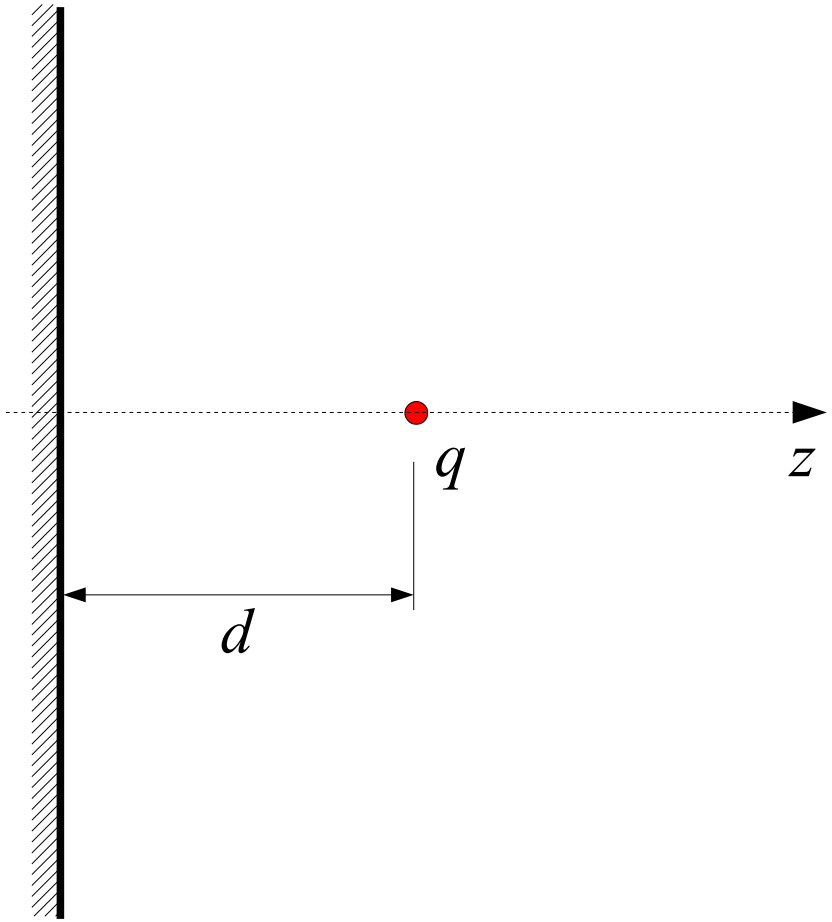
Método das Imagens

- Carga próxima a um plano condutor
- Linhas de carga e cilindros condutores
- Carga próxima à interface plana entre meios distintos

Carga próxima a um plano condutor



Carga próxima a um plano condutor

- Queremos determinar a função potencial φ no semiespaço direito, ou seja, $z > 0$, que satisfaz à equação:

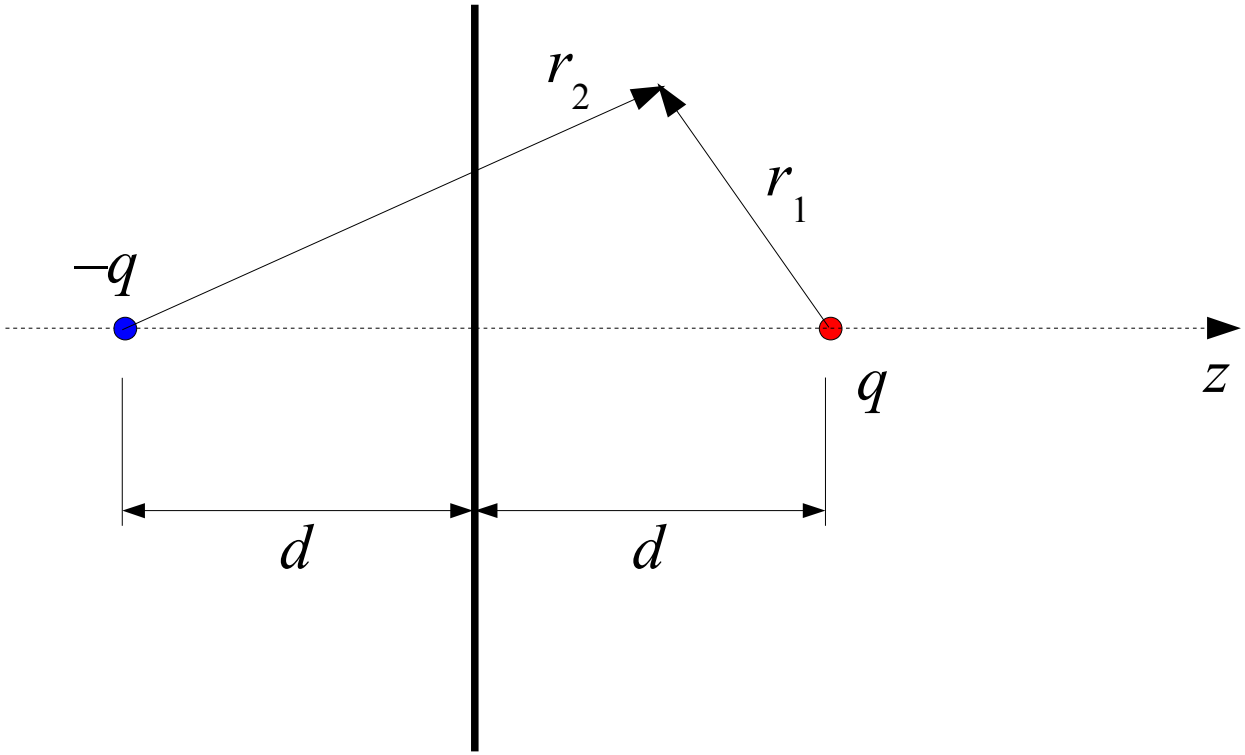
$$\nabla^2 \varphi = 0 : z > 0, (x, y, z) \neq (0,0,d)$$
$$\oiint_{\Sigma \supset (0,0,d)} \nabla \varphi \cdot d\vec{S} = -q/\varepsilon$$

- Condições de contorno:
  - $\varphi(\infty) = 0$
  - $\varphi(z=0) = 0$

Solução

$$\varphi = \frac{q}{4\pi\varepsilon r_1} - \frac{q}{4\pi\varepsilon r_2} \qquad z > 0$$

$$r_1 = \sqrt{x^2 + y^2 + (z-d)^2}; \quad r_2 = \sqrt{x^2 + y^2 + (z+d)^2}$$



Campo Elétrico

$$\vec{E} = \frac{q}{4\pi\varepsilon r_1^2}\hat{u}_{r_1} - \frac{q}{4\pi\varepsilon r_2^2}\hat{u}_{r_2} \qquad z>0$$

ou

$$\vec{E} = -\nabla\varphi = \frac{q}{4\pi\varepsilon}\left\{\frac{z-d}{\left[(z-d)^2+r^2\right]^{3/2}} - \frac{z+d}{\left[(z+d)^2+r^2\right]^{3/2}}\right\}\hat{u}_z +$$

$$\frac{q}{4\pi\varepsilon}\left\{\frac{r}{\left[(z-d)^2+r^2\right]^{3/2}} - \frac{r}{\left[(z+d)^2+r^2\right]^{3/2}}\right\}\hat{u}_r$$

Densidade de carga no plano

$$\vec{E}(z=0)=\frac{q}{4\pi\varepsilon}\left\{\frac{-d}{\left[d^2+r^2\right]^{3/2}}-\frac{d}{\left[d^2+r^2\right]^{3/2}}\right\}\hat{u}_z+$$

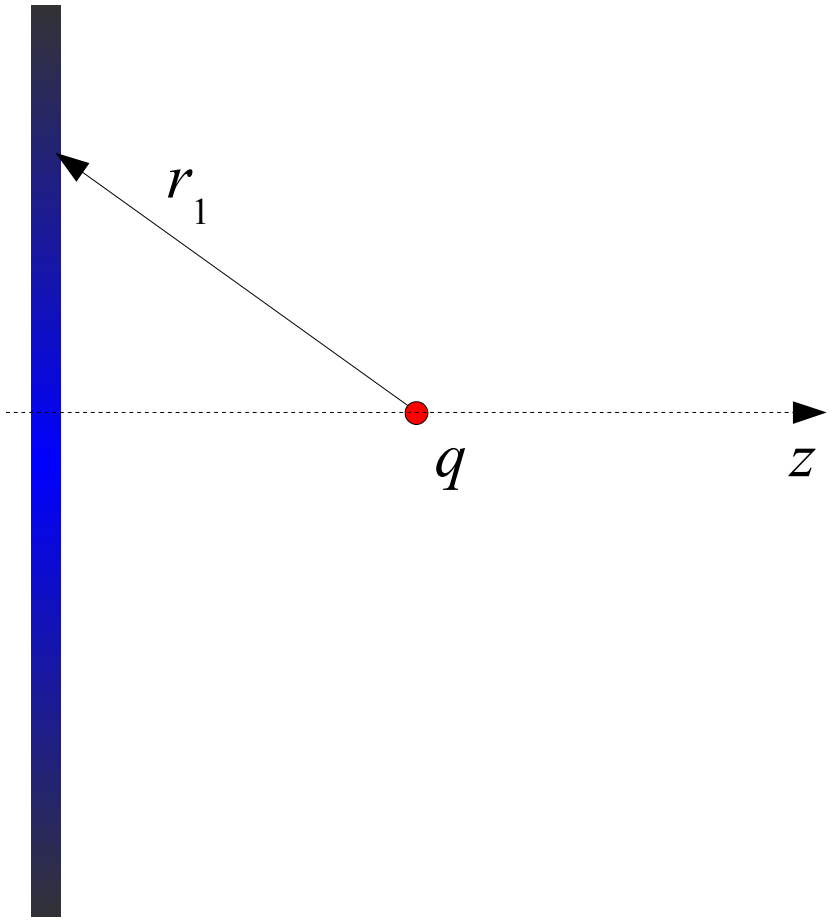
$$+\frac{q}{4\pi\varepsilon}\left\{\frac{r}{\left[d^2+r^2\right]^{3/2}}-\frac{r}{\left[d^2+r^2\right]^{3/2}}\right\}\hat{u}_r=$$

$$=\frac{-q}{4\pi\varepsilon}\left\{\frac{2d}{\left[d^2+r^2\right]^{3/2}}\right\}\hat{u}_z=\frac{-q\,d}{2\pi\varepsilon r_1^3}\hat{u}_z$$

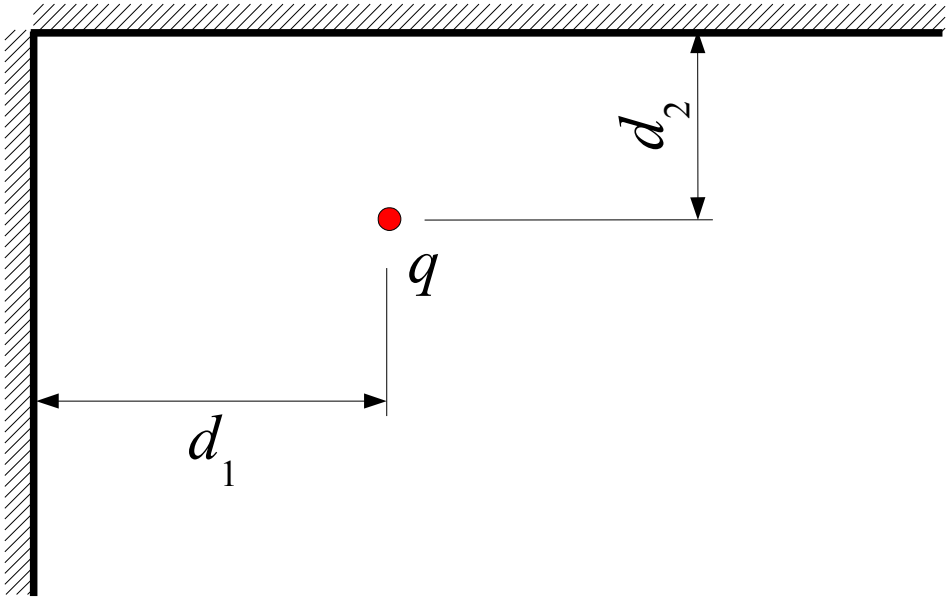
$$\rho_s=D_z\left(z=0^+\right)=\frac{-q\,d}{2\pi r_1^3}$$

$$Q_{plano}=\iint\rho_s\,dS=-q$$

Densidade de carga no plano



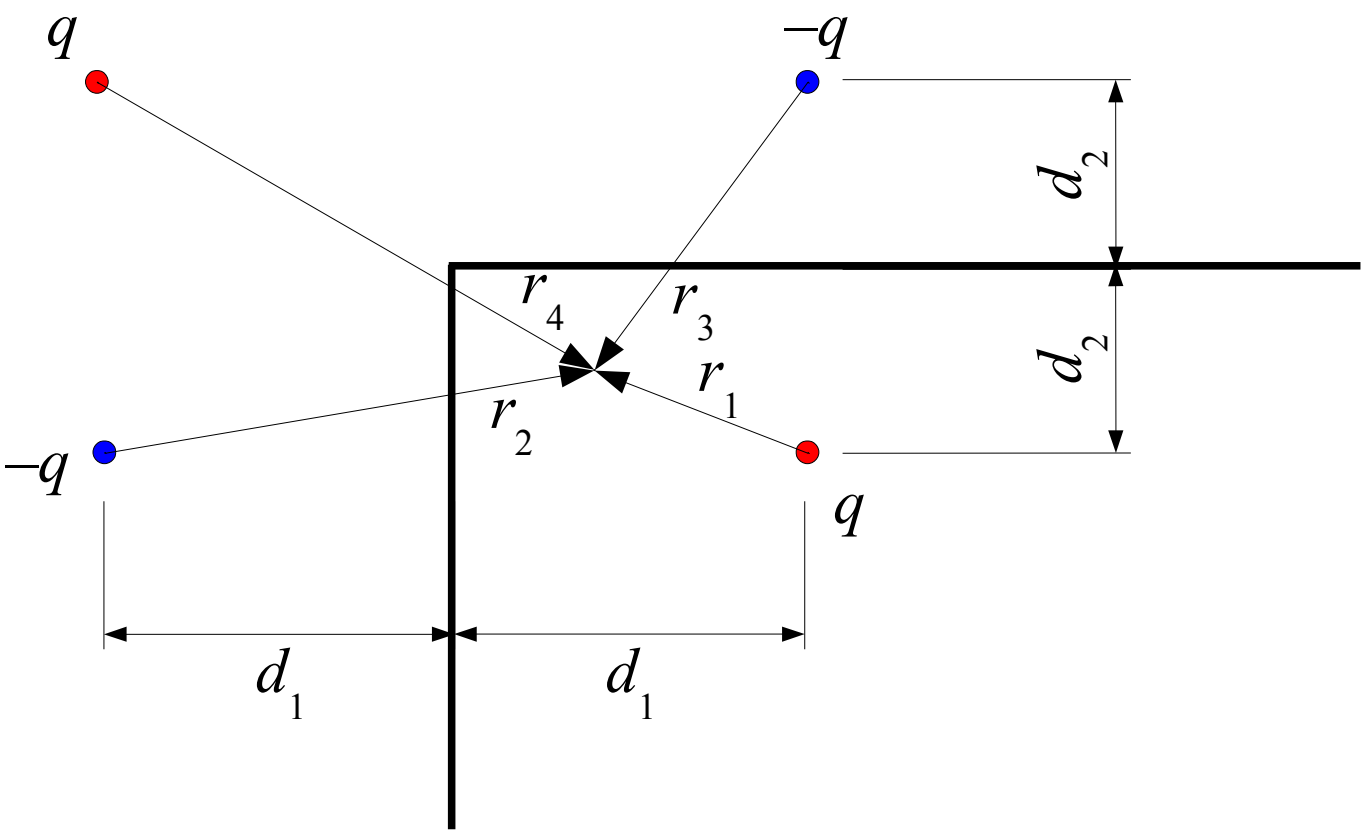
2 planos condutores ortogonais



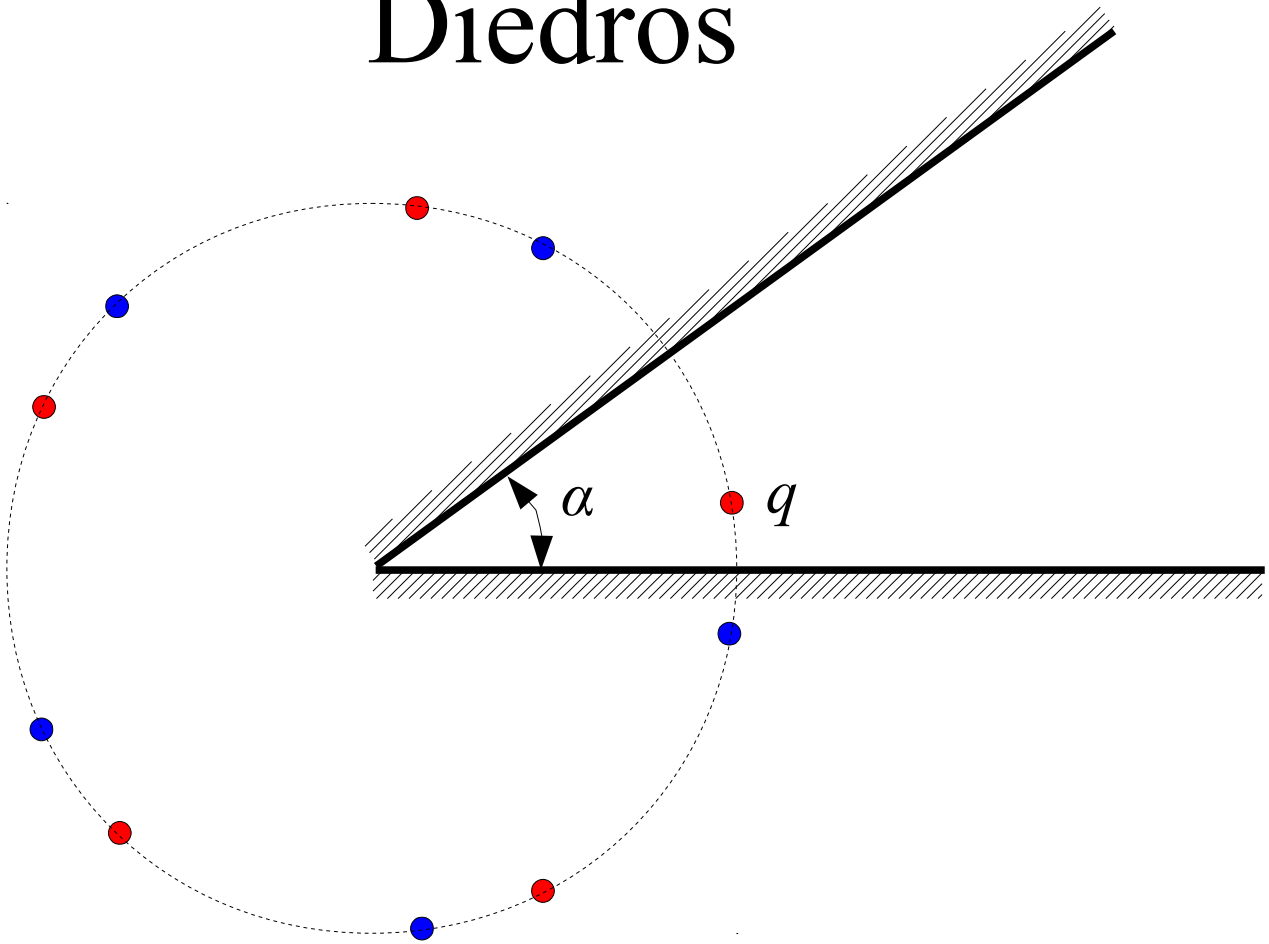
2 planos condutores ortogonais

$$\varphi = \frac{1}{4\pi\varepsilon}\sum_{i=1}^{4}\frac{q_i}{r_i} \quad x>0, y>0$$

$$q_i = \pm q\,; \quad r_i = \sqrt{(x \mp d_1)^2 + (y \mp d_2)^2 + z^2}$$



Diedros



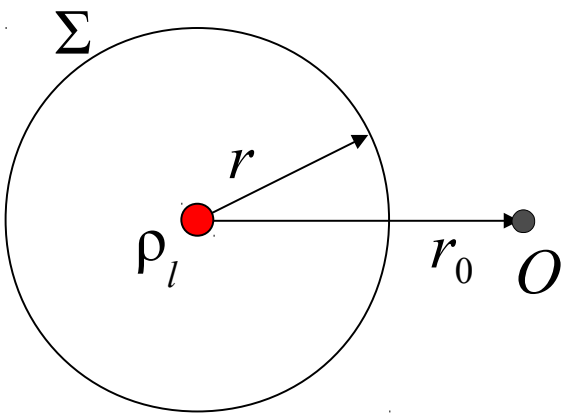
$$\alpha = \frac{\pi}{n} \Rightarrow 2n - 1 \quad \text{imagens}$$

Linha de Carga

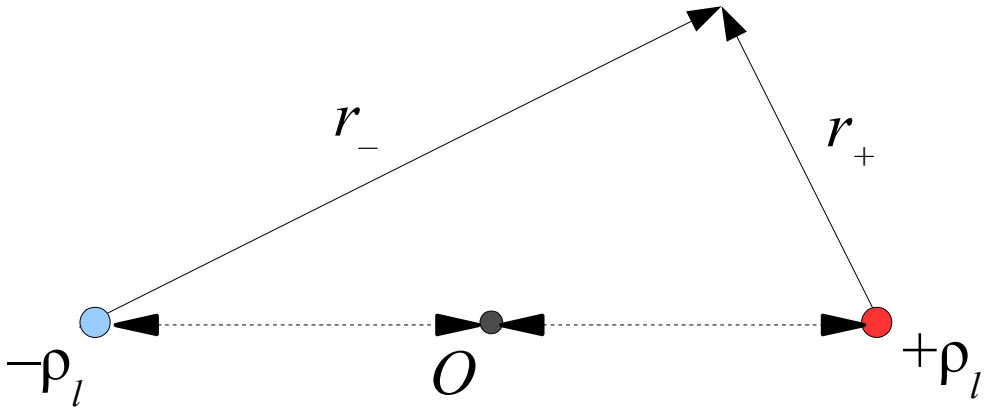
$$\rho_l \, \ell = \oiint_{\Sigma} \vec{D} \cdot d\vec{S} = D_r \, 2\pi r \, \ell \Rightarrow$$

$$\vec{D} = \frac{\rho_l}{2\pi r} \vec{u}_r \Rightarrow \vec{E} = \frac{\rho_l}{2\pi \varepsilon \, r} \vec{u}_r \Rightarrow$$

$$\varphi(r) - \varphi(r_0) = \int_r^{r_0} \vec{E} \cdot d\vec{\ell} = \frac{\rho_l}{2\pi\varepsilon} \ln\left(\frac{r_0}{r}\right)$$



Linhas de Cargas Opostas

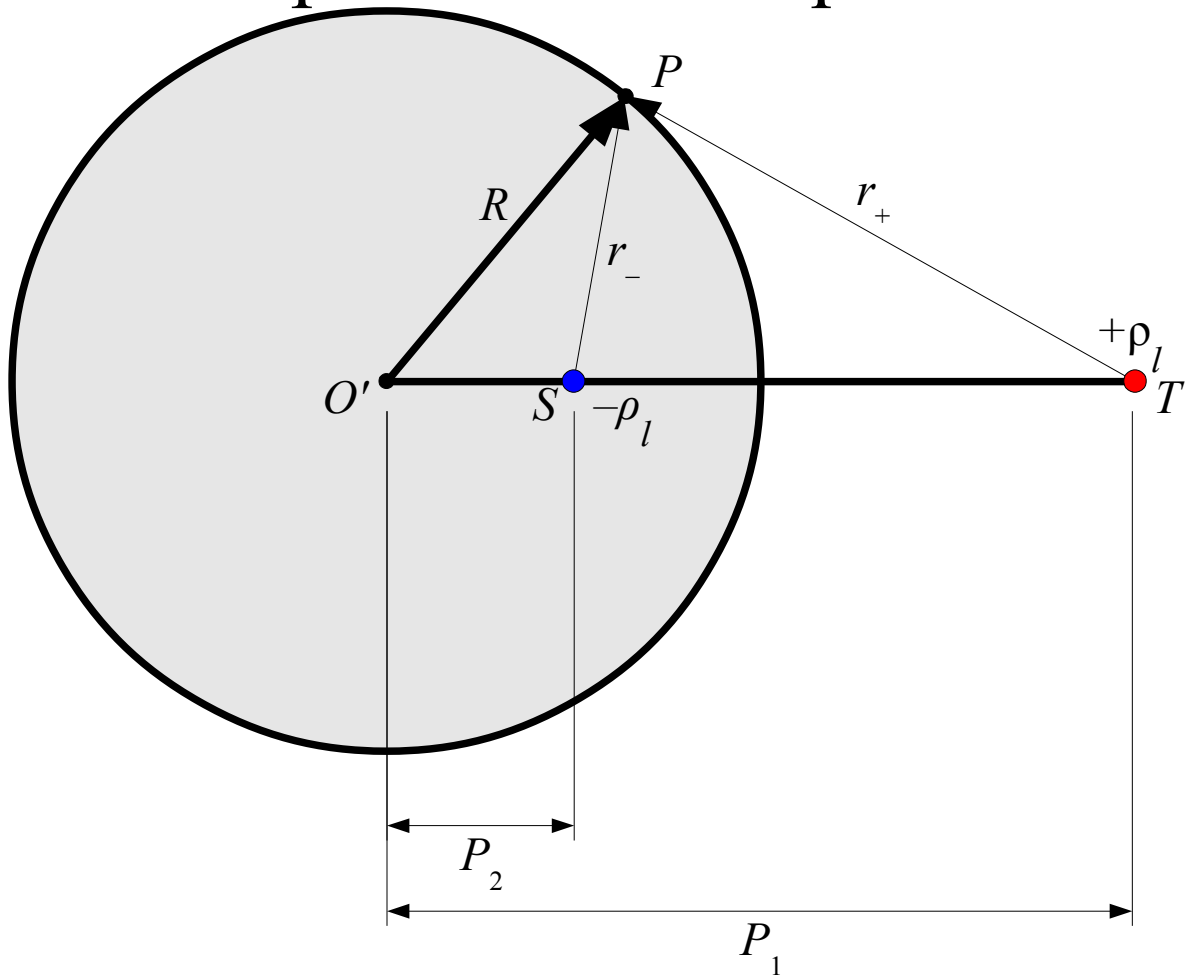


$$\varphi = \frac{\rho_l}{2\pi\varepsilon} \ln\left(\frac{r_-}{r_+}\right)$$

Equipotenciais: $\dfrac{r_-}{r_+} = cte$

Linhas de Cargas Opostas

- Equipotenciais: superfícies cilíndricas de raio R centradas em pontos O' tais que $R^2 = P_1 P_2$:



$$\phi = \frac{\lambda}{2\pi\varepsilon} \ln\left(\frac{r_-}{r_+}\right)$$

Demonstração

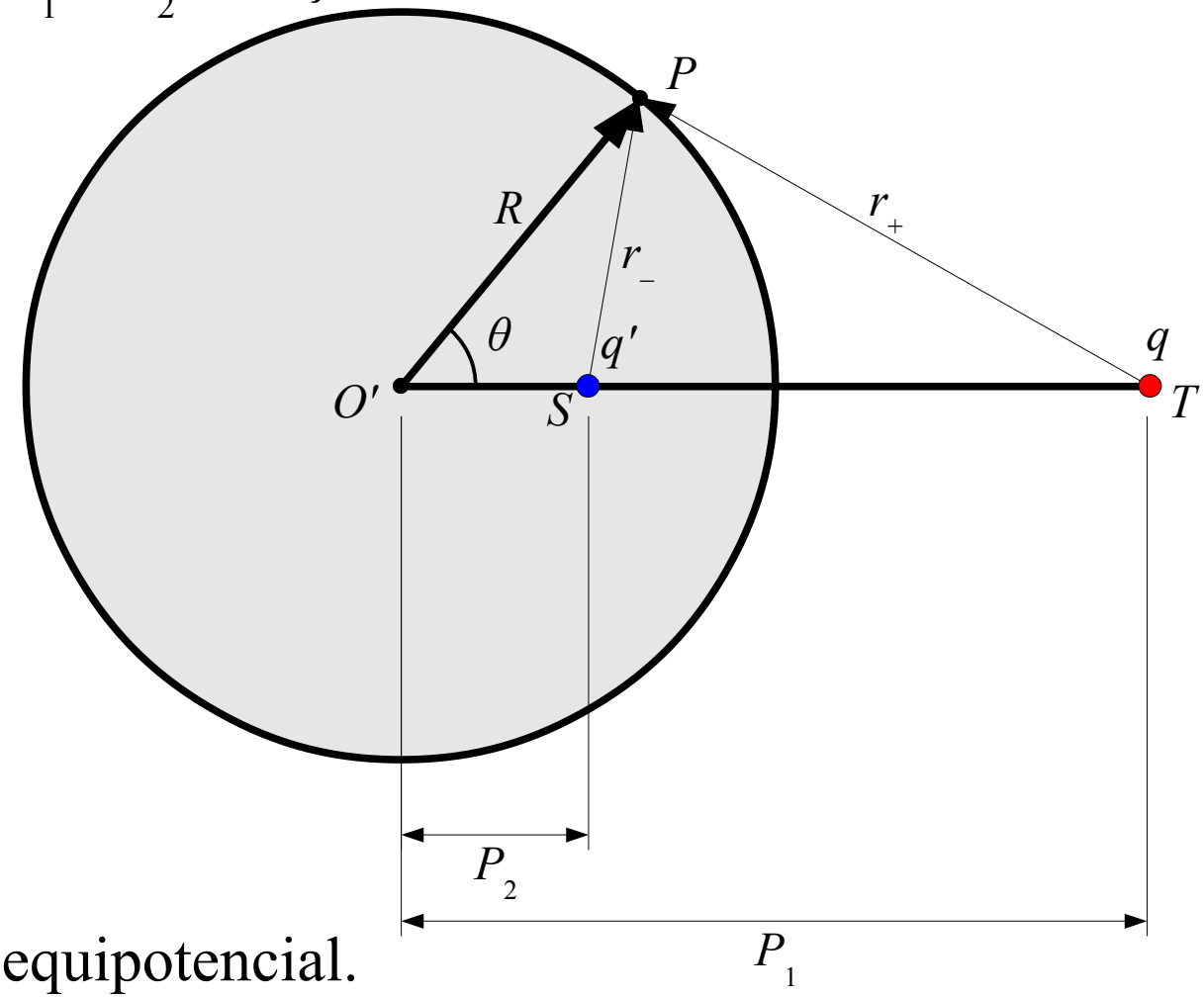
- Dados S e T, escolhe-se O' (à esquerda de S, e alinhado com ST),
- Para esse O', temos P_1 e P_2 e traçamos a circunferência de centro em O' com raio R:

$$P_1 P_2 = R^2$$

Vamos mostrar que para qualquer ponto nessa circunferência temos

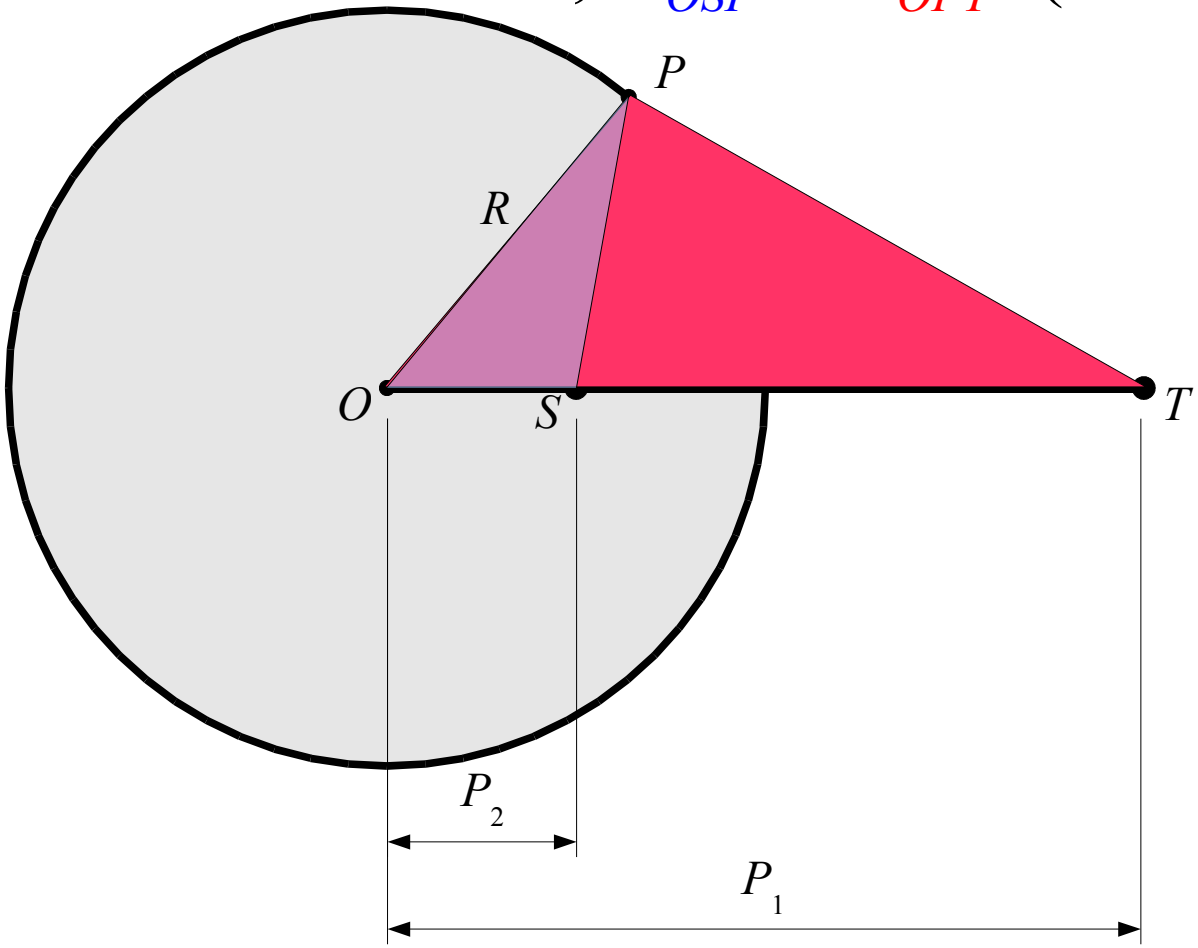
$$\frac{r_-}{r_+} = cte$$

E, portanto, ela é uma equipotencial.

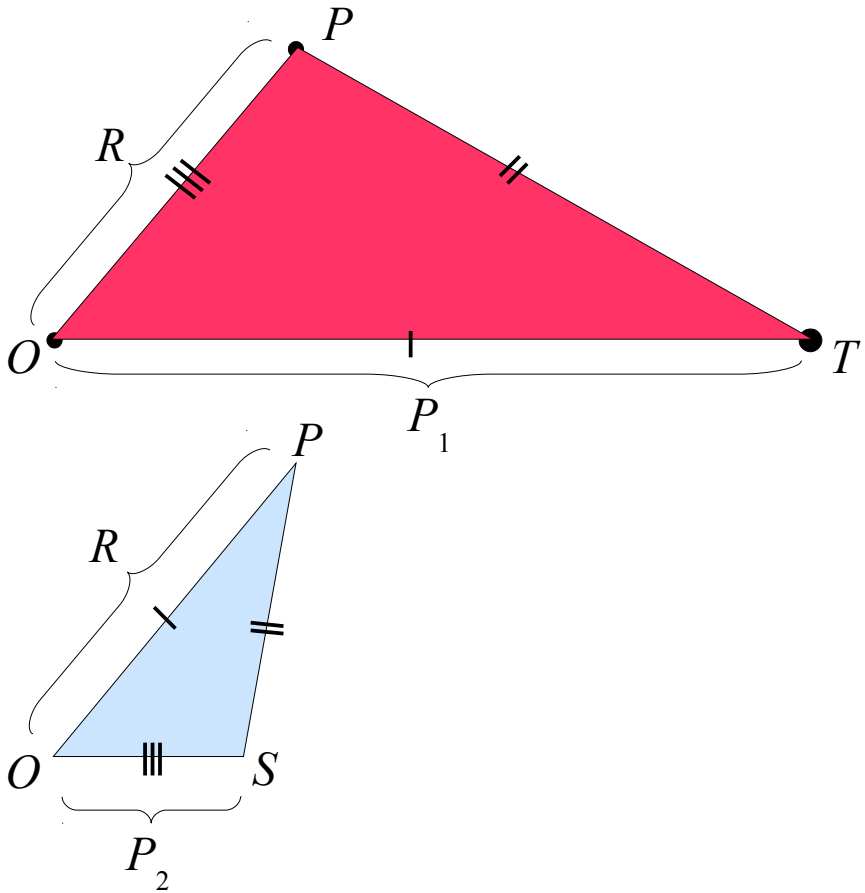


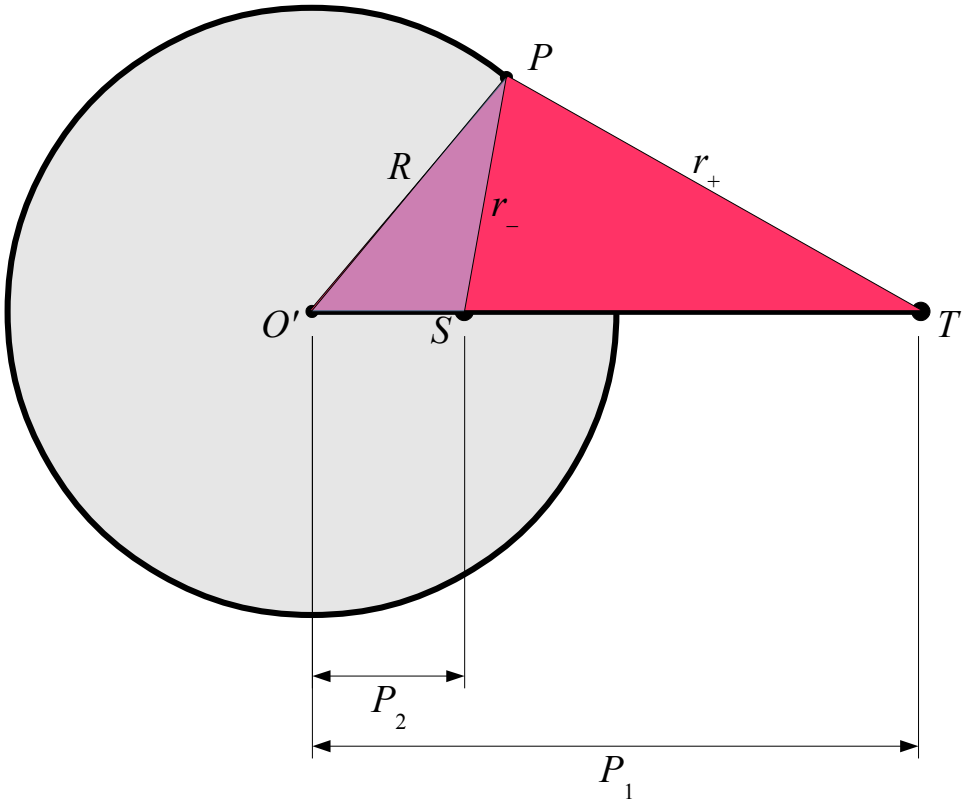
Demonstração:

- Se $P_1P_2=R^2$ então, para $\forall$ P sobre a circunferência de raio R, $\Delta_{OSP} \sim \Delta_{OPT}$:(LAL)



Demonstração:

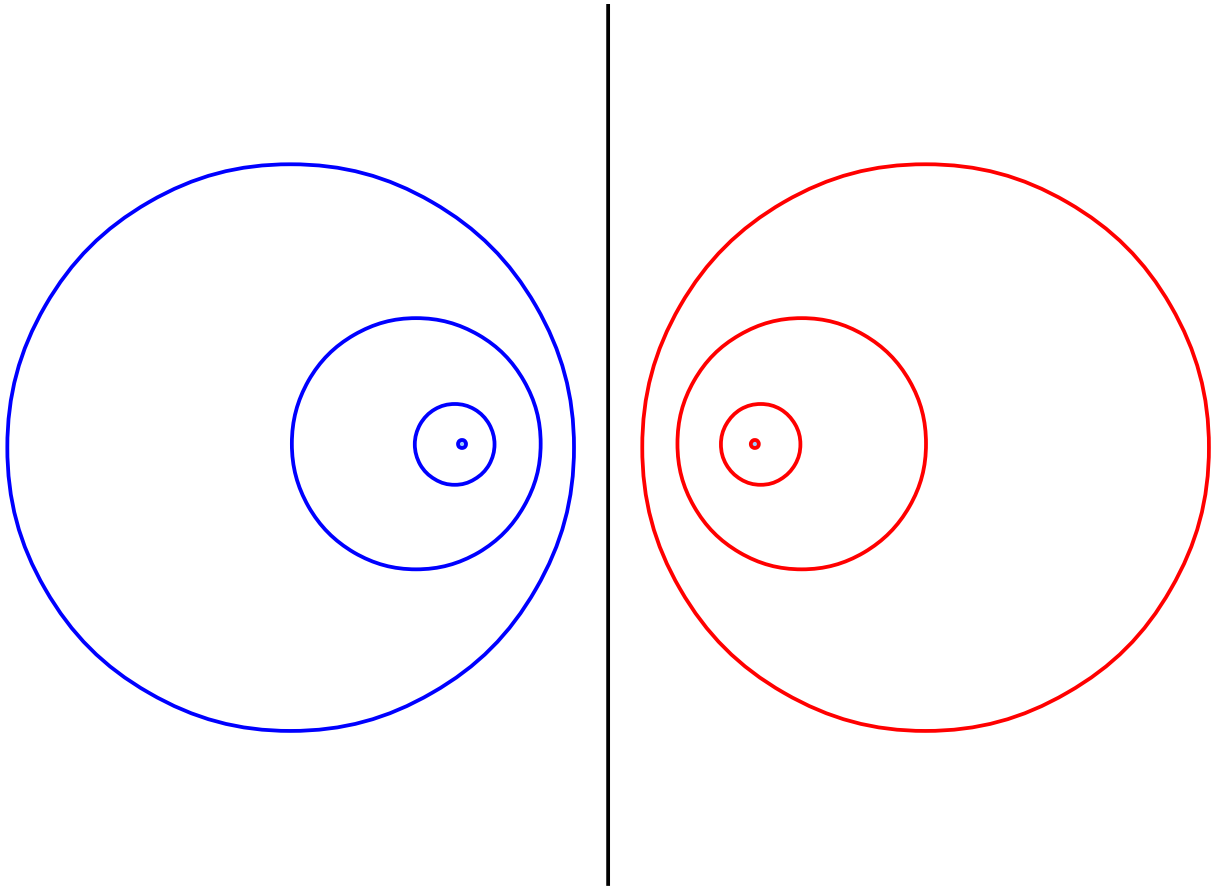




$$\frac{P_2}{R} = \frac{R}{P_1} \Rightarrow \frac{OS}{OP} = \frac{OP}{OT}$$

$$\frac{SP}{PT} = \frac{OP}{OT} \Rightarrow \frac{r_-}{r_+} = \frac{R}{P_1} = \text{cte}$$

Equipotenciais



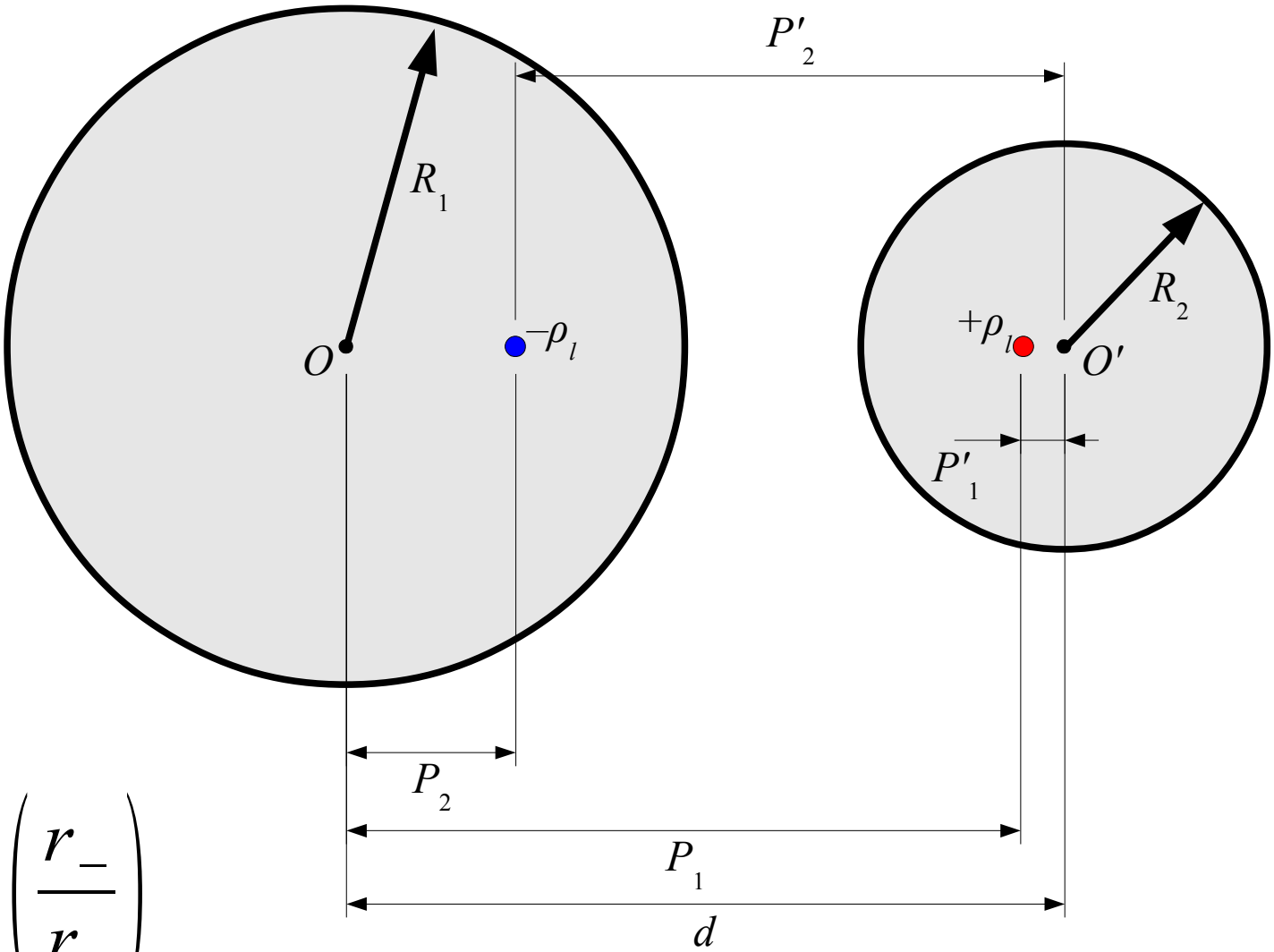
Cilindros Condutores Paralelos

$$R_1^2 = P_1 P_2 ,$$

$$R_2^2 = P'_1 P'_2 .$$

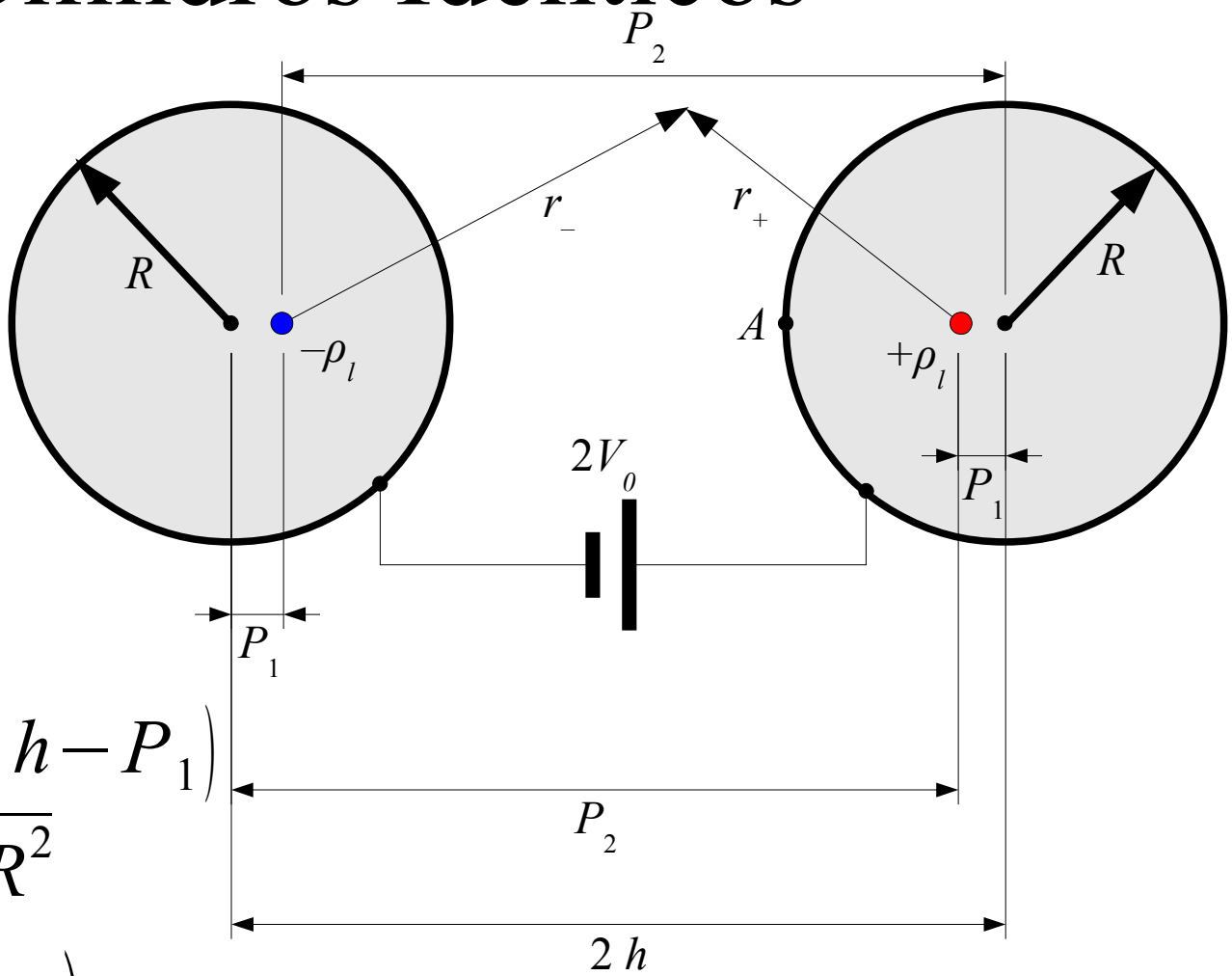
$$P'_1 = d - P_1$$

$$P'_2 = d - P_2 .$$



$$\varphi = \frac{\rho_l}{2\pi\varepsilon} \ln\left(\frac{r_-}{r_+}\right)$$

Cilindros Idênticos



$$R^2 = P_1 P_2 = P_1 \left(2h - P_1\right)$$

$$\Rightarrow P_1 = h - \sqrt{h^2 - R^2}$$

$$\varphi = \frac{\rho_l}{2\pi\varepsilon} \ln\left(\frac{r_-}{r_+}\right)$$

Resistência e Capacitância

$$\varphi(A)=\frac{\rho_l}{2\pi\varepsilon}\ln\left(\frac{2h-R-P_1}{R-P_1}\right)=\frac{\rho_l}{2\pi\varepsilon}\ln\left(\frac{2h-R-h+\sqrt{h^2-R^2}}{R-h+\sqrt{h^2-R^2}}\right)=$$

$$=\frac{\rho_l}{2\pi\varepsilon}\ln\left(\frac{h-R+\sqrt{h^2-R^2}}{R-h+\sqrt{h^2-R^2}}\right)=\frac{\rho_l}{2\pi\varepsilon}\ln\left(\frac{h}{R}+\sqrt{\frac{h}{R^2}^2-1}\right)=V_0\Rightarrow$$

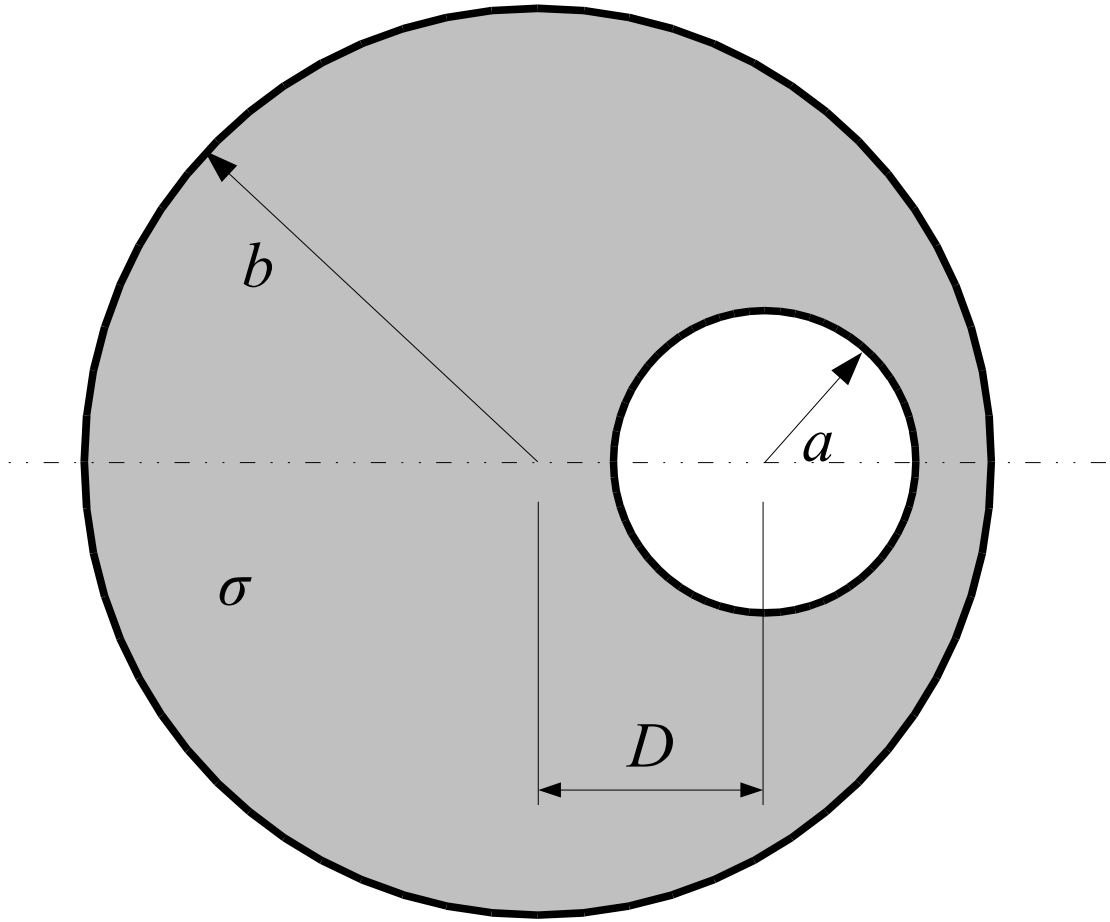
$$\rho_l=\frac{2\pi\varepsilon V_0}{\ln\left(\frac{h}{R}+\sqrt{\frac{h}{R^2}^2-1}\right)}=\frac{2\pi\varepsilon V_0}{\operatorname{arccosh}\left(\frac{h}{R}\right)}$$

Resistência e Capacitância

$$C = \frac{\rho_l \ell}{2 V_0} = \frac{\pi \varepsilon \ \ell}{\ln\left(\frac{h}{R} + \sqrt{\frac{h}{R^2}^2 - 1}\right)} = \frac{\pi \varepsilon \ \ell}{\operatorname{arccosh}\left(\frac{h}{R}\right)}$$

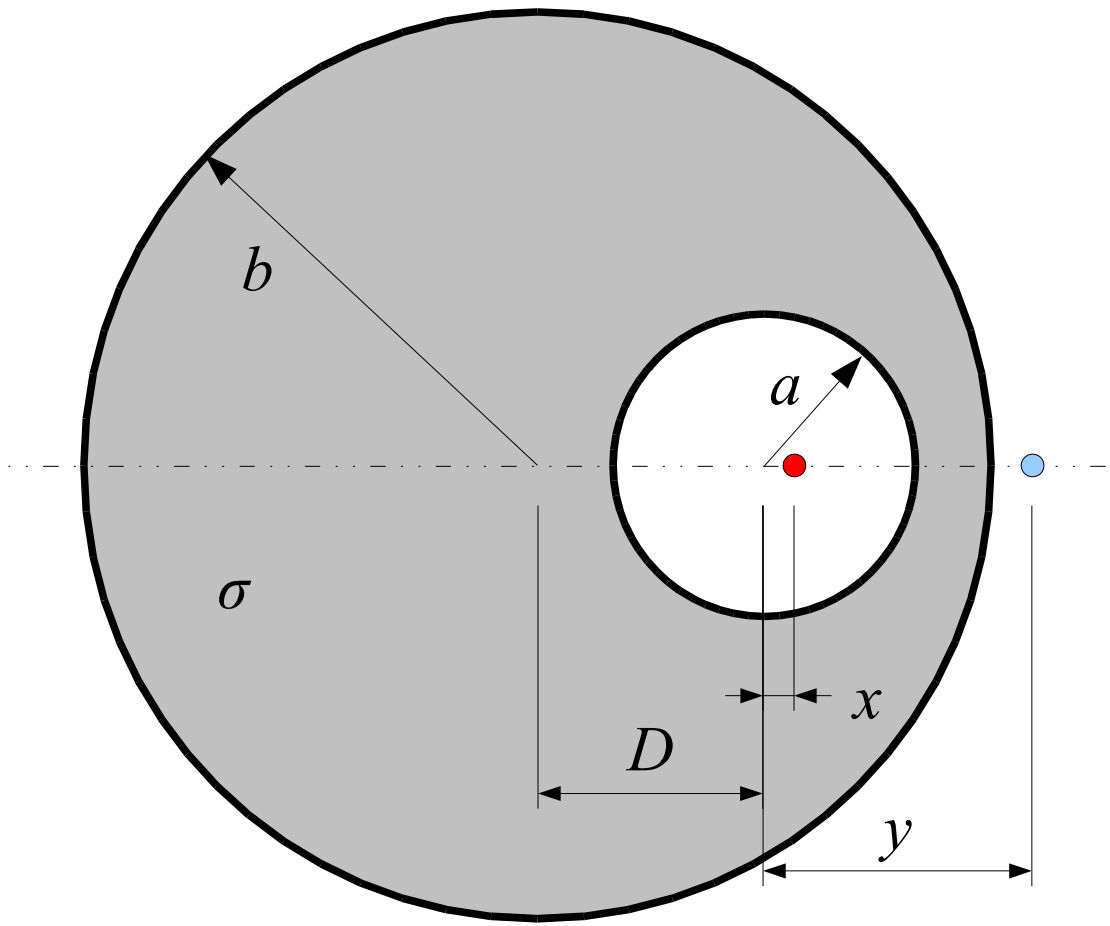
$$R = \frac{\ln\left(\frac{h}{R} + \sqrt{\frac{h}{R^2}^2 - 1}\right)}{\pi \sigma \ \ell} = \frac{\operatorname{arccosh}\left(\frac{h}{R}\right)}{\pi \sigma \ \ell}$$

Ex. 11 p. 406



$a = 2$ cm, $b = 6$ cm, $D = 3$ cm, $L = 100$ m

Ex. 11 p. 406



$$a^2 = x\,y$$

$$b^2 = (x+D)(y+D)$$

Ex. 11 p. 406

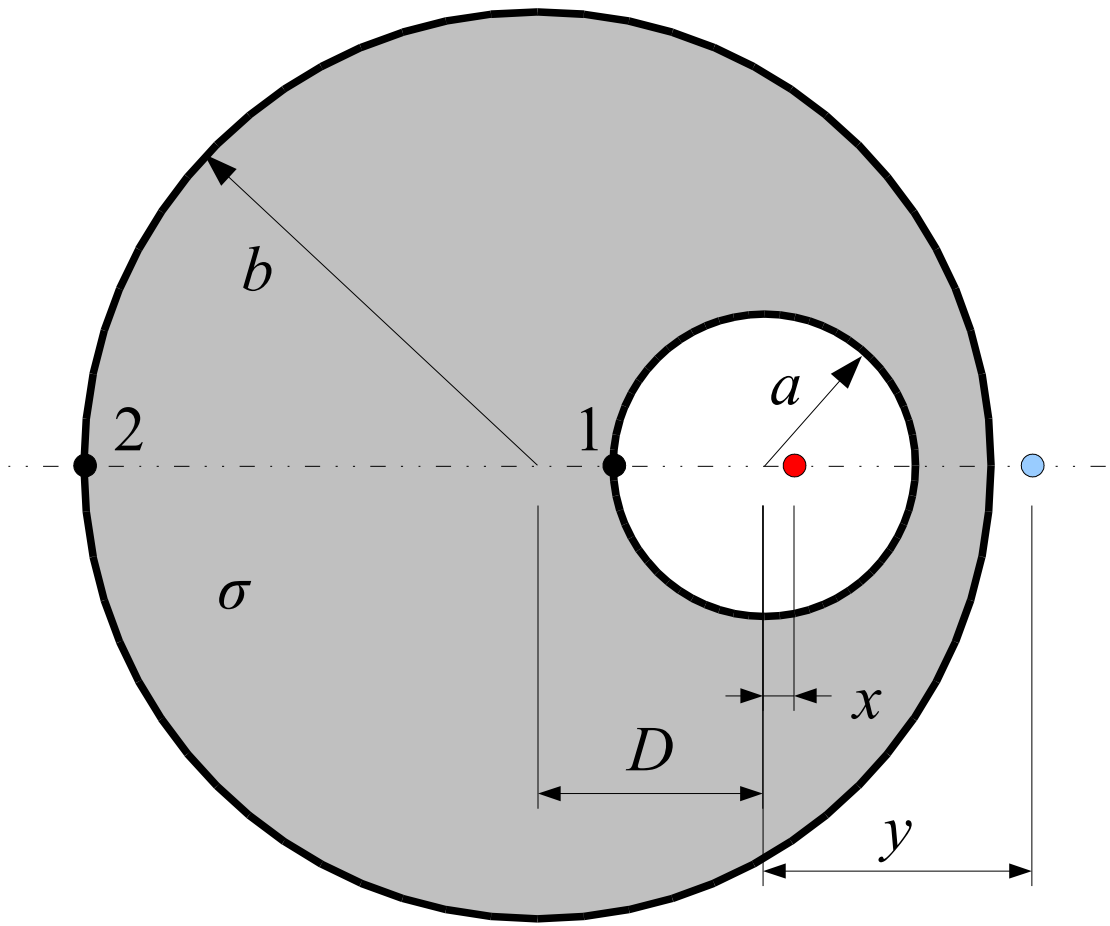
$$a^2 = x\,y$$
$$b^2 = (x+D)(y+D)$$

$$b^2 = (x+D)\left(\frac{a^2}{x}+D\right) \Rightarrow D\,x^2 + (a^2+D^2-b^2)\,x + D\,a^2 = 0$$

$$x = \frac{b^2-a^2-D^2 \pm \sqrt{(b^2-a^2-D^2)^2 - 4\,a^2\,D^2}}{2\,D}$$

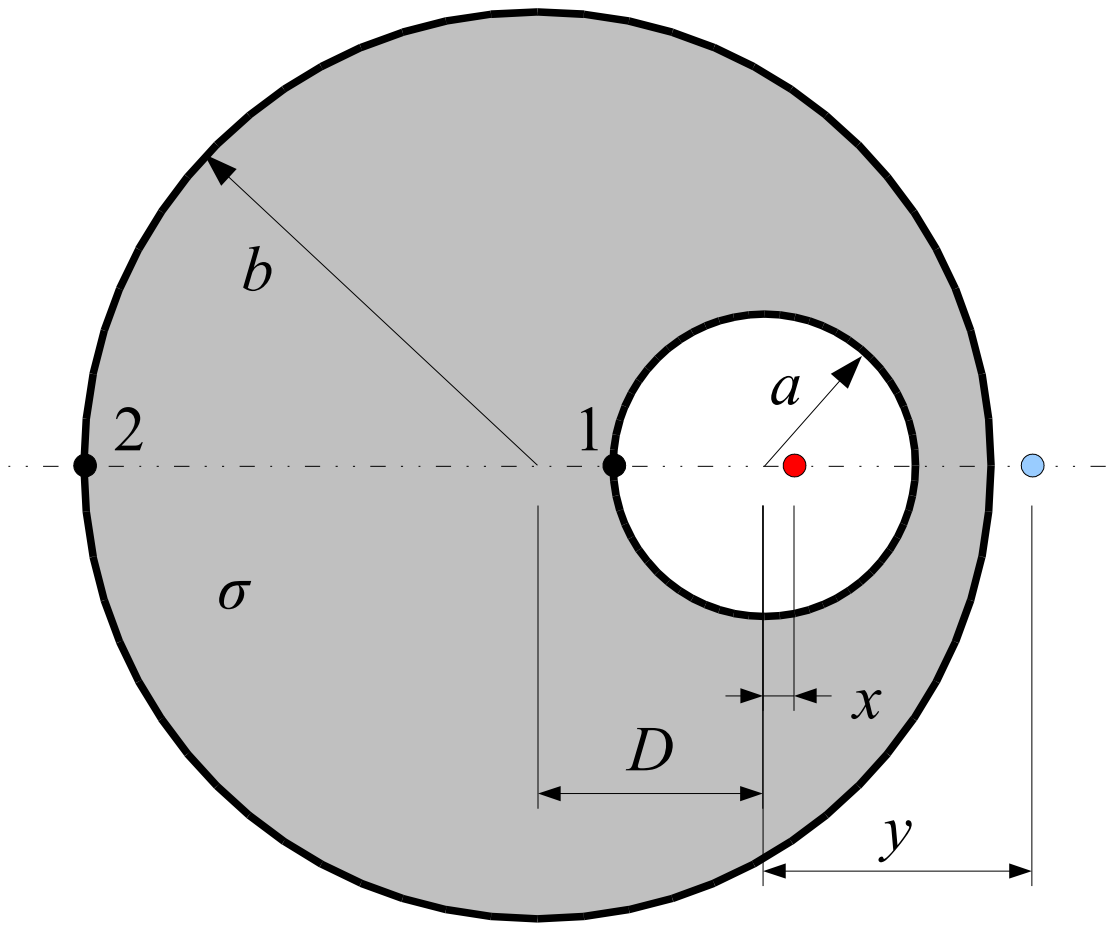
$$x = 0{,}563\,cm\,;\; y = 7{,}10\,cm$$

Ex. 11 p. 406



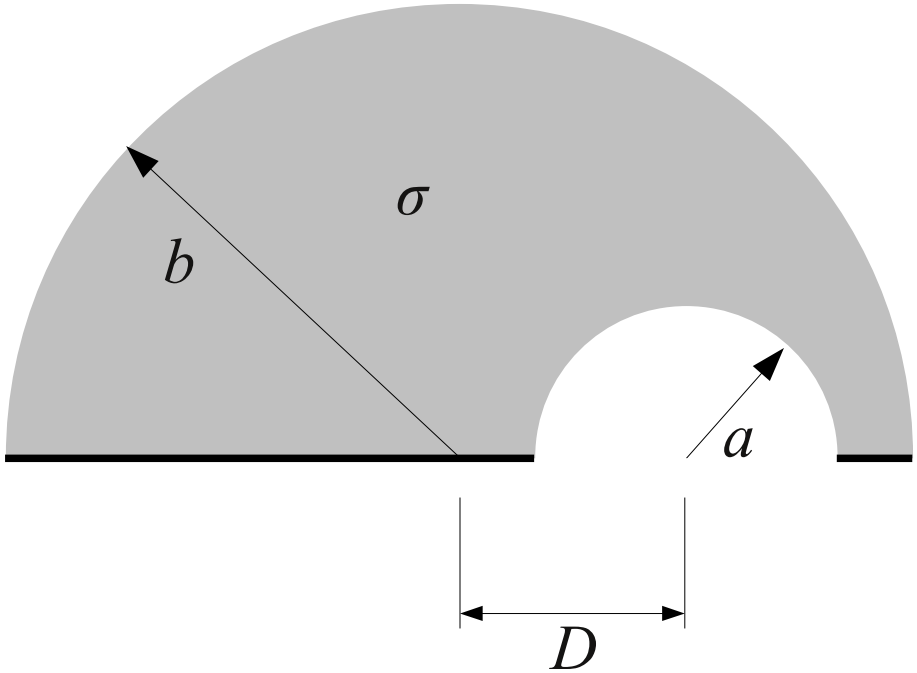
$$\varphi_2 = \frac{I/L}{2\pi\sigma}\ln\left(\frac{y+D+b}{x+D+b}\right) \qquad \varphi_1 = \frac{I/L}{2\pi\sigma}\ln\left(\frac{y+a}{x+a}\right)$$

Ex. 11 p. 406



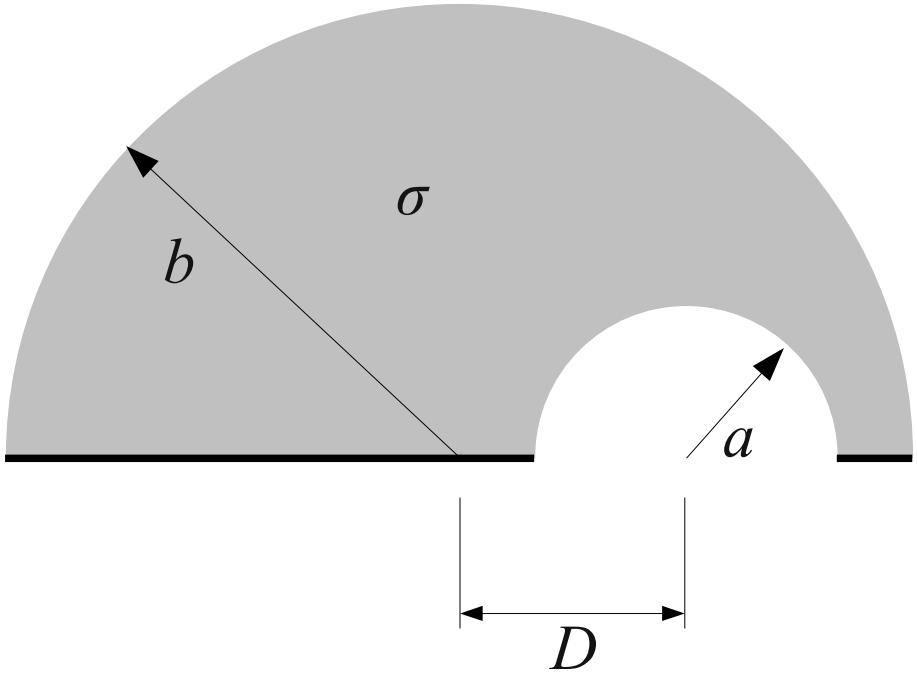
$$R = \frac{\varphi_1 - \varphi_2}{I} = 0{,}12\,\Omega$$

Ex. 11 p. 406



$a = 2$ cm, $b = 6$ cm, $D = 3$ cm, $L' = 2$ m

Ex. 14 p. 331



$$R = \frac{n_s}{n_p}\frac{1}{\sigma L} \qquad R' = \frac{n_p/2}{n_s}\frac{1}{\sigma L'}$$

$$\frac{n_s}{n_p} = R\sigma L \Rightarrow R' = \frac{1}{R\sigma L}\frac{1}{2\sigma L'} = 210\,\Omega$$