um só vetor $Ax \in \mathbf{R}^p$, isto é, temos uma função (ou transformação) definida no $\mathbf{R}^n$ e tomando valores no $\mathbf{R}^p$.

Sabemos também que para esta função, que de alguma forma está associada a um sistema *linear*, valem as propriedades:

1) $A(u+v) = Au + Av$,

2) $A(\lambda u) = \lambda(Au)$

e que estas propriedades foram essenciais para mostrar que as soluções do sistema homogêneo $Ax = 0$ formam um subespaço do $\mathbf{R}^n$.

Fourth Edition

LINEAR ALGEBRA
AND ITS APPLICATIONS

Gilbert Strang

Chapter

2 Vector Spaces

2.1 VECTOR SPACES AND SUBSPACES

A real vector space is a set of *vectors* ***together with rules for vector addition and multiplication by real numbers***. Addition and multiplication must produce vectors in the space, and they must satisfy the eight conditions.

DEFINITION A ***subspace*** of a vector space is a nonempty subset that satisfies the requirements for a vector space: ***Linear combinations stay in the subspace***.

(i) If we add any vectors x and y in the subspace, $x + y$ is *in the subspace*.

(ii) If we multiply any vector x in the subspace by any scalar c, cx is *in the subspace*.

The Column Space of A

We now come to the key examples, the **column space** and the **nullspace** of a matrix A. ***The column space contains all linear combinations of the columns of A.*** It is a subspace of $\mathbf{R}^m$. We illustrate by a system of $m = 3$ equations in $n = 2$ unknowns:

$$\textbf{Combination of columns equals } b \qquad \begin{bmatrix} 1 & 0 \\ 5 & 4 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}. \qquad (1)$$

With $m > n$ we have more equations than unknowns—and *usually there will be no solution*. The system will be solvable only for a very "thin" subset of all possible b's. One way of describing this thin subset is so simple that it is easy to overlook.

2A The system $Ax = b$ is solvable if and only if the vector b can be expressed as a combination of the columns of A. Then b is in the column space.

This description involves nothing more than a restatement of $Ax = b$, *by columns*:

$$\textbf{Combination of columns} \qquad u\begin{bmatrix}1\\5\\2\end{bmatrix} + v\begin{bmatrix}0\\4\\4\end{bmatrix} = \begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}. \qquad (2)$$

These are the same three equations in two unknowns. Now the problem is: Find numbers u and v that multiply the first and second columns to produce b. The system is solvable exactly when such coefficients exist, and the vector (u, v) is the solution x.

We can describe *all combinations* of the two columns geometrically: *$Ax = b$ can be solved if and only if b lies in the* ***plane*** *that is spanned by the two column vectors* (Figure 2.1). This is the thin set of attainable b. If b lies off the plane, then it is not a combination of the two columns. In that case $Ax = b$ has no solution.

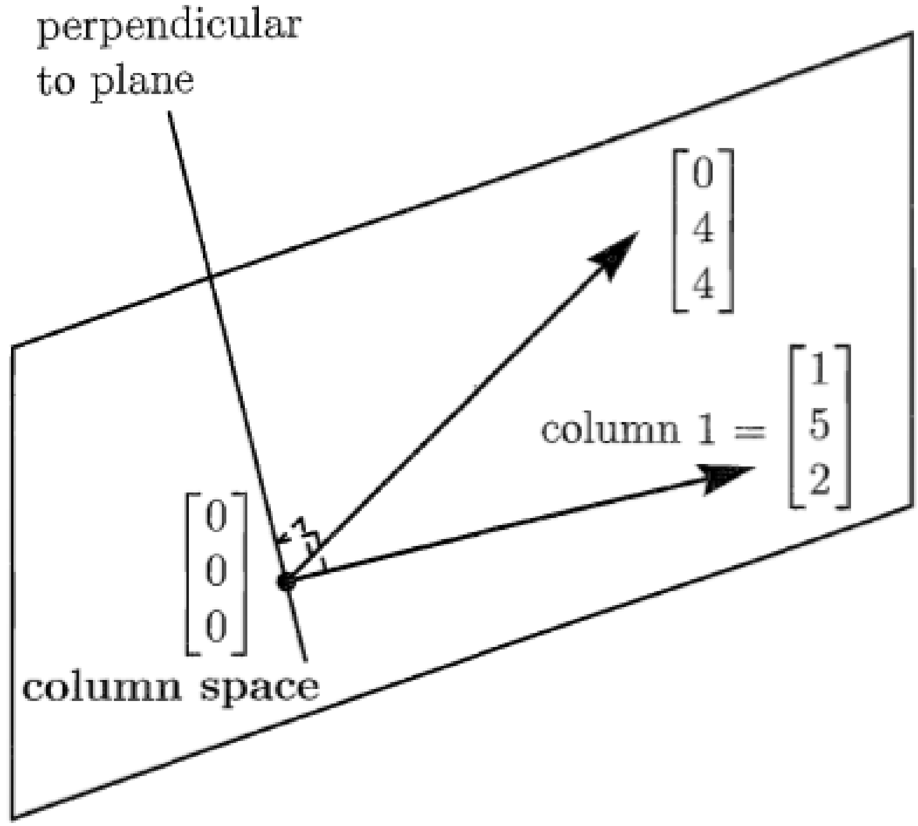


Figure 2.1 The column space $\boldsymbol{C}(A)$, a plane in three-dimensional space.

What is important is that this plane is not just a subset of $\mathbf{R}^3$; it is a subspace. It is the ***column space*** of A, consisting of ***all combinations of the columns***. It is denoted by $\boldsymbol{C}(A)$.

Requirements (i) and (ii) for a subspace of $\mathbf{R}^m$ are easy to check:

(i) Suppose b and b' lie in the column space, so that $Ax = b$ for some x and $Ax' = b'$ for some x'. Then $A(x + x') = b + b'$, so that $b + b'$ is also a combination of the columns. The column space of all attainable vectors b is closed under addition.

(ii) If b is in the column space $C(A)$, so is any multiple cb. If some combination of columns produces b (say $Ax = b$), then multiplying that combination by c will produce cb. In other words, $A(cx) = cb$.

The Nullspace of A

The second approach to $Ax = b$ is "dual" to the first. We are concerned not only with attainable right-hand sides b, but also with the solutions x that attain them. The right-hand side $b = 0$ always allows the solution $x = 0$, but there may be infinitely many other solutions. (There always are, if there are more unknowns than equations, $n > m$.) ***The solutions to $\mathbf{Ax} = \mathbf{0}$ form a vector space—the nullspace of $\mathbf{A}$.***

The ***nullspace*** of a matrix consists of all vectors x such that $Ax = 0$. It is denoted by $N(A)$. It is a subspace of $\mathbf{R}^n$, just as the column space was a subspace of $\mathbf{R}^m$.

Requirement (i) holds: If $Ax = 0$ and $Ax' = 0$, then $A(x + x') = 0$. Requirement (ii) also holds: If $Ax = 0$ then $A(cx) = 0$. Both requirements fail if the right-hand side is not zero! Only the solutions to a *homogeneous* equation ($b = 0$) form a subspace. The nullspace is easy to find for the example given above; it is as small as possible:

$$\begin{bmatrix} 1 & 0 \\ 5 & 4 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The first equation gives $u = 0$, and the second equation then forces $v = 0$. The nullspace contains only the vector $(0, 0)$. This matrix has "independent columns"—a key idea that comes soon.

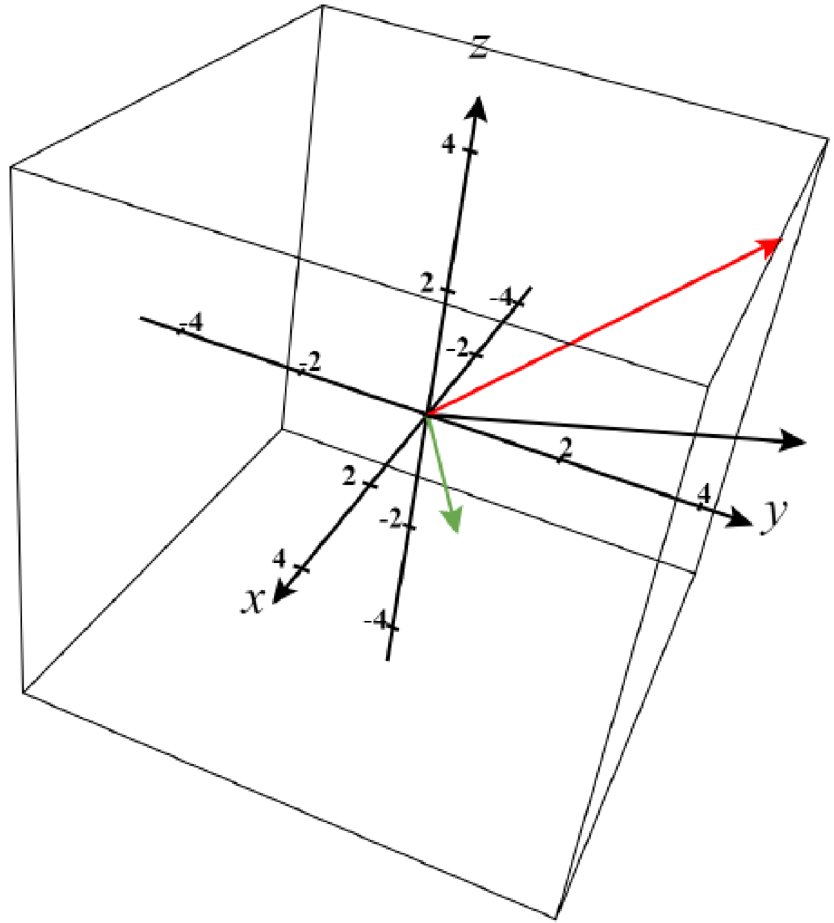
The situation is changed when a third column is a combination of the first two:

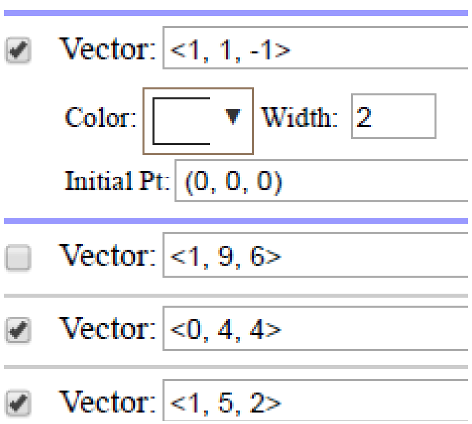
$$\textbf{Larger nullspace} \qquad B = \begin{bmatrix} 1 & 0 & 1 \\ 5 & 4 & 9 \\ 2 & 4 & 6 \end{bmatrix}.$$

B has the same column space as A. The new column lies in the plane of Figure 2.1; it is the sum of the two column vectors we started with. But the nullspace of B contains the vector $(1, 1, -1)$ and automatically contains any multiple $(c, c, -c)$:

$$\textbf{Nullspace is a line} \qquad \begin{bmatrix} 1 & 0 & 1 \\ 5 & 4 & 9 \\ 2 & 4 & 6 \end{bmatrix} \begin{bmatrix} c \\ c \\ -c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The nullspace of B is the line of all points $x = c$, $y = c$, $z = -c$. (The line goes through the origin, as any subspace must.) We want to be able, for any system $Ax = b$, to find $\boldsymbol{C}(A)$ and $\boldsymbol{N}(A)$: all attainable right-hand sides b and all solutions to $Ax = 0$.





https://www.monroecc.edu/faculty/paulseeburger/calcnsf/CalcPlot3D/

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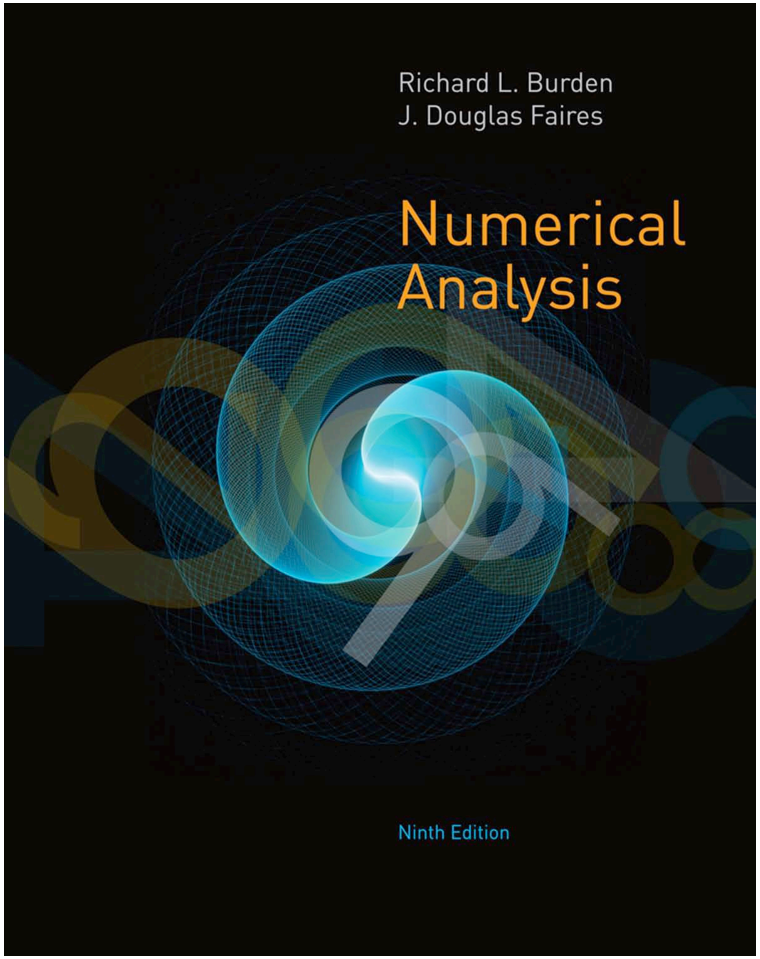


Numerical Analysis

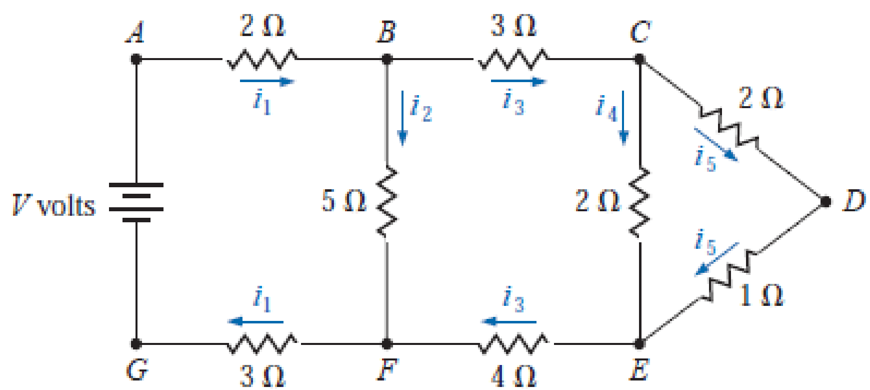
NINTH EDITION

Richard L. Burden
Youngstown State University

J. Douglas Faires
Youngstown State University

CHAPTER 6

Direct Methods for Solving Linear Systems



Equations

Kirchhoff's laws:

current law: $I_{\text{into junction}} = I_{\text{out of junction}}$

voltage law: $V_{gains} + V_{drops} = 0$ around any closed loop

$$5i_1 + 5i_2 = V,$$

$$i_3 - i_4 - i_5 = 0,$$

$$2i_4 - 3i_5 = 0,$$

$$i_1 - i_2 - i_3 = 0,$$

$$5i_2 - 7i_3 - 2i_4 = 0.$$

In this chapter we consider *direct methods* for solving a linear system of n equations in n variables. Such a system has the form

$$\begin{aligned}
E_1: &\quad a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1,\\
E_2: &\quad a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2,\\
&\qquad\qquad\qquad\qquad\vdots\\
E_n: &\quad a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n.
\end{aligned} \tag{6.1}$$

In this system we are given the constants a_{ij}, for each $i, j = 1, 2, \ldots, n$, and b_i, for each $i = 1, 2, \ldots, n$, and we need to determine the unknowns $x_1, \ldots, x_n$.

6.1 Linear Systems of Equations

We use three operations to simplify the linear system given in (6.1):

1. Equation E_i can be multiplied by any nonzero constant λ with the resulting equation used in place of E_i. This operation is denoted $(\lambda E_i) \rightarrow (E_i)$.
2. Equation E_j can be multiplied by any constant λ and added to equation E_i with the resulting equation used in place of E_i. This operation is denoted $(E_i + \lambda E_j) \rightarrow (E_i)$.
3. Equations E_i and E_j can be transposed in order. This operation is denoted $(E_i) \leftrightarrow (E_j)$.

By a sequence of these operations, a linear system will be systematically transformed into to a new linear system that is more easily solved and has the same solutions. The sequence of operations is illustrated in the following.

Illustration

$$
\begin{aligned}
E_1 &: \quad x_1 + x_2 \phantom{{}- x_3} + 3x_4 = 4,\\
E_2 &: \quad 2x_1 + x_2 - x_3 + x_4 = 1,\\
E_3 &: \quad 3x_1 - x_2 - x_3 + 2x_4 = -3,\\
E_4 &: \quad -x_1 + 2x_2 + 3x_3 - x_4 = 4,
\end{aligned}
$$

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$$
\begin{aligned}
E_1 &: \quad x_1 + x_2 \phantom{{}-x_3} + 3x_4 = 4, \\
E_2 &: \quad 2x_1 + x_2 - x_3 + x_4 = 1, \\
E_3 &: \quad 3x_1 - x_2 - x_3 + 2x_4 = -3, \\
E_4 &: \quad -x_1 + 2x_2 + 3x_3 - x_4 = 4,
\end{aligned}
$$

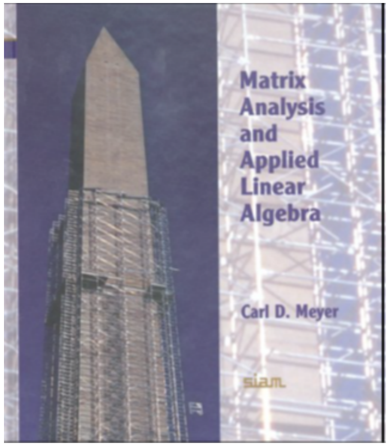
$(E_2 - 2E_1) \to (E_2)$, $(E_3 - 3E_1) \to (E_3)$, and $(E_4 + E_1) \to (E_4)$.

$(E_3 - 4E_2) \to (E_3)$

$(E_4 + 3E_2) \to (E_4)$.

$$
\begin{aligned}
E_1 &: \quad x_1 + x_2 + 3x_4 = 4, \\
E_2 &: \quad -x_2 - x_3 - 5x_4 = -7, \\
E_3 &: \quad 3x_3 + 13x_4 = 13, \\
E_4 &: \quad -13x_4 = -13.
\end{aligned}
\tag{6.3}
$$

The solution to system (6.3), and consequently to system (6.2), is therefore, $x_1 = -1$, $x_2 = 2$, $x_3 = 0$, and $x_4 = 1$. □



The earliest recorded analysis of simultaneous equations is found in the ancient Chinese book *Chiu-chang Suan-shu* (*Nine Chapters on Arithmetic*), estimated to have been written some time around 200 B.C. In the beginning of Chapter VIII, there appears a problem of the following form.

> *Three sheafs of a good crop, two sheafs of a mediocre crop, and one sheaf of a bad crop are sold for 39 dou. Two sheafs of good, three mediocre, and one bad are sold for 34 dou; and one good, two mediocre, and three bad are sold for 26 dou. What is the price received for each sheaf of a good crop, each sheaf of a mediocre crop, and each sheaf of a bad crop?*

Today, this problem would be formulated as three equations in three unknowns by writing

$$\begin{aligned} 3x + 2y + z &= 39, \\ 2x + 3y + z &= 34, \\ x + 2y + 3z &= 26, \end{aligned}$$

where x, y, and z represent the price for one sheaf of a good, mediocre, and bad crop, respectively. The Chinese saw right to the heart of the matter. They

Que tal resolver o sistema mais antigo que se tem notícia ?

a) com precisão infinita

b) com 1 casa decimal a cada operação

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Fim...

AULA 01