Dinamica Non Lineare di Strutture e Sistemi Meccanici

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Lezione 3

Elements of Stability Theory

Lagrangian formulation (recalling)

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_r}\right) - \frac{\partial T}{\partial q_r} + \frac{\partial V}{\partial q_r} = N_r \ , \ r = 1,2,...,n$$

System of second-order differential equations (holonomic constraints)

$$\ddot{\mathbf{q}} = \mathbf{h}(\mathbf{q}, \dot{\mathbf{q}}, t) \qquad \ddot{q}_r = h_r\left(q_1, q_2,...,q_n, \dot{q}_1, \dot{q}_2,...,\dot{q}_n, t\right)$$

Example: SDOF linear oscillator

$$\ddot{u} = \gamma(t) - \omega^2 u - 2\xi\omega\dot{u} \quad \text{with} \quad \gamma(t) = \frac{R(t)}{m}, \quad \omega = \sqrt{\frac{k}{m}}, \quad \xi = \frac{c}{2m\omega}$$

Example: MDOF linear system

$$\ddot{\mathbf{U}} = \mathbf{M}^{-1}\left[\mathbf{R}(t) - \mathbf{KU} - \mathbf{C\dot{U}}\right]$$

Elements of Stability Theory

Hamiltonian formulation (recalling)

Generalized momenta:
$$p_r = \frac{\partial T}{\partial \dot{q}_r}$$

Hamiltonian:
$$H = \sum_{r=1}^{n} \dot{q}_r\, p_r - T + V$$

System of first-order differential equations (holonomic constraints)

$$\dot{q}_r = \frac{\partial H}{\partial p_r}$$

$$\dot{p}_r = N_r - \frac{\partial H}{\partial q_r}$$

Elements of Stability Theory

Hamiltonian formulation (recalling)

Example: SDOF linear oscillator

$$p = \frac{\partial T}{\partial \dot{q}} = m\dot{q}$$

$$H = p\dot{q} - T + V = \frac{p^2}{2m} + \frac{kq^2}{2}$$

$$N = R(t) - c\dot{q} = R(t) - c\frac{p}{m}$$

$$\Rightarrow \quad \begin{cases} \dot{q} = \dfrac{1}{m}\,p \\[4mm] \dot{p} = R(t) - kq - \dfrac{c}{m}\,p \end{cases}$$

Elements of Stability Theory

Lagrangian formulation:
from second- to first-order system of differential equations through change of variables

$$\ddot{q}_r = h_r\left(q_1, q_2, ..., q_n, \dot{q}_1, \dot{q}_2, ..., \dot{q}_n, t\right)$$

$$y_r = q_r$$
$$y_{r+n} = \dot{q}_r$$
$$\implies \begin{cases} \dot{y}_r = y_{r+n} \\ \dot{y}_{r+n} = h_r\left(y_1, y_2, ..., y_{2n}, t\right) \end{cases}$$

$$\dot{\mathbf{y}} = \mathbf{g}\left(\mathbf{y}, t\right)$$

Example: SDOF linear oscillator

$$y_1 = q$$
$$y_2 = \dot{q}$$
$$\implies \begin{cases} \dot{y}_1 = y_2 \\ \dot{y}_2 = \gamma\left(t\right) - \omega^2 y_1 - 2\xi\omega y_2 \end{cases}$$
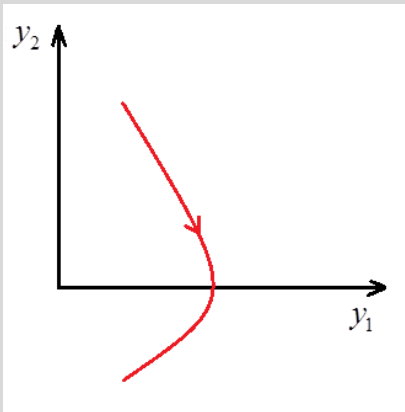
Elements of Stability Theory

Phase space

Autonomous systems

$$\dot{\mathbf{y}} = \mathbf{g}(\mathbf{y})$$

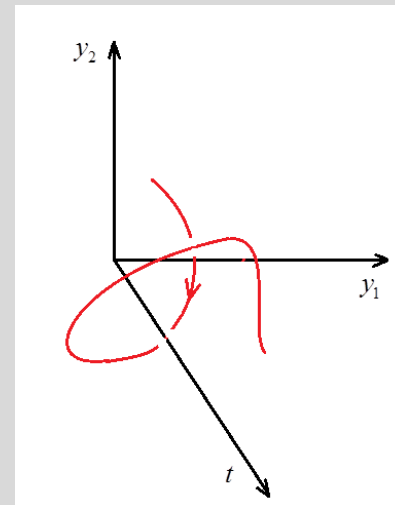n-dimensional space

$$y_1 \times y_2 ... \times y_{2n}$$

Non-autonomous systems

$$\dot{\mathbf{y}} = \mathbf{g}(\mathbf{y}, t)$$

$(n+1)$-dimensional space

$$y_1 \times y_2 ... \times y_{2n} \times t$$

Elements of Stability Theory

Singular phase points (equilibrium points) $\dot{\mathbf{y}} = \mathbf{g}(\mathbf{y}) = 0$

Regular phase points $\dot{\mathbf{y}} = \mathbf{g}(\mathbf{y}) \neq 0$

Phase trajectory tangent

$$\frac{dy_2}{dy_1} = \frac{g_2(y_1, y_2)}{g_1(y_1, y_2)} = \frac{g_2(y_1, y_2)}{y_2}$$

Tangent at singular phase points is indeterminate

$$\frac{dy_2}{dy_1} = \frac{g_2(y_1, y_2)}{g_1(y_1, y_2)} = \frac{0}{0}$$

Tangent at regular phase points with $g_1(y_1, y_2) = y_2 = 0$ and $g_2(y_1, y_2) \neq 0$ is orthogonal to the y_1 axis

Through a regular phase point passes just one phase trajectory (Theorem of Cauchy-Lipschitz)

Elements of Stability Theory

Non-perturbed solution: $\quad y_r = y_r^0(t), \quad r = 1, 2, ..., 2n$

Perturbed solution: $\quad y_r = y_r^0(t) + \delta y_r(t), \quad r = 1, 2, ..., 2n$

$$\delta \dot{y}_r = g_r\left(y_1^0 + \delta y_1, y_2^0 + \delta y_2, ..., y_{2n}^0 + \delta y_{2n}, t\right) - \dot{y}_r^0$$

Perturbation equations:
$$\delta \dot{y}_r = f_r\left(\delta y_1, \delta y_2, ..., \delta y_{2n}, t\right)$$
$$\boldsymbol{\delta \dot{y}} = \mathbf{f}\left(\boldsymbol{\delta y}, t\right)$$

$$\boldsymbol{\delta \dot{y}} = \mathbf{A}(t)\boldsymbol{\delta y} + \mathbf{N}(\boldsymbol{\delta y}, t) \quad \text{with} \quad \mathbf{A}(t) = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{y}}\right|_{\mathbf{0}} \text{ and } \mathbf{N}(\boldsymbol{\delta y}, t) = \mathbf{f}(\boldsymbol{\delta y}, t) - \mathbf{A}(t)\boldsymbol{\delta y}$$

Note: the non-perturbed solution corresponds to the trivial solution $\boldsymbol{\delta y} = \mathbf{0}$ of the perturbation equations

Elements of Stability Theory

Example: SDOF linear oscillator

$$\begin{cases} \delta \dot{y}_1 = \delta y_2 \\ \delta \dot{y}_2 = -\omega^2 \delta y_1 - 2\xi\omega\, \delta y_2 \end{cases}$$

$$\boldsymbol{\delta \dot{y}} = \mathbf{f}\left(\boldsymbol{\delta y}\right) \qquad \text{or} \qquad \boldsymbol{\delta \dot{y}} = \mathbf{A}\boldsymbol{\delta y}$$

with

$$\mathbf{A} = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{y}}\right|_{\mathbf{0}} = \begin{bmatrix} 0 & 1 \\ -\omega^2 & -2\xi\omega \end{bmatrix}$$

Elements of Stability Theory

A non-perturbed solution $\mathbf{y^0}(t)$ is stable if the distance $\boldsymbol{\delta}\mathbf{y}(t)$ to the perturbed solutions remains within prescribed bounds for all times and arbitrarily defined perturbations

Non-perturbed solution

Equilibrium $\qquad \mathbf{y^0} = const.$
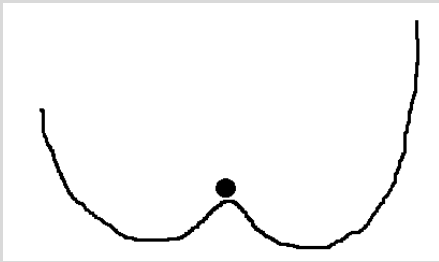
Motion $\qquad \mathbf{y^0}(t)$

"Type" of perturbation

Kinematical (initial conditions): $\boldsymbol{\delta}\mathbf{y}(0) \neq \mathbf{0}$

Topological (perturbation of parameters or perturbation of mathematical model)

Elements of Stability Theory

Perturbation "size"

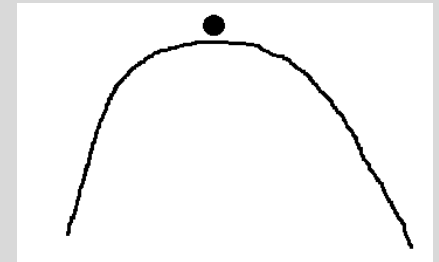Local $\quad \left\| \delta \mathbf{y}(0) \right\| < \delta$

Global
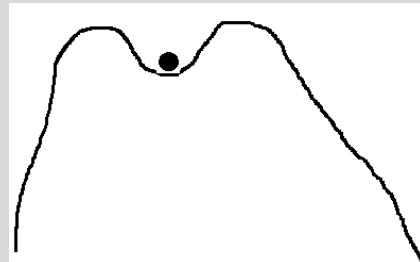
Global S
Local I

Global I
Local S
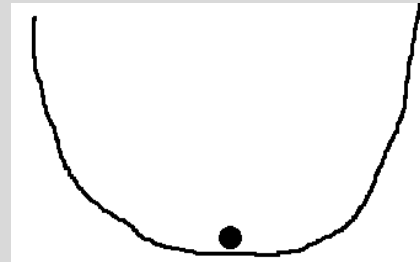
Global S
Local S

Global I
Local I

Elements of Stability Theory

Stability concept (Leipholz)

"Character" of perturbation
- Deterministic
- Stochastic

Example: <u>definition</u> of stability in the quadratic mean:

$$\lim_{\tau \to \infty} E_\tau \left\| \boldsymbol{\delta y}(t) \right\|^2 < \varepsilon \qquad \sigma_{\boldsymbol{\delta y}}^2 = \int_{-\infty}^{\infty} S_{\boldsymbol{\delta y}}(\omega) \, d\omega < \varepsilon$$
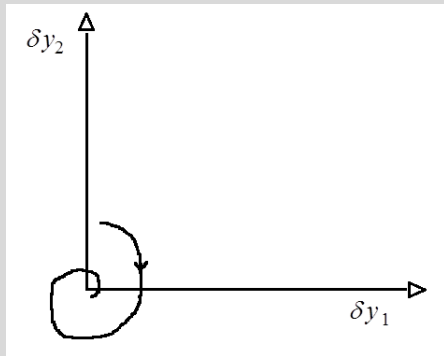
Stability x Confiability x Integrity

Elements of Stability Theory

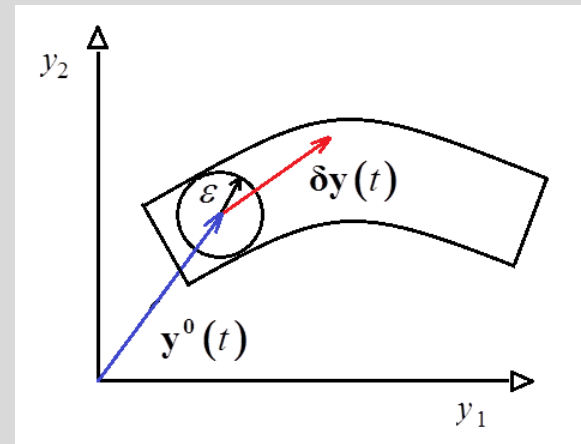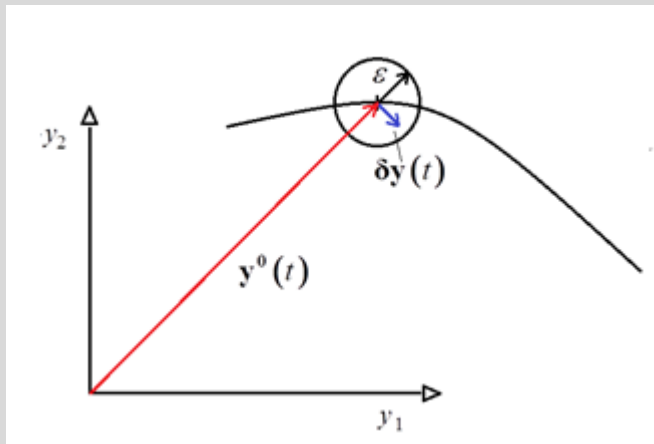Tendency of perturbed solution

Asymptotic

Non-asymptotic
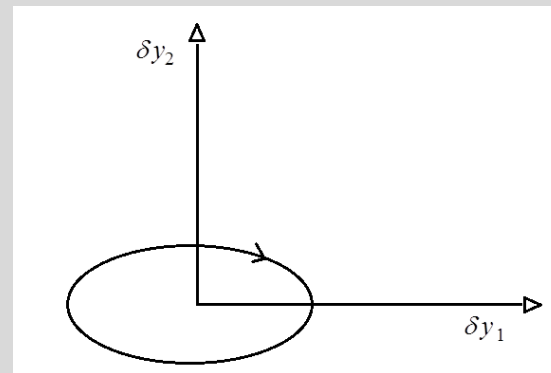
Elements of Stability Theory

Stability concept (Leipholz)

Admissible region for perturbed solution

Kinetic

Geometric

Elements of Stability Theory

Stability definitions

Liapunov

Stability of equilibrium of autonomous systems in the sense:
kinematical, local, deterministic, non-asymptotic, kinetic

Poincaré

Stability of motion of autonomous systems in the sense:
kinematical, local, deterministic, non-asymptotic,geometric

Particular case: orbital stability of periodic motions

Structural

Stability of equilibrium or motion in the sense:
topological, local, deterministic, asymptotic

Particular cases: parametric stability; Mathieu stability

Elements of Stability Theory

Given $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$, such that,
if $\left\| \boldsymbol{\delta y}(0) \right\| < \delta(\varepsilon)$ then $\left\| \boldsymbol{\delta y}(t) \right\| < \varepsilon$ for $t > 0$

Liapunov's methods

First method (indirect)
Second method (direct)

Elements of Stability Theory

Perturbation equation for the analysis of the stability of equilibrium of the trivial solution $\boldsymbol{\delta y} = \mathbf{0}$

$$\boldsymbol{\delta \dot{y}} = \mathbf{f}(\boldsymbol{\delta y}) = \mathbf{A}\boldsymbol{\delta y} + \mathbf{N}(\boldsymbol{\delta y})$$

with $\quad \mathbf{A} = \left.\dfrac{\partial \mathbf{f}}{\partial \mathbf{y}}\right|_{\mathbf{0}} \quad$ and $\quad \mathbf{N}(\boldsymbol{\delta y}) = \mathbf{f}(\boldsymbol{\delta y}) - \mathbf{A}\boldsymbol{\delta y}$

Consider the associated linearized problem

$$\boldsymbol{\delta \dot{y}} = \mathbf{A}\boldsymbol{\delta y}$$

Solução geral

$$\boldsymbol{\delta y} = \boldsymbol{\delta y_0} e^{\lambda t}$$

Elements of Stability Theory

$$\left(\mathbf{A} - \lambda\mathbf{I}\right)\delta\mathbf{y}_0 = \mathbf{0}$$

For non-trivial solutions it is required that

$$\left|\mathbf{A} - \lambda\mathbf{I}\right| = 0$$

It is the classic eigenvalue problem for matrix $\mathbf{A}$

$$b_0\lambda^{2n} + b_1\lambda^{2n-1} + \ldots + b_{2n-1}\lambda + b_{2n} = 0$$

In the general case, there exists $2n$ complex roots for the characteristic equation

$$\lambda_k = R_k + iI_k, \quad R_k \text{ and } I_k \text{ are real numbers}$$

Elements of Stability Theory

Liapunov's first method

Theorem 1 (Liapunov): If $R_k < 0 \quad \forall k = 1, 2 \ldots 2n \Rightarrow \boldsymbol{\delta y = 0}$ is L-stable

Theorem 2 (Liapunov): If $\exists R_k > 0 \quad \Rightarrow \boldsymbol{\delta y = 0}$ is L-unstable

Definition of L-critical case: there exists at least one eigenvalue with zero real part $R_k = 0$, yet none of them with positive real part.

Theorem 3 (Leipholz): In the critical case, if the multiplicity p_k of all the eigenvalues with null real part $\left(R_k = 0 \right)$ is equal to the rank decrement d_k of the matrix $\mathbf{A} - \lambda_k \mathbf{I}$, then the solution $\boldsymbol{\delta y = 0}$ is L-stable for the linear system. If $p_k > d_k$, then the solution $\boldsymbol{\delta y = 0}$ is L-unstable for the linear system.

Elements of Stability Theory

Theorem 4 (Routh-Hurwitz): If all principal minors of the matrix
B (below) are positive, then the solution $\boldsymbol{\delta y = 0}$ is L-stable.
The reciprocal is also true.

$$\mathbf{B} = \begin{bmatrix} b_1 & b_0 & 0 & 0 & 0 & 0 & ... & 0 \\ b_3 & b_2 & b_1 & b_0 & 0 & 0 & ... & 0 \\ ... & ... & ... & ... & ... & ... & ... & 0 \\ ... & ... & ... & ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & 0 & 0 & b_{2n} & b_{2n-1} & b_{2n-2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & b_{2n} \end{bmatrix}$$

$$b_{r>2n} = 0 \quad \text{and} \quad b_{r<0} = 0$$

Elements of Stability Theory

Theorem 5 (Liapunov): Except for the L-critical case, the conclusions drawn from Theorems 1 and 2 for the linearized system $\delta\dot{\mathbf{y}} = \mathbf{A}\delta\mathbf{y}$ can be extended to the non-linear system $\delta\dot{\mathbf{y}} = \mathbf{A}\delta\mathbf{y} + \mathbf{N}(\delta\mathbf{y})$

Dynamical systems theory

Theorem 5' (Hartman-Grobman): If a singularity of the linear system $\delta\dot{\mathbf{y}} = \mathbf{A}\delta\mathbf{y}$ is hyperbolic , then the linearized system is topologically equivalent to the non-linear system $\delta\dot{\mathbf{y}} = \mathbf{A}\delta\mathbf{y} + \mathbf{N}(\delta\mathbf{y})$ in the singularity neighbourhood, that is, between the phase space flows of the non-linear and the linear systems there exists a diffeomorphism (transformation that is continuous with continuous derivative)

Elements of Stability Theory

$$\begin{cases} \delta \dot{y}_1 = \delta y_2 \\ \delta \dot{y}_2 = -\omega^2 \delta y_1 - 2\xi\omega \delta y_2 \end{cases}$$

$$\delta \dot{\mathbf{y}} = \mathbf{A}\delta\mathbf{y}$$

$$\mathbf{A} = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{y}}\right|_0 = \begin{bmatrix} 0 & 1 \\ -\omega^2 & -2\xi\omega \end{bmatrix}$$

$$2\xi\omega \rightarrow b \qquad \omega^2 \rightarrow c$$

$$b \in \square \qquad c \in \square$$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -c & -b \end{bmatrix}$$

characteristic equation $\quad \lambda^2 + b\lambda + c = 0 \quad \Longrightarrow \quad \lambda = \dfrac{-b \pm \sqrt{b^2 - 4c}}{2}$

Elements of Stability Theory

Let $\boldsymbol{\delta x} = \mathbf{B}\boldsymbol{\delta y}$ such that $\boldsymbol{\delta \dot{y}} = \mathbf{A}\boldsymbol{\delta y} \Rightarrow \boldsymbol{\delta \dot{x}} = \mathbf{C}\boldsymbol{\delta x}$

with $\mathbf{C}$ being a Jordan canonical form

Remark: $\mathbf{B}$ must be such that $\mathbf{CB} = \mathbf{BA} \Rightarrow \mathbf{C} = \mathbf{BAB}^{-1}$

Case (a): $\lambda_1 \in \mathbb{R}, \lambda_2 \in \mathbb{R}, \lambda_1 \neq \lambda_2 \rightarrow \mathbf{C} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$

$b^2 - 4c > 0$

Case (b): $\lambda_1 = \lambda_2 = \lambda \in \mathbb{R} \rightarrow \mathbf{C} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$ ou $\mathbf{C} = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$

$b^2 - 4c = 0$

Case (c): $\lambda_1 = \lambda = \alpha + i\beta \in \mathbb{C}, \lambda_2 = \bar{\lambda} = \alpha - i\beta \in \mathbb{C} \rightarrow \mathbf{C} = \begin{bmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{bmatrix}$
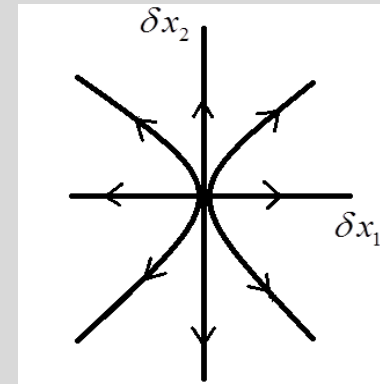
$b^2 - 4c < 0$

Elements of Stability Theory
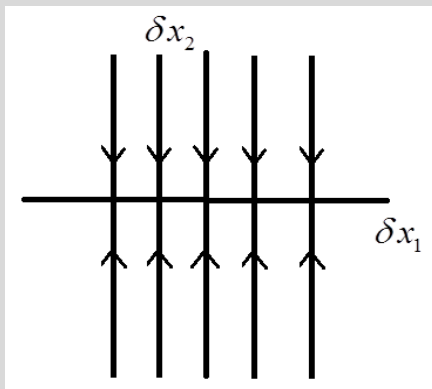
Case (a)

$$\delta x_i = \delta x_i^0 e^{\lambda_i t}$$

$$\frac{\partial(\delta x_2)}{\partial(\delta x_1)} = \left(\frac{\lambda_2}{\lambda_1}\right)\left(\frac{\delta x_2^0}{\delta x_1^0}\right) e^{(\lambda_2 - \lambda_1)t}$$
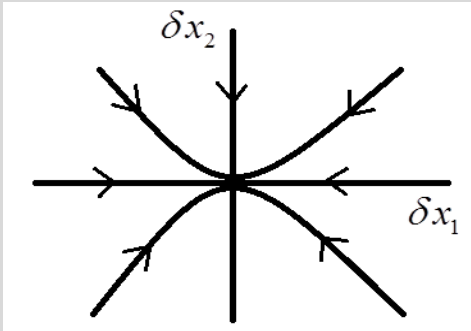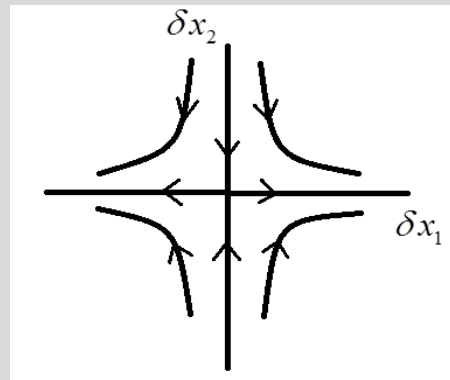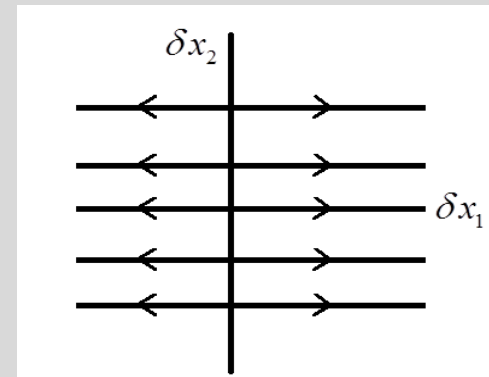

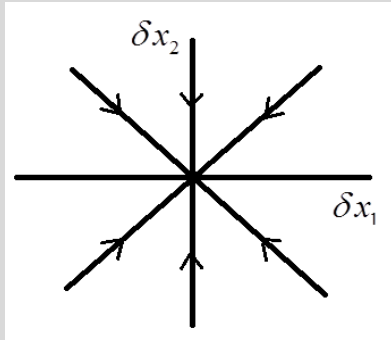$$\lambda_2 < \lambda_1 < 0$$

$$0 < \lambda_2 < \lambda_1$$

$$\lambda_2 < \lambda_1 = 0$$

$$\lambda_2 < 0 < \lambda_1$$

$$0 = \lambda_2 < \lambda_1$$

Elements of Stability Theory

Example: stability analysis for the solution $\boldsymbol{\delta y = 0}$ of a SDOF oscillator

Case (b1)
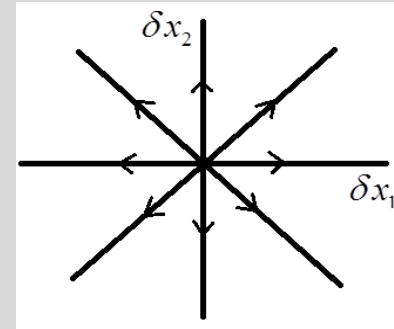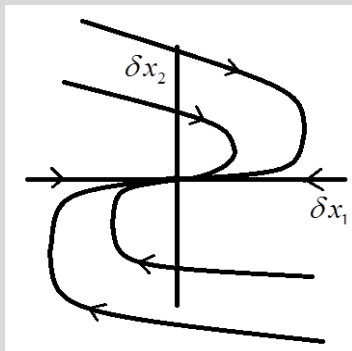
$$\delta x_i = \delta x_i^0 e^{\lambda t} \Rightarrow \frac{\partial(\delta x_2)}{\partial(\delta x_1)} = \left(\frac{\delta x_2^0}{\delta x_1^0}\right)$$
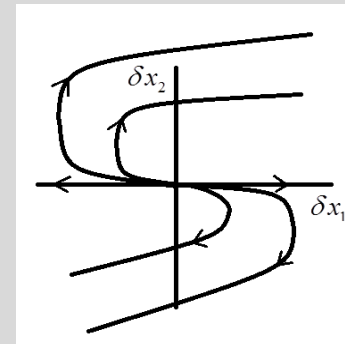
$$\lambda_2 = \lambda_1 < 0$$

$$\lambda_2 = \lambda_1 = 0$$

$$0 < \lambda_2 = \lambda_1$$
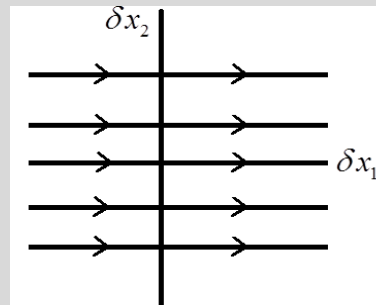
Case (b2)

$$\delta x_1 = \left(\delta x_1^0 + t\delta x_2^0\right)e^{\lambda t} \quad \delta x_2 = \delta x_2^0 e^{\lambda t}$$

$$\frac{\partial(\delta x_2)}{\partial(\delta x_1)} = \frac{\delta x_2^0}{\delta x_1^0 + \left(t+\frac{1}{\lambda}\right)\delta x_2^0} = \frac{1}{\frac{\delta x_1^0}{\delta x_2^0} + \left(t+\frac{1}{\lambda}\right)}$$

$$\lambda_2 = \lambda_1 < 0$$

$$0 < \lambda_2 = \lambda_1$$

Elements of Stability Theory

Example: stability analysis for the solution $\boldsymbol{\delta y = 0}$ of a SDOF oscillator
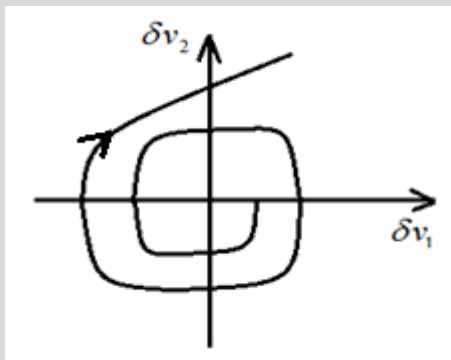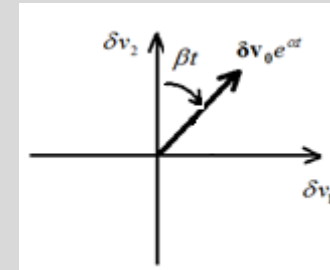
Case (c)

Change variables...

$$\boldsymbol{\delta\dot{x}} = \begin{bmatrix} \alpha+i\beta & 0 \\ 0 & \alpha-i\beta \end{bmatrix}\boldsymbol{\delta x}$$

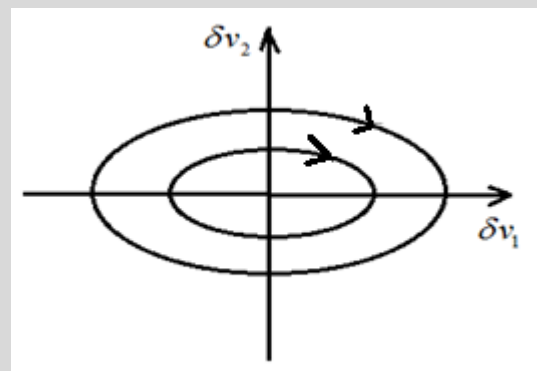$$\boldsymbol{\delta v} = \begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix}\boldsymbol{\delta x}$$

$$\boldsymbol{\delta\dot{v}} = \begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix}\boldsymbol{\delta\dot{x}} = \begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix}\begin{bmatrix} \alpha+i\beta & 0 \\ 0 & \alpha+i\beta \end{bmatrix}\begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix}^{-1}\boldsymbol{\delta v} = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}\boldsymbol{\delta v}$$

Define vector $\boldsymbol{\delta v} = \delta v_1 + i\delta v_2$ in Argand's plane ...

$$\boldsymbol{\delta\dot{v}} = (\alpha+i\beta)\boldsymbol{\delta v} \Rightarrow \boldsymbol{\delta v} = \boldsymbol{\delta v_0}e^{\alpha t}e^{i\beta t}$$

$\alpha > 0$

$\alpha = 0$
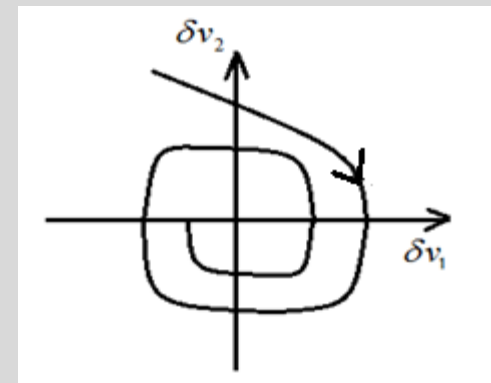
$\alpha < 0$

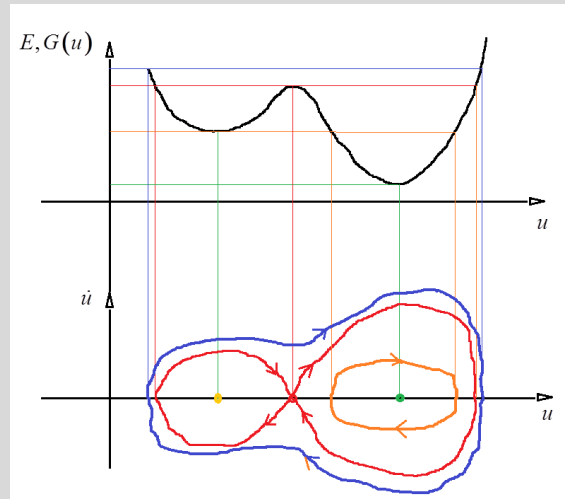# Elements of Stability Theory

Synthesis (Thompson)
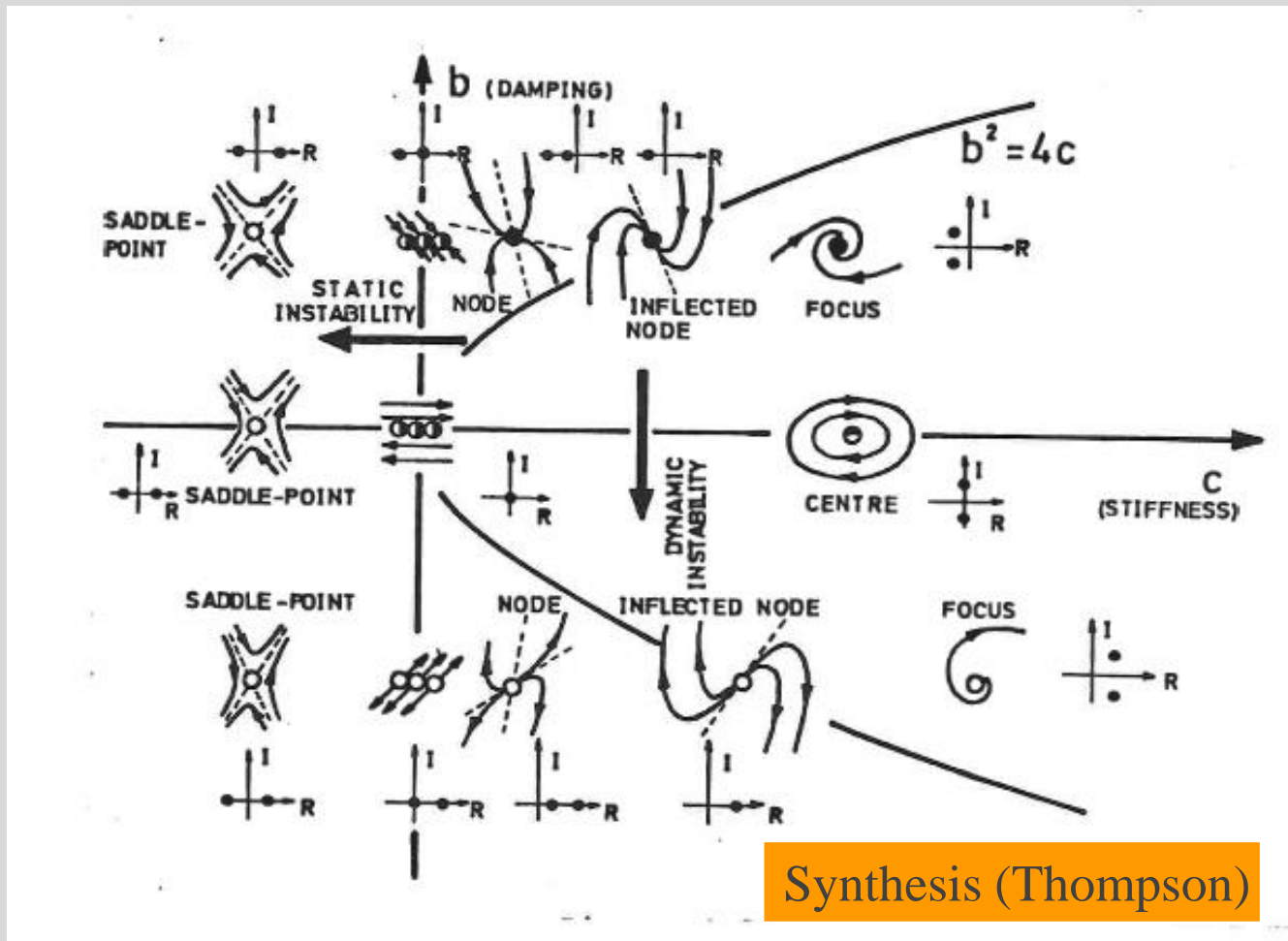
Elements of Stability Theory

$$\ddot{u} + g(u) = 0 \Rightarrow \ddot{u}\,du + g(u)\,du = 0 \Rightarrow \ddot{u}\,\dot{u}\,dt + g(u)\,du = 0$$

Integrating: $\underbrace{\dfrac{\dot{u}^2}{2}}_{\text{kinetic energy}} + \underbrace{\displaystyle\int_0^u g(\eta)\,d\eta}_{\text{potential energy}} = \underbrace{E}_{\text{mechanical energy}} = const.$

Define: $G(u) = \displaystyle\int_0^u g(\eta)\,d\eta \Rightarrow \dot{u} = \pm\sqrt{2[E - G(u)]} \Rightarrow T = 2\underbrace{\displaystyle\int_{u(0)}^{u(T/2)} \dfrac{du}{\sqrt{2[E - G(u)]}}}_{\text{period of motion}}$

saddle-node
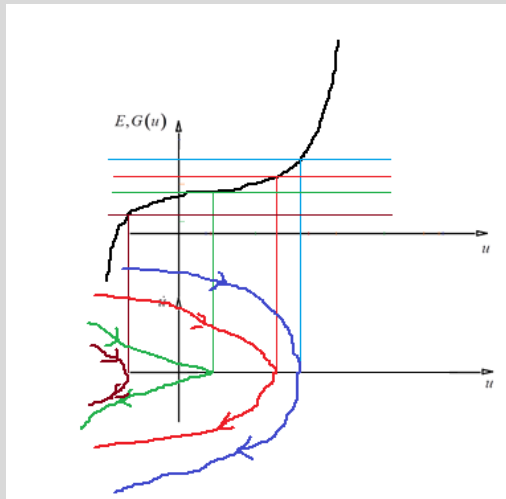
saddle & centres

Elements of Stability Theory

$$\boldsymbol{\delta\dot{y}} = \mathbf{f}(\boldsymbol{\delta y}) = \mathbf{A}\boldsymbol{\delta y} + \mathbf{N}(\boldsymbol{\delta y})$$

where $\quad \mathbf{A} = \dfrac{\partial \mathbf{f}}{\partial \mathbf{y}}\bigg|_{\mathbf{0}} \quad$ and $\quad \mathbf{N}(\boldsymbol{\delta y}) = \mathbf{f}(\boldsymbol{\delta y}) - \mathbf{A}\boldsymbol{\delta y}$

Theorem 6 (Liapunov): if there exists a function $\quad F(\boldsymbol{\delta y}) : E \to \square \quad$ such that:

$F \geq 0 \quad \forall \boldsymbol{\delta y}$

$F = 0 \Leftrightarrow \boldsymbol{\delta y} = \mathbf{0}$ then $\quad \boldsymbol{\delta y} = \mathbf{0}$ is L-stable

$$\dot{F} = \frac{\partial F}{\partial \delta y_r}\delta \dot{y}_r = \frac{\partial F}{\partial \delta y_r}f_r \leq 0$$

Elements of Stability Theory

Liapunov's second method

Theorem 7 (Liapunov): if there exists a function $F(\delta\mathbf{y}): E \to \square$ such that:

$$F \geq 0 \quad \forall \delta\mathbf{y}$$

$$F = 0 \Leftrightarrow \delta\mathbf{y} = \mathbf{0}$$

then $\delta\mathbf{y} = \mathbf{0}$ is asymptotically stable in Liapunov's sense

$$\dot{F} = \frac{\partial F}{\partial \delta y_r}\delta\dot{y}_r = \frac{\partial F}{\partial \delta y_r}f_r < 0$$

Theorem 8 (Chetayev): if there exists a function $F(\delta\mathbf{y}): E \to \square$ such that:
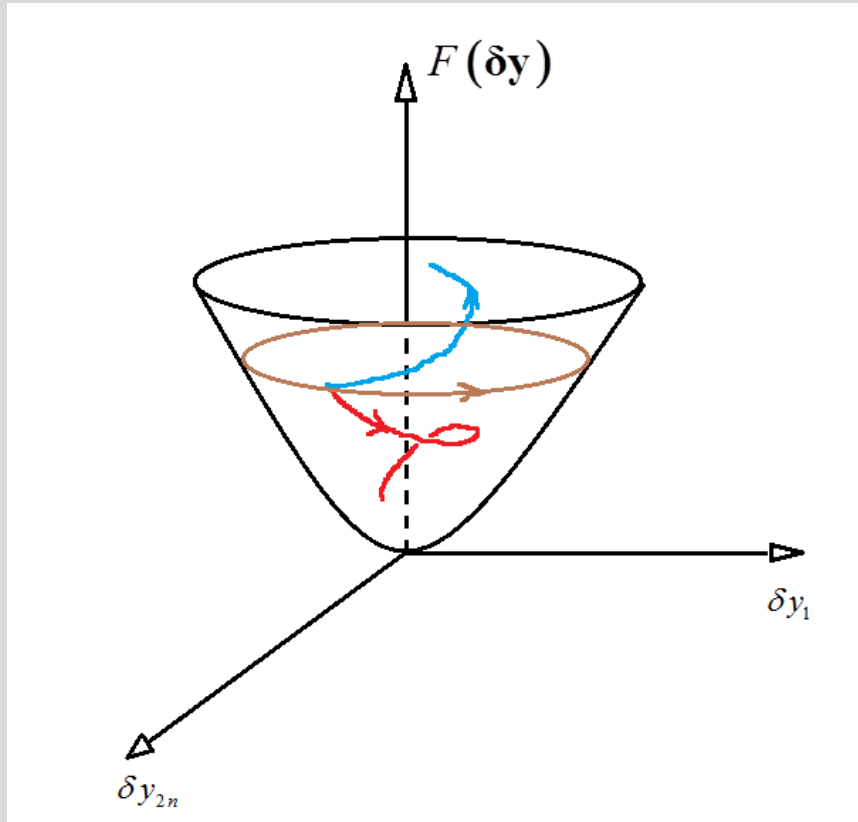
$$F \geq 0 \quad \forall \delta\mathbf{y}$$

$$F = 0 \Leftrightarrow \delta\mathbf{y} = \mathbf{0}$$

then $\delta\mathbf{y} = \mathbf{0}$ is L-unstable

$$\dot{F} = \frac{\partial F}{\partial \delta y_r}\delta\dot{y}_r = \frac{\partial F}{\partial \delta y_r}f_r > 0$$

Elements of Stability Theory

$F(\boldsymbol{\delta y})$ is called Liapunov's function

Elements of Stability Theory

Attractor

Subset of the phase space to which a solution of the dynamical system tends when $t \rightarrow \infty$ for initial conditions in a non-localized subset of the phase space (basin of attraction)

- Fixed point (stable equilibrium point): asymptotically stable singularity

- Limit cycle (periodic attractor): asymptotically stable orbit in the phase space with one dominating frequency or more than one commensurate dominating frequencies

- Limit torus: asymptotically stable manifold in the phase space, with more than one non-commensurate dominating frequency

- Strange attractor (chaos): coexistence of some of the previous attractors with non-compact (fractal) basins of attraction
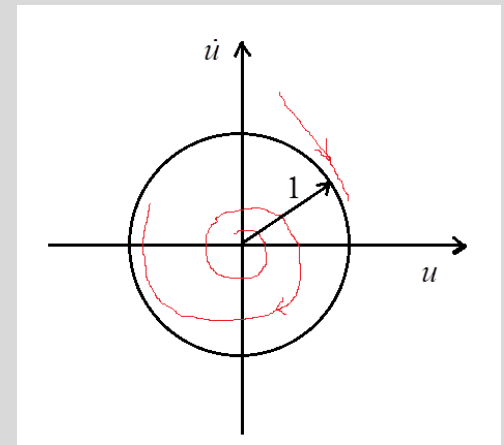
Elements of Stability Theory

Example: van der Pol equation

$$\ddot{u} - \dot{u} + u + \left(u^2 + \dot{u}^2 \right) \dot{u} = 0$$

Trivial solution $u(t) = 0$ is unstable

Periodic attractor $u(t) = \sin t$ is stable

Elements of Stability Theory

Hirsch & Smale: Differential Equations, Dynamical Systems
and Linear Algebra
Guckenheimer & Holmes: Nonlinear Oscillations, Dynamical Systems
And Bifurcation of Vector Fields

Elements of Stability Theory

- First Poincaré-Bendixson's Theorem:

If a phase trajectory C remains within a finite region without approaching a singularity, then C is a limit cycle or it tends to one.

- Second Poincaré-Bendixson's Theorem:

Given a region D of the phase space, bounded by two curves C' and C'', without a singularity in D, C' e C'', if all phase trajectories enter (exit) in D through the boundaries C' e C'', then there exists at least a stable (unstable) limit cycle in D.
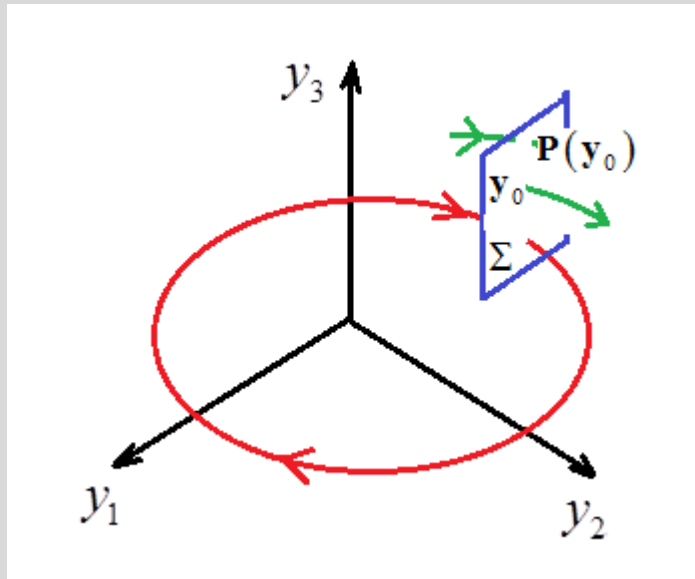
Elements of Stability Theory

- Let $\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y})$ be a flow of an autonomous system in $\square^{2n}$ and $\Sigma : \mathbf{f}(\mathbf{y}) \cdot \mathbf{N} \neq 0$ a section with normal $\mathbf{N}$. Consider the mapping $\mathbf{y}_0 \to \mathbf{P}(\mathbf{y}_0)$ defined by the intersection of the flow $\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y})$ with Σ. $\mathbf{P}(\mathbf{y}_0)$ is termed a "Poincaré's section" of the flow $\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y})$ through $\mathbf{y}_0$

- If the system is non-autonomous, defined by the flow $\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y}, t)$, an associated autonomous one $\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y})$ defined in $\square^{2n+1}$ can be proposed with the addition of $\dot{y}_{2n+1} = 1$, so that the Poincaré's sections can be defined orthogonally to the axis $y_{2n+1} = t$ at $t = t_0 + iT, \quad i = 1, 2, ...$

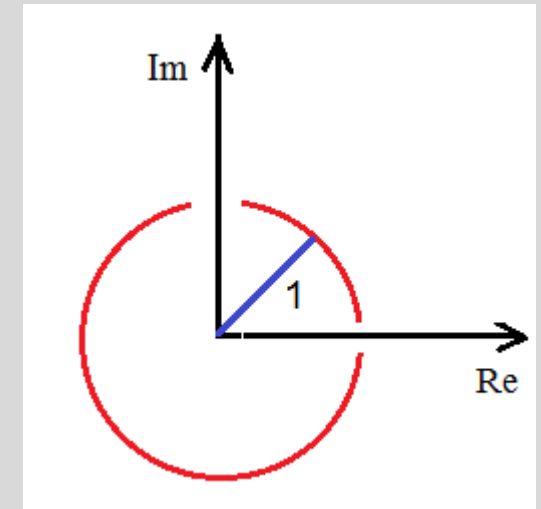# Elements of Stability Theory

Analyse the complex eigenvalues $\lambda_j = \mathrm{Re}_j + i\,\mathrm{Im}_j$ of linearized mapping $\mathbf{DP}(\mathbf{y}_0)$ to test stability.

Stability for $|\lambda_j| < 1$

Instability for $|\lambda_j| > 1$

Elements of Stability Theory

$$\ddot{u} + \left(-1 + u^2 + \dot{u}^2\right)\dot{u} + u = 0$$

$$\left.\begin{array}{l} y_1 = u \\ y_2 = \dot{u} \end{array}\right\} \Rightarrow \dot{\mathbf{y}} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}\mathbf{y} + \left\{\begin{array}{c} 0 \\ -\left(y_1^2 + y_2^2\right)y_2 \end{array}\right\}$$
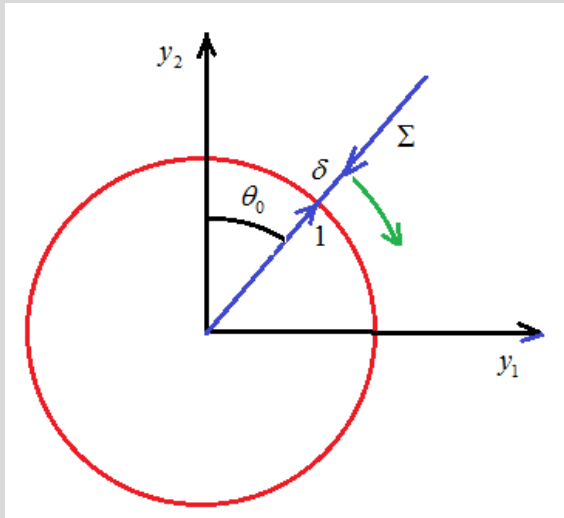
In polar co-ordinates

$$\left.\begin{array}{l} y_1 = r\sin\theta \\ y_2 = r\cos\theta \end{array}\right\} \Rightarrow r = 0 \text{ corresponds to an unstable focus}$$

$$\text{for } r \neq 0 \Rightarrow \left\{\begin{array}{l} \dot{r} = -r\left(r^2 - 1\right)\cos^2\theta \\ \dot{\theta} = 1 + \left(r^2 - 1\right)\sin\theta\cos\theta \end{array}\right.$$

It is readily seen that $r = 1$ and $\theta = t$ are a limit cycle

Elements of Stability Theory

Example of Poincaré's section (map)

$$r_0 = 1 + \varepsilon_0 \rightarrow r_j = 1 + \varepsilon_j \ \text{ for } \ \theta = \theta_0 + 2\pi j \quad j = 1, 2, \ldots$$

Mapping: $\quad \dot{r}_j = \dot{\varepsilon}_j = -\left(1 + \varepsilon_j\right)\left[\left(1 + \varepsilon_j\right)^2 - 1\right]\cos^2\theta_0$

$$\dot{\varepsilon}_j = -\left(2\varepsilon_j + 3\varepsilon_j^2 + \varepsilon_j^3\right)\cos^2\theta_0$$

Linearizing: $\ \dot{\varepsilon}_j = -\left(2\cos^2\theta_0\right)\varepsilon_j \Rightarrow \varepsilon_j = \varepsilon_0 e^{-4\pi j \cos^2\theta_0}$

Mapping in $\square^1$: $\ r_j \rightarrow r_{j+1} = P\left(r_j\right) = 1 + \left(r_j - 1\right)e^{-4\pi \cos^2\theta_0}$

$$\mathbf{DP} = \frac{dP\left(r_j\right)}{dr_j} = e^{-4\pi \cos^2\theta_0}$$

asymptotic stability for $\theta_0 \neq \dfrac{\pi}{2}$ or $\dfrac{3\pi}{2}$, since $\left|\lambda\right| < 1$

stability for $\theta_0 = \dfrac{\pi}{2}$ or $\dfrac{3\pi}{2}$, since $\dot{\varepsilon}_j = 0 \Rightarrow \varepsilon_j = \varepsilon_0$

Poincaré's section: $\ \theta = \theta_0$
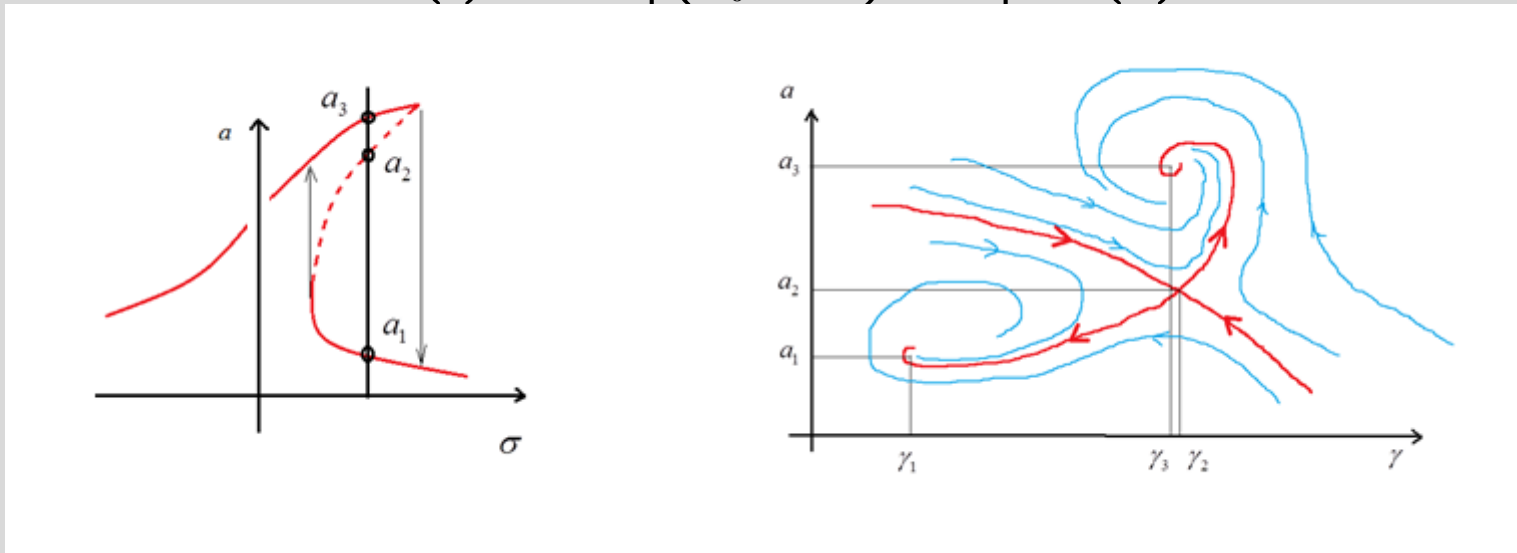
Elements of Stability Theory

Periodic attractor in non-autonomous dynamical system $\dot{\mathbf{y}} = \mathbf{g}(\mathbf{y}, t)$

Example: forced Duffing's equation

$$\ddot{u} + 2\varepsilon\mu\dot{u} + \omega_0^2 u + \varepsilon\alpha u^3 = \varepsilon k \cos(\omega_0 + \varepsilon\sigma)t \quad \text{with} \ \ 0 < \varepsilon << 1$$

There exist periodic attractors

$$u(t) = a\cos\left[(\omega_0 + \varepsilon\sigma)t + \gamma\right] + O(\varepsilon)$$

Estudo recai em estabilidade de singularidades...