Dinamica Non Lineare di Strutture e Sistemi Meccanici

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Lezione 2

Lagrangian formulation for discrete systems or discretized continua

Holonomic constraints $\mathbf{R} = \mathbf{R}(q_1, q_2, ..., q_n, t) \Rightarrow \dot{\mathbf{R}} = \frac{\partial \mathbf{R}}{\partial q_i}\dot{q}_i + \frac{\partial \mathbf{R}}{\partial t}$

Kinetic energy $T = \frac{1}{2}\int_\Omega \rho(\dot{\mathbf{R}} \cdot \dot{\mathbf{R}}) d\Omega = \frac{1}{2}A^{ij}\dot{q}_i\dot{q}_j + B^i\dot{q}_i + \frac{1}{2}C$

$$A^{ij}(q_1, q_2, ..., q_n, t) = \int_\Omega \rho\left(\frac{\partial \mathbf{R}}{\partial q_i} \cdot \frac{\partial \mathbf{R}}{\partial q_j}\right) d\Omega$$

$$B^{i}(q_1, q_2, ..., q_n, t) = \int_\Omega \rho\left(\frac{\partial \mathbf{R}}{\partial q_i} \cdot \frac{\partial \mathbf{R}}{\partial t}\right) d\Omega$$

$$C(q_1, q_2, ..., q_n, t) = \int_\Omega \rho\left(\frac{\partial \mathbf{R}}{\partial t} \cdot \frac{\partial \mathbf{R}}{\partial t}\right) d\Omega$$

Total potential energy $V(q_1, q_2, ..., q_n, t)$

Lagrangian formulation…

Lagrange's equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)-\frac{\partial L}{\partial q_i}=N_i$$

$$L=T\left(q_1,q_2,\ldots,q_n,\dot{q}_1,\dot{q}_2,\ldots,\dot{q}_n,t\right)-V\left(q_1,q_2,\ldots,q_n,t\right)$$

$$\Rightarrow \quad A^{ij}\ddot{q}_j+\left(\frac{\partial A^{ij}}{\partial q_k}-\frac{1}{2}\frac{\partial A^{jk}}{\partial q_i}\right)\dot{q}_j\dot{q}_k+\frac{\partial A^{ij}}{\partial t}\dot{q}_j$$

$$+\left(\frac{\partial B^i}{\partial q_j}-\frac{\partial B^j}{\partial q_i}\right)\dot{q}_j+\frac{\partial B^i}{\partial t}-\frac{1}{2}\frac{\partial C}{\partial q_i}+\frac{\partial V}{\partial q_i}=N_i$$

Lagrangian formulation…

Gyroscopic force (Coriolis)

$$\left(\frac{\partial B^i}{\partial q_j} - \frac{\partial B^j}{\partial q_i}\right)\dot{q}_j$$

$\Downarrow$

Gyroscopic "damping" matrix is anti-symmetric

Scleronomic systems

$$\mathbf{R} = \mathbf{R}\left(q_1, q_2, \ldots, q_n\right)$$

$$\frac{\partial A^{ij}}{\partial t} = 0; \quad B^i = 0; \quad C = 0 \quad \Rightarrow \quad A^{ij}\ddot{q}_j + \left(\frac{\partial A^{ij}}{\partial q_k} - \frac{1}{2}\frac{\partial A^{jk}}{\partial q_i}\right)\dot{q}_j\dot{q}_k + \frac{\partial V}{\partial q_i} = N_i$$

If the mass matrix is constant...

$$\frac{\partial A^{ij}}{\partial q_k} = 0 \quad \Rightarrow \quad A^{ij}\ddot{q}_j + \frac{\partial V}{\partial q_i} = N_i$$

Hamiltonian formulation…

Legendre's transformation $$f(x,y) \Rightarrow g(u,y) = ux - f(x,y)$$

with $u = \dfrac{\partial f}{\partial x}$

Observe that $$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = udx + \frac{\partial f}{\partial y}dy$$

$$dg = \frac{\partial g}{\partial u}du + \frac{\partial g}{\partial y}dy$$

$$= xdu + \overbrace{udx - \frac{\partial f}{\partial x}dx}^{0} - \frac{\partial f}{\partial y}dy$$

$$= xdu - \frac{\partial f}{\partial y}dy \Rightarrow x = \frac{\partial g}{\partial u} \quad \text{e} \quad \frac{\partial g}{\partial y} = -\frac{\partial f}{\partial y}$$

Hamiltonian formulation…

Duality

"Old"

$$f(x,y) = ux - g(u,y)$$

$$u = \frac{\partial f}{\partial x}$$

"New"

$$g(u,y) = ux - f(x,y)$$

$$x = \frac{\partial g}{\partial u}$$

Application to the Lagrangian

$$L(\mathbf{q},\dot{\mathbf{q}},t) \quad \Rightarrow \quad H(\mathbf{q},\mathbf{p},t) = \mathbf{p}^{\mathrm{T}}\dot{\mathbf{q}} - L(\mathbf{q},\dot{\mathbf{q}},t)$$

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \quad \Leftarrow \quad \text{with} \quad p_i = \frac{\partial L}{\partial \dot{q}_i}$$

Hamiltonian formulation…

$$H(\mathbf{q},\mathbf{p},t)=\mathbf{p}^{\mathrm{T}}\dot{\mathbf{q}}-L\left(\mathbf{q},\dot{\mathbf{q}},t\right)$$

$$dH=\frac{\partial H}{\partial q_i}dq_i+\frac{\partial H}{\partial p_i}dp_i+\frac{\partial H}{\partial t}dt=$$

$$=\dot{q}_i dp_i+\underbrace{p_i d\dot{q}_i-\frac{\partial L}{\partial \dot{q}_i}d\dot{q}_i}_{0}-\frac{\partial L}{\partial q_i}dq_i-\frac{\partial L}{\partial t}dt$$

$$\Rightarrow\quad\left\{\begin{aligned}\dot{q}_i&=\frac{\partial H}{\partial p_i}\\ \frac{\partial H}{\partial q_i}&=-\frac{\partial L}{\partial q_i}\\ \frac{\partial H}{\partial t}&=-\frac{\partial L}{\partial t}\end{aligned}\right.$$

Hamiltonian formulation…

Lagrange's equation with $N_i = 0$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \Rightarrow \left.\begin{aligned} \dot{p}_i &= \frac{\partial L}{\partial q_i} = -\frac{\partial H}{\partial q_i} \\ \dot{q}_i &= \frac{\partial H}{\partial p_i} \end{aligned}\right\}$$

First-order system

Hamilton's canonical equations

$$\dot{H} = \frac{dH}{dt} = \frac{\partial H}{\partial q_i}\dot{q}_i + \frac{\partial H}{\partial p_i}\dot{p}_i + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial H}{\partial p_i}\frac{\partial H}{\partial q_i} + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial t}$$

Hamiltonian formulation…

Lagrange's equation with $N_i \neq 0$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = N_i \Rightarrow \dot{p}_i = N_i + \frac{\partial L}{\partial q_i} = N_i - \frac{\partial H}{\partial q_i}$$

$$\dot{q}_i = \frac{\partial H}{\partial p_i}$$

Hamilton's canonical equations

$$\dot{H} = \frac{dH}{dt} = \frac{\partial H}{\partial q_i}\dot{q}_i + \frac{\partial H}{\partial p_i}\dot{p}_i + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial q_i}\frac{\partial H}{\partial p_i} + \frac{\partial H}{\partial p_i}\left(N_i - \frac{\partial H}{\partial q_i}\right) + \frac{\partial H}{\partial t} = N_i\frac{\partial H}{\partial p_i} + \frac{\partial H}{\partial t}$$

Hamiltonian formulation…

Scleronomic conservative system $\quad \mathbf{R} = \mathbf{R}(\mathbf{q}) \quad \text{e} \quad N_i = 0$

$$H = H(\mathbf{p},\mathbf{q}) \therefore \frac{\partial H}{\partial t} = 0 \Rightarrow \dot{H} = 0 \Rightarrow H = const.$$

$$L = T - V = \frac{1}{2} A^{ij} \dot{q}_i \dot{q}_j - V(\mathbf{q}) \Rightarrow \frac{\partial L}{\partial \dot{q}_j} = A^{ij} \dot{q}_i$$

$$H(\mathbf{q},\mathbf{p}) = \mathbf{p}^{\mathrm{T}} \dot{\mathbf{q}} - L(\mathbf{q},\dot{\mathbf{q}}) = p_j \dot{q}_j - L = \frac{\partial L}{\partial \dot{q}_j} \dot{q}_j - L$$

$$= A^{ij} \dot{q}_i \dot{q}_j - \frac{1}{2} A^{ij} \dot{q}_i \dot{q}_j + V = \frac{1}{2} A^{ij} \dot{q}_i \dot{q}_j + V$$

$$= T + V$$

Hamiltonian formulation…

Rheonomic system $\mathbf{R} = \mathbf{R}(\mathbf{q}, t)$ with $N_i = 0$ or $N_i \neq 0$

$$H = H(\mathbf{p}, \mathbf{q}, t) \text{ and } \dot{H} = N_i \frac{\partial H}{\partial p_i} + \frac{\partial H}{\partial t} \Rightarrow H \neq const.$$

$$L = T - V = \frac{1}{2} A^{ij} \dot{q}_i \dot{q}_j + B^i \dot{q}_i + \frac{1}{2} C - V(\mathbf{q}) \Rightarrow \frac{\partial L}{\partial \dot{q}_j} = A^{ij} \dot{q}_i + B^j$$

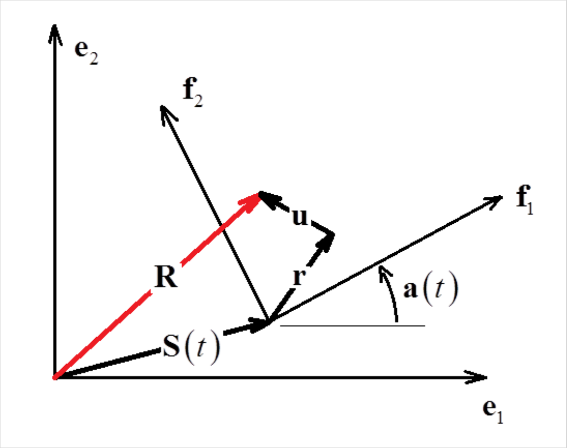
$$H(\mathbf{q}, \mathbf{p}) = \mathbf{p}^{\mathrm{T}} \dot{\mathbf{q}} - L(\mathbf{q}, \dot{\mathbf{q}}) = p_j \dot{q}_j - L = \frac{\partial L}{\partial \dot{q}_j} \dot{q}_j - T + V$$

$$= \left(A^{ij} \dot{q}_i \dot{q}_j + B^j \dot{q}_j \right) - T + V = \left(2T - C - B^j \dot{q}_j \right) - T + V$$

$$= \underbrace{T + V}_{\text{mechanic energy}} - \underbrace{C}_{\text{pseudo-potential energy}} - \underbrace{B^j \dot{q}_j}_{\text{Coriolis 'energy'}}$$

Total-Lagrangian Matrix Formulation

Consider a point of an elastic solid with scleronomic constraints
with respect to a relative reference frame, which, on its turn, is moving with respect to an “inertial” reference frame



Particular class of rheonomic constraint!

$$\mathbf{R} = \mathbf{S}(t) + \mathbf{r} + \mathbf{u}(\mathbf{q})$$

$$\mathbf{R} = R^i \mathbf{e}_i$$

$$\mathbf{r} = r^j \mathbf{f}_j = r^j \mathbf{a}^i_j(t) \mathbf{e}_i$$

$$\mathbf{u} = u^j \mathbf{f}_j = u^j \mathbf{a}^i_j(t) \mathbf{e}_i$$

$$R^i = S^i(t) + \mathbf{a}^i_j(t)\left[r^j + u^j(\mathbf{q})\right]$$

$$\mathbf{R} = \mathbf{S}(t) + \mathbf{a}(t)\left[\mathbf{r} + \mathbf{u}(\mathbf{q})\right]$$

Total-Lagrangian Matrix Formulation

Discretization $\mathbf{u}(\mathbf{q}) = \mathbf{H}\mathbf{q} + \boldsymbol{\Delta}(\mathbf{q})$ Geometric non-linearity

$\mathbf{H}$ is an interpolation matrix

Hence: $\mathbf{R} = \mathbf{S}(t) + \mathbf{a}(t)\left[\mathbf{r} + \mathbf{H}\mathbf{q} + \boldsymbol{\Delta}(\mathbf{q})\right]$

$$\delta\mathbf{R} = \mathbf{a}(t)\left[\mathbf{H} + \mathbf{N}\right]\delta\mathbf{q}, \text{ with } \mathbf{N} = \frac{\partial\boldsymbol{\Delta}}{\partial\mathbf{q}}$$

$$\dot{\mathbf{R}} = \dot{\mathbf{S}}(t) + \dot{\mathbf{a}}(t)\left[\mathbf{r} + \mathbf{H}\mathbf{q} + \boldsymbol{\Delta}\right] + \mathbf{a}(t)\left[\mathbf{H} + \mathbf{N}\right]\dot{\mathbf{q}}$$

$$\ddot{\mathbf{R}} = \ddot{\mathbf{S}}(t) + \ddot{\mathbf{a}}(t)\left[\mathbf{r} + \mathbf{H}\mathbf{q} + \boldsymbol{\Delta}\right] + 2\dot{\mathbf{a}}(t)\left[\mathbf{H} + \mathbf{N}\right]\dot{\mathbf{q}} + \mathbf{a}(t)\left[\mathbf{H} + \mathbf{N}\right]\ddot{\mathbf{q}} + \mathbf{a}(t)\mathbf{DN}\dot{\mathbf{q}}$$

$$\text{com } \mathbf{DN} = \sum_i \mathbf{NN_i}, \text{ e } \mathbf{NN_i} = \frac{\partial\mathbf{N}}{\partial q_i}\dot{q}_i$$

Total-Lagrangian Matrix Formulation

Linear elastic homogeneous and isotropic solid or rod

Theorem of virtual displacements/D'Alembert's principle $\delta W_{\text{int}} = \delta W_{ext}$

Continuous system

$$\delta W_{\text{int}} = \int_V \boldsymbol{\delta\varepsilon}^T \boldsymbol{\sigma} dV$$

$$\delta W_{ext} = \int_V \boldsymbol{\delta}\mathbf{R}^T \left(\mathbf{f}^{\mathbf{B}} - \rho \ddot{\mathbf{R}}\right) dV + \int_{S_f} \boldsymbol{\delta}\mathbf{R}^T \mathbf{f}^S dS$$

Discretized system

$$\delta W_{\text{int}} = \boldsymbol{\delta}\mathbf{q}^{\mathbf{T}} \mathbf{K}_{\mathbf{s}}(\mathbf{q})\mathbf{q}$$

$$\delta W_{ext} = \boldsymbol{\delta}\mathbf{q}^{\mathbf{T}} \left[\mathbf{P}_{\mathbf{s}} - \mathbf{M}_{\mathbf{s}}\ddot{\mathbf{q}} - \boldsymbol{\Delta}\mathbf{C}_{\mathbf{s}}\dot{\mathbf{q}} - \boldsymbol{\Delta}\mathbf{K}_{\mathbf{s}}\mathbf{q}\right]$$

$$\Downarrow$$

$$\boldsymbol{\delta}\mathbf{q}^{\mathbf{T}} \left\{\mathbf{M}_{\mathbf{s}}\ddot{\mathbf{q}} + \boldsymbol{\Delta}\mathbf{C}_{\mathbf{s}}\dot{\mathbf{q}} + \left[\mathbf{K}_{\mathbf{s}}(\mathbf{q}) + \boldsymbol{\Delta}\mathbf{K}_{\mathbf{s}}\right]\mathbf{q} - \mathbf{P}_{\mathbf{s}}\right\} = 0, \ \forall \boldsymbol{\delta}\mathbf{q}$$

$$\Downarrow$$

$$\mathbf{M}_{\mathbf{s}}\ddot{\mathbf{q}} + \boldsymbol{\Delta}\mathbf{C}_{\mathbf{s}}\dot{\mathbf{q}} + \left[\mathbf{K}_{\mathbf{s}}(\mathbf{q}) + \boldsymbol{\Delta}\mathbf{K}_{\mathbf{s}}\right]\mathbf{q} = \mathbf{P}_{\mathbf{s}}$$

Total-Lagrangian Matrix Formulation

Linear elastic homogeneous and isotropic solid or rod

Discretized system

$$\mathbf{M_s}\ddot{\mathbf{q}} + \Delta\mathbf{C_s}\dot{\mathbf{q}} + \left[\mathbf{K_s}\left(\mathbf{q}\right) + \Delta\mathbf{K_s}\right]\mathbf{q} = \mathbf{P_s}$$

$$\mathbf{M_s} = \int_V \left(\mathbf{H^T} + \mathbf{N^T}\right)\left(\mathbf{H} + \mathbf{N}\right) dV$$

$$\Delta\mathbf{C_s} = 2\int_V \rho\left(\mathbf{H^T} + \mathbf{N^T}\right)\mathbf{a^T}\dot{\mathbf{a}}\left(\mathbf{H} + \mathbf{N}\right) dV + \int_V \rho\left(\mathbf{H^T} + \mathbf{N^T}\right)\mathbf{DN}\,dV$$

$$\Delta\mathbf{K_s} = \int_V \rho\left(\mathbf{H^T} + \mathbf{N^T}\right)\mathbf{a^T}\ddot{\mathbf{a}}\mathbf{H}\,dV$$

$$\mathbf{P_s} = \int_V \left(\mathbf{H^T} + \mathbf{N^T}\right)\mathbf{a^T}\mathbf{f^B}\,dV + \int_{S_f} \left(\mathbf{H^T} + \mathbf{N^T}\right)\mathbf{a^T}\mathbf{f^s}\,dS$$

$$-\int_V \rho\left(\mathbf{H^T} + \mathbf{N^T}\right)\mathbf{a^T}\ddot{\mathbf{S}}\,dV - \int_V \rho\left(\mathbf{H^T} + \mathbf{N^T}\right)\mathbf{a^T}\ddot{\mathbf{a}}\left[\mathbf{r} + \boldsymbol{\Delta}\left(\mathbf{q}\right)\right] dV$$

Total-Lagrangian Matrix Formulation

2D linear elastic homogeneous and isotropic solid or rod

$$\mathbf{a} = \begin{bmatrix} \cos\theta(t) & -\sin\theta(t) \\ \sin\theta(t) & \cos\theta(t) \end{bmatrix}$$

$$\dot{\mathbf{a}} = \dot{\theta}\begin{bmatrix} -\sin\theta(t) & -\cos\theta(t) \\ \cos\theta(t) & -\sin\theta(t) \end{bmatrix} \Rightarrow \boxed{\mathbf{a}^{\mathrm{T}}\dot{\mathbf{a}} = \dot{\theta}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}$$

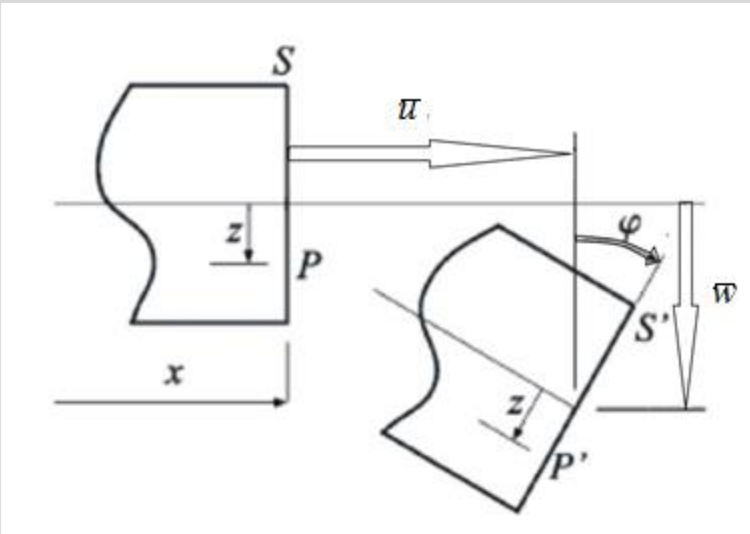
$$\ddot{\mathbf{a}} = \ddot{\theta}\begin{bmatrix} -\sin\theta(t) & -\cos\theta(t) \\ \cos\theta(t) & -\sin\theta(t) \end{bmatrix} - \dot{\theta}^2\begin{bmatrix} \cos\theta(t) & -\sin\theta(t) \\ \sin\theta(t) & \cos\theta(t) \end{bmatrix}$$

$$\Downarrow$$

$$\boxed{\mathbf{a}^{\mathrm{T}}\ddot{\mathbf{a}} = \ddot{\theta}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} - \dot{\theta}^2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}$$

Total-Lagrangian Matrix Formulation

Particular case: 2D linear Bernoulli-Euler prismatic rod element



$$u = \bar{u} - z\sin\varphi \cong \bar{u} - z\bar{w}'$$

$$w = \bar{w} + z(\cos\varphi - 1) \cong \bar{w}$$

$$\mathbf{u} = \boldsymbol{\Gamma}\bar{\mathbf{u}} = \boldsymbol{\Gamma}\,\mathbf{h}\mathbf{q} = \mathbf{H}\mathbf{q}$$

$\bar{\mathbf{u}}$

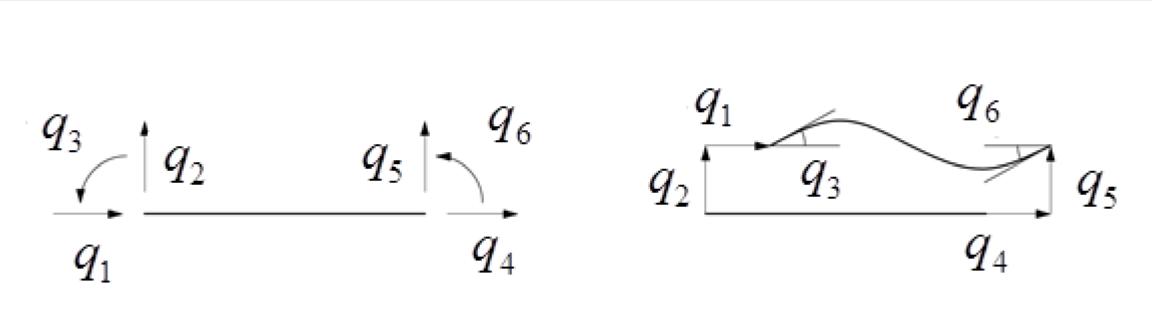
$$\mathbf{q} = \begin{bmatrix} q_1 & q_2 & q_3 & q_4 & q_5 & q_6 \end{bmatrix}^T$$

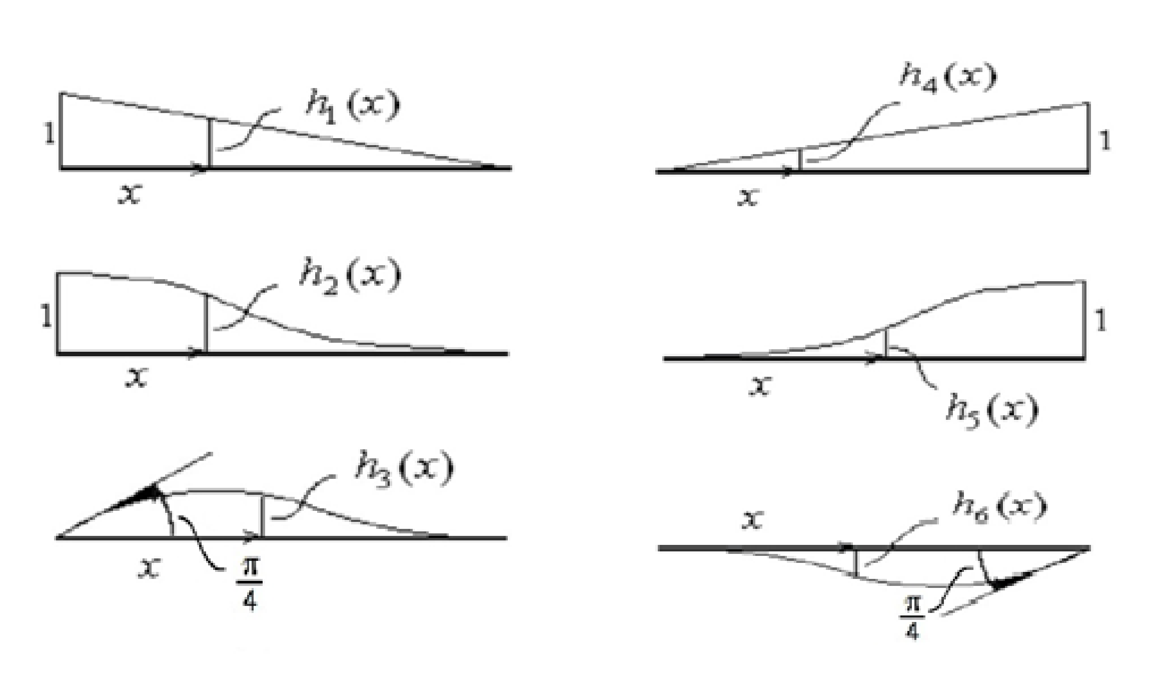
$$\boldsymbol{\Gamma} = \begin{bmatrix} 1 & -z\dfrac{\partial}{\partial x} \\ 0 & 1 \end{bmatrix} \qquad \mathbf{h} = \begin{bmatrix} h_1(x) & 0 & 0 & h_4(x) & 0 & 0 \\ 0 & h_2(x) & h_3(x) & 0 & h_5(x) & h_6(x) \end{bmatrix}$$

$$\mathbf{H} = \begin{bmatrix} h_1(x) & -zh_2'(x) & -zh_3'(x) & h_4(x) & -zh_5'(x) & -zh_6'(x) \\ 0 & h_2(x) & h_3(x) & 0 & h_5(x) & h_6(x) \end{bmatrix}$$

Total-Lagrangian Matrix Formulation

Particular case: 2D linear Bernoulli-Euler prismatic rod element





$$h_1(x) = 1 - \frac{x}{\ell}$$

$$h_2(x) = 1 - 3\frac{x^2}{\ell^2} + 2\frac{x^3}{\ell^3}$$

$$h_3(x) = x - 2\frac{x^2}{\ell} + \frac{x^3}{\ell^2}$$

$$h_4(x) = \frac{x}{\ell}$$

$$h_5(x) = 3\frac{x^2}{\ell^2} - 2\frac{x^3}{\ell^3}$$

$$h_6(x) = -\frac{x^2}{\ell} + \frac{x^3}{\ell^2}$$

Total-Lagrangian Matrix Formulation

Particular case: 2D linear Bernoulli-Euler prismatic rod element

$$\mathbf{R} = \mathbf{S}(t) + \mathbf{a}(t)[\mathbf{r} + \mathbf{u}]$$

Support excitation: $$\mathbf{S}(t) = \begin{Bmatrix} S_X(t) \\ S_Z(t) \end{Bmatrix} \qquad \mathbf{a} = \begin{bmatrix} \cos\theta(t) & -\sin\theta(t) \\ \sin\theta(t) & \cos\theta(t) \end{bmatrix}$$

Position vector in local system: $$\mathbf{r} = \begin{Bmatrix} x \\ z \end{Bmatrix}$$

$$\mathbf{u} = \mathbf{H}\mathbf{q}$$

Total-Lagrangian Matrix Formulation

Particular case: 2D linear Bernoulli-Euler prismatic rod element

$$\mathbf{M} = \int_V \rho \mathbf{H}^T \mathbf{H} dV$$

$$\mathbf{M} = \frac{\rho A \ell}{420} \begin{bmatrix} 140 & 0 & 0 & 70 & 0 & 0 \\ 0 & 156+504\eta & (22+42\eta)\ell & 0 & 54-504\eta & (-13+42\eta)\ell \\ 0 & (22+42\eta)\ell & (4+56\eta)\ell^2 & 0 & (13-42\eta)\ell & (-3-12\eta)\ell^2 \\ 70 & 0 & 0 & 140 & 0 & 0 \\ 0 & 54-504\eta & (13-42\eta)\ell & 0 & 156+504\eta & (-22-42\eta)\ell \\ 0 & (-13+42\eta)\ell & (-3-12\eta)\ell^2 & 0 & (-22-42\eta)\ell & (4+56\eta)\ell^2 \end{bmatrix}$$

$$\eta = \frac{I}{A\ell^2} << 1$$

Total-Lagrangian Matrix Formulation

Particular case: 2D linear Bernoulli-Euler prismatic rod element

Neglecting the mass moment of inertia

$$\mathbf{M} = \frac{\rho A \ell}{420}\begin{bmatrix} 140 & 0 & 0 & 70 & 0 & 0 \\ 0 & 156 & 22\ell & 0 & 54 & -13\ell \\ 0 & 22\ell & 4\ell^2 & 0 & 13\ell & -3\ell^2 \\ 70 & 0 & 0 & 140 & 0 & 0 \\ 0 & 54 & 13\ell & 0 & 156 & -22\ell \\ 0 & -13\ell & -3\ell^2 & 0 & -22\ell & 4\ell^2 \end{bmatrix}$$

Total-Lagrangian Matrix Formulation

Particular case: 2D linear Bernoulli-Euler prismatic rod element

$$\Delta\mathbf{C} = 2\int_V \rho\mathbf{H}^T\mathbf{a}^T\dot{\mathbf{a}}\mathbf{H}\,dV = 2\dot{\theta}\mathbf{C}_{\theta}$$

$$\mathbf{C}_{\theta} = \frac{\rho A\ell}{60}\begin{bmatrix} 0 & 21 & 3\ell & 0 & 9 & -2\ell \\ -21 & 0 & 0 & -9 & 0 & 0 \\ -3\ell & 0 & 0 & -2\ell & 0 & 0 \\ 0 & 9 & 2\ell & 0 & 21 & -3\ell \\ -9 & 0 & 0 & -21 & 0 & 0 \\ 2\ell & 0 & 0 & 3\ell & 0 & 0 \end{bmatrix}$$

Total-Lagrangian Matrix Formulation

Particular case: 2D linear Bernoulli-Euler prismatic rod element

$$\boxed{\Delta\mathbf{K} = \int_V \rho \mathbf{H}^T \mathbf{a}^T \ddot{\mathbf{a}} \mathbf{H} dV = \ddot{\theta}\mathbf{C}_\theta - \dot{\theta}^2 \mathbf{M}}$$

$$\mathbf{P} = \underbrace{\left(\int_V \mathbf{H}^T \mathbf{a}^T \mathbf{f}^B dV + \int_{S_f} \mathbf{H}^T \mathbf{a}^T \mathbf{f}^S dS \right)}_{\mathbf{P}_0} - \underbrace{\left(\int_V \rho \mathbf{H}^T \mathbf{a}^T \ddot{\mathbf{S}} dV + \int_V \rho \mathbf{H}^T \mathbf{a}^T \ddot{\mathbf{a}} \mathbf{r} dV \right)}_{\Delta\mathbf{P}}$$

$$\Delta\mathbf{P} = -\frac{\rho A \ell}{12} \begin{Bmatrix} 6\ddot{S}_X \cos\theta + 6\ddot{S}_Z \sin\theta \\ 6\ddot{S}_X \sin\theta - 6\ddot{S}_Z \cos\theta \\ \ell\ddot{S}_X \sin\theta - \ell\ddot{S}_Z \cos\theta \\ 6\ddot{S}_X \cos\theta + 6\ddot{S}_Z \sin\theta \\ 6\ddot{S}_X \sin\theta - 6\ddot{S}_Z \cos\theta \\ -\ell\ddot{S}_X \sin\theta + \ell\ddot{S}_Z \cos\theta \end{Bmatrix} + \frac{\rho A \ell}{60} \ddot{\theta} \begin{Bmatrix} 30 r_{0z} \\ 30 r_{0x} + 9\ell \\ 5\ell r_{0x} + 2\ell^2 \\ 30 r_{0z} \\ 30 r_{0x} + 21\ell \\ -5\ell r_{0x} - 3\ell^2 \end{Bmatrix} + \frac{\rho A \ell}{60} \dot{\theta}^2 \begin{Bmatrix} 30 r_{0x} + 10\ell \\ -30 r_{0z} \\ -5\ell r_{0z} \\ 30 r_{0x} + 20\ell \\ -30 r_{0z}\ell \\ 5\ell r_{0z} \end{Bmatrix}$$

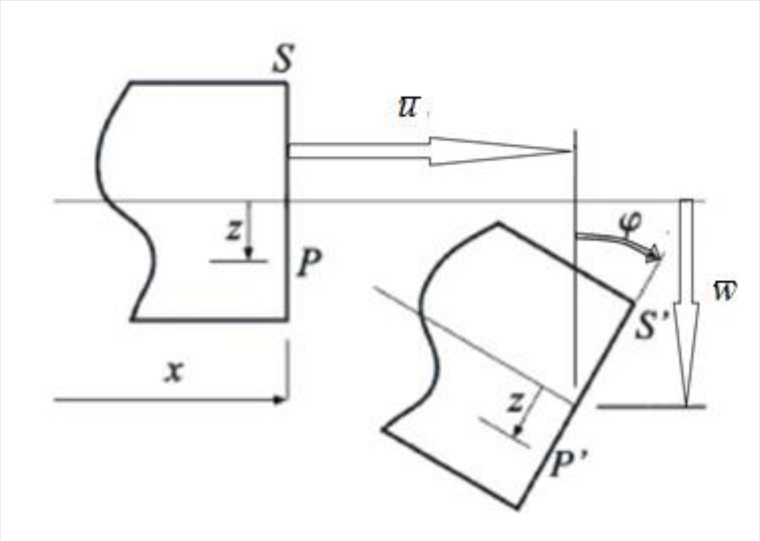
Total-Lagrangian Matrix Formulation

Particular case: 2D linear Bernoulli-Euler prismatic rod element

$$\mathbf{M}\ddot{\mathbf{q}} + \Delta\mathbf{C}\dot{\mathbf{q}} + \left(\mathbf{K} + \Delta\mathbf{K}\right)\mathbf{q} = \mathbf{P_0} + \Delta\mathbf{P}$$

Total-Lagrangian Matrix Formulation

Example: 2D non-linear Bernoulli-Euler prismatic rod element



$$u = \bar{u} - z\sin\varphi \cong \bar{u} - z\bar{w}'$$

$$w = \bar{w} + z(\cos\varphi - 1) \cong \bar{w} \boxed{- \frac{z}{2}\bar{w}'^2}$$

$$\varepsilon_x = u' + \frac{1}{2}u'^2 + \frac{1}{2}w'^2 \cong \bar{u}' - z\bar{w}'' \boxed{+ \frac{1}{2}\bar{w}'^2}$$

Brasil & Mazzilli, Appl. Mech. Rev., 46(11), S110-S117, 1993

Total-Lagrangian Matrix Formulation

Example: 2D non-linear Bernoulli-Euler prismatic rod element

Secant formulation in index notation (within Newton-Raphson procedure)

$$\boxed{M^{rs}\ddot{q}_s + D^{rs}\dot{q}_s + K_0^{rs}q_s = F^r}$$

Tangent formulation in index notation

$$\boxed{M_T^{rs}\delta\ddot{q}_s + D_T^{rs}\delta\dot{q}_s + K_T^{rs}\delta q_s = \delta F^r}$$

Brasil & Mazzilli, Appl. Mech. Rev., 46(11), S110-S117, 1993

Total-Lagrangian Matrix Formulation

Example: 2D non-linear Bernoulli-Euler prismatic rod element

$$\bar{w} = q_i \psi_i(x) \text{ where } \psi_1(x) = \psi_4(x) = 0 \text{ and } \psi_i(x) = h_i(x) \text{ for } i = 2,3,5,6$$

Neglecting the longitudinal inertial force, one arrives at constant axial strain within the rod element, so that

$$\bar{u} = q_i \phi_i(x) + \frac{1}{2} q_i q_j \left[\frac{x}{\ell} \alpha_{ij}(\ell) - \alpha_{ij}(x) \right]$$

where $\phi_i(x) = h_i(x)$ for $i = 1,4$ and $\phi_i(x) = 0$ for $i = 2,3,5,6$

and $\alpha_{ij}(x) = \int_0^x \psi_i'(\xi)\psi_j'(\xi)\, d\xi$

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Total-Lagrangian Matrix Formulation

Example: 2D non-linear Bernoulli-Euler prismatic rod element

Calling

$$\gamma^x = x_0 + x + q_i\phi_i(x) + \frac{1}{2}q_i q_j \beta_{ij}(x), \text{ where } \beta_{ij}(x) = \frac{x}{\ell}\alpha_{ij}(\ell) - \alpha_{ij}(x)$$

$$\delta^x = -q_i\psi_i'(x)$$

$$\gamma^z = z + q_i\psi_i(x)$$

$$\delta^z = 1 - \frac{1}{2}q_i q_j \psi_i'(x)\psi_j'(x)$$

the matrices of the secant equations of motion referred to the rod-element local system are:

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Total-Lagrangian Matrix Formulation

Example: 2D non-linear Bernoulli-Euler prismatic rod element

$$M^{rs} = \rho A \int_0^{\ell} \left(\gamma_{,r}^{x} \gamma_{,s}^{x} + \gamma_{,r}^{z} \gamma_{,s}^{z} \right) dx + \rho I \int_0^{\ell} \left(\delta_{,r}^{x} \delta_{,s}^{x} + \delta_{,r}^{z} \delta_{,s}^{z} \right) dx$$

$$D^{rs} = D_0^{rs} - 2\dot{\theta}\rho A \int_0^{\ell} \left(\gamma_{,r}^{x} \gamma_{,s}^{z} - \gamma_{,s}^{x} \gamma_{,r}^{z} \right) dx + \dot{q}_i \rho A \int_0^{\ell} \gamma_{,r}^{x} \gamma_{,si}^{x} dx + \dot{q}_i \rho I \int_0^{\ell} \delta_{,si}^{z} \delta_{,r}^{z} dx$$

$$K_0^{rs} = EA\ell \phi_r' \phi_s' + EI \int_0^{\ell} \psi_r''(x) \psi_s''(x) dx$$

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Total-Lagrangian Matrix Formulation

Example: 2D non-linear Bernoulli-Euler prismatic rod element

$$F^r = P^r + K_0^{rs} q_s - \frac{EA}{\ell}\left[\ell q_i \phi_i' + \frac{1}{2}\alpha_{ij}(\ell) q_i q_j\right]\left[\ell \phi_r' + \alpha_{kr}(\ell) q_k\right] + EI q_i \int_0^\ell \psi_i''(x)\psi_r''(x)\,dx$$

$$+\dot{\theta}^2 \rho A \int_0^\ell \left(\gamma_{,r}^x \gamma^x + \gamma_{,r}^z \gamma^z\right) dx + \dot{\theta}^2 \rho I \int_0^\ell \left(\delta_{,r}^x \delta^x + \delta_{,r}^z \delta^z\right) dx$$

$$-\ddot{\theta} \rho A \int_0^\ell \left(\gamma_{,r}^x \gamma^z - \gamma_{,r}^z \gamma^x\right) dx + \ddot{\theta} \rho I \int_0^\ell \left(\delta_{,r}^x \delta^z - \delta_{,r}^z \delta^x\right) dx$$

$$-\ddot{S}^x \rho A \left[\cos\theta \int_0^\ell \gamma_{,r}^x dx - \sin\theta \int_0^\ell \gamma_{,r}^z dx\right] - \ddot{S}^z \rho A \left[\sin\theta \int_0^\ell \gamma_{,r}^x dx + \cos\theta \int_0^\ell \gamma_{,r}^z dx\right]$$

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Total-Lagrangian Matrix Formulation

Example: 2D non-linear Bernoulli-Euler prismatic rod element

the matrices of the tangent equations of motion referred to the rod-element local system are:

$$\boxed{\begin{aligned} M_T^{rs} &= M^{rs} \\ D_T^{rs} &= D^{rs} \end{aligned}}$$

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Total-Lagrangian Matrix Formulation

Example: 2D non-linear Bernoulli-Euler prismatic rod element

$$K_T^{rs} = \frac{EA}{\ell}\left[\ell\phi_s' + \frac{1}{2}\alpha_{is}(\ell)q_i\right]\left[\ell\phi_r' + \alpha_{jr}(\ell)q_j\right]$$

$$+\frac{EA}{\ell}\left[\ell q_i\phi_i' + \frac{1}{2}\alpha_{ij}(\ell)q_i q_j\right]\alpha_{rs} + EI\int_0^\ell \psi_r''(x)\psi_s''(x)\,dx$$

$$-\dot{\theta}^2\rho A\int_0^\ell\left(\gamma_{,rs}^x\gamma^x + \gamma_{,r}^x\gamma_s^x + \gamma_{,r}^z\gamma_s^z\right)dx + \dot{\theta}^2\rho I\int_0^\ell\left(\delta_{,r}^x\delta_s^x + \delta_{,rs}^z\delta^z + \delta_{,r}^z\delta_s^z\right)dx$$

$$-\ddot{\theta}\rho A\int_0^\ell\left(\gamma_{,rs}^x\gamma^z + \gamma_{,r}^x\gamma_s^z - \gamma_{,r}^z\gamma_s^x\right)dx + \ddot{\theta}\rho I\int_0^\ell\left(\delta_{,r}^x\delta_s^z - \delta_{,rs}^z\delta^x - \delta_{,r}^z\delta_s^x\right)dx$$

$$+\ddot{S}^x\rho A\left[\cos\theta\int_0^\ell\gamma_{,rs}^x dx + \sin\theta\int_0^\ell\gamma_{,rs}^z dx\right] + \ddot{S}^z\rho A\left[-\sin\theta\int_0^\ell\gamma_{,rs}^x dx + \cos\theta\int_0^\ell\gamma_{,rs}^z dx\right]$$

$$+\ddot{q}_i\rho A\int_0^\ell\left(\gamma_{,rs}^x\gamma_{,i}^x + \gamma_{,r}^x\gamma_{,si}^x\right)dx + \ddot{q}_i\rho I\int_0^\ell\left(\delta_{,rs}^z\delta_{,i}^z + \delta_{,r}^z\delta_{,si}^z\right)dx$$

$$+\dot{q}_i\dot{q}_j\rho A\int_0^\ell\gamma_{,rs}^x\gamma_{,ij}^x dx + \dot{q}_i\dot{q}_j\rho I\int_0^\ell\delta_{,rs}^z\delta_{,ij}^z dx - 2\dot{\theta}\dot{q}_i\rho A\int_0^\ell\left(\gamma_{,rs}^x\gamma_{,i}^z - \gamma_{,is}^x\gamma_{,r}^z\right)dx$$

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Total-Lagrangian Matrix Formulation

Example: 2D non-linear Bernoulli-Euler prismatic rod element

$$\delta F^r = \delta P^r + 2\dot{\theta}\delta\dot{\theta}\rho A\int_0^\ell \left(\gamma^x_{,r}\gamma^x + \gamma^z_{,r}\gamma^z\right)dx + 2\dot{\theta}\delta\dot{\theta}\rho I\int_0^\ell \left(\delta^x_{,r}\delta^x + \delta^z_{,r}\delta^z\right)dx$$

$$+2\delta\dot{\theta}\rho A\int_0^\ell \left(\gamma^x_{,r}\gamma^z_s - \gamma^z_{,r}\gamma^x_s\right)dx + 2\delta\dot{\theta}\rho I\int_0^\ell \left(\delta^x_{,r}\delta^z_s - \delta^z_{,r}\delta^x_s\right)dx$$

$$+\delta\ddot{\theta}\rho A\int_0^\ell \left(\gamma^x_{,r}\gamma^z - \gamma^z_{,r}\gamma^x\right)dx + \delta\ddot{\theta}\rho I\int_0^\ell \left(\delta^x_{,r}\delta^z - \delta^z_{,r}\delta^x\right)dx$$

$$-\ddot{S}^x\delta\theta\rho A\left[-\sin\theta\int_0^\ell \gamma^x_{,r}dx + \cos\theta\int_0^\ell \gamma^z_{,r}dx\right] + \ddot{S}^z\delta\theta\rho A\left[\cos\theta\int_0^\ell \gamma^x_{,r}dx + \sin\theta\int_0^\ell \gamma^z_{,r}dx\right]$$

$$+\delta\ddot{S}^x\rho A\left[\cos\theta\int_0^\ell \gamma^x_{,r}dx - \sin\theta\int_0^\ell \gamma^z_{,r}dx\right] - \delta\ddot{S}^z\rho A\left[-\sin\theta\int_0^\ell \gamma^x_{,r}dx + \cos\theta\int_0^\ell \gamma^z_{,r}dx\right]$$

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