Keldysh Field Theory

Marcelo Janovitch B. Pereira

October 2019

Contents

1 Introduction

Equilibrium dynamics of a quantum system can be seen as an analytic continuation of the usual quantum mechanical unitary framework to imaginary time $\beta = 1/T$. This observation instantly hands path integral methods to systems in equilibrium. Nevertheless such methods do not account for the computation of the partition function of a general density matrix.

Although path integral methods are neither fundamental nor strictly necessary in thermodynamics they provide a very attractive toolkit. Path integrals are awesome in implementing, understanding and generalising symmetries in quantum systems. Moreover they provide a very powerful technique to construct effective descriptions: the renormalisation group.

The Keldysh-Schwinger[1] formalism which we'll discuss here is an approach to writing path integrals for general density matrix, i.e. non-equilibrium. Moreover, it can be used to describe time dependent hamiltonians and open quantum systems[Sieberer et al., 2016]..

A central idea lies in the construction of Keldysh path integrals: the evolution of a general density matrix always involve a doubled hilbert space in which half of the system evolve forward in time and the other half backwards.

We'll develop the intuition and the theory in the very simplest example, of a bosonic one-mode path integral and then apply it to a real problem: non-equilibrium quantum phase transitions in the Lipkin-Meshkov-Glick model.

This is an extensive topic and unavoidably I had to cut short on interesting stuff. Yet, the choice of restricting the extensive application to the LMG model is justified by some important goals. One is to provide an example which is simple enough to be tackled with minimum textbook knowledge. The second is that although the required theory is simple, the application involves state of the art questions, by which one can taste some paradigms of many-body phenomena. The examples restricted to non-equilibrium continuum degrees of freedom ended up being not chosen for their big complexity. I therefore chose to touch at the field theory case at equilibrium to display interesting symmetry properties.

2 Closed Time Contours

In this chapter we'll call attention to the fact that indeed time evolution of mixed states require both forward and backward evolution. We shall first point why this

[1]This formalism is due to Schwinger, Konstantinov, Perel, Kadanoff, Baym and Keldysh. Yet is always refered as Keldsyh or Schwinger-Keldysh

subtlety is usually overseen in equilibrium systems and then state the contour time evolution. A good deal of what is here will follow [Kamenev, 2011], until we have sufficient machinery to study the LMG model.

2.1 Time evolution in driven systems

Consider a closed system with explicitly time dependent hamiltonian $\mathcal{H}(t)$. It's evolution is given by von Neumann's equation viz.:

$$\partial_t \rho(t) = i[\mathcal{H}(t), \rho(t)] \tag{1}$$

Assuming we have information of the system only in the very distant past, i.e. $\rho_0 = \rho(-\infty)$ the solution of such equation is:

$$\rho(t) = \mathcal{U}_{t,-\infty} \rho(-\infty) \mathcal{U}_{-\infty,t} \tag{2}$$

Where we denote the operator of time evolution from time t to t' as $\mathcal{U}_{t,t'}$. Note that this operator, given we have explicit time dependence in $\mathcal{H}$, has to be a concatenation of time ordered small time slices $\delta_t \equiv (t-t')/N$:

$$\mathcal{U}_{t,t'} = \lim_{N\to\infty} e^{-i\mathcal{H}(t-\delta_t)\delta_t} e^{-i\mathcal{H}(t-2\delta_t)\delta_t} ... e^{-i\mathcal{H}(t-N\delta_t)\delta_t} e^{-i\mathcal{H}(t')\delta_t} \equiv \mathbb{T} \exp\left\{-i \int_{t'}^{t} \mathcal{H}(t) dt\right\} \tag{3}$$

And we denote the time ordering procedure by the symbol $\mathbb{T}$

2.2 The adiabatic assumption and equilibrium

Suppose we want to compute an expected value of an operator $\mathcal{O}$:

$$\langle \mathcal{O}(t) \rangle = \frac{1}{\mathrm{Tr}\{\rho(t)\}} \mathrm{Tr}\left\{ \underbrace{\mathcal{U}_{-\infty,t}}_{\text{forward}} \mathcal{O} \underbrace{\mathcal{U}_{t,-\infty}}_{\text{backward}} \rho(-\infty) \right\} \tag{4}$$

We now argue that if we assume that the system evolve adiabatically from $t = -\infty$ to $t = +\infty$ we only care about one of such branches of time evolution.
For concreteness we tackle the case of the evolution of the free ground state denoted by $|0\rangle$. This particular case corresponds to assuming that the system is asymptotically free of interactions. The interacting ground state is then given by:

$$|\mathrm{gs}\rangle \equiv \mathcal{U}_{-\infty,t} |0\rangle \tag{5}$$

Here comes the catch: we assume that at late time the system is again free, so that it returns to the free ground state up to a phase factor $\alpha \in \mathbb{R}$:

$$\mathcal{U}_{-\infty,+\infty} |0\rangle = e^{-\alpha} |0\rangle \tag{6}$$
$$\Rightarrow e^{-i\alpha} = \langle 0| \mathcal{U}_{-\infty,\infty} |0\rangle \tag{7}$$

With that in mind we can express:

$$\langle \mathcal{O} \rangle_{\text{gs}} = e^{i\alpha} e^{-i\alpha} \langle 0| \mathcal{U}_{-\infty,t} \mathcal{O} \mathcal{U}_{t,-\infty} |0\rangle \tag{8}$$
$$= e^{-i\alpha} \langle 0| \mathcal{U}_{-\infty,t} \mathcal{O} \mathcal{U}_{t,-\infty} e^{i\alpha} |0\rangle \tag{9}$$
$$= e^{-i\alpha} \langle 0| \mathcal{U}_{-\infty,t} \mathcal{O} \mathcal{U}_{t,-\infty} \mathcal{U}_{-\infty,\infty} |0\rangle \tag{10}$$
$$= e^{-i\alpha} \langle 0| \underbrace{\mathcal{U}_{-\infty,t}}_{\text{forward}} \mathcal{O} \underbrace{\mathcal{U}_{t,\infty}}_{\text{forward}} |0\rangle \tag{11}$$
$$\tag{12}$$

Therefore:

$$\boxed{\langle \mathcal{O}(t) \rangle_{\text{gs}} = \frac{\langle 0| \mathcal{U}_{-\infty,t} \mathcal{O} \mathcal{U}_{t,+\infty} |0\rangle}{\langle 0| \mathcal{U}_{-\infty,+\infty} |0\rangle}} \tag{13}$$

And we learn that under the assumption that the system is asymptotically free we only need one branch of time. Moreover: Wick rotate and we learn that the same hold's for an observable computed at a thermal state at $T = 1/\beta$ viz. $\langle \mathcal{O}(\beta) \rangle$. Of course, this is not valid out of equilibrium!

2.3 Back and forth: contour evolution

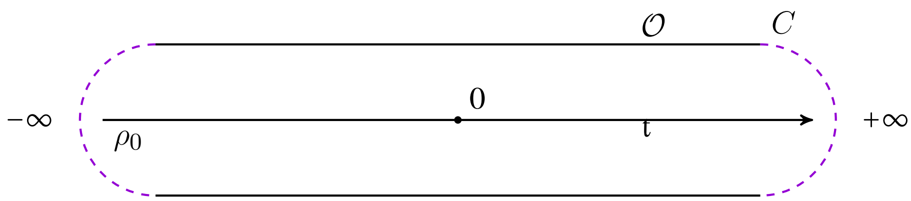
Let's write an expression which is valid out of equilibrium. We start by the first lesson of quantum mechanics:

$$\mathbb{1} = \mathcal{U}_{t,\infty} \mathcal{U}_{\infty,t} \tag{14}$$

Inserting this into our expression for $\langle \mathcal{O}(t) \rangle$ (4):

$$\langle \mathcal{O}(t) \rangle = \frac{\mathrm{Tr}\{\mathcal{U}_{-\infty,\infty} \mathcal{U}_{\infty,t} \mathcal{O} \mathcal{U}_{t,-\infty} \rho(-\infty)\}}{\mathrm{Tr}\{\rho(-\infty)\}} \tag{15}$$

The interpretation of such expression is the following: we come from $t = -\infty$, at t we get $\mathcal{O}$ inserted, evolve to $t = +\infty$ and then we close the contour going from ∞ to $-\infty$.



Motivated by this, we define the contour evolution operator and the corresponding partition function:

$$\mathcal{U}_C \equiv \mathcal{U}_{-\infty,\infty}\mathcal{U}_{\infty,-\infty} \tag{16}$$

$$\mathcal{Z} \equiv \frac{\mathrm{Tr}\{\mathcal{U}_C \rho(-\infty)\}}{\mathrm{Tr}\{\rho(-\infty)\}} \tag{17}$$

This is very often just $\mathbb{1}$ but a very useful one. Also we can define the generating functional for forward/backward(+/-) insertions of operators: let
$\mathcal{H}^{\pm}[V] \equiv \mathcal{H}(t) \pm \mathcal{O}V(t)$

$$\mathcal{Z}[V] \equiv \frac{\mathrm{Tr}\{\mathcal{U}_C[V]\rho(-\infty)\}}{\mathrm{Tr}\{\rho(-\infty)\}} \tag{18}$$

With this technology we can now compute insertions by functional derivatives:

$$\langle\mathcal{O}\rangle = \frac{i}{2}\frac{\delta}{\delta V}\mathcal{Z}[V]\bigg|_{V=0} \tag{19}$$

With that we close this section on time contours. We'll now delve into the derivation of a path integral representation of the partition function and generating function.

2.4 A more quantum information dressing

This idea of doubling the degrees of freedom to achieve the Keldysh path integral can be understood by using one of the most insightful mathematical results which appear often in the quantum information community: the Choi-Jamiokolski isomorphism[2]. The idea of such theorem is as follows. As we all know, the fundamental mathematical objects we use to do QM are:

$$|\psi\rangle \in \mathscr{H} \tag{20}$$

$$\langle\psi| \in \mathscr{H}^* \tag{21}$$

$$\sigma \in \mathscr{L}(\mathscr{H}) \tag{22}$$

[2] This theorem is valid in finite dimension and is easily interpreted in such context. Yet there's a measure theory result which generalises it to infinite dimensions, a Radon-Nykodin theorem for CPTP maps [Belavkin and Staszewski, 1986]

Where "*" denotes the dual of $\mathscr{H}$ and σ can be either a density matrix or a operator. To each $|\psi\rangle$ there's a natural correspondence to a $\langle\psi|$ so that $\mathscr{H}$ and $\mathscr{H}^*$ are isomorphic. Now, any operator/density matrix can be written as:

$$\sigma = \sum_{i,j} \sigma_{ij} |i\rangle \langle j| \in \mathscr{L}(\mathscr{H}) \tag{23}$$

It's therefore not surprising that there's a one-to-one correspondence between σ and a vector:

$$|\sigma\rangle = \sum_{ij} \sigma_{ij} |i\rangle |j\rangle \in \mathscr{H} \otimes \mathscr{H} \tag{24}$$

This idea is simple yet very deep. It says that, for example, any mixed-state in a given hilbert space can be translated in a pure state in an doubled hilbert space. Coherences get translated into entanglement for example! This theorem paves the way to an idea called *vectorization* which is thoroughly used in solving open quantum system dynamics and is actually in the core of Density Matrix Renormalisation Group(DMRG) techniques.
Now let's use this to have another perspective of the "forward" and "backward" evolution . First, jargon: we call $|\sigma\rangle$ the vectorized version of σ, i.e.: $vec(\sigma)$. Now we introduce without proof the following result:

$$vec(ABC) = (C^T \otimes A)vec(B) \tag{25}$$

Now we apply it to the solution of von Neumann's equation:

$$vec(U\rho(0)U^\dagger) = [(U^\dagger)^T \otimes U] |\rho(0)\rangle = (U^* \otimes U) |\rho(0)\rangle \tag{26}$$

We input a density matrix and what we get is a vector which lives in an doubled Hilbert space. The evolution structure we get is one that manifestly evolves half of the Hilbert space forward and the other as the complex conjugate of U. But $U^* = T^{-1}UT$ where T is the time reversal operators. It's also simple to show that $U^* \otimes U$ is unitary. Therefore we learn that once we want to evolve a general density matrix as a vector we need in turn to consider both forward and backward evolution.

3 The Keldysh path integral:

For the purpose of the study of the LMG model, as will become clear, one just needs the bosonic path integral. Therefore we'll concentrate our efforts in such case, we'll construct it using a coherent state approach. We'll also choose a specific initial state, where the system is at equilibrium. Although the derivation is much more general, this will highlight some interesting features.

3.1 Coherent States

We work in the second quantization(Fock space). For concreteness we only care about one mode with $[b, b^\dagger] = \mathbb{1}$. Also we define the coherent states as right eigenstates of b:

$$b\left|\phi\right\rangle = \phi\left|\phi\right\rangle \tag{27}$$

$$\left\langle\phi\right| b^\dagger = \left\langle\phi\right| \bar{\phi} \tag{28}$$

Where bar denotes complex conjugation.

Normal ordering: we say that the operator $\mathcal{O}$ which we assume to depend on $b, b^\dagger$ only, to be normal ordered if all $b^\dagger$'s are to the left and b's to the right. We denote a normaly ordered operators with $\mathcal{O}(b^\dagger, b)$.
If an operator is normal ordered, we can easily map it into a c-number function of $\phi, \bar{\phi}$[3]:

$$\left\langle\phi\right| \mathcal{O}(b^\dagger, b)\left|\phi'\right\rangle = \mathcal{O}(\bar{\phi}, \phi')\left\langle\phi|\phi'\right\rangle \tag{29}$$

The coherent states can be written by means of ladder operators:

$$\left|\phi\right\rangle = \sum_{n=0}^{\infty} \frac{\phi^n}{\sqrt{n!}}\left|n\right\rangle = e^{\phi b^\dagger}\left|0\right\rangle \tag{30}$$

Using that it's easy to compute the overlap of coherent states:

$$\left\langle\phi|\phi'\right\rangle = e^{\bar{\phi}\phi'} \tag{31}$$

Also we have the completeness relation:

$$\mathbb{1} = \int d[\bar{\phi}, \phi] e^{-|\phi|^2}\left|\phi\right\rangle\left\langle\phi\right| \tag{32}$$

$$d[\bar{\phi}, \phi] \equiv \frac{d(\Re\phi)d(\Im\phi)}{\pi} \tag{33}$$

Using the identity above we can now instead of using the number counting basis, take traces using the coherent states. For example:

$$\mathrm{Tr}\{\mathcal{O}\} = \int d[\bar{\phi}, \phi] e^{-|\phi|^2}\left\langle\phi\right| \mathcal{O}\left|\phi\right\rangle \tag{34}$$

[3]We ambiguously denote c-number functions and operator functions by the same $\mathcal{O}$, distinguishing 'em only by their arguments

We'll now show a useful identity:

$$f(\xi) \equiv \langle\phi|\, \xi^{b^\dagger b} \,|\phi'\rangle = e^{\bar{\phi}\phi'\xi} \tag{35}$$

For that purpose we first show another one:

$$g(b^\dagger b)b = bg(b^\dagger b - \mathbb{1}) \tag{36}$$

Proof: g($b^\dagger b$) is defined by it's spectral representation. This is valid since $b^\dagger b$ is hermition:

$$g(b^\dagger b) = \sum_{n=0}^{\infty} g(n) \,|n\rangle \langle n| \tag{37}$$

Then:

$$g(b^\dagger b)b = \sum_{n=0}^{\infty} g(n)\sqrt{n}\,|n-1\rangle \langle n| \tag{38}$$

But the term $n = 0$ is zero and we can write:

$$g(b^\dagger b)b = \sum_{n=1}^{\infty} g(n)\sqrt{n}\,|n-1\rangle \langle n| \tag{39}$$

Changing indices to $m = n - 1$, i.e. $n \to m + 1$:

$$g(b^\dagger b)b = \sum_{m=0}^{\infty} g(m-1)\,|m\rangle \langle m-1|\,\sqrt{m-1} \tag{40}$$

$$= \sum_{m=0}^{\infty} g(m-1)\,|m\rangle \langle m|\, b \tag{41}$$

Therefore:

$$g(b^\dagger b)b = g(b^\dagger b - \mathbb{1})b \tag{42}$$

We can now use this to prove (35). Differentiating with respect to ξ:

$$\partial_\xi f(\xi) = \langle\phi|\, b^\dagger b \xi^{b^\dagger b - \mathbb{1}} \,|\phi'\rangle = \langle\phi|\, \bar{\phi}\xi^{b^\dagger b} b\phi' \,|\phi'\rangle = \bar{\phi}\phi' f(\xi) \tag{43}$$

Where we used the latter identity. Now, since we know that $f(1) = e^{\bar{\phi}\phi'}$, we solve the differential equation yielding:

$$f(\xi) = e^{\bar{\phi}\phi'\xi} \tag{44}$$

3.2 The partition function in closed time

We shall now get to the nitty gritty of this chapter, which is obtaining a path integral representation for a closed time contour partition function. We'll work the case that

$$\mathcal{H} = \omega_0 b^\dagger b \tag{45}$$

The spirit of such derivation is basically the same for the usual path integral for amplitudes in QM. Yet, we'll do it explicitly to highlight the important differences which will appear. Such derivation consisists in expanding our formal $\mathcal{U}_C$ as a Trotter decomposition and inserting multiple coherent states between time slices.
Consider:

$$\mathcal{U}_C \equiv \mathcal{U}_{-\infty,\infty}\mathcal{U}_{\infty,-\infty} \tag{46}$$

$$\mathcal{Z} \equiv \frac{\mathrm{Tr}\{\mathcal{U}_C\rho(-\infty)\}}{\mathrm{Tr}\{\rho(-\infty)\}} \tag{47}$$

Where:

$$\mathcal{U}_{-\infty,\infty} = \lim_{t_1,t_N\to\infty} \mathbb{T}e^{-i(t_N-t_1)H} \tag{48}$$

$$\mathcal{U}_{\infty,-\infty} = \lim_{t_{2N},t_{N+1}\to\infty} \mathbb{T}e^{+i(t_{N+1}-t_{2n})H} \tag{49}$$

$$\tag{50}$$

Now we slice each of the time intervals in N pieces $(t_1 - t_N)/N = \delta t = (t_{2N} - t_{N+1})/N$ and we'll eventually take $N \to \infty$ but in a particular way: since the size of the interval will also go to infinity we want that once both limits are taken $\delta t N$ remains constant.

In the following lines we'll take $2N = 6$ to make our point thus dropping the limits. In that manner, we have:

$$\mathrm{Tr}\{\mathcal{U}_C\rho_0\} = \mathrm{Tr}\left\{e^{i\delta tH}e^{i\delta tH}\mathbb{1}e^{-i\delta tH}e^{-i\delta tH}\rho_0\right\} \tag{51}$$

Now, if we insert 5 sets of completeness relations (32) and take the trace represented by a sixth according to (34):

$$\mathrm{Tr}\{\mathcal{U}_C\rho_0\} = \int\prod_{j=1}^{6} d[\bar{\phi}_j,\phi_j]e^{-|\phi_j|^2} \tag{52}$$

$$\times \langle\phi_6| e^{i\delta tH} |\phi_5\rangle \langle\phi_5| e^{i\delta tH} |\phi_4\rangle \langle\phi_4| \mathbb{1} |\phi_3\rangle \langle\phi_3| e^{-i\delta tH} |\phi_2\rangle \langle\phi_2| e^{-i\delta tH} |\phi_1\rangle \langle\phi_1| \rho_0 |\phi_6\rangle \tag{53}$$

Now, since at the end of the day we‘ll take limits, we regard δ_t as small:

$$\langle\phi_j| e^{\pm i\delta t H(b^\dagger,b)} |\phi_{j-1}\rangle \approx (1 \pm iH(\bar{\phi}_j,\phi_{j-1}))\langle\phi_j|\phi_{j-1}\rangle \tag{54}$$

$$= [1 \pm iH(\bar{\phi}_j,\phi_{j-1})]e^{\bar{\phi}_j\phi_{j-1}} \tag{55}$$

$$\approx e^{\pm i\delta t H(\bar{\phi}_j,\phi_{j-1})}e^{\bar{\phi}_j\phi_{j-1}} \tag{56}$$

Moreover:

$$\langle\phi_1|\rho_0|\phi_6\rangle = \langle\phi_1| e^{-\beta(\omega_0-\mu)H(b^\dagger,b)}|\phi_6\rangle \tag{57}$$

$$= \exp\left\{\bar{\phi}_1\phi_6\Gamma\right\} \tag{58}$$

$\Gamma \equiv e^{-\beta(H-\mu N)} = e^{-\beta(\omega_0-\mu)}$
Substituting in our expression for the trace we get:

$$\begin{aligned}\mathrm{Tr}\{\mathcal{U}_C\rho_0\} = \int\left(\prod_{j=1}^{6} d[\bar{\phi}_j,\phi_j]e^{-|\phi_j|^2}\right)e^{\bar{\phi}_4\phi_3}e^{\bar{\phi}_1\phi_6\Gamma}\times \\ \times\exp\left\{i\delta t\left(\sum_{j=5}^{6}H(\bar{\phi}_j,\phi_{j-1}) - i\frac{\bar{\phi}_j\phi_{j-1}}{\delta t} - \sum_{j=2}^{3}H(\bar{\phi}_j,\phi_{j-1}) + i\frac{\bar{\phi}_j\phi_{j-1}}{\delta t})\right)\right\}\end{aligned} \tag{59}$$

As in the usual derivation of the path integral in QM we can see in the second line the Legendre transform of the hamiltonian popping up for both time branches which defines the action when we take limits. Yet there are some differences from the usual QM case.

To make them clear we now use the explicit form of $\mathcal{H}$, which is gaussian [4] In that case we can write this expression as a quadratic form:

$$\mathrm{Tr}\{\mathcal{U}_C\rho_0\} = \int\left(\prod_{j=1}^{6} d[\bar{\phi}_j,\phi_j]\right)\exp\left\{\bar{\phi}_j G^{-1}_{j,j'}\phi_{j'}\right\} \tag{60}$$

And the partition function:

$$\mathcal{Z} = \frac{1}{\mathrm{Tr}\{\rho_0\}}\int\left(\prod_{j=1}^{6} d[\bar{\phi}_j,\phi_j]\right)\exp\left\{\bar{\phi}_j G^{-1}_{j,j'}\phi_{j'}\right\} \tag{61}$$

[4] At most quadratic in bosonic operators.

Now, looking at the structure of (59) we have terms which are in the lower diagonal and terms in the main diagonal plus the two skew-diagonal terms: one due to the thermal state and another due to the midpoint in time. Moreover, between the lower diagonal terms the ones in the time-backwards branch have a + sign, so we call 'em h_+ and the time-forward one a − and we name 'em h_-, $h_\pm = 1 \mp i\omega_0 \delta t$. So, in matrix form:

$$G^{-1} = \left(\begin{array}{ccc|ccc} -1 & 0 & 0 & 0 & 0 & \Gamma \\ h_+ & -1 & 0 & 0 & 0 & 0 \\ 0 & h_+ & -1 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & h_- & -1 & 0 \\ 0 & 0 & 0 & 0 & h_- & -1 \end{array}\right) \tag{62}$$

What is important about this object is that this is **not** block-diagonal. Due to the skew diagonal terms the forward and backward branches do not evolve independently. Yet, the partition function is still computed as we are used to, i.e. the gaussian integral is performed using:

$$\mathcal{Z} = \frac{1}{\mathrm{Tr}\{\rho_0\}} \int \left(\prod_J d[\bar{\phi}_j, \phi_j] \right) \exp\left\{ \bar{\phi}_j G^{-1}_{j,j'} \phi_{j'} \right\} = \frac{1}{\mathrm{Tr}\{\rho_0\} \det\{-iG^{-1}\}} \tag{63}$$

Which also allow us to take arbitrary N and let $N \to \infty$. This is done by looking at the general form of the action:

$$S[\bar{\phi}, \phi] = \sum_{j=2}^{2N} \delta t_j \left[i\bar{\phi}_j \frac{\phi_j - \phi_{j-1}}{\delta t} - H(\bar{\phi}_j, \phi_{j-1}) \right] + i\bar{\phi}_1 \left[\phi_1 - i\rho(\omega_0)\phi_{2N} \right] \tag{64}$$

Where $\delta t_j = \delta t$ for the first N and $\delta t_j = -\delta t$ for the other half.
Taking the continuum limit we get:

$$S[\bar{\phi}(t), \phi(t)] = \int_C dt \bar{\phi}(t) G^{-1} \phi(t) \tag{65}$$

And we **denote** the inverse propagator: $i\partial_t - \omega_0$.
Now, instead of using this time contour deal we can simplify our life by introducing a pair of fields ϕ^+, ϕ^- each for a time branch so that we have an ordinary integral:

$$S[\bar{\phi}, \phi] = \int_{-\infty}^{\infty} dt [\bar{\phi}^+ (i\partial_t - \omega_0)\phi^+ - \bar{\phi}_- (i\partial_t - \omega_0)\phi^-] \tag{66}$$

Time to be sincere: the above expression is total bullshit. Our notation for the inverse propagator forcibly split the fields in two uncorrelated branches which is clearly not true for what we‘ve seen in the explicit matrix structure of G^{-1} in the finite N case.

At least in this set of variables we have no good "free-theory" and this is the next goal: construct a suitable representation, free of time contours and properly defined Green‘s functions. This will spare us from looking back to the discrete case everytime. Intuitively the propagators are related to two point functions, so it‘ll be instructive to look at the computation of the correlator between $\phi, \bar{\phi}$.

3.3 Green’s Functions

Let’s go back to the discrete case and compute the two point function:

$$\left\langle \phi_j \bar{\phi}_{j'} \right\rangle = \int \mathcal{D}[\bar{\phi}, \phi] \phi_j \bar{\phi}_{j'} \exp\left\{ i \left(\sum_{k,k'}^{2N} \bar{\phi}_k G^{-1}_{k,k'} \phi_{k'} \right) \right\} \tag{67}$$

Where our integration measure is $\mathcal{D} \equiv \prod_{j=2}^{2N} d[\bar{\phi}_j, \phi_j]$ and one can guess that quantity is just $iG_{j,j'}$. In total agreement with our gaussian integration formula the determination of such object consist in inverting the matrix G^{-1}. We shall do it again for $2N = 6$

$$iG = \frac{1}{\det\{-iG^{-1}\}} \left(\begin{array}{ccc|ccc} 1 & h_- h_+^2 \Gamma & h_+^2 \Gamma & h_+^2 \Gamma & h_+ \Gamma & \Gamma \\ h_- & 1 & h_- h_+^2 \Gamma & h_- h_+^2 \Gamma & h_- h_+ \Gamma & h_- \Gamma \\ h_-^2 & h_- & 1 & h_-^2 h_+^2 \Gamma & h_-^2 h_+ \Gamma & h_-^2 \Gamma \\ \hline h_-^2 & h_- & 1 & 1 & h_-^2 h_+ \Gamma & h_-^2 \Gamma \\ h_-^2 h_+ & h_- h_+ & h_+ & h_+ & 1 & h_-^2 h_+ \Gamma \\ h_-^2 h_+^2 & h_- h_+^2 & h_+^2 & h_+^2 & h_+ & 1 \end{array} \right) \tag{68}$$

Looking again to the quartiers of such object we note: the upper diagonal block equals the lower one if we make $h_+ \leftrightarrow h_-$. The relation between the skew-diagonal blocks is similiar but up to the statistical factor Γ.
In that manner we then write:

$$iG = \frac{1}{\det\{-iG^{-1}\}} \left(\begin{array}{c|c} iG^{\mathbb{T}} & iG^{<} \\ \hline iG^{>} & iG^{\tilde{\mathbb{T}}} \end{array} \right) \tag{69}$$

These functions are indeed correlation functions between the $\phi^{\pm}$ fields we introduce to account for forward and backward branches, that is, the full 2-point function can be decomposed in 4 parts, one for each 2-point function between $\phi^{\pm}$ fields. $\mathbb{T}$ and $\tilde{\mathbb{T}}$ indicate time ordered and anti time ordered while $<$ indicates the correlators of $\phi^{+}\bar{\phi}^{-}$ and similarly to $>$. G can be regarded as a covariance matrix between two theories. For gaussian systems covariance matrices are all one needs to know to figure out everything about the theory. For arbitrary $2N$ we have the following formulas:

$$\left\langle \phi^{+}\bar{\phi}^{+}\right\rangle = iG^{\mathbb{T}}_{j,j'} = \frac{h_{-}^{j-j'}}{\det\{-iG^{-1}\}} \times \begin{cases} 1 \text{ if } j \geq j' \\ \Gamma(h_{+}h_{-})^{N-1} \text{ if } j < j' \end{cases} \tag{70}$$

$$\left\langle \phi^{-}\bar{\phi}^{-}\right\rangle = iG^{\tilde{\mathbb{T}}}_{j,j'} = \frac{h_{-}^{j-j'}}{\det\{-iG^{-1}\}} \times \begin{cases} \Gamma(h_{+}h_{-})^{N-1} \text{ if } j > j' \\ 1 \text{ if } j \leq j' \end{cases} \tag{71}$$

$$\left\langle \phi_{j}^{+}\bar{\phi}_{j'}^{-}\right\rangle = iG^{>} = \frac{\Gamma h_{+}^{j'-1} h_{-}^{j-1}}{\det\{-iG^{-}1\}} \tag{72}$$

$$\left\langle \phi_{j}^{+}\bar{\phi}_{j'}^{-}\right\rangle = iG^{<} = \frac{h_{+}^{N-j} h_{-}^{N-j'}}{\det\{-iG^{-}1\}} \tag{73}$$

Looking at these elements we can take a proper $N \to \infty$ limit. First note that $h_{\pm}^{j} = (1 \pm i\omega_0 \delta t)^{j}$ but $N \to \infty$ means $\delta t << 1$ so $h_{\pm}^{j} \xrightarrow{N\to\infty} e^{\pm i\omega_0 \delta t j} = e^{\pm i\omega_0 t}$ and $\delta t j$ is just a piece of time t since we sliced the interval homegeneously and the same holds if we reason for j', t'. Moreover $(h_{+}h_{-})^{N} = (1 + \omega_0^2 \delta t^2)^{N} \xrightarrow{N\to\infty} 1$. Yet, the determinant is just $1 - \Gamma$ which combined with the statistical factor Γ gives the Bose-Einstein statistics:

$$n_B = \frac{\Gamma}{1-\Gamma} = \frac{e^{-\beta(\omega_0-\mu)}}{1 - e^{-\beta(\omega_0-\mu)}} \tag{74}$$

All these observations lead to the two point functions in the continuum limit:

$$\left\langle \phi^{+}(t)\bar{\phi}^{-}(t')\right\rangle = iG^{<}(t,t') = n_B e^{-i\omega_0(t-t')} \tag{75}$$

$$\left\langle \phi^{+}(t)\bar{\phi}^{-}(t')\right\rangle = iG^{<}(t,t') = (1+n_B) e^{-i\omega_0(t-t')} \tag{76}$$

$$\left\langle \phi^{+}(t)\bar{\phi}^{+}(t')\right\rangle = iG^{\mathbb{T}}(t,t') = \Theta(t-t')iG^{>}(t,t') + \Theta(t'-t)iG^{<}(t,t) \tag{77}$$

$$\left\langle \phi^{-}(t)\bar{\phi}^{-}(t')\right\rangle = iG^{\tilde{\mathbb{T}}}(t,t') = \Theta(t'-t)iG^{>}(t,t') + \Theta(t-t')iG^{<}(t,t) \tag{78}$$

Where Θ is the Heaviside function. One can also write the expected values above as path integrals, but as mentioned before and confirmed by our continuum limit the forward and backward branches $\phi^{\pm}$ are not independent. By noting the following identity between these functions we'll be in position to construct the desired representation for Keldysh path integrals:

$$G^{\mathbb{T}}(t,t') + G^{\tilde{\mathbb{T}}}(t,t') - G^{<}(t,t') - G^{>}(t,t') = 0 \tag{79}$$

Which is not valid for $t = t'$ since the Θ function has different limits from left and right. The discrete version would lead to a $\delta_{j,j'}$ at the RHS but in the continuum what we get is a singular line, whose integral over time is zero rather than one and therefore it is not a delta function. Yet, this also highlights that for most purposes the singularity is irrelevant. This relation between Green's functions is the first Ward identity we encounter in Keldysh theory. This relation is in fact true for any Keldysh theory, it's a redundancy inherent to the doubling of degrees of freedom although the derivation we made used the explicit form of the G's which are written for an equilibrium situation.

3.4 Keldysh variables

We shall now perform a rotation in our phase space, introducing "classical"(c) and "quantum"(q) variables[5]:

$$\phi^c = \frac{1}{\sqrt{2}}(\phi^+ + \phi^-) \tag{80}$$

$$\phi^q = \frac{1}{\sqrt{2}}(\phi^+ - \phi^-) \tag{81}$$

Using these we now rewrite our Green's functions:

$$\left\langle \phi^+(t)\bar{\phi}^+(t') \right\rangle = \frac{1}{2}\left(\left\langle \phi^c(t)\bar{\phi}^c(t') \right\rangle + \left\langle \phi^q(t)\bar{\phi}^q(t') \right\rangle + \left\langle \phi^c(t)\bar{\phi}^q(t') \right\rangle + \left\langle \phi^q(t)\bar{\phi}^c(t') \right\rangle \right) \tag{82}$$

$$\left\langle \phi^-(t)\bar{\phi}^-(t') \right\rangle = \frac{1}{2}\left(\left\langle \phi^c(t)\bar{\phi}^c(t') \right\rangle + \left\langle \phi^q(t)\bar{\phi}^q(t') \right\rangle - \left\langle \phi^c(t)\bar{\phi}^q(t') \right\rangle - \left\langle \phi^q(t)\bar{\phi}^c(t') \right\rangle \right) \tag{83}$$

$$\left\langle \phi^+(t)\bar{\phi}^-(t') \right\rangle = \frac{1}{2}\left(\left\langle \phi^c(t)\bar{\phi}^c(t') \right\rangle - \left\langle \phi^q(t)\bar{\phi}^q(t') \right\rangle - \left\langle \phi^c(t)\bar{\phi}^q(t') \right\rangle + \left\langle \phi^q(t)\bar{\phi}^c(t') \right\rangle \right) \tag{84}$$

$$\left\langle \phi^-(t)\bar{\phi}^+(t') \right\rangle = \frac{1}{2}\left(\left\langle \phi^c(t)\bar{\phi}^c(t') \right\rangle - \left\langle \phi^q(t)\bar{\phi}^q(t') \right\rangle + \left\langle \phi^c(t)\bar{\phi}^q(t') \right\rangle - \left\langle \phi^q(t)\bar{\phi}^c(t') \right\rangle \right) \tag{85}$$

If for instance we sum every term we get:

$$\left\langle \phi^c(t)\bar{\phi}^c(t') \right\rangle = \frac{1}{2}\left(\left\langle \phi^+(t)\bar{\phi}^+(t') \right\rangle + \left\langle \phi^-(t)\bar{\phi}^-(t') \right\rangle + \left\langle \phi^+(t)\bar{\phi}^-(t') \right\rangle + \left\langle \phi^-(t)\bar{\phi}^+(t') \right\rangle \right) = \underbrace{iG^< + iG^>}_{\equiv iG^K}$$

[5] I personally don't like these names. The "classical" in fact is an "average" between forward and backward branches whilst "quantum" is the difference between them. These terms are correct at $T = 0$, but for T$\neq$ 0 the "quantum" accounts both for classical and quantum noise and can lead to confusion.

Defining the Keldysh Green's function as the classical-classical correlator. Similar reasoning for the others leads to a new set of Green's functions structured as a covariance matrix:

$$i\mathbf{G} = \left(\begin{array}{c|c} iG^K(t,t') & iG^R(t,t') \\ \hline iG^A(t,t') & G^Q(t,t') \end{array}\right) \tag{86}$$

Where:

$$iG^R(t,t') = \left\langle \phi^c(t)\bar{\phi}^q(t') \right\rangle = \frac{i}{2}\left(G^{\mathbb{T}} - G^{\tilde{\mathbb{T}}} + G^> - G^<\right) = \Theta(t-t')(G^> - G^<) \tag{87}$$

$$iG^A(t,t') = \left\langle \phi^c(t)\bar{\phi}^q(t') \right\rangle = \frac{i}{2}\left(G^{\mathbb{T}} - G^{\tilde{\mathbb{T}}} - G^> + G^<\right) = \Theta(t'-t)(G^< - G^>) \tag{88}$$

$$iG^Q(t,t') = \left\langle \phi^q(t)\phi^q(t') \right\rangle = \frac{i}{2}\left(G^{\mathbb{T}} + G^{\tilde{\mathbb{T}}} - G^< - G^>\right) = 0 \tag{89}$$

In the last one we used our Ward identity (79) and we conclude that in Keldysh variables the convariance matrix is just:

$$i\mathbf{G} = \left(\begin{array}{c|c} iG^K(t,t') & iG^R(t,t') \\ \hline iG^A(t,t') & 0 \end{array}\right) \tag{90}$$

Now the covariance is built by correlators between classical and quantum fields. "R" stands for retarded and "A" for advanced. Although we pointed that these are sometimes ill defined for $t = t'$, if we note that $G^R + G^A = G^{\mathbb{T}} - G^{\tilde{\mathbb{T}}}$ and rewind to the discretized form we end up learning that indeed:

$$G^R(t,t) + G^A(t,t) = 0 \tag{91}$$

The case is different for $G^A - G^R$. This difference will yield us an important result. For that, we explicitly write:

$$G^R(t,t') = -i\Theta(t-t')(e^{-i\omega_0(t-t')}) \tag{92}$$

$$G^A(t,t') = i\Theta(t'-t))e^{-i\omega_0(t-t')} \tag{93}$$

$$G^K(t,t') = -i(1+2n_B(\omega_0))e^{-i\omega(t-t')} \tag{94}$$

Note that all temperature information resides in G^K. We'll now look at the Fourier transform of such objects, with respect to $\tau \equiv t - t'$ to the energy E representation. First, we note that G^R and G^A fourier transforms are exactly the kind of object we have when deriving the Feynman propagator for the a scalar field theory. Their fourier transforms are just:

$$\tilde{G}^R(E) = \frac{1}{E - \omega_0 + i\epsilon} \tag{95}$$

$$\tilde{G}^A(E) = \frac{1}{E - \omega_0 - i\epsilon} \tag{96}$$

$$\tag{97}$$

And we note that once we take $\epsilon \to 0$ we have:

$$\tilde{G}^R(E) - \tilde{G}^A(E) = 2\pi\delta(E - \omega_0) \tag{98}$$

The transform of G^K is pretty easy, it's just a δ:

$$\tilde{G}^K(E) = -i2\pi(1 + 2n_B(\omega_0))\delta(E - \omega_0) = -i2\pi(1 + 2n_B(\omega))\delta(E - \omega_0) \tag{99}$$

Let's now rewrite:

$$1 + 2n_B(\omega) = \frac{e^{-\beta(\omega-\mu)} + 1}{1 - e^{-\beta(\omega-\mu)}} = \frac{\cosh(\beta/2(\omega - \mu))}{\sinh(\beta/2(\omega - \mu))} = \coth(\beta/2(\omega - \mu)) \tag{100}$$

Now, comparing (98) with (99) and using (100) we obtain:

$$\boxed{\tilde{G}^K(E) = \coth\frac{\omega - \mu}{2T}\left(\tilde{G}^R(E) - \tilde{G}^A(E)\right)} \tag{101}$$

This result is called fluctuation-dissipation relation and it's a witness of thermal equilibrium. This is indeed a Ward-Takahashi identity comming from a symmetry which implements thermal equilibrium. This can be derived in a more general framework by a symmetry constructed with the KMS condition[6]:

$$\mathcal{T}_\beta\phi_\sigma(t) = \bar{\phi}_\sigma(-t - \sigma\beta/2) \tag{102}$$

To wrap up we shall use our covariance matrix to write the Keldysh action in it's whole glory. For that we just need the inverse of $\mathbf{G}$ and we can state:

[6] For more details check [Sieberer et al., 2016]

$$\boxed{S[\phi^c,\phi^q] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} dtdt'(\bar{\phi}^c(t),\bar{\phi}^q(t'))\begin{pmatrix} 0 & (G^A(t,t'))^{-1} \\ (G^R(t,t'))^{-1} & (G^K(t,t'))^{-1}\end{pmatrix}\begin{pmatrix}\phi^c(t') \\ \phi^q(t')\end{pmatrix}} \tag{103}$$

Or using spinor notation:

$$\boxed{S[\phi^c,\phi^q] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} dtdt'\mathbf{\Psi}(t)\mathbf{G}^{-1}(t,t')\mathbf{\Psi}(t')} \tag{104}$$

Using the fourier transform one can show that:

$$(G^R)^{-1} = \delta(t-t')(i\partial_t - \omega_0 \pm i\epsilon) = (G^A)^{-1} \tag{105}$$

To find the inverse of G^K we'd need a technology not developed so far which is the Wigner transform. We restrict ourselves to state the result which is $2i\epsilon F$, where $\epsilon \to 0$ and F is the Wigner transform. The fact that this term goes to zero is indeed true for free theories in general but not for interacting ones[7].

The "0" at the classical quartier of the action is basically the statement that the contour over the classical fields doesn't break time reversal and must be zero then.

3.5 A more high energy physics dressing

In this subsection we shall comment on two more elegant and mathematically robust ways to look at the Keldysh-Schwinger formalism. We'll refrain ourselves from a thoroughly exposition, leaving it for references.
In the bosonic case the Wigner-Weyl approach to quantum mechanics is suited to introduce Keldysh integrals, as hinted by the latter usage of the Wigner quasi-probability. In such formalism one assign to quantum mechanical states and operators objects called quasi-probability distributions. This objects are Schwarz class functions, normalised by one but can assume negative values. For the Wigner-Weyl formulation of Keldysh path integral the appendix of [Foss-Feig et al., 2017] provides a concise treatment.

The last approach is a very elegant and more field theoretical one. In the work of [Haehl et al., 2016] they argue that in fact this thing of doubling the degrees of freedom which one does to achieve the SK path integral leads to a redundancy. In the continuum case one can then introduce a gauge parameter and require path integral

[7]Central limit theorem says that gaussianity goes hand in hand with scale invariance. It's not surprising that gaussian systems generalize their properties easily when we add more modes

invariance. This leads to the introduction of a ghost anti-ghost pair $c, \bar{c}$. One then reformulates the theory in terms of superfields:

$$\boldsymbol{\Psi} = (\phi_c, \phi_q, c, \bar{c}) \tag{106}$$

Every Schwinger-Keldysh theory set in this framework will then conserve a set of BRST charges $\mathcal{Q} = \{\mathcal{Q}_{SK}, \bar{\mathcal{Q}}_{SK}\}$.
In the simplest case of a scalar field in equilibrium, the SK action is just:

$$S_K = -\int d^d k \frac{\sqrt{-g}}{2} \left(\partial_\mu \bar{\phi}_+ \partial^\mu \phi_+ - \partial_\mu \bar{\phi}_- \partial^\mu \phi_-\right) \tag{107}$$

$$S_K = -\int d^d k \frac{\sqrt{-g}}{2} \left(\partial_\mu \bar{\phi}_c \partial^\mu \phi_q + \partial_\mu \bar{\phi}_q \partial^\mu \phi_c\right) \tag{108}$$

Now if one performs the field transformation:
$\{\phi_+ \to \phi_+ + \chi, \phi_- \to \phi_- + \chi\} = \{\phi_c \to \phi_c + \chi, \phi_q \to \phi_q\}$. If one then requires that the path integral is invariant by ξ and take ghost charges: $gh(\phi_{c/q}) = 0$, $gh(c) = 1$ and $gh(\bar{c}) = -1$, the action expanded action is:

$$S_K = -\int d^d k \frac{\sqrt{-g}}{2} \left(\partial_\mu \bar{\phi}_c \partial^\mu \phi_q + \partial_\mu \bar{\phi}_q \partial^\mu \phi_c + \bar{c}\nabla^2 c + c\nabla^2 \bar{c}\right) \tag{109}$$

Besides the charges inherent to the SK construction aforementioned this action also conserves other set of charges related to the $\mathcal{T}_\beta$ symmetry $\mathcal{Q} = \{\mathcal{Q}_{KMS}, \bar{\mathcal{Q}}_{KMS}\}$
In fact this approach is very efficient when it comes to renormalisation group methods and it paves the way to the computation of out-of-time ordered correlators. Such objects are very important in either black-hole and quantum information communities, as they are used to quantify quantum chaos and local operators spread in time, as was shown in a remarkable work from [Maldacena et al., 2016].

3.6 The layman‘s Keldysh Path Integral

Ok, we've learned so far how to construct a Keldsyh path integral from scratch. Now, it'll be useful to summarise the way we take a simple hamiltonian in a closed system and write the Keldysh action right away without passing through the closed time contour.
Suppose we have our normal ordered $\mathcal{H}(b^\dagger, b)$. Then, from that we obtain the scalar $\mathcal{H}(\bar{\phi}, \phi)$. We can call it actually just $\mathcal{H}(\phi)$. One then substitute for $\phi^\pm$. In that manner, to avoid contour integral in time one instead write it as and ordinary integral by making:

$$\int_{\mathcal{C}} dt\mathcal{H}(\phi) \to \int dt[\mathcal{H}(\phi^+) - \mathcal{H}(\phi^-)] \tag{110}$$

In general, the integration boundaries of the RHS integral depend on the initial conditions. One then introduces intermediate variables in which the Legendre transform is simpler $\psi \pm \varphi \equiv \phi^{\pm}$:

$$S[\psi, \varphi] = 2i \int dt(\bar{\varphi}\partial_t \psi - \varphi \partial_t \bar{\psi}) - \int dt[\mathcal{H}(\psi + \varphi) - \mathcal{H}(\psi - \varphi)] \tag{111}$$

From these point there are two options. Either one write the final form of the action in terms of Keldysh variables:

$$\phi^c = \frac{1}{\sqrt{2}}(\phi^+ + \phi^-) \tag{112}$$

$$\phi^q = \frac{1}{\sqrt{2}}(\phi^+ - \phi^-) \tag{113}$$

By relating then to ψ, φ it's easy to obtain the the action as we introduced earlier. Nonetheless we shall instead convey to [Titum and Maghrebi, 2019]'s notation, where they instead use x and p. To do that one simply make the change:

$$\phi^{c/q} = \frac{1}{\sqrt{2}}(x_{c/q} + ip_{c/q}) \tag{114}$$

So if we have an (scalar) hamiltonian $\mathcal{H}(x, p)$ our Keldysh action is:

$$\boxed{S_K = \int dt(p_q \partial_t x_c - p_c \partial_t x_q) - \int dt \left[\mathcal{H}\left(\frac{x_c + x_q}{\sqrt{2}}, \frac{p_c + p_q}{\sqrt{2}}\right) - \mathcal{H}\left(\frac{x_c - x_q}{\sqrt{2}}, \frac{p_c - p_q}{\sqrt{2}}\right)\right]} \tag{115}$$

which we'll use to construct the SK action for the LMG model. Yet, to introduce the initial condition in a more general way it'll prove convenient to write it in terms of a Wigner function. For the case of mixed states, the Wigner function is given by the Wigner transform of the density matrix:

$$\boxed{\mathcal{W}(x, p) \equiv \frac{1}{\pi} \int_{-\infty}^{\infty} \langle x + y | \, \rho \, | x - y \rangle \, e^{-2ipy} dy} \tag{116}$$

One can think about the above expression as an analog of a wave function for the density matrix: it's a projection along values of the position operator weighted by momenta.

In that manner one can represent a general Keldysh path integral in x, p variables as:

$$\boxed{\mathcal{Z} = \int \mathcal{D}[x_{c/p}(t), p_{c/p}(t)]\mathcal{W}(x_c, p_c)e^{iS_K}} \tag{117}$$

Where one takes the initial state to be represented by classical fields

4 The Lipkin-Meshkov-Glick model

Ultracold atoms and trapped ions became ubiquitous terms in quantum physics during the last years, one may argue that these terms may magically bring money to your research. But there's in fact a reason for all this fuzz: such techniques go beyond the usual idea of an experimental set up, in which the goal is to to reproduce in a isolated manner the phenomena of nature. What's different in these platforms is that control is so tight that instead of simulating nature one can rather simulate theory. Toy models dear to theoreticians are promoted to the status of technology yet at the quantum level. This is the case of a lot of quantum spin chains which are prominent set ups to study genuinely quantum effects in many-body systems.

One of the most paradigmatic of such models is the Lipkin-Meshkov-Glick model. It's one of the few which display a quantum phase transition, is long range interacting although easy enough to admit exact treatment and whose phase diagram is complex enough to be tweaked in a variety of ways.

4.1 Basics of the LMG model

Our application of choice is the quenched dynamics of the LMG model, which is a prototype of long-range spin interacting model.

$$\mathcal{H} = \frac{J}{N}\sum_{i<j}^{N}\left[\gamma_x \sigma_i^x \sigma_j^x + \gamma_y \sigma_i^y \sigma_j^y\right] - \Delta \sum_{i=1}^{N} \sigma_i^z \tag{118}$$

where we fixed $i < j$ to avoid overcounting interactions. We also introduce a $1/N$ normalising factor where N is the system size. The thermodynamic limit stands for taking $N \to \infty$.
This model is well known and it's phase diagram thoroughly studied. It exhibits a phase transition for $\Delta = J \max\{\gamma_x, \gamma_y\}$. It's also an integrable model and can be analytically solved by the Bethe Ansatz in the $N \to \infty$ limit. Yet for our purposes we'll take another path.

Let's first note that this model can easily be cast in a much simpler and elegant way. Consider the total spin operators:

$$S_\alpha \equiv \frac{1}{2}\sum_{i=1}^{N}\sigma_i^\alpha \qquad\qquad \alpha = x, y, z \tag{119}$$

Introducing it in the hamiltonian we have:

$$\mathcal{H} = \frac{J}{2N}\left[\gamma_x S_x^2 + \gamma_y S_y^2\right] + \Delta S_z \tag{120}$$

Where the spin values are $s : N/2, N/2-1, ..., -N/2-1, -N/2$. In the disordered phase, below criticallity, it's magnetization profile is such that $1/N\langle S_x\rangle = 0 = 1/N\langle S_y\rangle$ and in the ordered phase at least one of these are non-vanishing [Botet and Jullien, 1983], but in both cases they have no scaling behaviour. In contrast, at criticallity, the squared magnetization exhibits the following scaling:

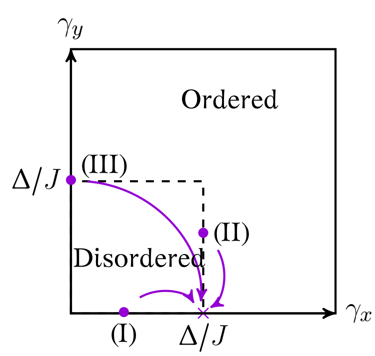
$$\frac{1}{N}\left\langle S_y^2\right\rangle \propto \frac{1}{N^{1/3}} \tag{121}$$

$$\frac{1}{N}\left\langle S_x^2\right\rangle \propto N^{1/3} \tag{122}$$

Assuming $\gamma_x > \gamma_y$. If the inequality is reversed, the exponents are too.

4.2 Quench-Dynamics and numerical approach

Quenched dynamics go hand-in-hand with experimental implementations. The idea of a quench is the following: suppose we prepare the system in the ground state of the Hamiltonian (120) for a parameter Δ_i, say $\mathcal{H}(\Delta_i)$. Then, since we have a good experimental control of the magnetic fields we suddenly change it to Δ_f. Now, we let the ground state of $\mathcal{H}(\Delta_i)$ evolve according to $\mathcal{H}(\Delta_f)$. You can guess that interesting physics come into play when we cross phase transitions using quenches. The quenches we'll investigate are depicted in the following diagram:



To study the *post quench* dynamics, we shall use information about the system both in the *pre-quench* and in the *target/bulk*. By *post quench* I mean the transient behaviour the system, once prepared in a initial condition, will undergo when we suddenly change a parameter. This transient regime as we'll see, is neither of the ones vastly studied in the literature but can be studied by means of them in the Keldysh formalism.
We shall first develop some intuition and understand what we want to achieve by looking at the exact diagonalization given by numerical simulations of the problem:

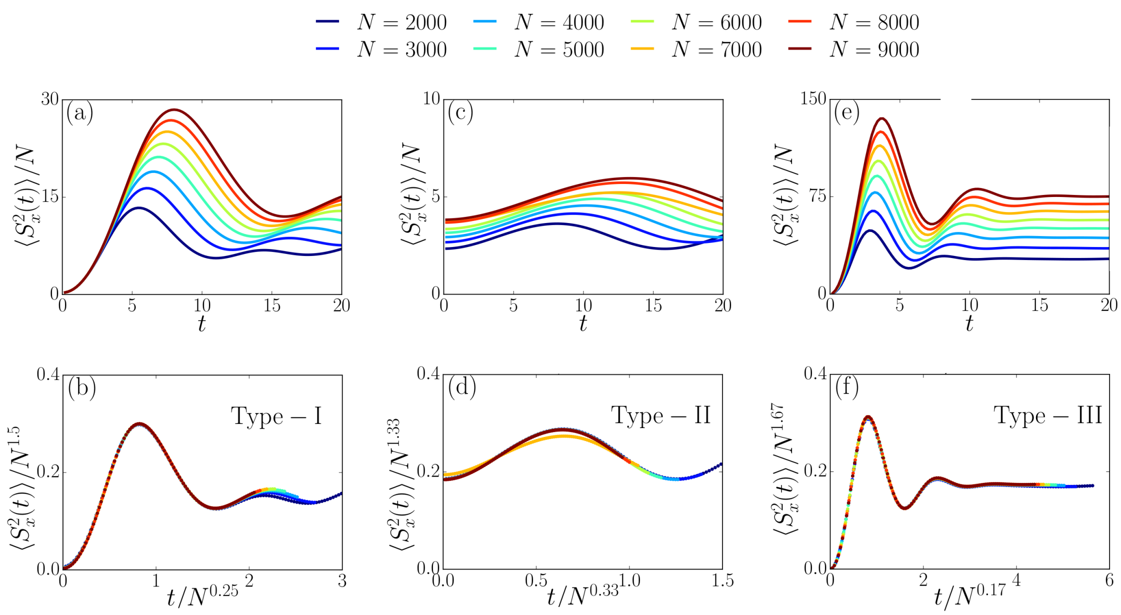


Figure 1 – Stolen from [Titum and Maghrebi, 2019]

In the figure above, at the upper row, the magnetization in the x direction is computed for different system sizes. At t=0 the system is in a different state for each type of quench and it's subsequently evolved by the quenched bulk. At bottom row we can see a neat trick to obtain the critical exponents of such profile: find what's the rescale factor needed to make all the curves coincide, both in horizontal and vertical axis. This procedure is the same of finding α, ζ such that:

$$\frac{\langle S_x^2\rangle}{N} = N^\alpha f\left(\frac{t}{N^\zeta}\right) \tag{123}$$

We can observe that the type (II) amount to $\alpha = 1/3$, which coincides with the critical behaviour already commented. The reason is that we haven't done much in that quench: we quenched from a critical point where $\gamma_x > \gamma_y$ to another one with the

same behaviour. Yet, for both other quenches different critical exponents arise, hinting for an non-equilibrium behaviour. Moreover, observe that although the shape of the second column curves of types (I) and (III) are similar, their scaling coefficients are in stark disagreement.
We shall now construct a bosonic hamiltonian which approximates the LMG model in the $N \to \infty$ limit. This will render us an action, suited for a renormalisation procedure which will trace out irrelevant physics. At the end we want to obtain the exponents and understand better how this phenomena emerge from the microscopic description.

4.3 Holstein-Primakoff transformation and large N approximations

We'll employ now a series of transfomations and approximations to the LMG hamiltonian (120). The transformations we'll need are the following:

$$S_z = \frac{N}{2} - a^\dagger a \tag{124}$$

$$S_- = S_x - iS_y = \sqrt{N - a^\dagger a}a \approx \sqrt{N}\left(1 - \frac{a^\dagger a}{2N}\right)a \tag{125}$$

$$S_+ = S_x + iS_y = a^\dagger\sqrt{N - a^\dagger a} \approx \sqrt{N}a^\dagger\left(1 - \frac{a^\dagger a}{2N}\right) \tag{126}$$

Such approximations are valid for $1/N << 1$. Although these mappings seem rather arbitrary there's a reason they are valid: they are compatible with the commutation relations of both sides of the equations [Landi, 2019].
Let's check it for a specific case. We know that $[S_+, S_-] = 2S_z$. Then, computing it in terms of Holstein-Primakoff bosons:

$$[S_+, S_-] = \sqrt{N - a^\dagger a}\,\underbrace{aa^\dagger}_{1+a^\dagger a}\,\sqrt{N - a^\dagger a} - a^\dagger\sqrt{N - a^\dagger a}\sqrt{N - a^\dagger a}a \tag{127}$$

$$= (1 + a^\dagger a)(N - a^\dagger a) - a^\dagger(N - a^\dagger a)a \tag{128}$$

$$= (N - a^\dagger a)(1 + a^\dagger a) - (N - a^\dagger a)a^\dagger a - a^\dagger a \tag{129}$$

$$= N - 2a^\dagger a = 2\left(\frac{N}{2} - a^\dagger a\right) = 2S_z \tag{130}$$

Now, within the approximation for the thermodynamic limit we shall transform the LMG hamiltonian. For concreteness let's transform the first term keeping track of the N's;

$$\frac{1}{N}S_x^2 = \frac{1}{N}(S_+ + S_-)^2 = \frac{1}{4}\left[\left(1 - \frac{a^\dagger a}{2N}\right)a + a^\dagger\left(1 - \frac{a^\dagger a}{2N}\right)\right]^2 \tag{131}$$

$$= \frac{1}{4}\left[(a + a^\dagger) - \frac{(a^\dagger aa + a^\dagger a^\dagger a)}{2N}\right]^2 \tag{132}$$

$$= \frac{1}{4}(a + a^\dagger)^2 - \frac{1}{2N}\left[(a + a^\dagger)(a^\dagger aa + a^\dagger a^\dagger a) + (a^\dagger aa + a^\dagger a^\dagger a)(a + a^\dagger)\right] + 1/N^2(a^\dagger aa + a^\dagger a^\dagger a)^2$$

We now discard the $1/N^2$ term since it decays too fast.

$$\frac{4}{N}S_x^2 = (a + a^\dagger)^2 - \frac{1}{2N}\left[(a + a^\dagger)(a^\dagger aa + a^\dagger a^\dagger a) + (a^\dagger aa + a^\dagger a^\dagger a)(a + a^\dagger)\right] \tag{133}$$

The latter is further simplified for the following reason: let's bring $a^\dagger a$ to the front in every term of the last two. Using the canonical commutation relations we have for example $aa^\dagger a = a^\dagger aa + a$. That is, every time we pull $a^\dagger a$ to the front we get an a extra. But this extra terms will be suppressed by the $1/N$ because since the total spin scales with N, $a^{(\dagger)} \propto \sqrt{N}$ in the worst case. Such re-orderings once put together will yield terms which are $\mathcal{O}(a^{(\dagger)2})/N$, that is, which will be either zero or constants therefore being reabsorbed. For that reason we commute stuff and keep only the following terms:

$$\frac{4}{N}S_x^2 = (a + a^\dagger)^2 - \frac{1}{2N}a^\dagger a(2a^\dagger a^\dagger + 2aa + 2a^\dagger a + 2aa^\dagger) \tag{134}$$

$$= (a + a^\dagger)^2 - \frac{1}{N}a^\dagger a(a + a^\dagger)^2 \tag{135}$$

A similar reasoning applies to S_y and we have:

$$\frac{1}{N}S_x^2 = \frac{1}{4}\left[(a + a^\dagger)^2 - \frac{1}{N}a^\dagger a(a + a^\dagger)^2\right] \tag{136}$$

$$\frac{1}{N}S_y^2 = -\frac{1}{4}\left[(a - a^\dagger)^2 - \frac{1}{N}a^\dagger a(a - a^\dagger)^2\right] \tag{137}$$

Finally, together with the expression for S_z we have:

$$\mathcal{H} = \mathcal{H}_0 + V \tag{138}$$

$$\mathcal{H}_0 = -\frac{J}{2}\left[\gamma_x(a + a^\dagger)^2 - \gamma_y(a - a^\dagger)^2\right] - \Delta(N - 2a^\dagger a) \tag{139}$$

$$V = \frac{J}{2N}a^\dagger a\left[\gamma_x(a + a^\dagger)^2 - \gamma_y(a - a^\dagger)^2\right] + \frac{\mathcal{O}(a^{(\dagger)2})}{N} \tag{140}$$

Now, if we switch to $a = \frac{1}{\sqrt{2}}(x + ip)$ we get our effective hamiltonian:

$$\boxed{\mathcal{H} = (\Delta - J\gamma_x)x^2 + (\Delta - J\gamma_y)p^2 + \frac{J}{2N}(x^2 + p^2)(\gamma_x x^2 + \gamma_y p^2)} \tag{141}$$

Whose free part is just a harmonic oscillator with mass $m = [2(\Delta - J\gamma_y)]^{-1}$ and frequency $\Omega^2 = 4(\Delta - J\gamma_x)(\Delta - J\gamma_y)$

4.4 Effective Field Theory: Keldysh action for the LMG model

We shall now write our path integral for this model . First, since ordering came up to be unimportant we can simply associate the operator (141) with the scalar $\mathcal{H}(x, p)$. The Keldysh path integral we seek is given by the non-equilibrium partition function (117). For that we write the hamiltonian term of the action for(141):

$$\mathcal{H}\left(\frac{x_c + x_q}{\sqrt{2}}, \frac{p_c + p_q}{\sqrt{2}}\right) - \mathcal{H}\left(\frac{x_c - x_q}{\sqrt{2}}, \frac{p_c - p_q}{\sqrt{2}}\right) = 2(\Delta - J\gamma_x)x_c x_q + 2(\Delta - J\gamma_y)p_c p_q \tag{142}$$

$$+ \frac{J}{2N}[(x_c^2 + x_q^2 + p_c^2 + p_q^2)(\gamma_x x_c x_q + \gamma_y p_c p_q) + (x_c x_q + p_c p_q)(\gamma_x(x_c^2 + x_q^2) + \gamma_y(p_c^2 + p_q^2))] \tag{143}$$

We are of course in position to write the action now. But our goal is not a bare action, which is, ironically, barely tractable. We seek an effective description and for that we‘ll use arguments of scale. To do that we shall use some known results about the near-critical LMG model

$$\frac{1}{N}\left\langle S_x^2 \right\rangle \propto N^{1/3} \tag{144}$$

$$\frac{1}{N}\left\langle S_y^2 \right\rangle \propto N^{-1/3} \tag{145}$$

$$\text{for } \gamma_x > \gamma_y \tag{146}$$

Of course, if the inequality is reversed the behaviours of S_x^2, S_y^2 are too. Moreover, away from criticallity there‘s no scaling phenomena, that is, both share a N^0 behaviour. Now, according to equation (136) and (137) we can obtain the scaling behaviour of x^2 and p^2 under the same assumptions:

$$x^2 \propto N^{1/3} + \mathcal{O}(N^{-1}) \tag{147}$$

$$p^2 \propto N^{-1/3} + \mathcal{O}(N^{-1}) \tag{148}$$

$$\text{for } \gamma_x > \gamma_y \tag{149}$$

Since the action describes the *target* state of the system, which in all cases satisfy $\gamma_x > \gamma_y$, we see that p^n/N terms are suppressed for $n \geq 2$ and $N \to \infty$ against their x^2 counterpart. Moreover, we take $\Delta - J\gamma_y > 0$ so that p field is massive. With that reasoning the hamiltonian term is just:

$$2(\Delta - J\gamma_x)x_c x_q + 2(\Delta - J\gamma_y)p_c p_q + \frac{J\gamma_y}{N}(x_c^2 + x_q^2)x_c x_q + \text{supressed terms} \tag{150}$$

So we have that:

$$S_K = \int_0^\infty dt\Big[p_q\partial_t x_c - x_q\partial_t p_c - 2(\Delta - J\gamma_x)x_c x_q - 2(\Delta - J\gamma_y)p_c p_q - \frac{J\gamma_y}{N}(x_c^2 + x_q^2)x_c x_q\Big] \tag{151}$$

Before further simplifying the action, note that if we equate classical and quantum fields what we have is lagrangian for a Harmonic oscillator plus a x^4 potential. This is the limit of the equilibrium theory.
Moving on with S_K, we can modify the Legendre transform part for:

$$\int_0^\infty x_q\partial_t p_c = -x_{q0}p_{c0} - \int_0^\infty dt p_c\partial_t x_q \tag{152}$$

and we get:

$$S_K = -x_{q0}p_{c0} - \int_0^\infty dt\Big[p_q\partial_t x_c + p_c\partial_t x_q - 2(\Delta - J\gamma_x)x_c x_q - 2(\Delta - J\gamma_y)p_c p_q - \frac{J\gamma_y}{N}(x_c^2 + x_q^2)x_c x_q\Big]$$

Thanks to our approximations, the $p_{c/q}$ degrees of freedom are at most quadratic and we can perform the p path integral using the saddle point equation:

$$\frac{\delta S_K}{\delta p_{c/q}} = 0 \Rightarrow \partial_t p_{c/q} = 2(\Delta - J\gamma_y)p_{c/q} \tag{153}$$

Then, defining $K^{-1} \equiv 2(\Delta - J\gamma_y)$, $r \equiv 2(\Delta - J\gamma_x)$ and $u \equiv J\gamma_x$ the action writes:

$$S_K = -x_{q0}p_{c0} + \int_0^\infty [K\dot{x}_q\dot{x}_c - r x_q x_c - \frac{u}{N}(x_c^2 + x_q^2)x_c x_q] \tag{154}$$

We have performed the path integral in p **excluding** dp_{c0} from the integration measure which leaves us with:

$$\mathcal{Z} = \int \mathcal{D}[x_{c/q}(t)]dp_{c0}\mathcal{W}(x_{c0}, p_{c0})e^{iS_K} \tag{155}$$

We shall now perform a trick which will free us from remaining p_{c0}'s and at the same time x_{q0}. This is desirable because $\mathcal{W}$ is a function of classical fields only and we want momenta integrated out. So, our final effective action should have all it's boundary terms inside $\mathcal{W}$ and only classical fields.

$$\int dt\dot{x}_q\dot{x}_c = -x_{q0}\dot{x}_{c0} - \int dt x_q\dot{x}_c \tag{156}$$

Now, let's rewrite again our path integral, where we factor out of the integration measure dx_{q0} too:

$$\mathcal{Z} = \int \mathcal{D}[x_{c/q}(t)]dx_{q0}e^{-ix_{q0}(p_{c0}-K\dot{x}_{c0})}dp_{c0}\mathcal{W}(x_{c0},p_{c0})e^{iS_K} \tag{157}$$

Where we redefined our integration measure without the dx_{q0} and also our action:

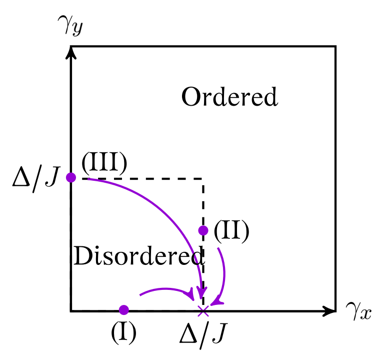
$$S_K = -\int_0^\infty dt \quad Kx_q\ddot{x}_c + rx_qx_c - \frac{u}{N}(x_c^2+x_q^2)x_cx_q \tag{158}$$

Integration over the factored-out degrees of freedom leads to setting $p_{c0} = K\dot{x}_{c0}$ all around. The final result for the partition function is therefore:

$$\boxed{\mathcal{Z} = \int \mathcal{D}[x_{c/q}]\mathcal{W}(x_{c0},K\dot{x}_{c0})e^{iS_K}} \tag{159}$$

4.5 Scaling Analysis

Let's remind first the quench dynamics presented in the phase diagram:



The action encodes the *target*(bulk) scalings while the Wigner function encodes the *pre-quench* ones. Based on knowledge of the individual phases, we want to find out what is the scaling behaviour *post quench*, where there's a competition of the initial state preparation and the bulk dynamics. We can write the scaling dependence manifestly as

$$\mathcal{W}(x,p) \to \mathcal{W}(x^2 N^{-\alpha_0}, p^2 N^{\alpha_0}) \tag{160}$$

The differences in the quenches are in the initial state for we also quench to the same *target*. For type (I), we are deep in the disordered phase, away for criticallity and in that regime the magnetization vanishes in either x or y, rendering $\alpha_0 = 0$ [Botet and Jullien, 1983]. For (II), $\gamma_x > \gamma_y$ so $\alpha = -1/3$ and for (III) we have $\gamma_x < \gamma_y$ then $\alpha_0 = 1/3$, according to (147) and (148). To further simplify the analysis we can set $K \equiv 1$. Moreover, we shall focus henceforth in the analysis of types (I) and (III) since at case (II) the initial state and the bulk are in the same critical line, this case is more like a control case.

Now, since $\gamma_x < \gamma_y$ in the bulk we can expect that the post quench state will behave as $1/N \langle S_x^2 \rangle \propto \langle x_c^2 \rangle N^\alpha$ for $\alpha > 0$. This renders the scaling dimension $[x_c] = \alpha/2$. We now seek a scale invariant desription of the phenomena we know it's emergent[8] so, to scale away this N factor we can introduce a variable:

$$X_c \equiv x_c N^{-\alpha/2} \tag{161}$$

Now, we shall do the same for time by introducing:

$$T = tN^{-\zeta} \tag{162}$$

From that we can construct:

$$\dot{X} = \frac{d}{dT} X = \dot{x} N^{\alpha/2-\zeta} \tag{163}$$

In terms of these new variables, the Wigner function writes:

$$\mathcal{W}(x_{c0}^2, \dot{x}_{c0}) = \mathcal{W}(X_{c0}^2 N^{\alpha-\alpha_0}, \dot{X}_{c0} N^{\alpha-2\zeta+\alpha_0}) \tag{164}$$

Opposite to the bulk, in both types (I) and (III) the X_{c0} is supressed, $X_{c0} \approx 0$. Yet, we can take valuable information from imposing scale invariance of $\dot{X}_{c0}$:

[8] This is similar to what particle physicists do when they require that added terms in the Lagrangian must satisfy symmetry principles. Here the symmetry is scale invariance and we require that the free and classicaly-dominant parts satisfy it. It's indeed quite phenomenological

$$\alpha - 2\zeta + \alpha_0 = 0 \tag{165}$$
$$\zeta = \frac{\alpha + \alpha_0}{2} \tag{166}$$

Now let's look at the action. We remember that $r = 2(\Delta - J\gamma_x)$ indicates distance from the critical point[9], so since we are dealing with the critical point $r = 0$ and as mentioned we took $K = 1$:

$$S_K = -\int_0^\infty dt \quad x_q \ddot{x}_c - \frac{u}{N}(x_c^2 + x_q^2) x_c x_q \tag{167}$$

But we have two interacting terms, one cubic in x_c and other in x_q. We in principle don't know if their scalings are the same so we should rewrite these term as:

$$S_K = -\int_0^\infty dt \quad x_q \ddot{x}_c - \frac{u_c}{N} x_c^3 x_q - \frac{u_q}{N} x_q^3 x_c \tag{168}$$

Now, we do the following substitutions:

$$dt \to dT N^\zeta \tag{169}$$
$$x_q \to X_q N^{\gamma/2} \tag{170}$$
$$\ddot{x}_c \to \ddot{X}_c N^{\alpha/2 - 2\zeta} \tag{171}$$
$$\tag{172}$$

So:

$$S_K = -\int_0^\infty dT \quad X_q \ddot{X}_c N^{\alpha/2 - \zeta + \gamma/2} - u_c X_c^3 X_q N^{3/2\alpha + \zeta + \gamma/2 - 1} - u_q X_c^2 X_q^3 N^{\alpha/2 + \zeta + 3\gamma/2 - 1} \tag{173}$$

The first term can be set scale invariant if we impose:

$$\gamma = 2\zeta - \alpha \tag{174}$$
$$[x_q] = \zeta - \frac{\alpha}{2} \tag{175}$$

Doing the same at the classical vertex term (u_c) and using (166):

[9] The authors point out that the same analysis can be used for $r \neq 0$

$$1 = 2\zeta + \alpha \tag{176}$$
$$\alpha = \frac{1 - \alpha_0}{2} \tag{177}$$
$$\zeta = \frac{1 + \alpha_0}{4} \tag{178}$$

Now, this does not necessarily imply that u_q is scale invariant too. In fact using these results we find

$$[u_q] = -2\alpha_0 - 1 \tag{179}$$

Indeed, we shouldn't expect that the quantumness survive going to higher scales.

4.6 Analysis of the LMG results

Let's recap what we did. We started with a description in terms of quantum and classical fields. The classical fields stand for the "average" of forward and backward branches of evolution. The quantum ones are the differences between forward and backward branches, so that x_q^2 captures the fluctuations away from equilibrium. We did some approximations concerning the known behaviour at criticallity. This allowed us to write an action at most quadratic in momenta. We could then integrate out these degrees of freedom and write an scale invariant quadratic term and classical interaction vertex thus rendering quantum corrections.

Let's summarise the critical behaviour obtained in the last section:

	Dimension	**Type (I)** $\alpha_0 = 0$	**Type (II)** $\alpha_0 = 1/3$	**Type (III)** $\alpha_0 = -1/3$
$[x_c^2]$	$(1 - \alpha_0)/2$	$1/2$	$1/3$	$2/3$
$[x_q^2]$	α_0	0	$1/3$	$-1/3$
$[t]$	$(1 + \alpha_0)/4$	$1/4$	$1/3$	$1/6$
$[u_q]/N$	$-3/2(1 + \alpha_0)$	$-3/2$	$-5/2$	-1

Among all these approximations and reasonings it's quite remarkable that the results agree with exact numerics which predicted $\alpha = 0.5, 0.33, 0.66$ for types (I) (II) (III) respectively. Moreover they shed light on how physics in the higher scale emerge from a comprehensive microscopic description of the density matrix.

It's known that thermal phase transitions occur with exponent $\alpha_0 = 1/2$, which coincides with the universality class of $\langle S_x^2 \rangle / N$ at type (I) hinting for an effective finite temperature! The type (II) quench remains in the same universality class it began, the QPT's UC (T=0). Both of the latter display a quantum vertex $[u_q]/N$ which is damped faster than $1/N$. Yet, the type (III) is damped as $1/N$ which is slower and could be taken as a perturbative correction, moreover the $[x_c^2]$ scaling is neither in QPT nor in TPT universality classes. These indicate that effective dynamics are of non-equilibrium.

5 Conclusion

The Keldysh-Schwinger formalism has something in common with mean-field theory: building a description based on mean values and fluctuations. Nevertheless, mean field departs from fluctuations computed at thermal equilibrium whilst Keldysh formalism is based on a prescription for a general density matrix and does not assume small fluctuations, it just equally embrace them within the formalism. Indeed, comparative analysis has proven the value of Keldysh methods in non-equilibrium [Maghrebi and Gorshkov, 2016] and what we've shown highlights it's validity at criticallity. MF is known to breakdown miserably at criticallity, since the hypothesis that fluctuations remain small is contradicted.

The basic ideas of the theory shown here have suffered a rework in the frame of BRST symmetries as we've briefly seen and have been used extensively for example in black hole physics. Beyond that, other very interesting applications might be worth mentioning.

SK theory can be used in the context of topological field theory to study transport in (2+1)d fermionic systems which display the quantum Hall effect. Such lattice systems are usually studied by an effective description, in which one translates the topological structure which gives rise to anyonic statistics into a gauge boson coupled to the fermionic chain. Fermions are then integrated out giving rise to the Chern-Simons action. Yet, in non-equilibrium situations where new phenomena appears one uses SK partition function and let forward and backward evolving parts couple to different gauge bosons. This leads to a effective description as shown in the work of [Glorioso et al., 2019].

To conclude, I think a lot of the ideas presented here are complementary the particle physicist's view of the renormalisation group studied in class. It brings different insights on what physics is all about: scale. In the context of many-body physics one is often interested in questions such as which microscopic phenomena lead to the macroscopic or mesoscopic observed ones. In that sense our example of the LMG model has a lot of interesting features: it displays how three distintic behaviours can emerge from the same system in the thermodynamics limit and is very amenable to experimental implementation.

References

[Belavkin and Staszewski, 1986] Belavkin, V. and Staszewski, P. (1986). *"A Radon-Nikodym theorem for completely positive maps"*. *Reports on Mathematical Physics*, 24(1):49 – 55.

[Botet and Jullien, 1983] Botet, R. and Jullien, R. (1983). Large-size critical behavior of infinitely coordinated systems. *Phys. Rev. B*, 28:3955–3967.

[Foss-Feig et al., 2017] Foss-Feig, M., Niroula, P., Young, J. T., Hafezi, M., Gorshkov, A. V., Wilson, R. M., and Maghrebi, M. F. (2017). Emergent equilibrium in many-body optical bistability. *Physical Review A*, 95(4).

[Glorioso et al., 2019] Glorioso, P., Gromov, A., and Ryu, S. (2019). Effective response theory for floquet topological systems.

[Haehl et al., 2016] Haehl, F., Loganayagam, R., and Rangamani, M. (2016). Schwinger-keldysh formalism i: Brst symmetries and superspace. *Journal of High Energy Physics*, 2017.

[Kamenev, 2011] Kamenev, A. (2011). *Field Theory of Non-Equilibrium Systems*. Cambridge University Press.

[Landi, 2019] Landi, G. (2019). "statistical mechanics lec. 8: Lipkin-meshkov-glick model".

[Maghrebi and Gorshkov, 2016] Maghrebi, M. F. and Gorshkov, A. V. (2016). Nonequilibrium many-body steady states via keldysh formalism. *Physical Review B*, 93(1).

[Maldacena et al., 2016] Maldacena, J., Shenker, S. H., and Stanford, D. (2016). A bound on chaos. *Journal of High Energy Physics*, 2016(8).

[Sieberer et al., 2016] Sieberer, L. M., Buchhold, M., and Diehl, S. (2016). Keldysh field theory for driven open quantum systems. *Reports on Progress in Physics*, 79(9):096001.

[Titum and Maghrebi, 2019] Titum, P. and Maghrebi, M. F. (2019). Non-equilibrium criticality in quench dynamics of infinite-range spin models.