Journal of Sailboat Technology, Article 2014-01.

BIFILAR SUSPENSION MEASUREMENT OF BOAT INERTIA PARAMETERS

Peter F. Hinrichsen

Manuscript received September 12, 2013; revision received November 29, 2013; accepted December 5, 2013.

Abstract: Measurements of the inertia parameters (Gregory, 2006) of a keelboat hull using a bifilar suspension (Newman and Searle, 1951) are described. Bifilar yaw moment measurement normally entails accurate measurement of the length *l* and spacing 2*d* of the suspension, and of T_y the period of pure yaw oscillation (Miller, 1930). The primary difficulty with a bifilar suspension is avoiding unwanted modes of oscillation, specifically sway when measuring yaw. However, for an athwartships suspension, the sway motion is that of a simple pendulum of period T_s and observation of the combined motion allows the yaw gyradius $k_y \equiv k_{zz}$ to be determined as $k_y = (T_y/T_s)d$. Thus only the ratio of the periods and the suspension spacing need to be measured. Measurements of the normal mode periods of the double pendulum motion (Rafat, Wheatland et al., 2009) when the hull is displaced in surge allow for the pitch gyradius $k_p \equiv k_{yy}$ and the height l_2 of the center of mass to be determined. The latter can be confirmed by measuring the incline angle of the hull when a weight is suspended from the stern and/or the bow. Repeating yaw measurements with the hull tilted, and then with the bifilar suspension fore and aft to measure the roll gyradius, $k_r \equiv k_{xx}$, allows for the angle ψ of the inertia ellipsoid (Wells 1967) principal x axis to the hull x axis to be calculated. Although the present keelboat measurements were made using ultrasonics (Daedalon, 1991) and photogates (Pasco, 2000), such measurements can now be more easily made using MEMs gyros, such as that in the iPhone (xSensor, 2010). This is illustrated by the measurements on a model keelboat.

Keywords: Experimental methods, hull, keel boats, model testing, motions.

NOMENCLATURE

a	The height of the axis above the center of mass
$\vec{a}_{cm}$	Linear acceleration of the center of mass
a, b, and c	Semi axes of an ellipsoid
A_1, A_2	Amplitudes of first and second normal modes of double pendulum oscillations
A, A_s, A_y	Amplitudes of sway and yaw oscillation.
b	Yaw period linear time variation coefficient
b_b, b_s	Vertical distance of incline mass below suspension point P at bow and at stern
$2d$	Spacing between the suspension lines
$\vec{F}$	Net force on an object
g, g'	Acceleration due to gravity and g corrected for air buoyancy
h_i	Perpendicular distance of the mass element m_i from the rotation axis
I	Moment of Inertia
I_{hp}, I_{hy},	Hull shell pitch and yaw moments of inertia
I_{kp}, I_{ky},	Keel pitch and yaw moments of inertia
I_{px}, I_{py}, I_{pz}	Moments of inertia of the hull plus keel about the principal x, y and z axes
I_{xx}, I_{yy}, I_{zz}	Moments of inertia of the hull plus keel about the hull x, y and z axes
I_{xy}, I_{xz}, I_{yz}	Products of inertia ($I_{xy} = I_{yx}$ etc.)
I_ϕ	Moment of inertia about an axis in the xz plane at an angle ϕ to the x axis

ISSN 1548-6559

k, k_0	Gyradius, i.e. radius of gyration, gyradius for zero added masses
k_{kp}, k_{hp}, k_p	Pitch gyradii of the keel, the hull shell and the hull plus keel
k_{px}, k_{py}, k_{pz}	Gyradii of the hull plus keel about the principal x, y and z axes
K_{xx}, k_{yy}, k_{zz}	Gyradii of the hull plus keel about the hull x, y and z axes
k_{xs}, k_{ys}, k_{zs}	The roll, pitch and yaw gyradii of the frame about the hull center of mass
$k_p \equiv k_{yy}$	Pitch gyradius of the hull.
$k_r \equiv k_{xx}$	Roll gyradius of the hull.
$k_y \equiv k_{zz}$, k_{hy}	Yaw gyradii of the hull and hull shell
$k_{\phi A}$, $k_{\phi B}$	Yaw gyradii about axes at pitch angles ϕ_A, ϕ_B, to the hull z axis.
$k_{\varphi A}$, $k_{\varphi B}$	Yaw gyradii about axes at roll angles φ_A, φ_B to the hull z axis
K_{P2}	The ratio $K_{P2} = k_p / l_2$ of the pitch gyradius to the center of mass depth
l_0	Effective length of compound pendulum suspension $l_0 = \sqrt{l_1^2 - d^2}$
l, l_1	Length of the suspension lines.
l_2	Vertical distance of the CM below the hull suspension points
L_{12}	The ratio $L_{12} = l_1 / l_2$ of the suspension length to the center of mass depth
L_0	Hull LOA
L_s	Horizontal distance of the incline mass m_d from the suspension point P
M	Mass of the hull plus suspension beam etc.
M'	Mass of the air displaced by the hull
m_d	Inclining mass
m_i	Mass element.
m	Mass added fore and aft at λ_f and λ_a
n	Number of oscillations per beat
R_1 , R_2, R_y	Period ratios $R_1 = T_1/T_s$ $R_2 = T_2/T_s$ and $R_y = T_y/T_s$
T	Measured period of oscillation
T_0	Undamped period of oscillation
T_{y0}	Averaged yaw period
T_1, T_2	Periods of first and second normal modes of double pendulum oscillation
T_a, T_b=T_m, T_e	Periods of the average, beat and envelope functions for combined yaw sway
T_s, T_y	Periods of sway and yaw oscillation
x_k, z_k	Keel center of mass horizontal and vertical positions relative to the keel datum
$\overline{x}$, $\overline{z}$	Hull plus keel center of mass positions relative to the hull datum point
y, y_0	Yaw sway displacement and initial displacement
$\vec{\alpha}$	Angular acceleration
ϕ, ϕ_A, ϕ_B	Angles, in the xz plane, of the rotation axis to the hull z axis.
ϕ_p	Hull pitch angle
ϕ_b, ϕ_s	Hull incline angle with mass m_d at the bow, and at the stern
ϕ_1	Angle of the suspension line to the vertical.
ϕ_2	Angle to the vertical of the line from the hull suspension point P to the hull CM
φ	Roll angle
φ_1, φ_2	Phases of normal modes one and two of the double pendulum oscillation
φ_m, φ_s, φ_y	Phases of the period modulation, the sway and the yaw oscillation.
β	Exponent of the damping power law dependence on amplitude.
λ	Horizontal distance of the center of mass from the hull datum point (HDP)
λ_f, λ_a	Horizontal distances fore and aft of the masses m added fore and aft
$\theta_0(t)$	Time varying angular amplitude of yaw oscillation
ρ, ρ_a	Densities of the hull and the surrounding air
τ_m, τ_s, τ_y	Damping constant of the period modulation, the sway, and the yaw motions

ISSN 1548-6559

$\vec{\Gamma}$, Γ_D	Torque, Drag torque
ω_1, ω_2	Angular frequencies of first and second normal modes of double pendulum oscillation
θ_y	Yaw deflection angle
ψ	Angle between the hull x axis and the principal x axis

INTRODUCTION

The subject of this paper is the measurement of the inertial properties of model or full size sailboats. A clear understanding of Moments and Products of Inertia is essential to any study of the motions of a hull (Wells, 1967; Gregory, 2006). Modern tank testing is now done in both flat water and waves, which induce pitch and roll (ie., rotational motions) and thus the response of the model depends on its moments of inertia about these axes, although for many tests the model is only free in pitch. The steering characteristics of a hull will also be influenced by the moment of inertia about the yaw axis. Thus the inertia parameters have to be calculated or measured prior to testing in waves.

The dinghy sailing community first became interested in moment of inertia, or gyradius measurement, in the 1960s when many classes were switching from wood to glass construction (Lamboley, 1971; Wells, 1971). One of the concerns was that glass boats could be built with light ends and would gain an advantage. This could lead to the construction of excessively expensive and fragile boats to the detriment of the class; therefore, many classes looked for simple and effective ways to control this tendency. The best known method, although not the only one, is the test developed in 1971 by Gilbert Lamboley (Lamboley, 1971) for the Finn Class.

All sailors who have fought to keep up their speed in light airs when a motor boat wake causes the boat to pitch know that pitching can have a devastating effect on the boat's speed, and they will make every effort to minimize pitching. One of the parameters which affect the pitching response of a boat is its moment of inertia about the pitch axis, and the general wisdom is that it should be a minimum. The theory of the motion of boats in a seaway (Van Duyne, 1972; Dorn, 1974; Kiss, 1987; Ales and McGettigan, 1981) is beyond the scope of this paper; however, detailed knowledge of the inertia parameters of the boat is essential for modeling the response.

When sailing the crew is not an insignificant part of the total mass, and together with the mast and sails cannot be considered as rigid bodies. The mass distribution of modern canting keel boats is variable and not symmetrical about the center plane as assumed here. Furthermore, when immersed in water the added hydrodynamic mass effects (Newman, 1977) will significantly increase the effective inertia properties, so the computation of the hull motions is complex.

Most measurements of sailboat and tank test model moments of inertia are made using compound pendulum methods (Roy, 1984) which only measure the pitch gyradius. The purpose of the present paper is to show that the use of a bifilar suspension has the advantage of being more precise (Card, 2000), as well as allowing for all of the elements of the inertia tensor to be measured with one simple setup. Thus the methods for measuring the pitch, the yaw and the roll gyradii as well as the orientation of the principal axes (Gregory, 2006) are described as well as proof of principle measurements on a model hull. The pendulum methods are included for comparison with the bifilar suspension. The sensitivity and precision as well as some of the corrections and limitations are described.

INERTIAL PROPERTIES OF RIGID BODIES

For linear motion, the acceleration of a rigid body produced by a force $\vec{F}$ is given in both magnitude and direction by Newton's law $\vec{F} = M\,\vec{a}_{cm}$, where $\vec{a}_{cm}$ is the acceleration of the center of mass and *M* is the total mass. This does not depend on how the mass is distributed within the body or on the point of application of the force. However, if the force does not act through the center of mass, it also produces an angular acceleration $\vec{\alpha}$ given by $\vec{\Gamma} = I\vec{\alpha}$ where $\vec{\Gamma}$ is the torque about the center of mass. The moment of inertia (Wells, 1967) *I* of a body of mass $M = \Sigma m_i$ about a given axis is

$$I = \sum_i m_i h_i^2 = \int_v \rho\left(x, y, z\right)\, h^2\, dv = Mk^2 \qquad (1)$$

where the sum is over all masses m_i, and h_i is the perpendicular distance of m_i from the axis. The gyradius *k* is the root mean square radius about this axis. Thus the moment of inertia *I* and the gyradius *k* depend on both the location and the orientation of the axis about which the body rotates. Fortunately, the moment of inertia about axes of a given direction is a minimum for the axis through the center of mass, and if known, *I* about any parallel axis can be easily calculated (Gregory, 2006), so we will specify moments of inertia about axes through the center of mass. Unfortunately, unlike the mass which is a scalar, i.e. independent of orientation, the moment of inertia depends on the orientation of the rigid body and is a 3 x 3 tensor (Baierlein, 1983; Gregory, 2006).

It can be shown that the inertia properties of a rigid body of arbitrary shape (so even a hull with a canted keel or a proa) can be represented by an equivalent ellipsoid with semi axes a, b, and c of appropriate length and orientation. Thus there is always a set of body coordinates, called the principal axes for which the diagonal elements of the inertia tensor are zero (Gregory, 2006). Although the methods described could be extended to the general case, the present paper is limited to considering hulls with reflection symmetry about their center plane, and for such hulls two principal axes – x and z – lie in the center plane with the third, y axis, perpendicular to that plane. Thus the orientation of the principal axes are determined by the angle ψ that the principal x axis makes with the hull x axis. It is therefore easy to visualize what one might call a "mass equivalent ellipsoid," which would have semi axes $a^2 = 5/2(k_{py}^2 + k_{pz}^2 - k_{px}^2)$ etc. Then for the present keelboat hull with $k_{px} = 0.625$m, $k_{py} = 1.054$m and $k_{pz} = 1.156$m the mass equivalent ellipsoid would have a = 2.27m, b = 1.24m and c = 0.64m as shown in Figure 1a. An ellipsoid of this shape and the same mass as the hull would have identical inertial properties to those of the hull.

However, the moment of inertia about any axis through the center of mass can also be visualized by the inertia ellipsoid (Wells, 1967), which has its axes along the principal axes and has x_p, y_p, z_p intercepts of k_{px}^{-1}, k_{py}^{-1} and k_{pz}^{-1}, as shown in Figure 1b. The gyradius about any axis through the center of mass is then the inverse of the radius of the ellipsoid in that direction.

ISSN 1548-6559

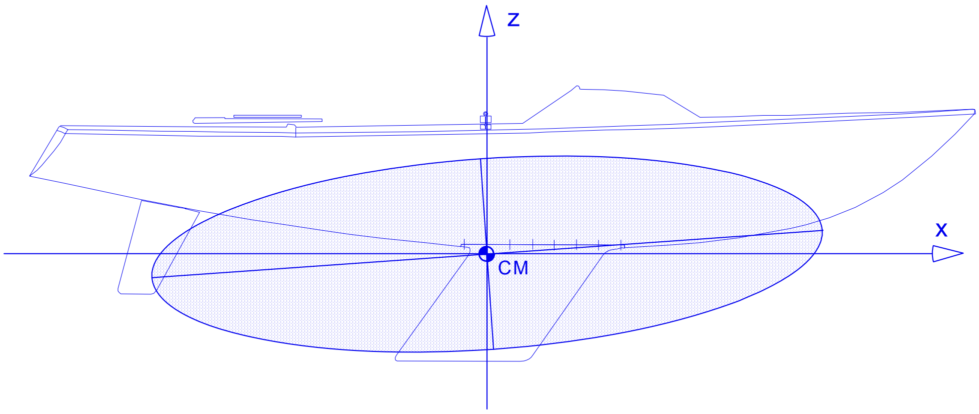

Figure 1a. The "mass equivalent ellipsoid" of a keelboat hull.

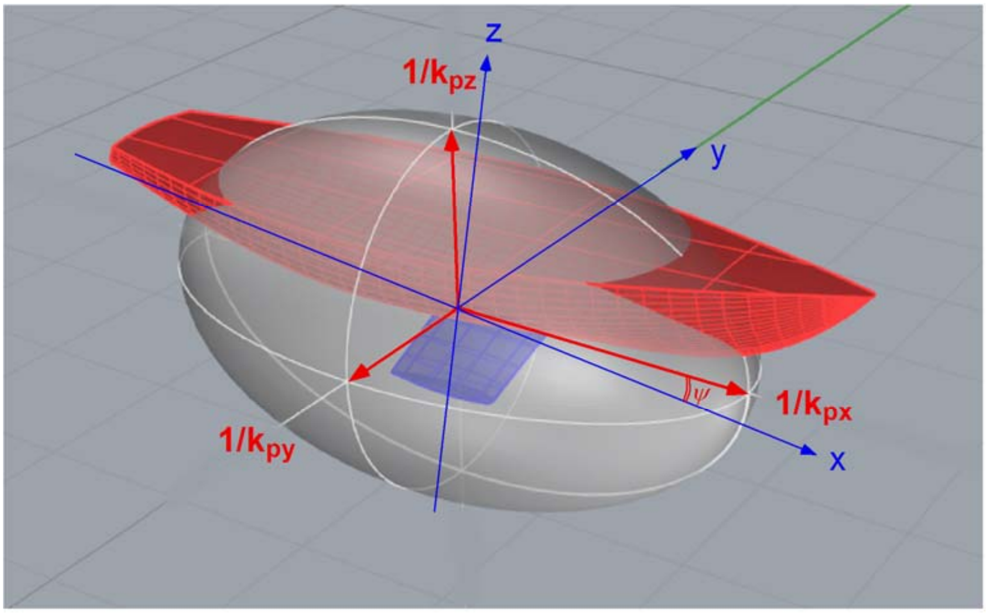

Figure 1b. The inertia ellipsoid of a keelboat hull.

For rotations about an arbitrary axis through the center of mass the inertia tensor

$$I = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{xy} & I_{yy} & I_{yz} \\ I_{xz} & I_{yz} & I_{zz} \end{pmatrix} \qquad (2)$$

which in general has off diagonal elements, called products of inertia, but is symmetrical, i.e. $I_{xx} = \sum_i m_i \left(y_i^2 + z_i^2 \right)$ and $I_{xy} = I_{yx} = \sum_i m_i x_i y_i$ etc. Thus to completely specify the inertia properties of a rigid body one has to know the mass *M* and the six independent elements of the inertia tensor. However, it is always possible to choose a set of coordinates, called the principal axes, such that the three products of inertia vanish (Gregory, 2006). One then has to determine the directions of the three principal axes.

Fortunately for us one can assume that most rigid sailboat hulls are symmetrical about their *xz* center plane. The principal pitch axis is then perpendicular to, and the yaw and roll

ISSN 1548-6559

principal axes are in, the center plane. So one only has to determine the angle ψ that the principal yaw axis makes with the hull z axis. This can be determined from measurements of the gyradii about three axes, at known angles in the center plane, i.e. with the hull level, pitched forward and pitched aft, and/or the roll axis.

In terms of the principal moments of inertia, the measured moments I_{xx}, I_{zz} about the hull x and z axes and I_ϕ about an axis in the center plane at an angle ϕ to the z axis are

$$\begin{aligned} I_{xx} &= I_{px} Cos^2\psi + I_{pz} Sin^2\psi \\ I_{zz} &= I_{px} Sin^2\psi + I_{pz} Cos^2\psi \\ I_\phi &= I_{px} Sin^2\left(\phi+\psi\right) + I_{pz} Cos^2\left(\phi+\psi\right) \end{aligned} \qquad (3)$$

These equations can be solved (Green, 1927; Miller, 1930) for ψ and the principal gyradii k_{px}, and k_{pz} to give

$$\begin{aligned} Tan2\psi &= \frac{\left(k_\phi^2 - k_{xx}^2 Sin^2\phi - k_{zz}^2 Cos^2\phi\right)}{\left(k_{xx}^2 - k_{zz}^2\right) Sin\phi Cos\phi} \\ k_{px} &= \sqrt{\frac{k_{xx}^2 - k_{zz}^2 Tan^2\psi}{1 - Tan^2\psi}}, \quad k_{pz} = \sqrt{\frac{k_{zz}^2 - k_{xx}^2 Tan^2\psi}{1 - Tan^2\psi}} \end{aligned} \qquad (4)$$

MEASUREMENT METHODS

Measurement of the inertial properties of a hull requires that a torque be applied to it and that the resulting angular acceleration be measured. They cannot be measured by any static method. The most convenient way of conducting such measurements is to apply an oscillatory torque of known, preferably linear, dependence on the angular displacement and observe the resulting angular oscillation when the hull is displaced. The angular acceleration, and hence the moment of inertia, can then be derived from the period of oscillation. Torsion bars (Turner, 1950), or springs (Wells, 1971) at known moment arms, are often used to supply the varying torque but these methods, such as the pitch moment of Inertia test devised by Ted Wells (Wells, 1971) for the Snipe class, and the Yaw test used for a time by the Star class require careful calibration of the spring system.

The gravitational force on a pendulum provides a torque about a horizontal axis (proportional to the Sine of the angular displacement and hence approximately linear for small angles) and is the basis of the Lamboley (Lamboley, 1971), Oskar Weber's Dragon Class incline-swing (Watts, 1986) and the tilt-swing test used for the Ynglings (Hinrichsen, 2004) at the 2004 Olympics. The double pendulum (Wells, 1967) and the bifilar suspension also rely on gravity to provide a known torque (Newman and Searle, 1951), but in the latter case, about a vertical axis.

The Lamboley test (Lamboley, 1971), which is used by the Finn and Europe classes and has been tried by a number of other classes, determines the pitch gyradius and requires the measurement of the periods of oscillation of the hull about two horizontal athwartships axes in order to determine *a,* the height of the axis above the center of mass, and the pitch gyradius k_P. The result depends on the difference of the squares of the two periods, thus its accuracy is limited. Furthermore, the measurement of two periods is time consuming and requires the hull to be raised and lowered twice. The incline-swing (Watts, 1986) and tilt-swing tests require the measurement of an inclination angle and one period of oscillation, i.e. still two separate measurements. The IMS rule at one time considered introducing Watt Webb's in the water pitching test (McCurdy, 1990).

High precision Laboratory instruments measure the period of oscillation of objects mounted on platforms supported on air bearings and attached to inverted torsional suspensions, but they are very expensive and not transportable. Torsional suspensions have wide applications and have been used for measurements on full size aircraft (Turner, 1950), but again require calibration.

The bifilar suspension (Newman and Searle, 1951), initially described by Isaac Newton, is similar in principle but has the advantages of simplicity and not having to calibrate the torsional rigidity. They have been used since the 1920s to measure the yaw gyradius of full size aircraft (Green, 1927), and are currently used for measurements on Unmanned Air Vehicles (Jardin and Mueller, 2009) and tank test ship models (Kiss, 1987; Card, 2000). A bifilar suspension was successfully used for exploratory measurements of the yaw gyradii k_y of Flying Dutchman hulls at the 1984 Olympics (Hinrichsen, 1986) and 1990 FD World Championships, and has since been used on Sailboards, 470s, a laser (Waine, 1988), an International 14 and, as reported here, on a Yngling hull.

The bifilar method has the two major advantages, one that the yaw gyradius is directly proportional to the period of oscillation and two that the center of gravity is in the plane of the suspension and can be assumed to be in the symmetry plane of the hull, so only one period of oscillation has to be measured. Thus the precision is typically three times better than for pitch measurements. However, in order to achieve this, some skill is required to avoid extraneous oscillations. Rather than avoid the sway, the present paper proposes a simple method of using the combined yaw and sway oscillations to improve the precision of the measurements. As higher accuracy and precision are demanded, the theoretical description of the measurements must become more sophisticated, and so a number of corrections will be described.

YAW OSCILLATION OF A BIFILAR SUSPENSION

For measurement of the yaw gyradius k_y the hull is suspended by two wires of length l, a horizontal athwartships distance $2d$ apart, as shown in Figure 2. The hull should be level, with the centerline halfway between the two wires. For the measurement of the period of yaw oscillation, T_y, ideally the hull is rotated level without displacing its center and then released. The subsequent motion is an oscillatory rotation about the vertical yaw axis. For small angular oscillations $\theta << k_y l / d^2$ the period T_y is (Newman and Searle, 1951)

$$T_y = \frac{2\pi k_y}{d}\sqrt{\frac{l}{g}} \qquad (5)$$

where "g" is the local acceleration due to gravity. Thus measurements of the suspension length l, the spacing $2d$, and the period T_y give the yaw gyradius k_y as:

$$k_y = \left\{ \frac{d}{2\pi} \sqrt{\frac{g}{l}} \right\} . T_y \qquad (6)$$

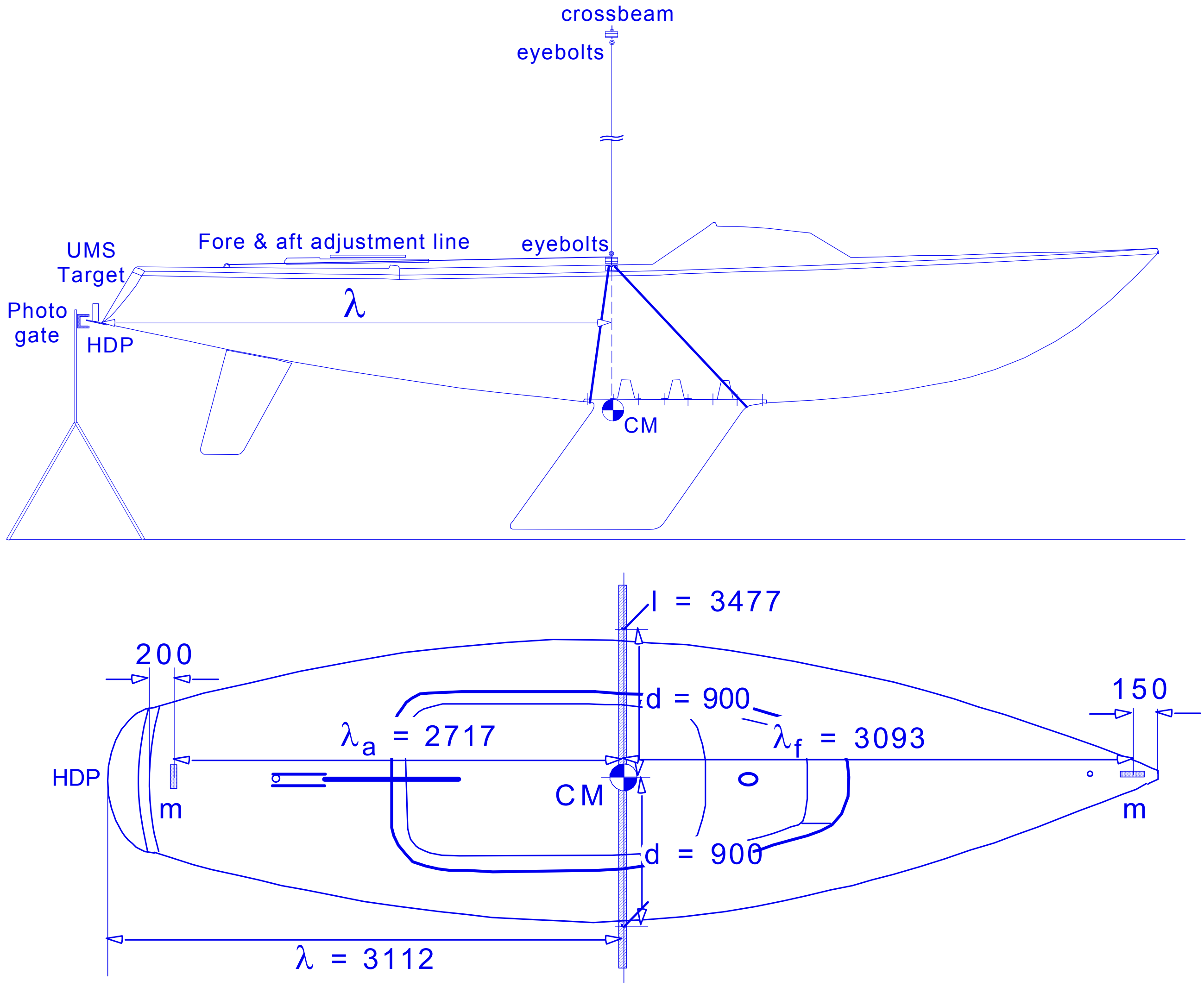

Figure 2. For the yaw test the hull was suspended symmetrically and level by two lines of length *l* and athwartships spacing 2*d*. The positions λ_a and λ_f of the masses *m* added for sensitivity tests are also shown.

The main disadvantage of this method is that it is difficult to completely eliminate the other oscillations of the hull such as sway, and the coupled surge-pitch double pendulum motion. Careful release of the hull is required in order to minimize these motions and a laser beam or plumb bob over the center of the suspension aids in avoiding linear displacement of the hull. If measurements are made at the stern using a vertical photo gate at the equilibrium position, sway, i.e. a side to side oscillation of the whole boat, affects the timing, as the horizontal motion of the stern (due to the combined yaw plus sway) does not generally have exactly the same time interval between alternate swings through the equilibrium position, as does the pure yaw oscillation. Although a few percent of sway is not noticeable, its effect on the precision of the yaw period measurement is significant.

ISSN 1548-6559

The athwartships sway oscillation has the period of an ideal simple pendulum, as the hull does not rotate in roll. The period T_S of small amplitude sway oscillation is (Newman and Searle, 1951)

$$T_s = 2\pi\sqrt{\frac{l}{g}} \qquad (7)$$

One can think of this as a pendulum clock for timing the yaw oscillation. There is a simple relation between the yaw period T_y, the sway period T_s and the yaw gyradius k_y, which is given by

$$k_y = \left\{\frac{T_y}{T_s}\right\}.\ d \qquad (8)$$

Note that the suspension length l and the gravitational acceleration g as well as buoyancy effects due to the air, which reduces the downward force on the hull and thus both modes of oscillation, are eliminated from this ratio. However, Figure 8 shows that the damping of the sway and the yaw oscillations are different, and may contribute to the small period variations with amplitude shown in Figure 23.

THE YAW-SWAY PERIOD RATIO METHOD

The yaw gyradius can thus be determined from the spacing "2*d*" and the ratio of the yaw to sway periods T_y/T_S. The latter can be very precisely determined from the beat pattern of the combined motion of the stern, or the bow. To optimize this, the amplitudes of the yaw and sway motions at the stern should be about equal. This can be achieved by holding the stern stationary and displacing the bow laterally; however, the ratio of the periods can still be determined even for relatively small sway amplitudes.

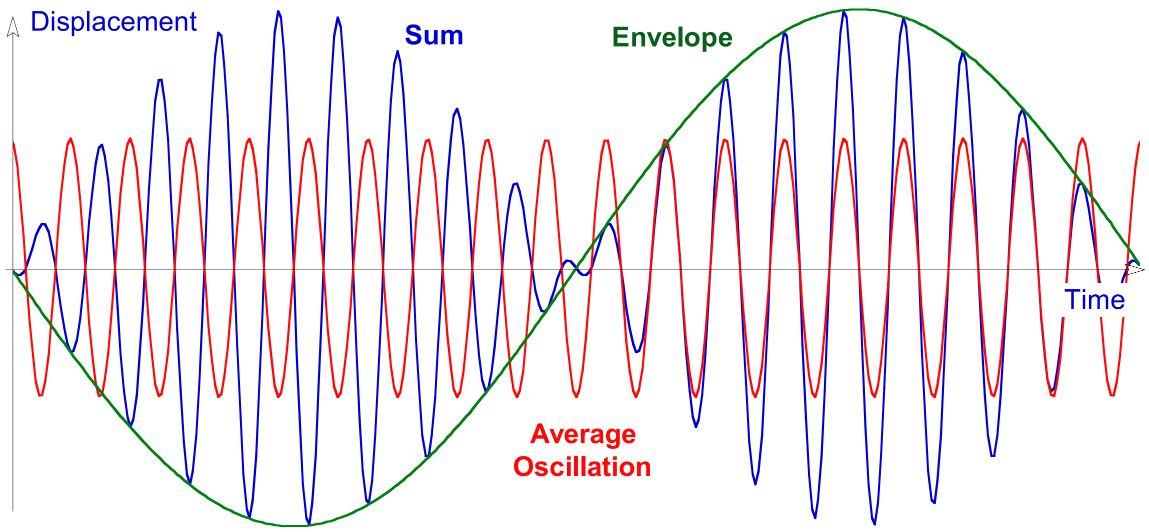

Figure 3. The sum of two equal amplitude oscillations of periods 4.39 and 3.83 s is equivalent to an average oscillation of period 4.09 s modulated by an envelope function of period 30.0 s, see equation (9).

ISSN 1548-6559

For equal, undamped yaw and sway with amplitudes A_y $(=\lambda\theta_y)$ = A_s = A and phases φ_y and φ_s, the motion is

$$\begin{aligned} y &= A_y Sin\left(\frac{2\pi}{T_y}t+\varphi_y\right)+A_s Sin\left(\frac{2\pi}{T_s}t+\varphi_s\right) \\ &= 2ACos\left(\frac{\pi\left(T_y-T_s\right)}{T_yT_s}t+\frac{\varphi_y-\varphi_s}{2}\right)Sin\left(\frac{\pi\left(T_y+T_s\right)}{T_yT_s}t+\frac{\varphi_y+\varphi_s}{2}\right) \end{aligned} \quad (9)$$

Equation (9) shows that for equal amplitudes A, i.e. 100% modulation, the sum of the two oscillations is an “average” Sine function of frequency $\left(T_y+T_s\right)/2T_yT_s$, i.e. period

$$T_a = 2T_yT_s\Big/\left(T_y+T_s\right) \quad (10)$$

modulated by an “envelope” Cosine function of frequency $\left|T_y-T_s\right|/2T_yT_s = 1/2\left|1/T_s-1/T_y\right|$ i.e. half the frequency difference, then the beat period T_b = T_e /2 is

$$T_b = T_e/2 = T_yT_s\Big/\left|T_y-T\right| \quad (11)$$

as the envelope function produces two maxima, or beats, per period. Thus, for equal yaw and sway, the stern, or bow, starts with negligible oscillation, builds up to a maximum amplitude 2A and then decreases to negligible amplitude, after which the cycle repeats itself as shown in Figure 3. The number “n” of oscillations per beat is then

$$\begin{aligned} n &= \frac{T_b}{T_a} = \frac{T_yT_s}{T_y-T_s}\Bigg/\frac{2T_yT_s}{T_y+T_s} \\ &= \frac{\left(T_y/T_s+1\right)}{2\left(T_y/T_s-1\right)} \quad or \quad \frac{\left(T_s/T_y+1\right)}{2\left(T_s/T_y-1\right)} \end{aligned} \quad (12)$$

depending on whether $T_y > T_s$ or $T_y < T_s$ so

$$k_y = \frac{T_y}{T_s}d = \frac{2n+1}{2n-1}d \quad or \quad \frac{2n-1}{2n+1}d \quad (13)$$

depending on whether $k_y > d$ or $k_y < d$. For suspensions attached to the hull at the sheer $k_y > d$ and this can be confirmed by calculating k_y with approximate values in equation (5).

Thus by just observing and counting the number of oscillations per beat, one can determine the period ratio and hence the yaw gyradius from a simple measurement of 2d. For example for the periods of 4.39 and 3.83 seconds, as illustrated in Figure 3, n = 7.33, while counting maxima would give 15 for 2 beats. The calculated ratio would be 1.143 instead of 1.146, i.e. an error of -0.3 percent. Even counting 7 per beat would give 1.154 or an error of only +0.7 percent. Once an initial estimate is obtained in this way the spacing

ISSN 1548-6559

2d could be adjusted to be closer to twice the calculated gyradius, thus making the beat period much larger and the error in the counted number of maxima negligible. Note if $d = k_y$ then $T_y = T_s$ and the beat period becomes infinite, i.e. the hull will just rotate about the stationary stern. A typical yaw sway beat pattern for a laser hull (Waine, 1988) on a bifilar suspension is shown in Figure 4.

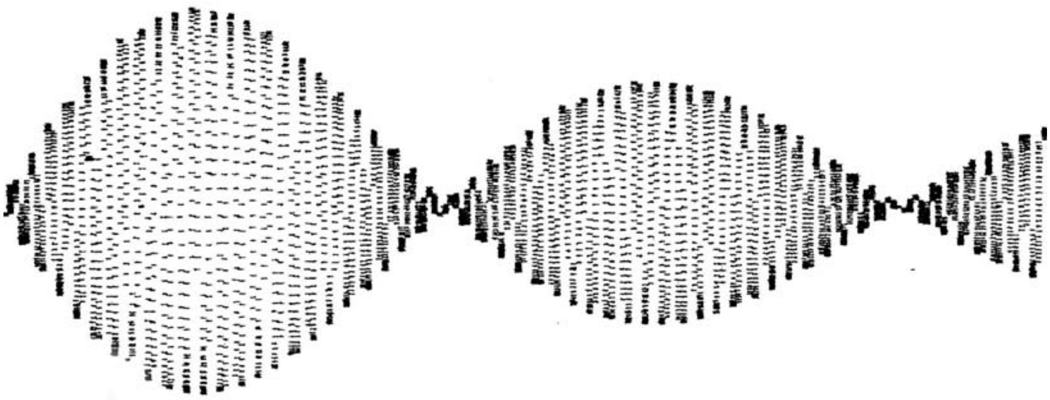

Figure 4. The position of the bow of a laser dinghy hull (Waine, 1988) undergoing yaw and sway on a bifilar suspension. Note $n = 25$ so $k_y = 1.04\ d$.

MEASUREMENTS OF A HULL YAW GYRADIUS

The pitch gyradii of all the Yngling hulls at the 2004 Olympic Games were measured and it was of interest to compare this data with a measurement of the yaw gyradius for one of these hulls. The pitch and yaw gyradii are both measures of the fore and aft weight distribution but the effect of the keel is expected to be somewhat different about these two axes. Furthermore it is of interest to see what precision can be achieved using the bifilar suspension and to investigate the problems which can arise when applying this technique to a 630 kg keelboat hull.

Figure 5. The hull suspended for the yaw test. The two cross beams with eyebolts to which the supporting lines are attached can be seen. The vertical photogate used for timing is mounted on the tripod at the stern (the rudder was reversed before measurement!).

As shown in Figure 5 the hull was suspended from a gantry by two equal lines of length l = 3477±6 mm. It was subsequently found that the lines stretched significantly and also suffered from creep, thus limiting the precision of the direct measurement, but not of the ratio measurement. For any future measurements a very low stretch line with negligible torsional rigidity should be used, preferably with bearings at each end. For mechanical reasons the spacing 2d = 1800 mm was chosen to be about equal to the beam of the hull.

The weight and dimensions of the crossbeam or frame must be measured and then the beam firmly fixed to the hull. Note that although a relatively heavy beam was used for the present measurements, only a spacer wire between brackets firmly seated at the gunwales is required in order to maintain the spacing 2d. Such an arrangement makes a negligible contribution to the measured gyradius. If for safety reasons the lifting sling is left in place, it must be loosened sufficiently as to not affect the torsional rigidity of the suspension. A laser pointer or plumb bob over the center of the beam aids in avoiding linear displacements when rotating the hull.

PHOTOGATE YAW PERIOD MEASUREMENTS

A "computerized photogate timer" (Pasco, 2000) was used to continuously measure the yaw period four times per oscillation of the stern, as well as the speed through the photogate in order to determine the amplitude. Yaw period data were recorded with both large and small sway in order to observe its effect on the precision. A typical set of data are shown in Figure 6 and were fitted with a variable offset decaying sinusoidal function.

$$T_y = T_{y0}\left(1+bt\right) + \exp\left(\frac{-t}{\tau_m}\right) Sin\left(\frac{2\pi t}{T_m} + \varphi_m\right) \qquad (14)$$

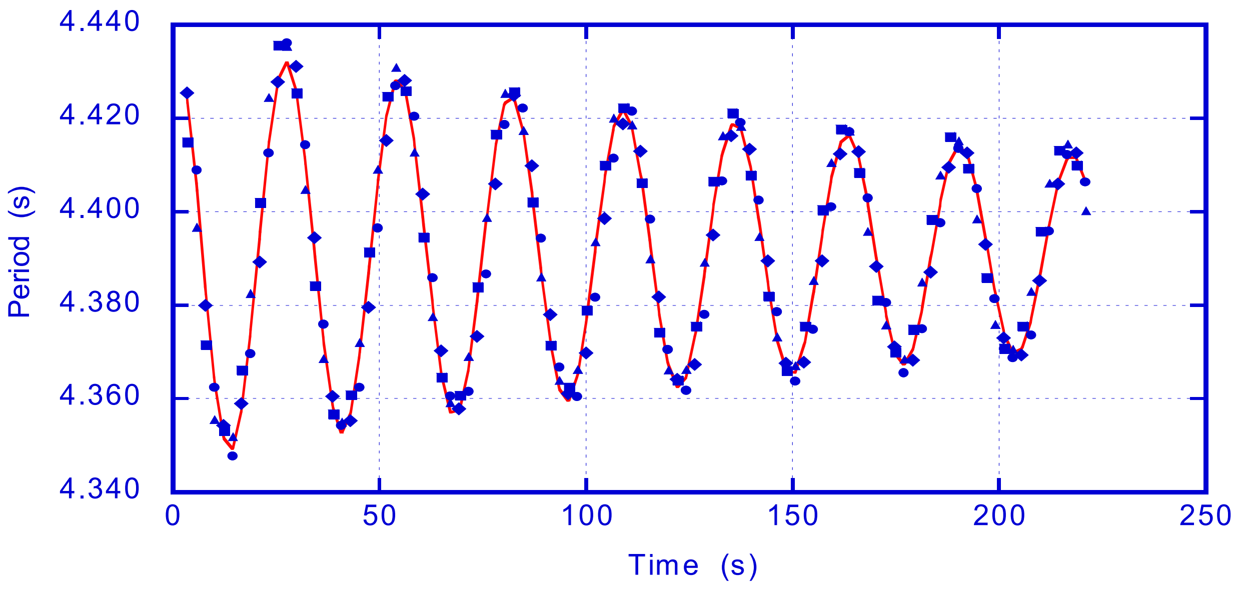

Figure 6. The oscillation period of yaw motion as measured at the stern by a vertical photogate. The systematically decreasing modulation (~1.8 to 0.9%) due primarily to some sway motion is apparent. A variable offset decaying sinusoidal function, equation (14), was fitted to the data to remove this sway modulation.

The yaw period variation with time, due to decaying yaw amplitude, is accounted for by the constant b, while τ_m accounts for the decay of the period modulation due to the decay of the sway motion. The averaged yaw periods T_{y0} deduced from the fits are listed in Table 1 together with the modulation period $T_m = T_b$ from which the suspension length *l* and the yaw gyradius k_y can be deduced.

KEELBOAT MEASUREMENTS WITH MASSES ADDED AT THE BOW AND STERN

In order to determine the sensitivity of these measurements to mass at the ends, a series of measurements were made with masses of 1.2 kg placed 2717 mm aft and 3088 mm forward of the hull center. The results of these measurements are shown in Figure 7 and Table 1. The line in Figure 7 is not a fit to the data, but is calculated from the hull yaw gyradius $k_y(0)$ = 1056 mm for m = 0, and the mass M of the hull. The excellent agreement with the data confirms the precision of the technique to be $\delta k_y/k_y \sim 0.2\%$.

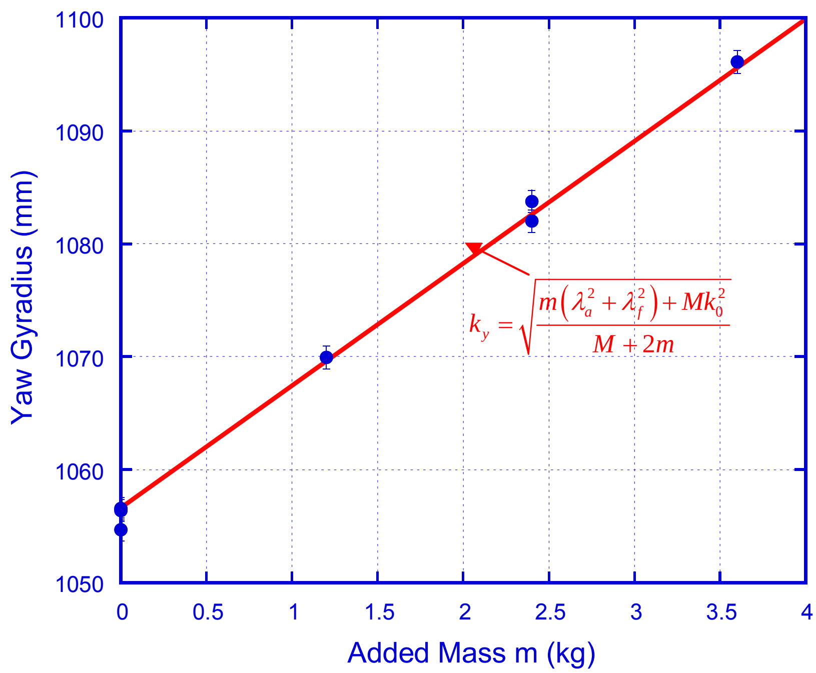

Figure 7. The variation of the measured gyradius with added mass at the bow and stern, see Figure 2. The line is not a fit, it is the theoretical prediction.

Table 1. Photogate yaw periods.

Added mass kg	Yaw Period Sec	Yaw Gyradius mm
0	4.391	1056.6
0	4.390	1056.4
0	4.383	1054.7
1.2	4.446	1069.9
2.4	4.504	1083.8
2.4	4.496	1082.0
3.6	4.555	1096.1

ISSN 1548-6559

KEELBOAT ULTRASONIC MEASUREMENTS

The position of a light 50 mm diameter cylinder mounted at the stern was measured at 20 ms intervals using an ultrasonic system "UMS" (Daedalon, 1991), similar to the rangefinder on a Polaroid camera with a resolution of about ±0.1mm. A typical set of data is shown in Figure 8 and it will be noted that both the amplitude and the modulation decrease with time thus showing that the damping of the sway and the yaw are significant and different. The first 50 s of the data were modeled by the sum of two exponentially damped oscillations, namely

$$y = y_0 + A_y Exp\left(\frac{-t}{\tau_y}\right) Sin\left(\frac{2\pi t}{T_y} + \varphi_y\right) + A_s Exp\left(\frac{-t}{\tau_s}\right) Sin\left(\frac{2\pi t}{T_s} + \varphi_s\right) \qquad (15)$$

where y_0 is the offset, A_y, τ_y, T_y, φ_y and A_s, τ_s, T_s, φ_s are the amplitude, damping, period and phase of the yaw and sway oscillations respectively.

The TableCurve 2D least squares fitting program (TableCurve2D, 1994) was used to fit this function to the data (Figure 9), and the results are listed in Table 2. The data were also analyzed using an FFT routine (Lungu, 2012). The period ratio can be extracted with a high degree of precision, however, second order effects such as damping, added aerodynamic mass, etc. limit the ultimate precision and accuracy. Creep and elastic extension of the suspension lines used, as well as the variation of the periods with amplitude (see below) would account for the difference in yaw gyradius derived from equations (6) and (8).

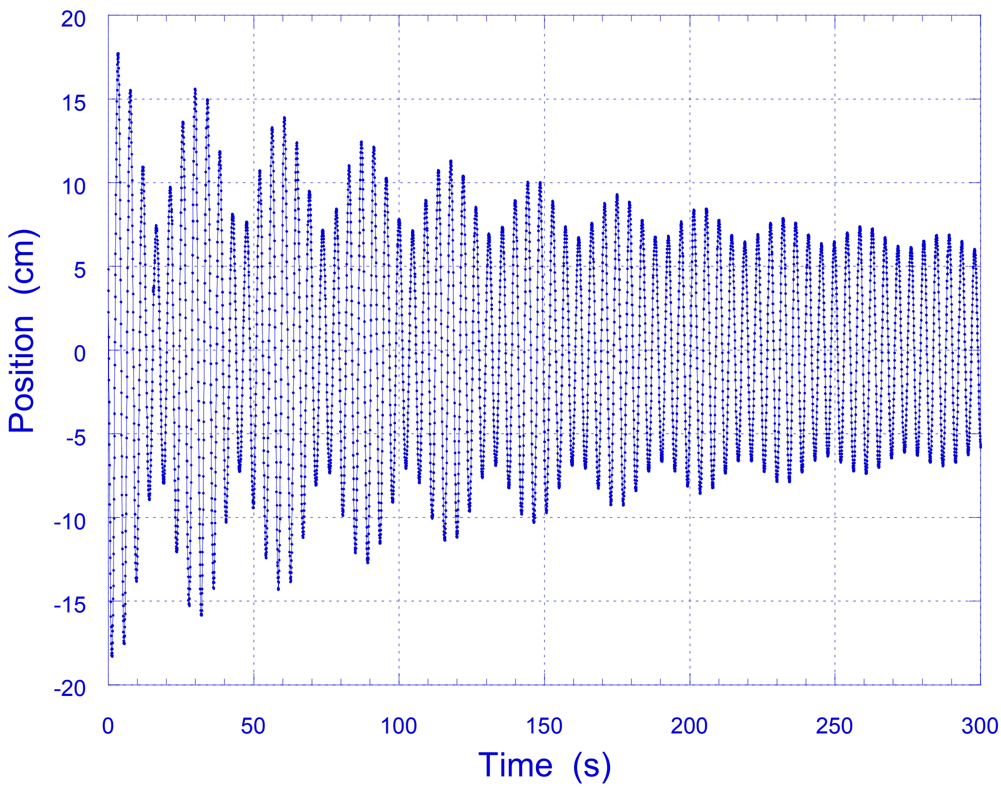

Figure 8. The position of the stern as a function of time when the hull is released with both yaw and sway displacement.

Table 2. Ultrasonic yaw data.

Parameter	Units	UMS 1	UMS 4
Yaw period	s	4.394	4.407
Sway period	s	3.802	3.820
Ratio T_y/T_s		1.156	1.154
K_y from ratio	mm	1040	1038
K_y from Yaw period	mm	1057	1060

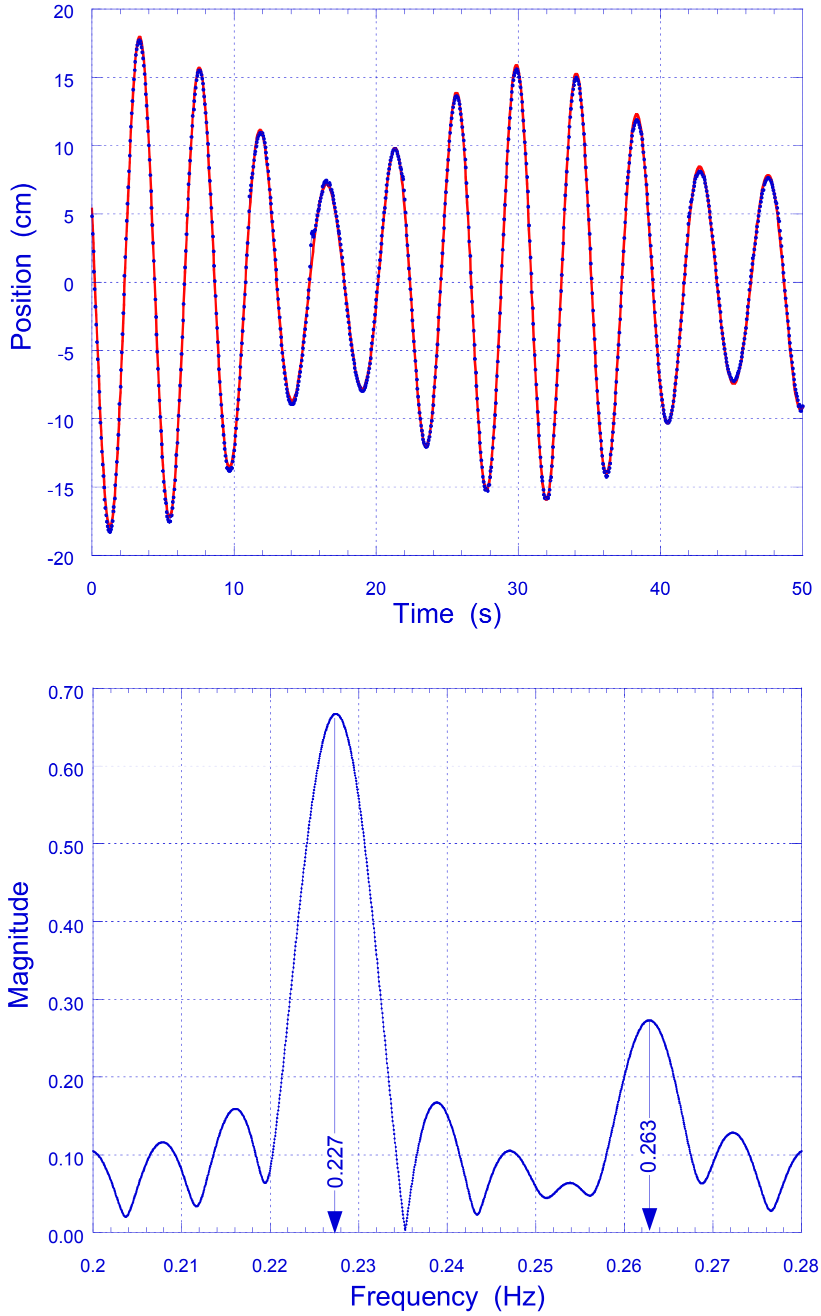

Figure 9. The least squares fit of equation (15) to the first 50s of the UMS data, and an FFT of this data.

MODEL HULL MEASUREMENTS

Subsequent to the above measurements on the full size keelboat hull, the techniques described below were envisaged and MEMs gyros and accelerometers, such as those in the iPhone (xSensor, 2010), became generally available. Therefore, as proof of principle a series of measurements were made on the 13.18 kg model hull of LOA = 1.87 m, beam = 0.288 m, shown in Figure 10.

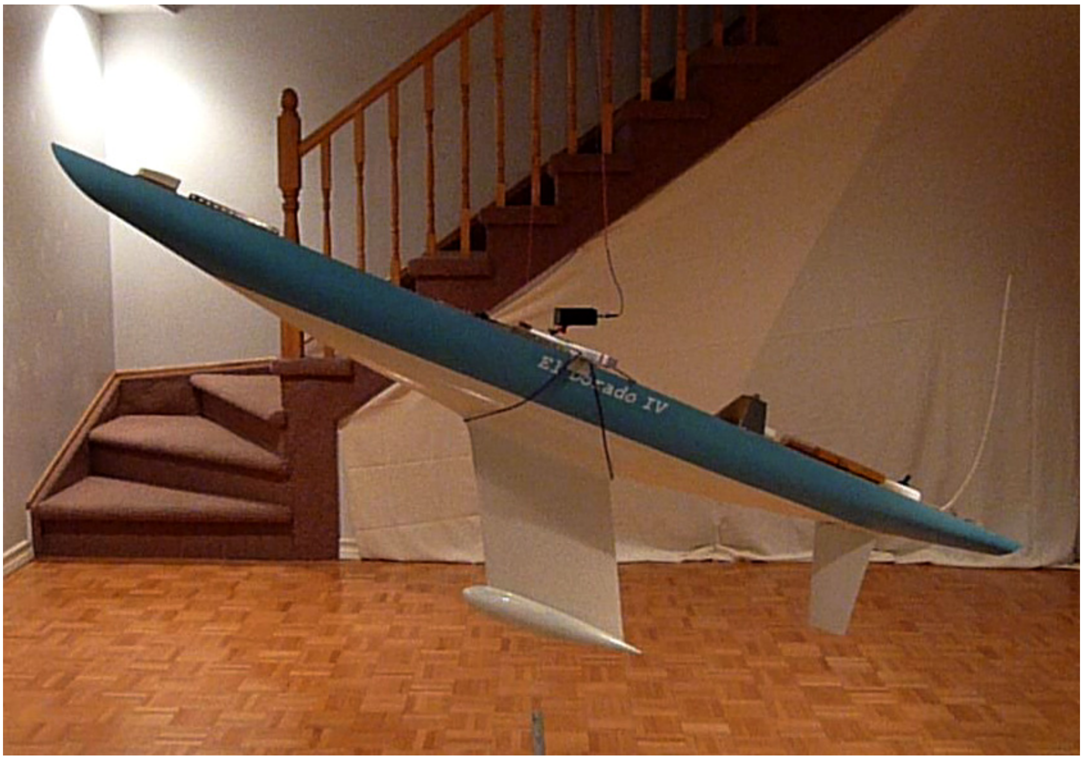

Figure 10. The model hull hanging bow up on the bifilar suspension.

Table 3. Model hull data.

Hull		Suspension			Gyradii		
	ϕ_p(deg)	l_1	d	l_2	k_y	k_p	k_r
Athwartships	0.0	504	280	249	309	360	
Athwartships	0.0	1006	280	244	308	353	
Athwartships	0.0	1084	260	243	313	355	
Athwartships	0.0	1560	280	244	309	357	
Bow down	-31.1	1012	280	262	294	367	
Stern Down	22.8	1012	280	254	290	360	
Fore & Aft	0.0	1012	280	251	310		204

For the model hull suspended horizontally by an athwartships suspension the separate yaw and sway, the combined yaw-sway, the double pendulum motion, and the mode 1 and mode 2 oscillations separately, were recorded using an iPhone running xSensor Pro (xSensor, 2010). Measurements were made with five suspension lengths, then with the bow down and with the bow up as shown in Figure 10. The center of mass height was

ISSN 1548-6559

checked by hanging a weight at the transom and measuring the angle of inclination.

The roll gyradius measurements were made with the suspension fore and aft, i.e. in the hull center plane (see Figure 12). The periods of oscillation were derived by fitting the time data to damped sine functions or from Fourier transforms (Lungu, 2012) similar to those in Figure 14. The model hull data presented in Table 3 were derived using the period ratio equations (8), (17), and (18).

The yaw gyradii data are consistent as the difference between the fore and aft and level data are due to some added cross pieces. The bow down and bow up yaw gyradii combined with the level and roll gyradii fit an xz inertia ellipse (see Figure 20) and lead to $\psi = -5.50^o$ and -5.54^o.

PITCH-SURGE DOUBLE PENDULUM

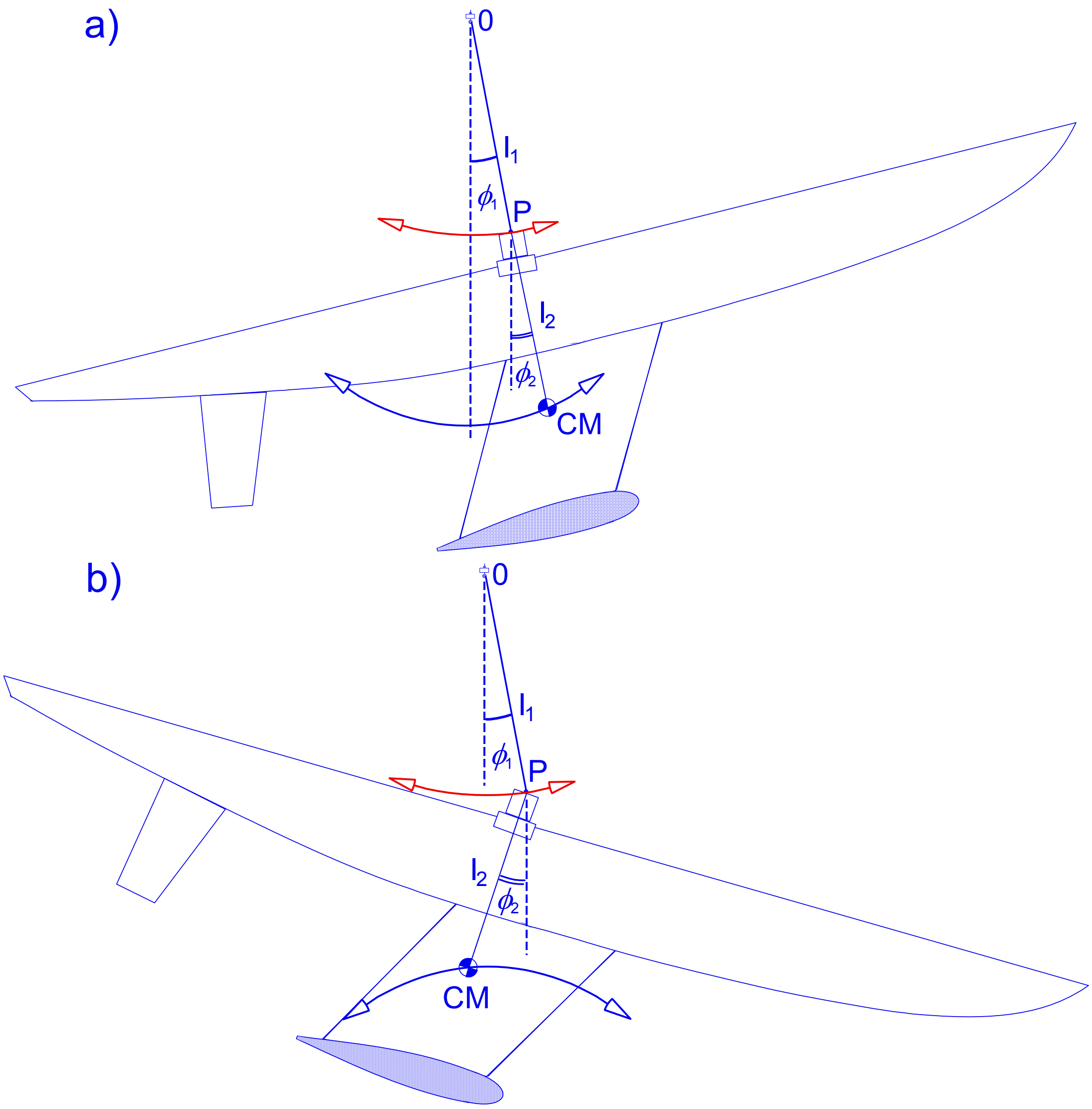

Figure 11. a) Mode 1 and b) mode 2 of the hull on an athwartships suspension, oscillating in pitch-surge-heave.

A hull on an athwartships bifilar suspension can also be excited in pitch-surge oscillation and measurement of the frequencies, or the periods T_1 and T_2, of the two normal modes of this coupled oscillation, allows both the vertical position l_2 of the center of mass below the suspension point P, and the pitch gyradius k_p to be determined from a single set of data. For mode 1 the hull oscillation about point P is in phase with the pendular oscillation about the suspension point O (Figure 11a), while in mode 2 they are out of phase by 180^o (Figure 11b). Furthermore, a measurement of the sway period T_s of the suspension allows k_p and l_2 to be determined purely in terms of these three periods. Thus the yaw as well as the pitch gyradius and center of mass height can be measured without removing the hull from the bifilar suspension.

The problem of the double pendulum has an interesting history (Auerbach and Host, 1930). The Emperor's bell (Kaiserglocke) of Cologne Cathedral was installed in 1885, but the bell did not ring reliably, as the clapper swung together with the bell in one of the normal modes of this double pendulum. The problem was analyzed by Von Veltmann (Veltmann, 1876) and later by G. Hamel (Hamel, 1912) and corrections applied.

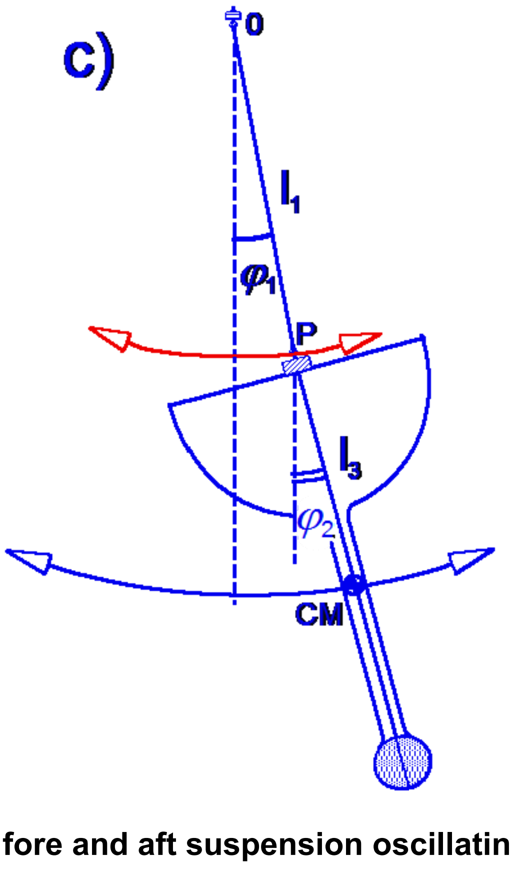

Figure 12. The hull on a fore and aft suspension oscillating in roll-sway-heave.

For a hull of mass M, pitch gyradius k_p and center of mass a distance l_2 below the lower bearing at P, suspended by lines of length l_1, Lagrange's equations lead to analytically intractable equations, which can be modeled using Mathworks and Simulink (Mathworks) to solve the equations numerically, including nonlinear effects, and derive the pitch gyradius and center of mass position.

However, in the limit of small angles one can substitute sinusoidal oscillations into Lagrange's equations (Spiegel, 1967), which leads to a quadratic equation with solutions ω_1 and ω_2 for the angular frequencies of the two normal modes:

$$\omega_{1,2}^2 = \frac{g}{2l_1k_p^2}\left\{\left(l_1l_2 + l_2^2 + k_p^2\right) \pm \sqrt{\left\{l_1l_2 + l_2^2 + k_p^2\right\}^2 - 4l_1l_2k_p^2}\right\} \qquad (16)$$

Rearranging leads to l_2 and k_p in terms of the periods T_1, T_2 and $T_s = 2\pi\sqrt{l_1/g}$, or the ratios $R_1 = T_1/T_s$ and $R_2 = T_2/T_s$ as:

$$l_2 = \frac{g}{4\pi^2T_s^2}\left(T_s^2 - T_1^2\right)\left(T_2^2 - T_s^2\right) = l_1\left(1 - R_1^2\right)\left(R_2^2 - 1\right) \qquad (17)$$

and

$$k_p = \frac{g}{4\pi^2T_s^2}\sqrt{T_1^2T_2^2\left(T_s^2 - T_1^2\right)\left(T_2^2 - T_s^2\right)}$$

$$\frac{k_p}{l_1} = \sqrt{R_1^2R_2^2\left(1 - R_1^2\right)\left(R_2^2 - 1\right)} \qquad (18)$$

An initial displacement of the hull in heave, while keeping it approximately level, excites both modes equally. Alternatively, after a little practice one can resonantly excite only mode 1 and time it, and then only mode 2, so such measurements can be made with a simple timer.

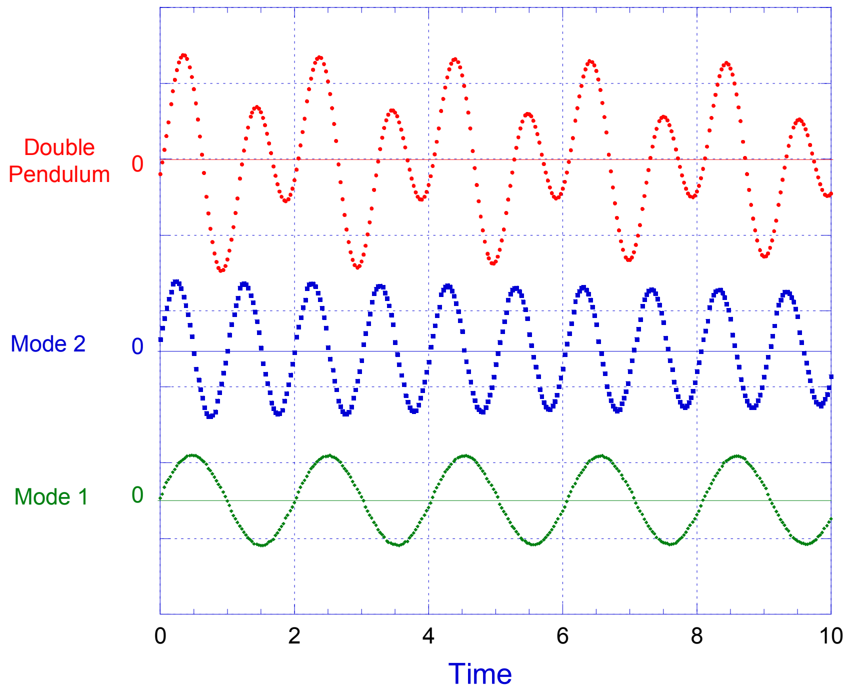

Figure 13. The pitch-surge-heave double pendulum motion data for the model hull on a 504 mm bifilar suspension, when displaced horizontally in surge, then separately excited in mode 1 and mode 2.

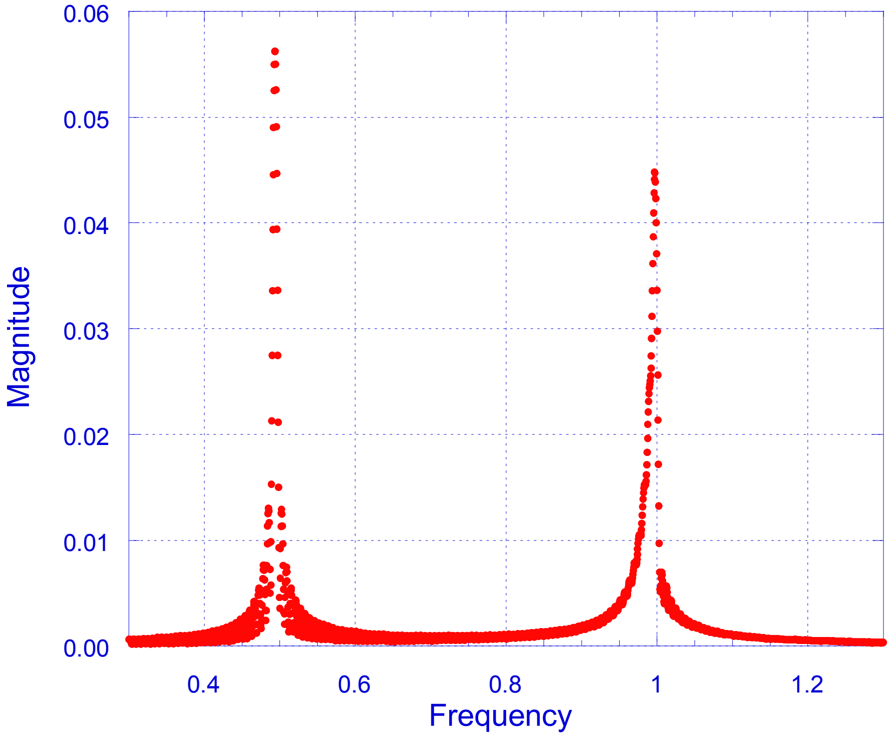

Figure 14. The Fourier transform of the double pendulum data in Figure 13 provides a rapid method of deriving the precise normal mode frequencies.

These motions can be recorded by a MEMS gyro (iPhone 4) and the periods T_1, T_2 derived. Then T_s is measured by displacing the hull in sway. Thus the three parameters l_2, k_p and k_y can be determined from three measurements without changing the suspension, which is a major advantage for measurements of heavy hulls.

A typical set of data for the model hull excited in the double pendulum mode is shown in Figure 13. A non linear fit to two damped sine functions similar to equation (15) can be used to extract precise values for T_1 and T_2. However, Fourier analysis of the data, as shown in Figure 14, leads directly to the required frequencies of the two normal modes.

The double pendulum technique, with the bifilar suspension in the center plane of the hull as shown in Figure 12, was used to measure the roll gyradius k_r of the model hull.

Although in practice one would only make one set of measurements, as a proof of principle measurements were made with the model hull suspended level with four suspension lengths at d = 280 mm, and one with d = 260 mm. The data are shown in Figure 15 together with the theoretical predictions of equations (5), (7) and (16) for k_y = 303 mm, k_p = 352 mm and l_2 = 253 mm.

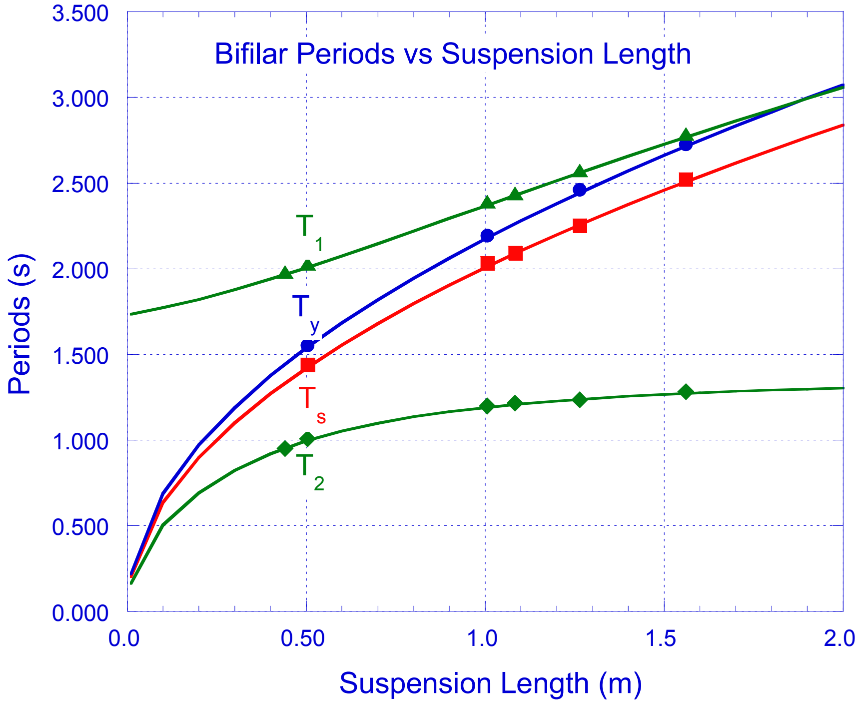

Figure 15. The model hull oscillation periods T_s of Sway, T_y of Yaw, T_1 and T_2 of modes 1 and 2, as a function of the bifilar suspension length l_1. The curves are theoretical fits with d = 0.280 m, k_p = 0.352m, k_y = 0.303m and l_2 = 0.253m.

OPTIMIZATION

Equation (16) can be re written as:

$$T_{1,2} = \sqrt{l_2}\sqrt{\frac{8\pi^2 L_{12} K_{P2}^2}{g\left\{\left(1+L_{12}+K_{P2}^2\right) \pm \sqrt{\left\{1+L_{12}+K_{P2}^2\right\}^2 - 4L_{12}K_{P2}^2}\right\}}} \qquad (19)$$

where $K_{P2} = k_p/l_2$ is a non-dimensional characteristic of the hull and $L_{12} = l_1/l_2$ specifies the suspension. The variation of the two periods with l_1 is thus given by the second square root, leaving the periods to scale with $\sqrt{l_2}$. Differentiating equation (19) to derive $\partial k_p/\partial T_1$ and $\partial k_p/\partial T_2$, and then assuming that the uncertainties ±δT in both periods are the same, leads to $\partial k_p/\partial T = \sqrt{\left(\partial k_p/\partial T_1\right)^2 + \left(\partial k_p/\partial T_2\right)^2}$ which is plotted versus L_{12} in Figure 16. For keelboats $K_{P2} \approx 1.5$ while for dinghies $K_{P2} \approx 2.5$ to 4.5. It can be seen that there is a minimum in these curves and the value of L_{12} for the minimum is plotted versus K_{P2} in Figure 17. Thus Figure 17 together with preliminary estimates of k_p and l_2 allow one to choose a value of the suspension length l_1 which optimizes the precision of the double pendulum measurements. The optimum suspension lengths for the keel boat and model hull would have been 1.7 m and 0.5 m respectively.

ISSN 1548-6559

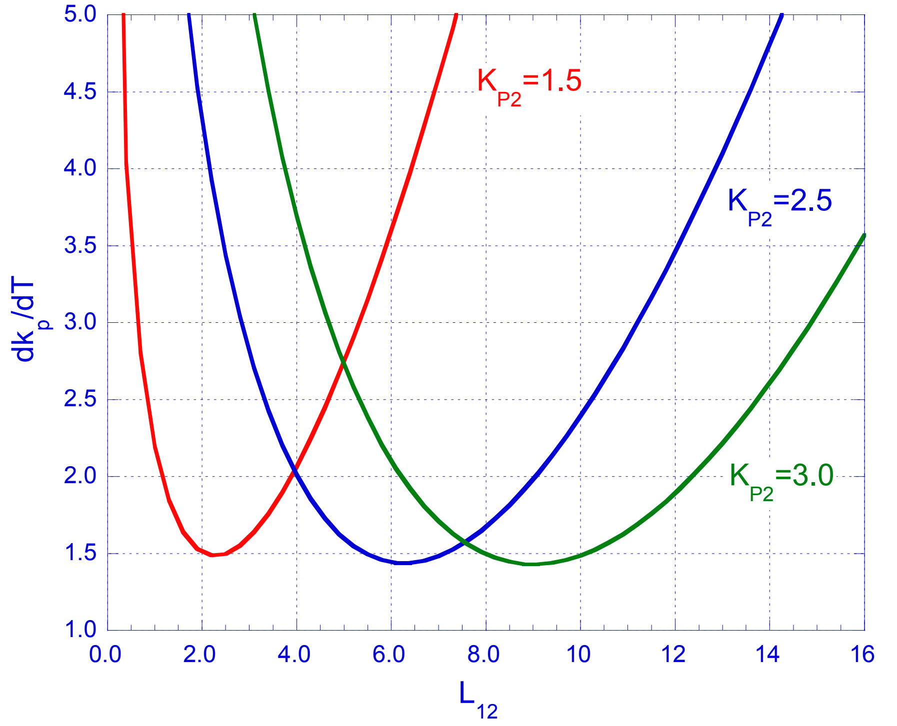

Figure16.The variation of the derivative of the pitch gyradius with period $\partial k_p/\partial T = \sqrt{(\partial k_p/\partial T_1)^2 + (\partial k_p/\partial T_2)^2}$ **versus the ratio** $L_{12} = l_1/l_2$ **.**

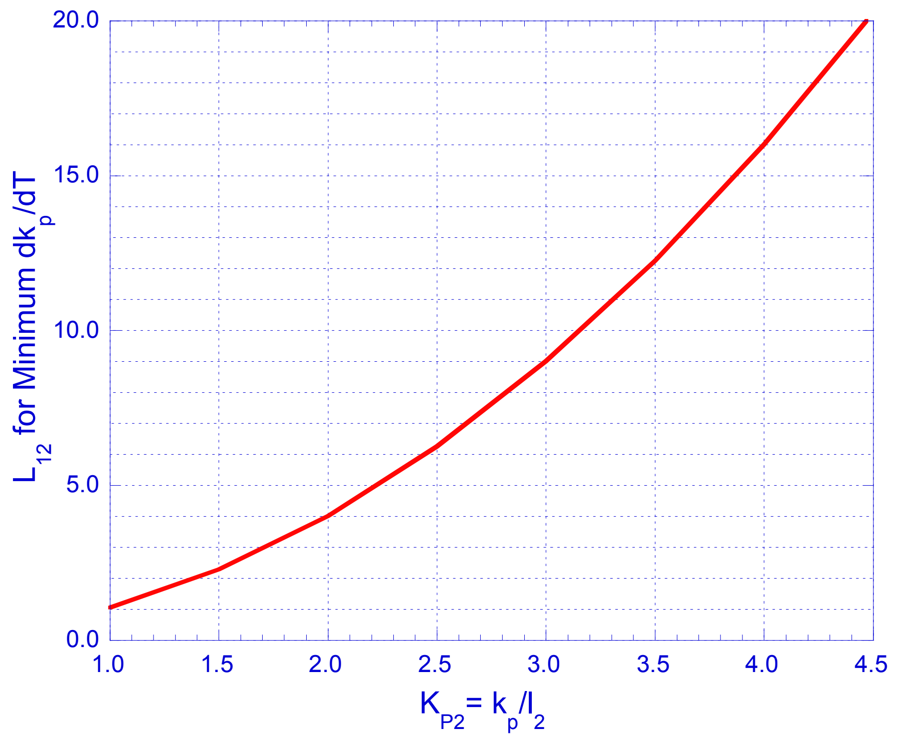

Figure 17.The value of $L_{12} = l_1/l_2$ **for minimum** $\partial k_P/\partial T$ **as a function of** $K_{P2} = k_P/l_2$ **.**

SENSITIVITY TO ADDED MASSES

In order to test the sensitivity of the double pendulum test, the periods T_1, T_2, T_y and T_s were measured with a series of small masses m successively added at both λ_f = 883 mm and λ_a = 859 mm fore and aft on the deck of the model hull. The results (see Figure 18) are in good agreement with the theoretical predictions and indicate that changes of less than 1 percent in the yaw or pitch gyradii can be detected. It was interesting to note that the primary effect of these added masses was to change the pitch and yaw gyradii with only minor effects on the center of mass height, and this led to significant changes in T_1, with little effect on T_2. In order to confirm this, a mass was added first on the deck above the center of mass and then under the base of the keel. The data were in excellent agreement with the calculated changes in k_y, k_p and l_2.

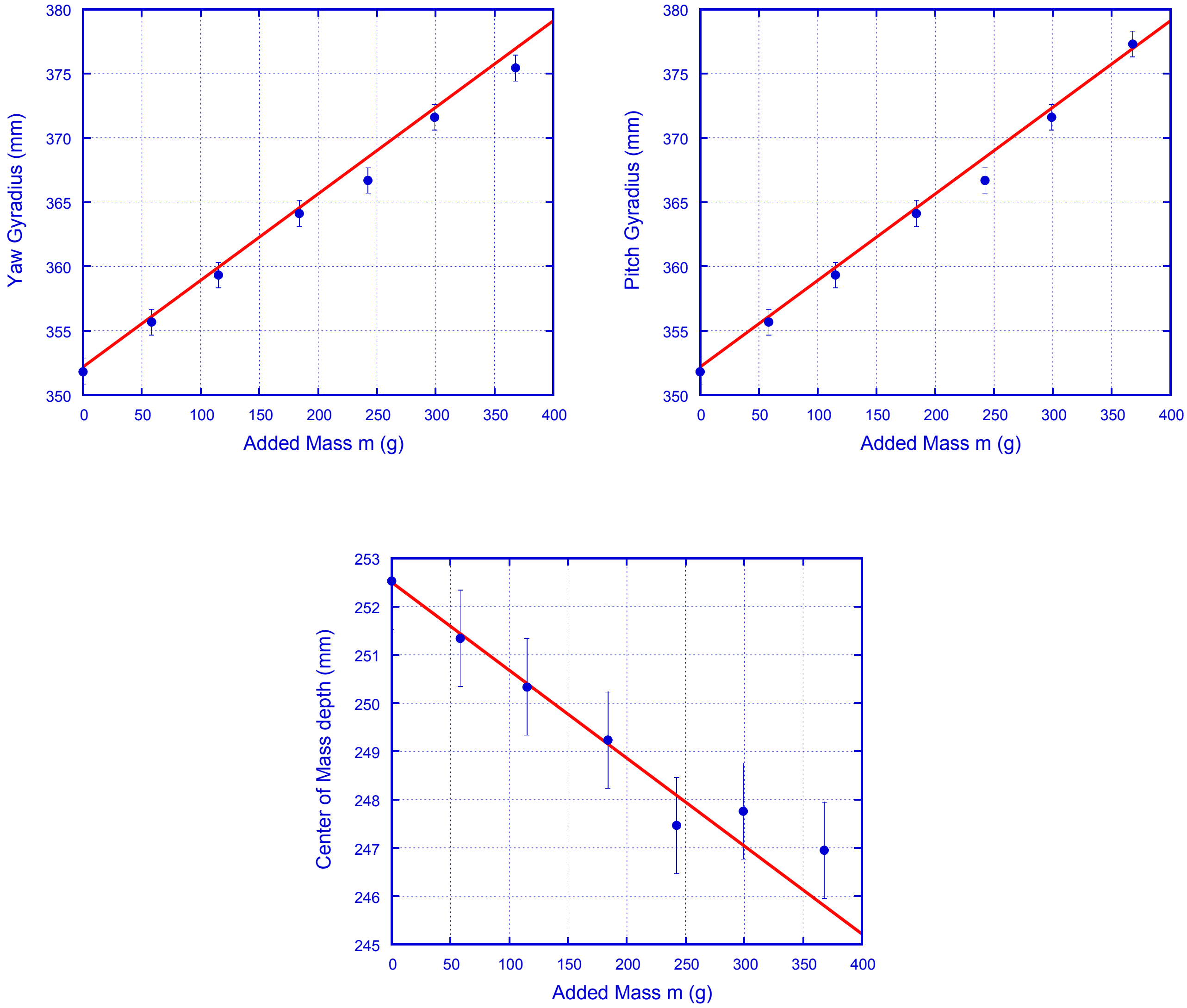

Figure 18. The variation of the measured yaw gyradius k_y, pitch gyradius k_p and center of mass depth l_2 of the model hull with masses added on the deck at the bow and stern. The lines are the theoretical predictions.

INCLINE TEST

The distance l_2 of the center of mass below the suspension point P can easily be checked by hanging a mass m_d at either the bow or the stern and measuring the angle of inclination, as is done in the Dragon test (Watts, 1986). The center of mass position l_2 is given by:

$$l_2 = \frac{m_d}{M}\left\{\frac{L_s}{Tan\phi_s} - b_s\right\} \qquad (20)$$

where b_s and b_b are the depths of the inclining mass suspension points below the point P, when the hull is horizontal. The precision can be improved by repeating the inclining with the mass m_d at the bow. Then

$$l_2 = \frac{m_d}{M}\frac{\left(L_0 - b_b Tan\phi_b - b_s Tan\phi_s\right)}{\left(Tan\phi_b + Tan\phi_s\right)} \qquad (21)$$

in terms of the overall length L_0, so the precise fore and aft location L_s of the point P is not then required.

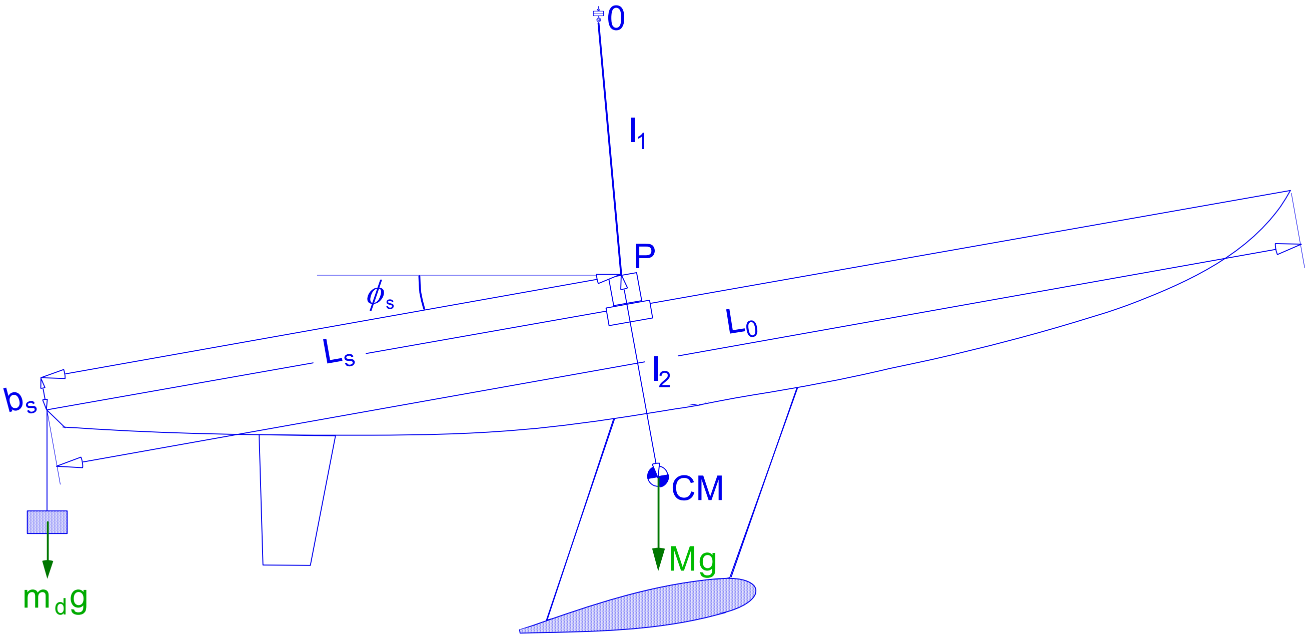

Figure 19. The hull inclined at pitch angle ϕ_s by a mass m_d hung at a distance L_s from the pivot point P.

COMPOUND PENDULUM TEST

If the hull is suspended by two sets of inverted "V" lines as shown in Figure 21 with the suspension axis fore and aft, or athwartships, the roll gyradius, or the pitch gyradius, can be determined from the period of hull oscillation as a compound pendulum (Newman and Searle, 1951) which is:

ISSN 1548-6559

$$T = 2\pi\sqrt{\frac{a^2 + k^2}{ag}} \qquad (22)$$

with the center of mass a distance $a = (l_0 + l_2)$ below the suspension axis, see Figure 21. So

$$k = \sqrt{\frac{T^2 ga}{4\pi^2} - a^2} \qquad (23)$$

Thus if l_2 is known from previous measurements then the gyradius can be calculated. The roll gyradius can be measured with an A-G:C-E suspension, as shown in red in Figure 21, while the pitch gyradius can be measured with an A-C:G-E suspension.

The compound pendulum differentials are:

$$\frac{\partial k}{\partial a} = \frac{\left(k^2 - a^2\right)}{2ka}, \quad \frac{\partial k}{\partial T} = \frac{\sqrt{ag}}{2\pi k}\sqrt{k^2 + a^2} \qquad (24)$$

Thus ideally if l_0 is chosen so $a \approx k$ then $\partial k / \partial a \approx 0$ and the measured gyradius $k = T^2 g / 8\pi^2$ is essentially independent of $a = (l_0 + l_2)$. Furthermore $\partial k / \partial T \approx \sqrt{kg / 4\pi^2}$ so a single period measurement leads to a quite precise gyradius. Although this is possible for the pitch gyradius, as in general $k_p > l_2$, it is less practical for the roll gyradius as normally $k_r < l_2$, but the precision will still be improved by keeping l_0 as small as possible. Compound pendulum measurements at two values of l_0 allow both k and l_2 to be determined, i.e. a Lamboley test. Measurements of the model hull roll gyradius were made with l_0 = 441 mm and 305 mm, and led to a roll gyradius of 205 mm.

PRINCIPAL AXES

The assumed symmetry of the hull leads to the hull y axis being the principal pitch axis of the inertia ellipsoid (see Figure 1) and the principal yaw and roll axes being at some angle ψ in the xz hull center plane. The stern and bow sections of the hull are significantly different and the centers of mass of the hull shell and of the keel are therefore not necessarily on the hull z axis through the combined center of mass. The principal z axis is therefore at an angle ψ to the hull z axis.

Measurements of the yaw and roll gyradii k_y, k_r, and k_ϕ, i.e. about an axis inclined at an angle ϕ to the hull z axis (as shown in Figure 20) allow the principal gyradii k_{py}, k_{pr} and the angle ψ to be calculated using equations (4).

In principle, precise yaw measurements of k_y, $k_{\phi A}$ and $k_{\phi B}$ about two extra axes inclined at significantly different angles ϕ_A and ϕ_B can be used instead of the "sway-roll-heave double pendulum" to determine the roll gyradius k_r, which is then given by:

$$k_r = \sqrt{\frac{\left(k_{\phi A}^2 Sin2\phi_B - k_{\phi B}^2 Sin2\phi_A\right)}{2Sin\left(\phi_A - \phi_B\right) Sin\phi_A Sin\phi_B} + \frac{k_z^2}{Tan\phi_A Tan\phi_B}} \qquad (25)$$

Note that if $\phi_A = (\phi_B+\pi/2)$ then equation (25) reduces to

$$k_r^2 = k_{\phi A}^2 + k_{\phi B}^2 - k_z^2 \qquad (26)$$

For the present keelboat hull if the three gyradii are measured to ±1 mm with the hull level, and tilted plus and then minus 30°, the estimated precision of the calculated roll gyradius is ±7 mm or 1 percent, but deteriorates rapidly for smaller inclination angles ϕ. The advantage, however, is that the same athwart ship bifilar suspension can be used for all three measurements. If the displacement Δ_A of the suspension point P along the deck and the angle ϕ_A of inclination are measured these can be used as another check on the center of mass position as $l_2 = \Delta_A/\text{Tan}\ \phi_A$.

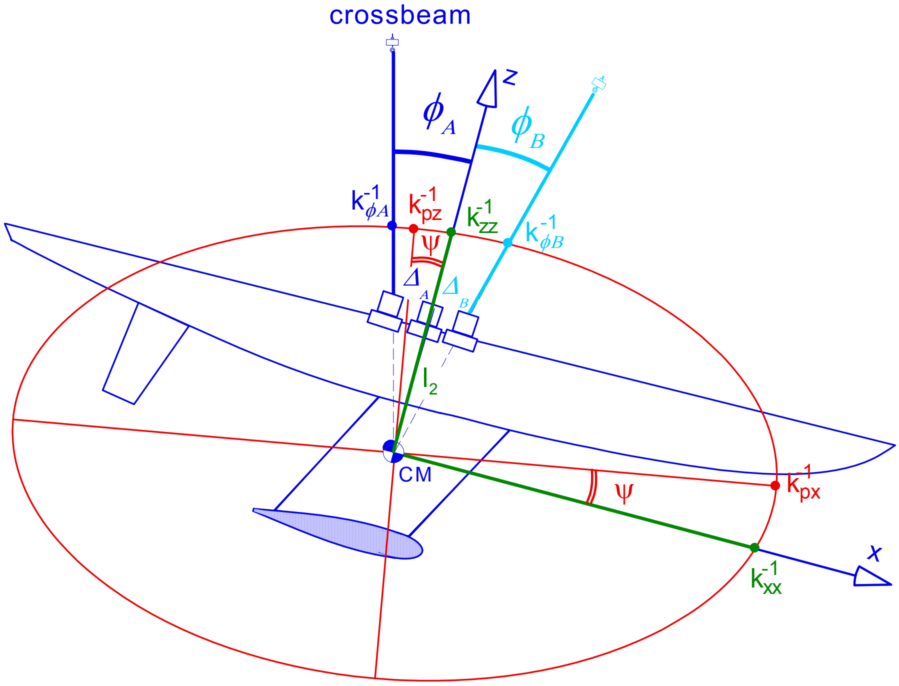

Figure 20. The hull suspended for a yaw test about an axis inclined at an angle ϕ_A (and subsequently at ϕ_B) to the hull z axis in order to determine ψ, the inclination of the principal axes of the inertia ellipsoid.

RECOMMENDED SUSPENSION SYSTEM

The system used to attach the suspension to the hull rotates with it and therefore is included in the measured gyradii. For dinghy hulls light hooks on the gunwales which tension a spacing wire of length 2d can be used.

However, for keelboat hulls a more substantial system is required and corrections for this support system have to either be calculated from the dimensions of the system or from independent moment of inertia measurements. In order to minimize calculation the same system should be used for all the measurements, and in order to avoid calculating products of inertia it should be symmetrical if possible. To minimize this correction, it should be recalled that mass added at the gyradius from the center of mass has a

ISSN 1548-6559

negligible effect on the measured gyradius.

A square frame of side 2*d* with suspension points at the corners and the centers of the sides (see Figure 21) would facilitate all of the measurements described here. The roll k_{xs}, pitch k_{ys} and yaw k_{zs} gyradii of such a square frame about the hull CM are:

$$k_{xs} = k_{ys} \approx \sqrt{l_2^2 + 2d^2/3} \qquad (27)$$
$$k_{zs} \approx 2d/\sqrt{3}$$

Then preliminary estimates of the hull gyradii can be used to choose an optimum value for the spacing *d*.

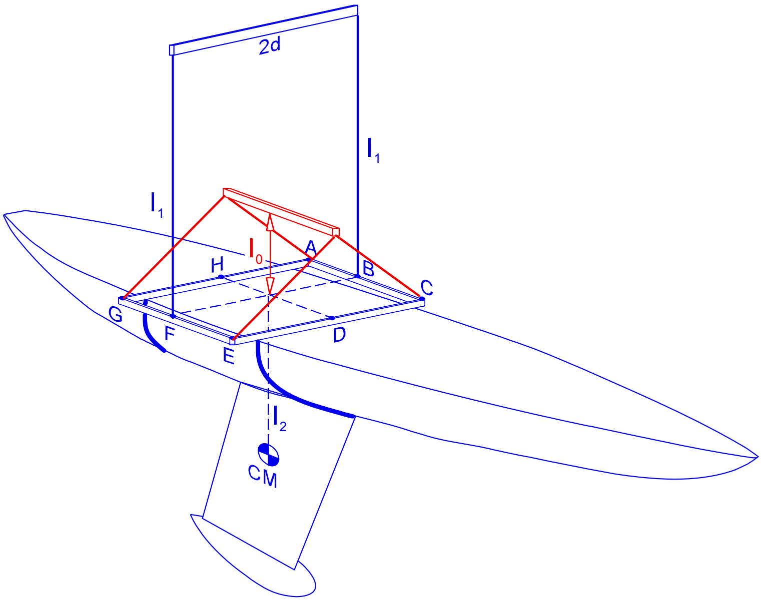

Figure 21. The recommended keelboat support system is a light square frame with suspension points A to H at the corners and the centers of each side.

Table 4. Suspension options.

Suspension points	Yaw	Pitch	Roll	Center of Mass
B-F	k_{zz}	k_{yy}		l_2
H-D	k_{zz}		k_{xx}	l_2
A-G	$k_{\phi A}$	k_{yy}		l_2
C-E	$k_{\phi B}$	k_{yy}		l_2
A-C	$k_{\varphi A}$		k_{xx}	l_2
G-E	$K_{\varphi B}$		k_{xx}	l_2
A-C+G-E		k_{yy}		
A-G+C-E			k_{xx}	

Note: Pitch inclinations ϕ are in the xz center plane
while roll inclinations φ are in the yz athwartships plane.

For a typical keelboat hull, such as the Yngling, this leads to a choice of 2*d* only slightly greater than the beam, which is ideal from a mechanical point of view. Although mounting the frame parallel to the water plane is preferable, if a cabin prevents this it is not essential. Once the inertia ellipsoid is established in the frame of reference of the square frame the gyradii along the hull water plane axes can easily be derived using equation (3).

Using such a square frame, the measurements that can be made at spacing 2*d* are listed in Table 4. Suspensions from A-C and/or G-E could be used to confirm the symmetry about the center plane of the hull. The A-G+C-E suspension shown in red in Figure 21 can be used to measure the roll gyradius k_r once l_2 is known from other measurements, such as $l_2 = d/\mathrm{Tan}\phi_A$ from suspension A-G. The correction to the present keelboat yaw gyradius measurements due to the substantial crossbeam plus webbing was 0.7 percent.

PRECISION AND CORRECTIONS

It is necessary to apply a number of corrections to the raw data in order to obtain the actual moment of inertia of the hull and these will be briefly listed. See the previous section for the correction for the suspension frame. The derivation of equation (5) does not take into account the generally negligible kinetic energy of the heave oscillation accompanying the yaw oscillation. The correction to the yaw oscillation period T_y is, in general, negligible but can be included in numerical solutions of the full equations of motion (Jardin, 2009).

Hull leveling

If the hull is not leveled in pitch and in roll the measured gyradius at a pitch angle $\phi_p = (\phi + \psi)$ and roll angle φ to the principal z axis is given by:

$$k^2(\phi_p,\varphi) = \frac{k^2_{px} Tan^2\phi_p Cos^2\varphi}{1+Tan^2\phi_p Cos^2\varphi} + \frac{k^2_{py} Tan^2\varphi Cos^2\phi_p}{1+Tan^2\varphi Cos^2\phi_p} + \frac{k^2_{pz}}{1+Tan^2\phi_p + Tan^2\varphi} \qquad (28)$$

For a perfectly symmetrical hull with $\psi = \phi_p = \varphi = 0$ both $\partial k/\partial\phi = 0$ and $\partial k/\partial\varphi = 0$. If the pitch and yaw gyradii are approximately equal, and this is usually so for hulls, then $\partial k/\partial\varphi$ essentially remains zero. For the present keelboat hull $\partial k/\partial\varphi = 0.03$ mm/deg, so roll leveling is not critical. However for finite angles ψ between the principal and the hull z axes $\partial k/\partial\phi$ increases in magnitude as shown in Figure 22.

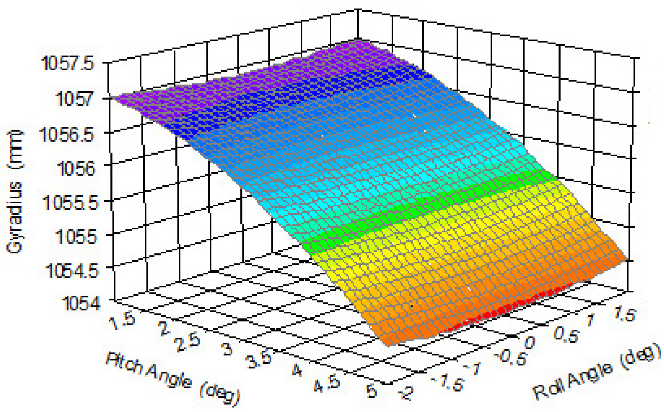

Figure 22. The measured gyradius as a function of the pitch angle ϕ_p and roll angle φ with the principal z axis for ψ = 3 degrees.

For ψ = 3 degrees, k_{px} ~ 623 mm (estimated by modeling the keelboat hull shell as a semi ellipsoid) and k_{pz} = 1058 mm then $\partial k/\partial \phi$ = 0.6 mm/deg, but increases with pitch angle so that at $\phi = 18^0$ it is 3 mm/deg. Pitch angles must therefore be measured precisely when determining ψ.

Air damping and added aerodynamic mass

The hull is swung in air which, therefore, affects the measurements in a number of ways. The buoyancy force acts to reduce the effective value of the gravitational acceleration to $g' = (1-\rho_a/\rho)\, g$, with the air density ρ_a ~ 0.0012 gm/ml and the hull density ρ ~ 0.7 to 1 gm/ml. This correction is typically of the order 0.1 percent. This is comparable with the variation of g with latitude (Wells, 1971), which should also be taken into account. The entrapped air will also make a small contribution. However, the significant effects of the air are the damping, which causes the amplitude and hence the period to change, and the added aerodynamic mass effects (Gracey, 1941; Newman, 1977; Brennen, 1982) which are due to the inertia added to the hull because an accelerating body accelerates some volume of the surrounding air. Essentially the kinetic energy of the motion must include that of the surrounding air and this can be taken into account by adding the mass M' of the displaced air multiplied by a shape and velocity dependent added aerodynamic mass coefficient. These coefficients can be calculated for linear motion of simple shapes such as ellipsoids, but for oscillatory motion of arbitrary shapes they have to be measured empirically.

The damping of both the yaw and sway of the keelboat and model hull oscillations was analyzed in detail and found to be complex. To first approximation the keelboat had decay time constants of τ_y ~ 640s and τ_s ~ 124s for yaw and sway. This damping can be caused by the air (Nelson and Olsson, 1986) but also by friction (Squire, 1986) at the bearings and flexure of the hull (Peters and Pritchett, 1977; Peters, 2003)' and fittings. Linear frictional damping does not change the period (Squire, 1986) and even for large linear velocity dependent damping the period change is negligible (Peters, 2003). For linear damping with decay constant τ the period is $T = T_0 \Big/ \sqrt{1+\left(T/2\pi\tau\right)^2}$, so for the observed $T/\tau = 6.3 \times 10^{-3}$ the correction is $T_0 = T\left(1-5x10^{-7}\right)$.

The Reynold's Number (Newman, 1977) of the motion varies from zero to about 10^5 so the flow would transition from laminar to turbulent and the drag force would transition from linear to quadratic each quarter cycle. However, the primary difficulty in modeling this situation is that after the first quarter oscillation, the hull moves through disturbed air which has a component of velocity in the opposite direction and the shed vortices can feed back some energy to the motion (Dynnikov and Dynnikova, 2008). The effect of nonlinear damping can to some extent be taken into account by numerical modeling using Simulink (Jardin and Mueller, 2009). The hull oscillation damping curves θ_0(t) were empirically modeled by assuming that an exponential decay constant τ was a power function of the amplitude θ_0, i.e. $\tau\left(\theta_o\right) \propto \theta_0^{(1-\beta)}$ where β is a constant which can be derived from a fit to the data. Typically β = 0.3, which corresponds to an average damping torque $\Gamma_D \propto \dot{\theta}^{(1+\beta)}$, i.e. between linear and quadratic dependence on the angular velocity.

ISSN 1548-6559

Period variation with amplitude

The pendulum period given by equation (7) is the small amplitude limit and due to nonlinearity the period is expected to vary with amplitude ϕ (radians) as $T_s(\phi)=T_s(0)\left(1+\phi^2/16+\ldots\right)$ (Baker and Blackburn, 2005). Bifilar oscillations have a similar theoretical dependence on amplitude (Cromer, 1995; Jardin and Mueller, 2009). However, for both the present yaw measurements and those of pitch oscillations of both dinghy and keelboat hulls, the variation of the period with amplitude is found to be approximately linear and greater than this equation predicts (see Figure 23). The rapid decrease in the Dragon and Yngling pitch periods below 1 degree are probably due to striction in the bearing, as similar effects have been observed for pendulums with sharp point bearings, and this effect limited the precision of the model hull measurements. The design of the three degree-of-rotation bearings is therefore crucial if high precision measurements are required. Extrapolating to zero amplitude is not feasible with the present period data, as for small amplitude it varies as the logarithm of the amplitude. The surrounding air could be the cause of the linear period increase observed at larger amplitudes in both the pitch and yaw data.

However, as shown in Figures 7 and 18 relative measurements can be very precise if taken at the same amplitude. The sway period was also amplitude dependent and contributed to the amplitude variation of the T_y/T_s ratio with sway amplitude, as shown in Figure 24. These effects limit the achievable precision and contribute to the uncertainties in the present measurements.

The effects of various misalignments and approximations such as those due to the finite suspension line mass, elasticity and torque, non-parallel or unequal suspension wires, non-level support, or hull, and off center hull have been calculated (Cromer, 1995) and shown to have negligible effect provided reasonable care is taken. They can, however, lead to small parasitic oscillations.

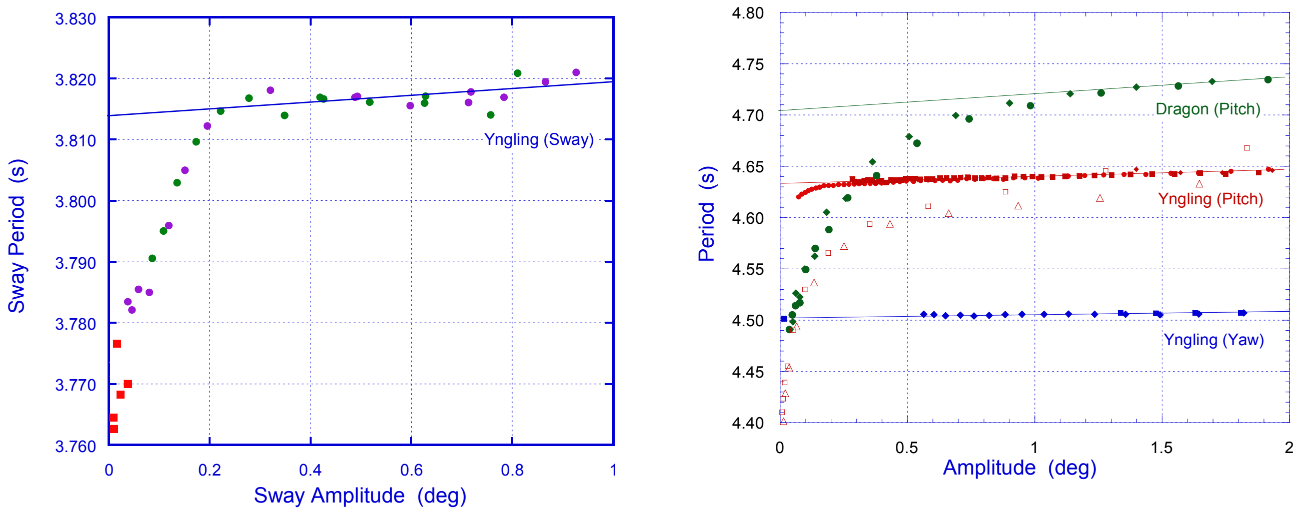

Figure 23. The variation of the Dragon pitch, the Yngling pitch (two hulls), Yaw and sway periods with amplitude.

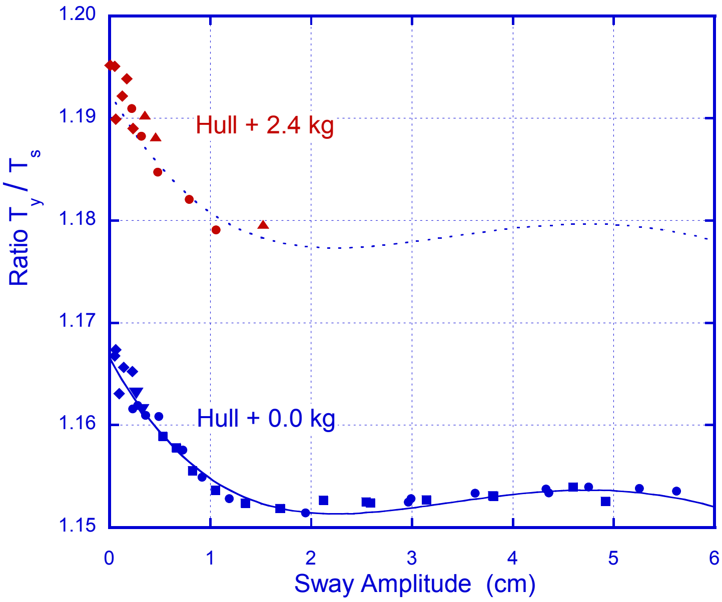

Figure 24. The variation of the keelboat T_y/T_s ratio with sway amplitude for the bare hull, and with 2.4 kg of added masses.

Yaw-Sway-Pitch coupling

When the suspended horizontal hull is displaced in the plane of the suspension, it swings in sway (with some heave) and does not develop any other oscillations as the suspension forms a parallelogram, so the hull remains level. When the horizontal hull is rotated about the vertical symmetry axis it oscillates in yaw (again with some heave), but without developing any other oscillations. The suspension lines again make the same angle with the vertical, although in opposite directions, so the two suspension points are at the same height and the hull does not roll. Thus in both cases, although the hull oscillates vertically, it remains horizontal with no torques about a horizontal axis. If the hull is only displaced perpendicular to the plane of the suspension, i.e. in pitch-surge, then it oscillates as a double pendulum and does not develop any sway or yaw.

However, when the hull is displaced in both sway and yaw, the two suspension wires are no longer at the same angle to the vertical, so the hull is no longer perfectly horizontal. For the keelboat suspension (l = 3.477 m and d = 0.900 m) rotated in yaw by 10 degrees and displaced in sway by 20 cm, the difference between the vertical displacement and the sum of those for pure sway and pure yaw is less than 0.07 percent. The roll angle is less than 0.06 degrees (0.001 rad) so the vertical displacement of the center of mass due to roll is negligible ($<5\times10^{-4}$ mm compared to 5.8 mm for the sway). Thus the effect on the sway and yaw periods was negligible.

However, although the roll is constrained by the suspension, the hull is still free to pitch. The coupling to pitch motion is due to a small net horizontal torque, as the torques due to the two suspension lines no longer cancel. This effect is very small and the motion is, in general, a combination of the yaw and sway oscillations as described above. In the present keelboat and model hull measurements no significant pitching was observed to develop except when the model hull suspension lines were lengthened to 1560 mm. It can be seen from Figure 15 that at this length the yaw period T_y approaches that of the double

ISSN 1548-6559

pendulum mode 1 oscillation T_1, which may be a sufficient, but not necessary condition for the pitching to develop.

T. R. Kane and Gan-Tai Tseng (Kane and Tseng, 1967) have shown that due to the nonlinearities in the Lagrangian equations for the four degrees of freedom of the bifilar suspension, for certain combinations of the suspension line lengths l, their spacing $2d$ and the gyradii, k_y and k_p, of the suspended hull, the motion of the hull after simultaneous displacement in sway and yaw develops pitching oscillation. The initial sway-yaw motion decreases in amplitude and the hull begins to pitch in a beat pattern which is a combination of at least two frequencies which are not those of the normal pitching modes. This pitching builds up as the sway-yaw decreases, after which the sway-yaw increases as the pitching beats decrease and the cycle repeats but with less amplitude due to damping. This interesting motion resembles that of a Wilberforce pendulum (Berg and Marshall 1991).

This motion, although predicted by T. R. Kane and Gan-Tai Tseng, still requires further analysis in order to determine the precise conditions under which it occurs, and thus should be avoided. The period of yaw oscillation T_y is shifted by this coupling to the pitching oscillation and cannot, therefore, be used for accurate gyradius measurement. However, it only occurs with simultaneous sway and yaw excitation, so individual sway and yaw measurements are still accurate and it can be avoided by changing the geometry of the suspension.

COMPARISON OF PITCH AND YAW GYRADII

It is generally assumed that the yaw and pitch gyradii of hulls are almost equal, as the beam and depth are comparable and significantly smaller than the LOA. This assumption was confirmed by yaw and pitch gyradii measurements on some 22 Flying Dutchman dinghy hulls in 1984 (Hinrichsen, 1991). However for keelboats, and especially those with modern deep bulb keels, the keel makes a much larger contribution to the pitch moment of inertia than to the yaw moment of inertia; firstly, because the keel pitch gyradius is larger than the yaw gyradius and secondly, due to the small horizontal but large vertical displacement of the keel center of mass from that of the hull.

The details of the keelboat keel design and specifications were available as well as measurements of its center of mass and pitch gyradius. After correction for the swing system, the measured keel pitch gyradius was k_{kp} = 365±15 mm and the center of mass position of x_k = 693±34 mm and z_k = 286±13 mm relative to the keel datum point were in good agreement with the calculated values.

The measured pitch and yaw gyradii and the center of mass of the "hull plus keel" combined with the calculated values for the keel (see Table 3) allow one to deduce the pitch and yaw gyradii of the hull (plus the rudder and fittings) as k_{hp} = 1448 mm and k_{hy} = 1459 mm. They are within 0.8 percent of each other. Thus the assumption that the pitch and yaw gyradii of the hull are essentially the same is confirmed.

However, when the keel is included the pitch and yaw moments of inertia are I_p = 832 kg·m^2 and I_y = 696 kg·m^2. The pitch and yaw moments of the hull alone are I_{hp} = 664 kg·m^2 (80%) and I_{hy} = 674 kg·m^2 (97%) and for the keel alone I_{kp} = 35 kg·m^2 (4%) and I_{ky} = 21 kg·m^2 (3%). The horizontal separation of the hull and keel centers of mass is

58 mm, and so only adds 0.5 kg·m^2 to the boat yaw moment of inertia. In contrast, the keel center of mass is 923 mm below that of the hull and so this adds 133 kg·m^2 to the pitch moment of inertia of the boat, i.e. four times as much as just the keel moment of inertia itself. The positions of the hull and keel centers of mass suggest that the principal yaw axis could be at an angle $\psi \sim \tan^{-1}(58/923) = 3.6^0$.

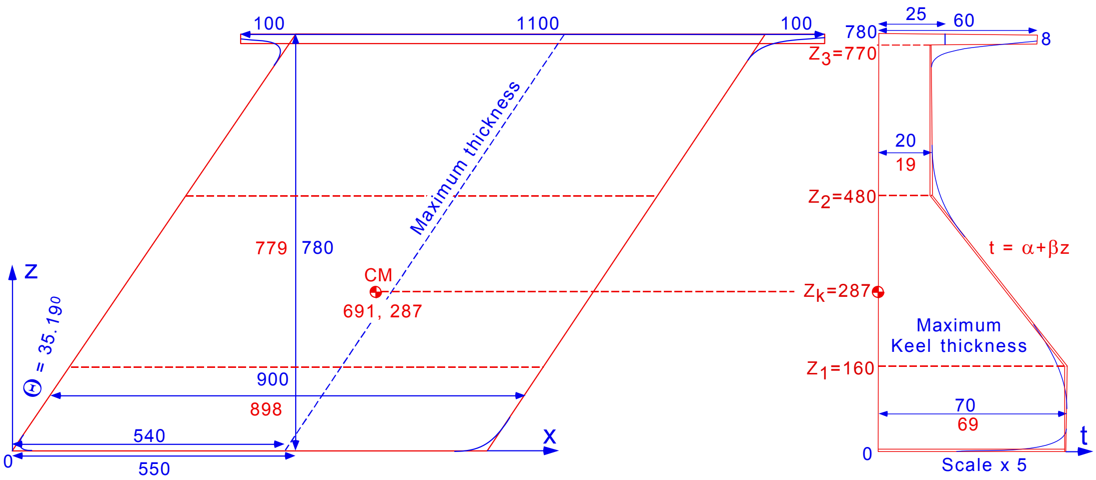

Figure 25. The keel outline and maximum width t versus depth z. Note the x5 scale for the width t.

Figure 26. A keel suspended from a swing bearing for center of mass and pitch gyradius measurement.

The net effect of the keel is thus to contribute an extra 169 kgm^2 (20%) to the pitch moment but only 23 kg·m^2 (3%) to the yaw moment. The keel contribution to the total moments of inertia is thus about six times greater in pitch than in yaw. In the case of a bulb keel one would expect this fraction to be even higher.

Table 4. Keelboat hull and keel data.

Quantity	Hull +Keel	Keel (calc.)	Hull (calc.)
Mass *M* kg	622.7	306.2	316.5
Center of Mass $\bar{x}$ mm	3112	3141	3083
Center of Mass $\bar{z}$ mm	-465	-934	-11
Pitch Gyradius k_p mm	1156	339	1448
Yaw Gyradius k_y mm	1057	263	1459

Note: *Center of Mass is relative to the Hull Datum point, see HDP in Figure 2.*

CONCLUSION

The four elements of the inertia tensor and the position of the center of mass of a symmetrical hull can be inexpensively measured by determining the yaw and double pendulum oscillation periods of the hull on a bifilar suspension. Observation of the yaw-sway beat pattern provides a simple method of determining the yaw gyradius in terms of just the suspension spacing, hence eliminating other parameters and sources of error. The variation of the oscillation periods with amplitude, although small, are greater than predicted for both the bifilar yaw and double pendulum pitch oscillations, and limit the accuracy and precision of these hull gyradius measurements but also apply to other methods of measurement. It is believed that the use of precision bearings would have significantly improved these measurements. However, the period ratio method, at a given amplitude, can detect yaw gyradius changes of 0.1 percent. The keelboat measurements were made with just some spectra line, crossbeams and a timer triggered by a photogate as well as an ultrasonic range measurement system. However, such measurements can now be made using something as simple as an iPhone with the xSensor App (xSensor, 2010), as was used for the measurements on the model hull. The iPhone simultaneously records both the three components of the acceleration as well as the rotations about the three axes and can therefore monitor the presence of any parasitic oscillations which could affect the precision.

Once the elements of the inertia ellipsoid about the center of mass are known, the moment of inertia about an axis at any orientation and through any point in space can be calculated (Wells 1967), and the moments of inertia of other components such as the mast, sails and crew can be added once their inertia parameters are known (Hinrichsen, 2002).

ACKNOWLEDGEMENTS

The present keelboat measurements would not have been possible without the very generous cooperation of Hein Ruyten who made the keelboat hull and the facilities of his boatyard available for the day. I am also grateful to Hans Huber for the loan of the model hull. Dr. Matt Jardin kindly forwarded his papers on moment of inertia measurements of a UAV, as well as pointed me to the NACA reports on full-scale aircraft measurements. I would like to thank Andrew Williams of 3Dmeasure.com for graciously supplying Figure (1). A shorter version of this paper was presented at the 21st Chesapeake Sailing Yacht Symposium, Annapolis, Maryland, March 2013.

ISSN 1548-6559

REFERENCES

Ales, M. R. and McGettigan, J. L. (1981). "An Experimental Analysis of the Effects of Pitch Gyradius on Ship Motion in Head Seas." Division of Engineering and Weapons, United States Naval Academy, Annapolis.

Auerbach, F. and Host, W. (1930). *Handbuch der Physikalischen und Techinschen Mechanik*. Leipzig, Verlag von Johann Ambrosius Barth.

Baierlein, R. (1983). *Newtonian Dynamics*, New York, McGraw-Hill Book Co.

Baker, G. L. and Blackburn, J. A. (2005). "The Pendulum - A Case Study in Physics", Oxford University Press.

Berg, R. E. and Marshall, T. S. (1991). "Wilberforce Pendulum Oscillations and Normal Modes." *American Journal of Physics* **59** (January), 18.

Brennen, C. E. (1982). "A Review of Added Mass and Fluid Inertial Forces." Naval Civil Engineering Laboratory, CR 82.010, 55.

Card, T. (2000). "Bifilar Swinging Experiment." St Johns, Newfoundland, National Research Council Canada, Institute for Marine Dynamics, 28.

Cromer, A. (1995). "Many Oscillations of a Rigid Rod." *American Journal of Physics*, **63** (2 February), 9.

Daedalon (1991). "EG-90 Ultrasonic Measurement System.", 50.

Dorn, W. G. V. (1974). *Oceanography and Seamanship*. New York, Dodd, Mead & Company.

Dynnikov, Y. and Dynnikova, G. (2008). "Application of Viscous Vortex Domains Method for Solving Flow-Sructure Problems." Moscow, Institute of Mechanics of Lemonosov Moscow State University, 6.
http://istina.msu.ru/media/publications/articles/7f0/342/3841370/fullpaper_dyndyn.pdf

Gracey, W. (1941). "The Additional-Mass Effect of Plates as Determined by Experiment." Report National Advisory Committee for Aeronautics (707), 11.

Green, M. W. (1927). "Measurement of the Moments of Inertia of Full Scale Aircraft." Technical Notes National Advisory Committee for Aeronautics (265), 18.

Gregory, R. D. (2006). *Classical Mechanics*. Cambridge, Cambridge University Press.

Hamel, G. (1912). *Element Mechanik.* Leipzig.

Hinrichsen, P. F. (1986). "Weight Distribution." Yachts and Yachting, 23-.

ISSN 1548-6559

Hinrichsen, P. F. (1991). "Gyradius Measurement of Olympic Class Dinghies and Keelboats." *Proceedings of the Tenth Cheasapeake Sailing Yacht Symposium*, Annapolis, Maryland, USA.

Hinrichsen, P. F. (2002). "Swinging in Marblehead." Finnfare(March), 4.

Hinrichsen, P. F. (2004). "Keelboat Swing Tests." ISAF Guide for Measurement. J. P. Marmier. Southampton, ISAF**,** 39-168.

Jardin, M. R. (2009). "Improving Mass Moment of Inertia Measurements." MATLAB Digest. Maine, Mathworks**,** 8.

Jardin, M. R. and Mueller, E. R. (2009). "Optimized Measurements of Unmanned-Air-Vehicle Mass Moment of Inertia with a Bifilar Pendulum." *Journal of Aircraft*, **46,** (May-June), 3.

Kane, T. R. and Tseng, G.-T. (1967). "Dynamics of the Bifilar Pendulum." *Int. J. Mech.Sci.* **9**, 83-96.

Kiss, T. K. (1987). "An Experimental Analysis of the Effects of Pitch Gyradius on Added Ship Resistance in Waves." Division of Engineering and Weapons. Annapolis, Maryland, 21402, United States Naval Academy**,** 28.

Lamboley, G. (1971). "Influence of the Weight Distribution in a Boat, Proposed Means of Control," International Finn Association, c/o IYRU London.

Lamboley, G. (1971). "Weight Distribution." *Yachts and Yachting***,** 790-793 and 921-924.

Lungu, G. (2012). "A Fourier Transform Model in Excel.", www.excelunusual.com.

Mathworks "Matlab & Simulink." Natick, Massachusetts, U.S.A.

McCurdy, R. S. (1990). "Feasability Study of the Measurement of Mass Moment of Inertia in Pitch for Cruiser/Racer Yachts." *Proceedings of the New England Sailing Yacht Symposium*, 23 March 1990, Coast Guard Academy.

Miller, M. P. (1930). "An Accurate Method of Measuring the Moments of Inertia of Airplanes." Technical Notes National Advisory Committee for Aeronautics(351), 26.

Nelson, R. A. and Olsson, M. G. (1986). "The Pendulum-Rich Physics from a Simple System." *American Journal of Physics* **54** (2 February), 9.

Newman, F. H. and Searle, V. H. (1951). *The General Properties of Matter*. Edward Arnold & Co. London

Newman, J. N. (1977). *Marine Hydrodynamics*. The MIT Press, Cambridge, MA.

Pasco (2000). *DataStudio*. Pasco Scientific, San Francisco. www.pasco.com

Peters, R. D. (2003). "Nonlinear Damping of the 'Linear' Pendulum." from arxiv.org/pdf/physics/0306081.

Peters, R. D. and Pritchett, T. (1977). "The Not-so-simple Harmonic Oscillator." *American Journal of Physics* **65** (11), 1067-1073.

Rafat, M. Z., Wheatland, M. S., et al. (2009). "Dynamics of the Double Pendulum with Distributed Mass." *American Journal of Physics* **77** (3).

Roy, D. (1984). "Balance for Measuring Moments of Inertia of Ship Models." VMI L127. Institute for Marine Dynamics, National Research Council Canada, Ottawa**,** 42.

Spiegel, M. R. (1967). *Theoretical Mechanics*. Schaum Series, McGraw-Hill, New York.

Squire, P. T. (1986). "Pendulum Damping." *American Journal of Physics* **54** (11 November), 984-991.

TableCurve2D, J. S. S. (1994). *TableCurve2D*, Systat Software Inc., www.sigmaplot.com

Turner, H. L. (1950). "Measurement of the Moment of Inertia of an Airplane by a Simplified Method." National Advisory Committee for Aeronautics. NACA TN 2201, Washington, 49.

Van Duyne, C. (1972). "Measuring Weight Distribution: Lamboley." Yachting**,** 35-194.

Veltmann, W. (1876). "Ueber die Bewegung einer Glocke." *Dinglers polytechnisches Journal* **22**, 481-494.

Waine, A. (1988). "A Method for the Measurement of Damping in the Bifilar Suspension System," Middlebury College, Vermont, U.S.A.

Watts, T. (1986). "International Dragon Class Swing Tests." London, International Yacht Racing Union, London**,** 17.

Wells, D. A. (1967). *Lagrangian Dynamics*. Schaum Series, McGraw-Hill, New York.

Wells, T. (1971). "Moment of Inertia." Snipe Bulletin**,** 13.

xSensor (2010). "iPhone App." Crossbow Technologies Inc., www.iTunes.com.