Reductions

Computable function f :

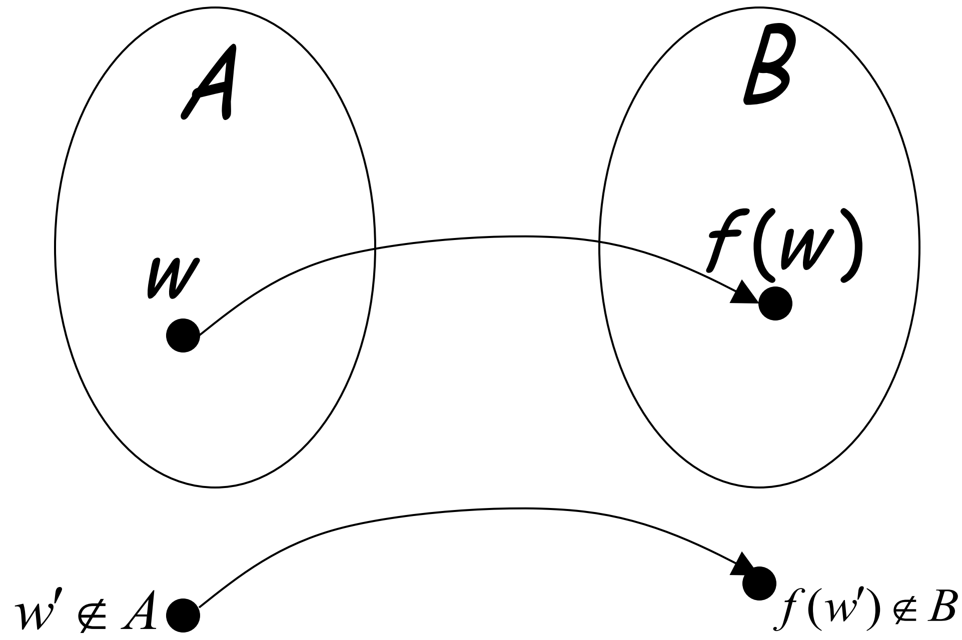
There is a deterministic Turing machine M which for any input string w computes $f(w)$ and writes it on the tape

Problem X is reduced to problem Y

If we can solve problem Y
then we can solve problem X

Definition:

Language A
is reduced to
language B



There is a computable function f (*reduction*) such that:

$$w \in A \Leftrightarrow f(w) \in B$$

Theorem 1:

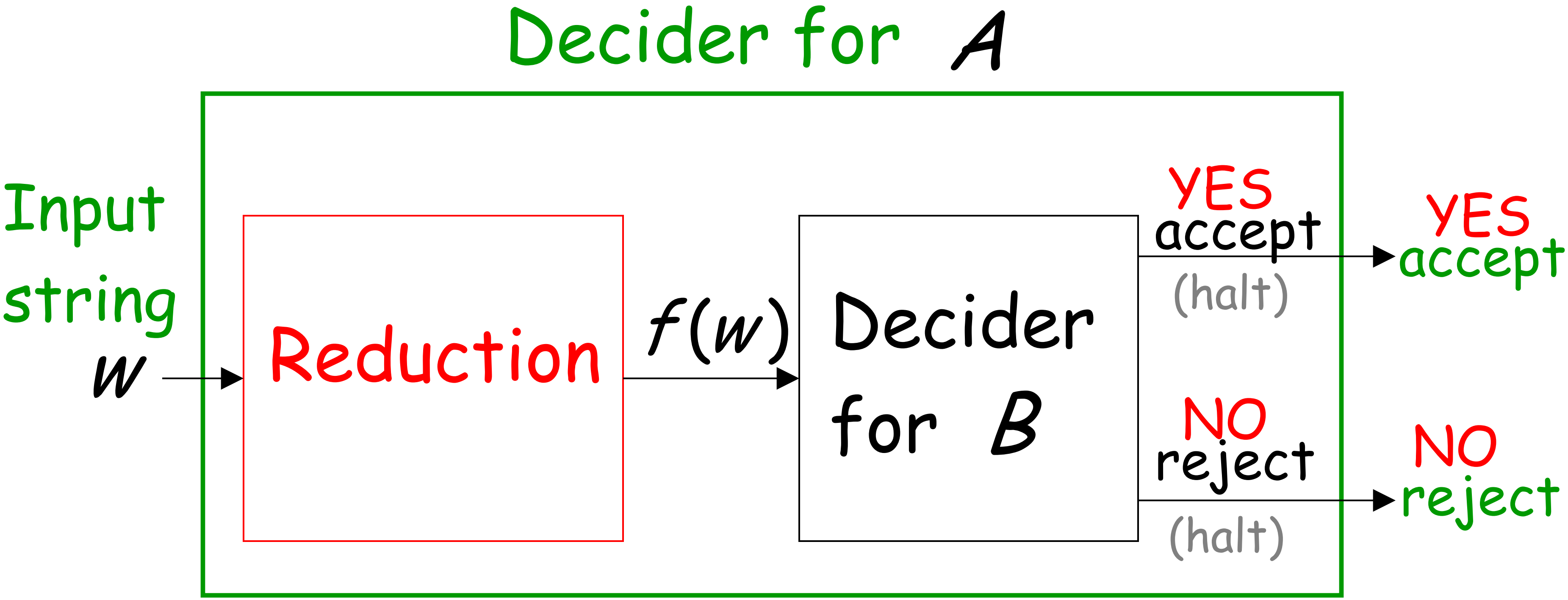
If: Language A is reduced to B
and language B is decidable

Then: A is decidable

Proof:

Basic idea:

Build the decider for A
using the decider for B



From reduction: $w \in A \Leftrightarrow f(w) \in B$

END OF PROOF

Example:

$$EQUAL_{DFA} = \{\langle M_1, M_2 \rangle : M_1 \text{ and } M_2 \text{ are DFAs that accept the same languages}\}$$

is reduced to:

$$EMPTY_{DFA} = \{\langle M \rangle : M \text{ is a DFA that accepts the empty language } \emptyset\}$$

We only need to construct:

$\langle M_1, M_2 \rangle$ → **Reduction** Turing Machine for reduction f → $f(\langle M_1, M_2 \rangle) = \langle M \rangle$ DFA

$$\langle M_1, M_2 \rangle \in EQUAL_{DFA} \quad \Leftrightarrow \quad \langle M \rangle \in EMPTY_{DFA}$$

Let L_1 be the language of DFA M_1

Let L_2 be the language of DFA M_2

$\langle M_1, M_2 \rangle$ → **Reduction** Turing Machine for reduction f → $f(\langle M_1, M_2 \rangle) = \langle M \rangle$ DFA

construct DFA M

by combining M_1 and M_2 so that:

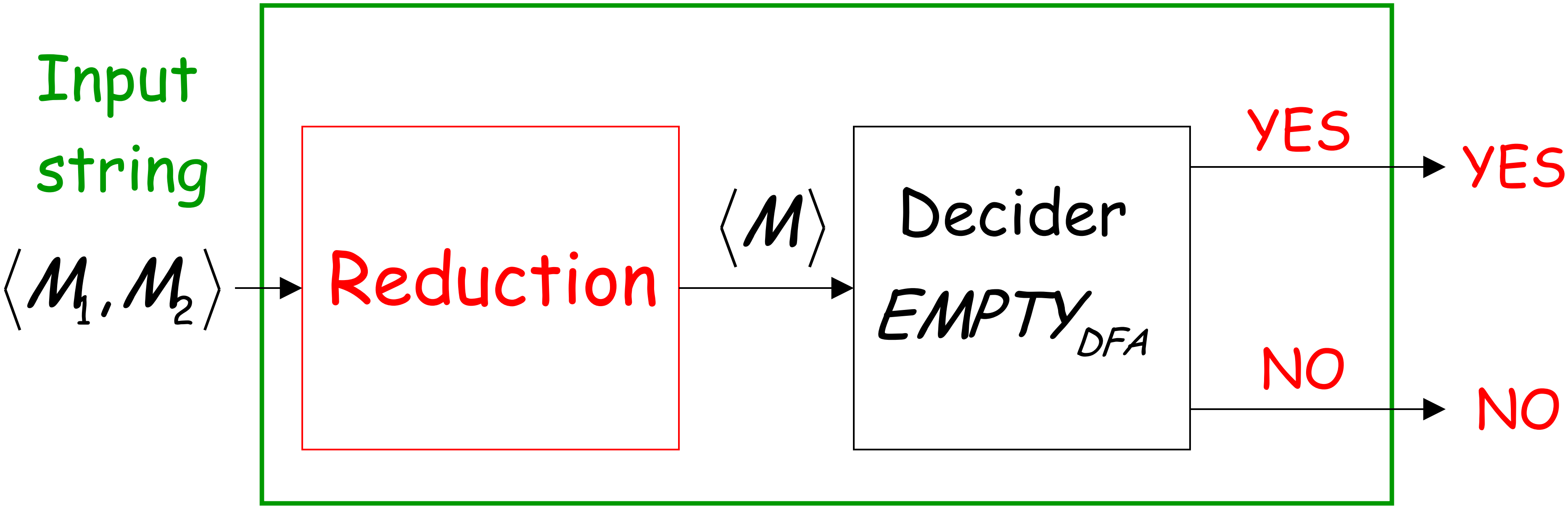
$$L(M) = (L_1 \cap \overline{L_2}) \cup (\overline{L_1} \cap L_2)$$

$$L(M) = (L_1 \cap \overline{L_2}) \cup (\overline{L_1} \cap L_2)$$

$$L_1 = L_2 \quad \Leftrightarrow \quad L(M) = \emptyset$$

$$\langle M_1, M_2 \rangle \in EQUAL_{DFA} \quad \Leftrightarrow \quad \langle M \rangle \in EMPTY_{DFA}$$

Decider for $EQUAL_{DFA}$



Theorem 2:

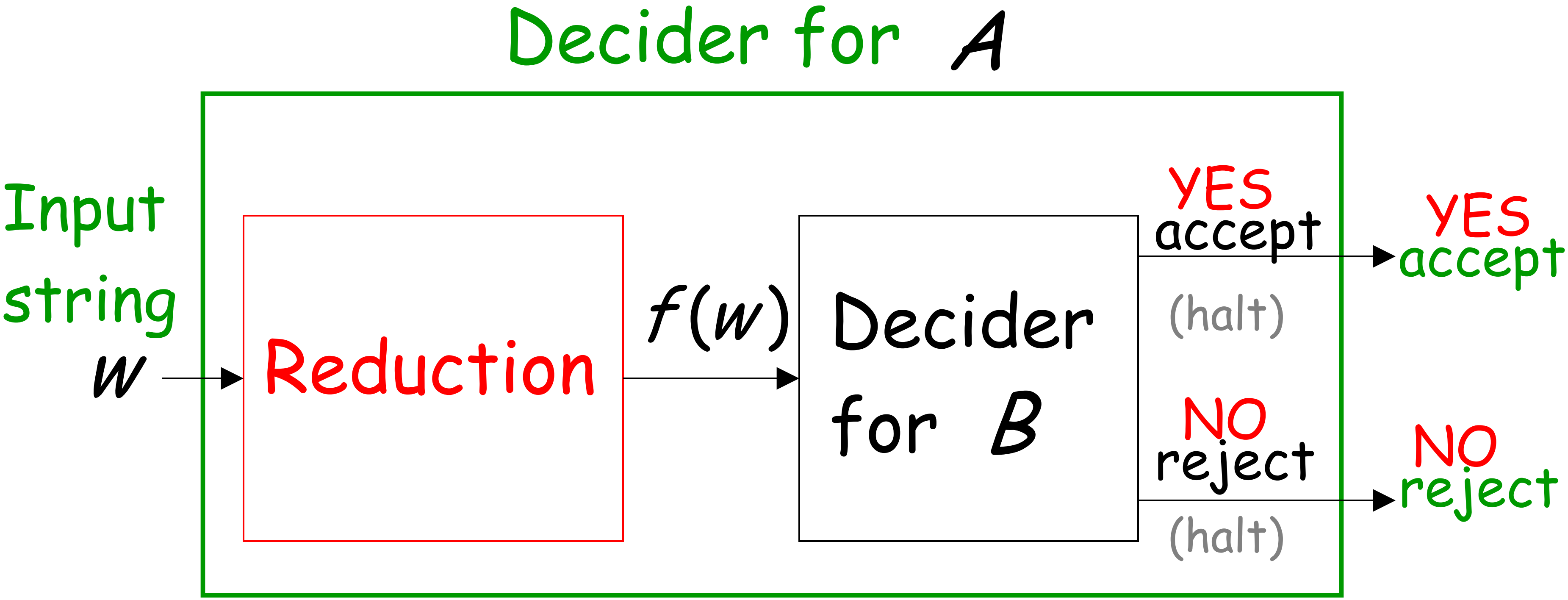
If: Language A is reduced to B
and language A is undecidable

Then: B is undecidable

Proof: Suppose B is decidable
Using the decider for B
build the decider for A

Contradiction!

If B is decidable then we can build:



$$w \in A \Leftrightarrow f(w) \in B$$

CONTRADICTION!

END OF PROOF

Observation:

To prove that language B is undecidable
we only need to reduce
a known undecidable language A to B

State-entry problem

Input:
- Turing Machine M
- State q
- String w

Question: Does M enter state q while processing input string w ?

Corresponding language:

$$STATE_{TM} = \{\langle M, w, q \rangle : M \text{ is a Turing machine that enters state } q \text{ on input string } w\}$$

(while processing)

Theorem: $STATE_{TM}$ is undecidable

(state-entry problem is unsolvable)

Proof: Reduce

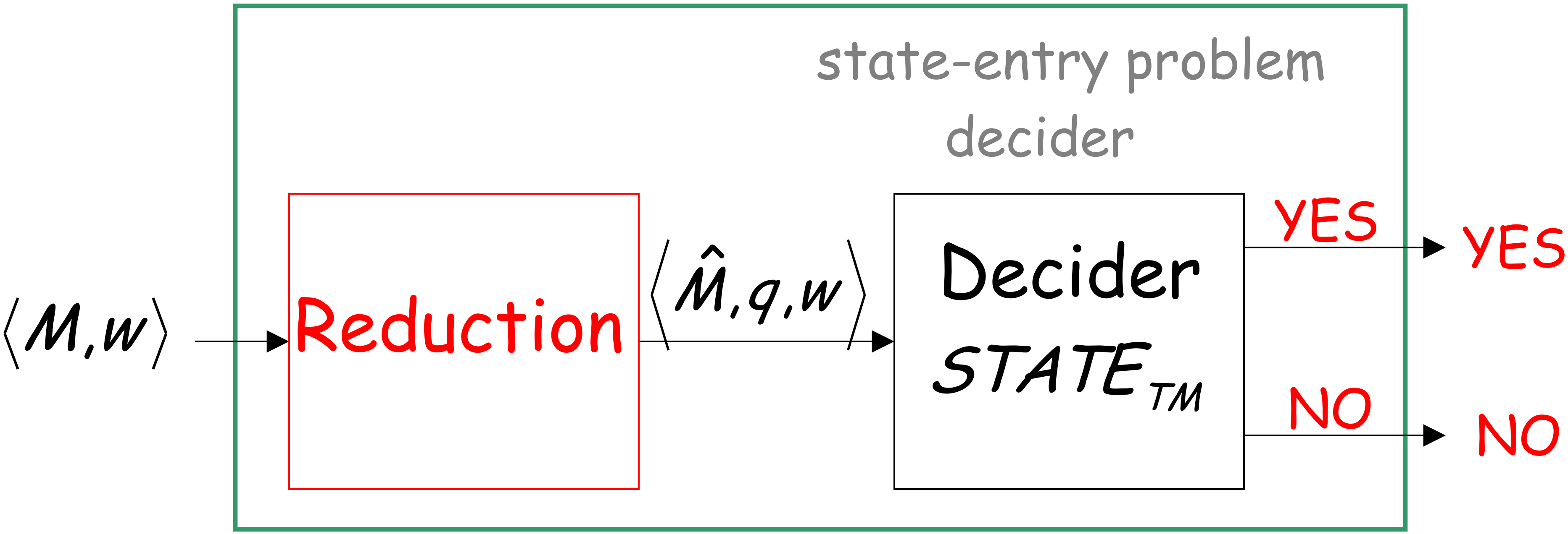
$HALT_{TM}$ (halting problem)

to

$STATE_{TM}$ (state-entry problem)

Halting Problem Decider

Decider for $HALT_{TM}$



Given the reduction,
if $STATE_{TM}$ is decidable,
then $HALT_{TM}$ is decidable

A contradiction!
since $HALT_{TM}$
is undecidable

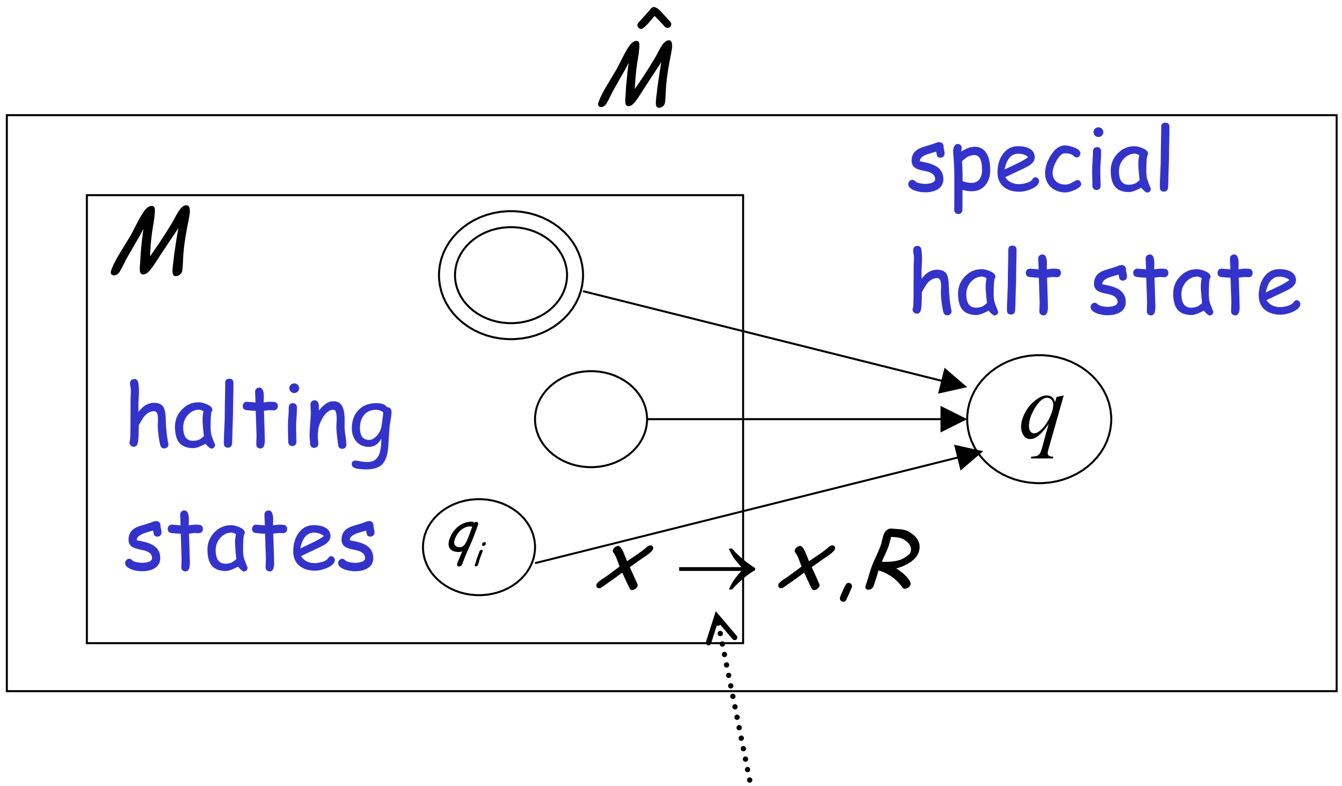
We only need to build the reduction:

$\langle M, w \rangle$ → **Reduction** → $\langle \hat{M}, q, w \rangle$

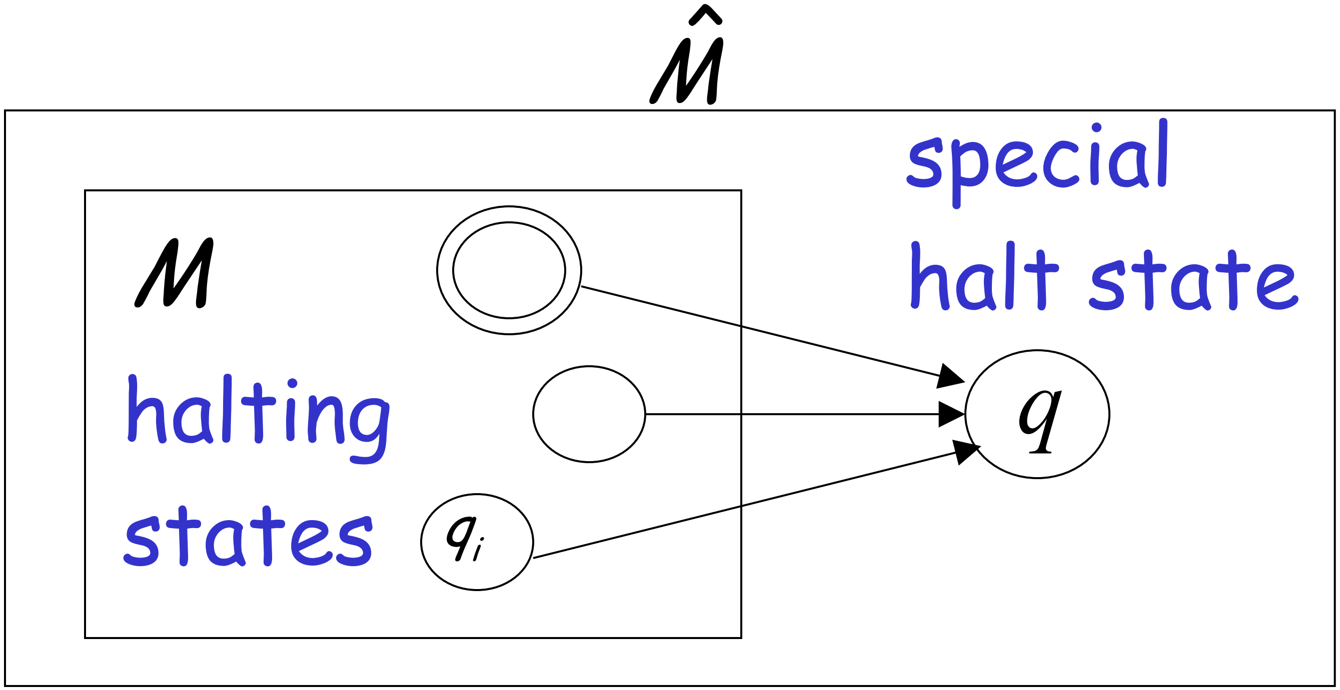
So that:

$$\langle M, w \rangle \in HALT_{TM} \iff \langle \hat{M}, w, q \rangle \in STATE_{TM}$$

For the reduction, construct $\hat{M}$ from M:



A transition for every unused tape symbol x of q_i



M halts $\Leftrightarrow$ $\hat{M}$ halts on state q

Therefore: M halts on input w

$\Updownarrow$

$\hat{M}$ halts on state q on input w

Equivalently:

$$\langle M, w \rangle \in HALT_{TM} \Longleftrightarrow \langle \hat{M}, w, q \rangle \in STATE_{TM}$$

END OF PROOF

Blank-tape halting problem

Input: Turing Machine M

Question: Does M halt when started with a blank tape?

Corresponding language:

$$BLANK_{TM} = \{\langle M \rangle : M \text{ is a Turing machine that halts when started on blank tape}\}$$

Theorem: $BLANK_{TM}$ is undecidable

(blank-tape halting problem is unsolvable)

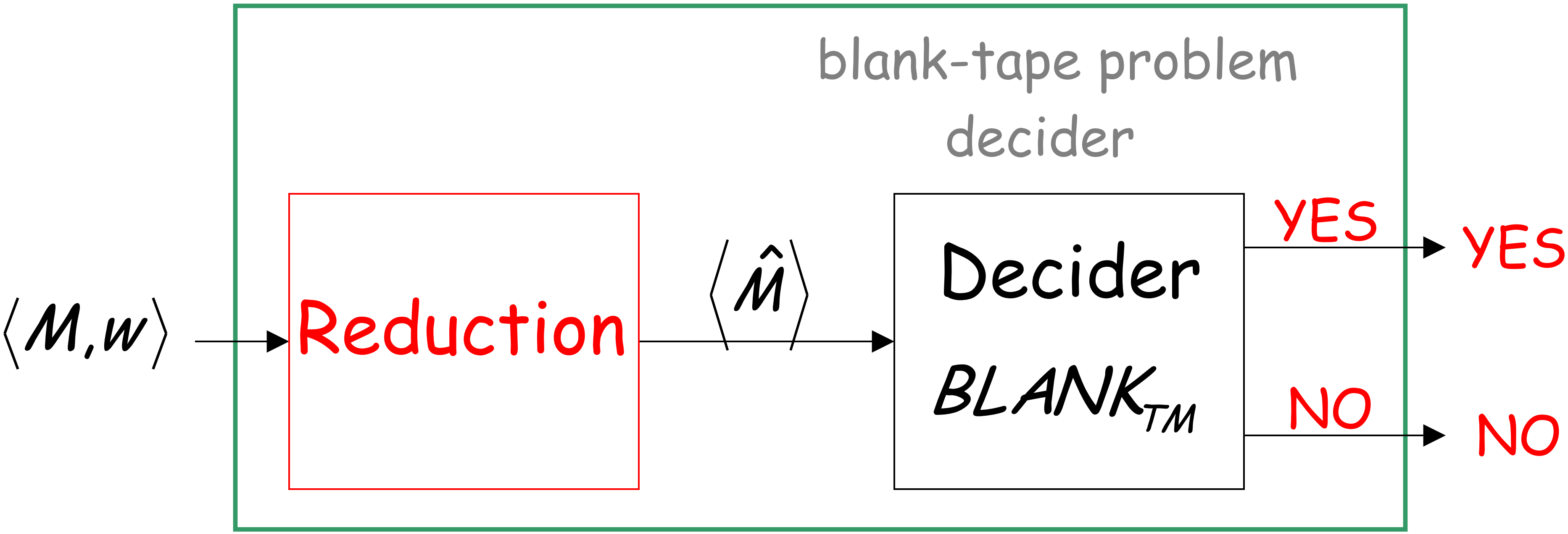
Proof: Reduce

$HALT_{TM}$ (halting problem)

to

$BLANK_{TM}$ (blank-tape problem)

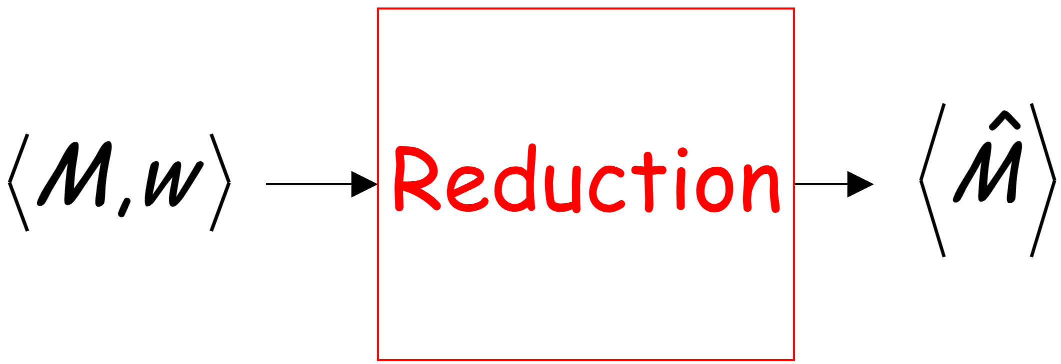
Decider for $HALT_{TM}$



Given the reduction,
If $BLANK_{TM}$ is decidable,
then $HALT_{TM}$ is decidable

A contradiction!
since $HALT_{TM}$
is undecidable

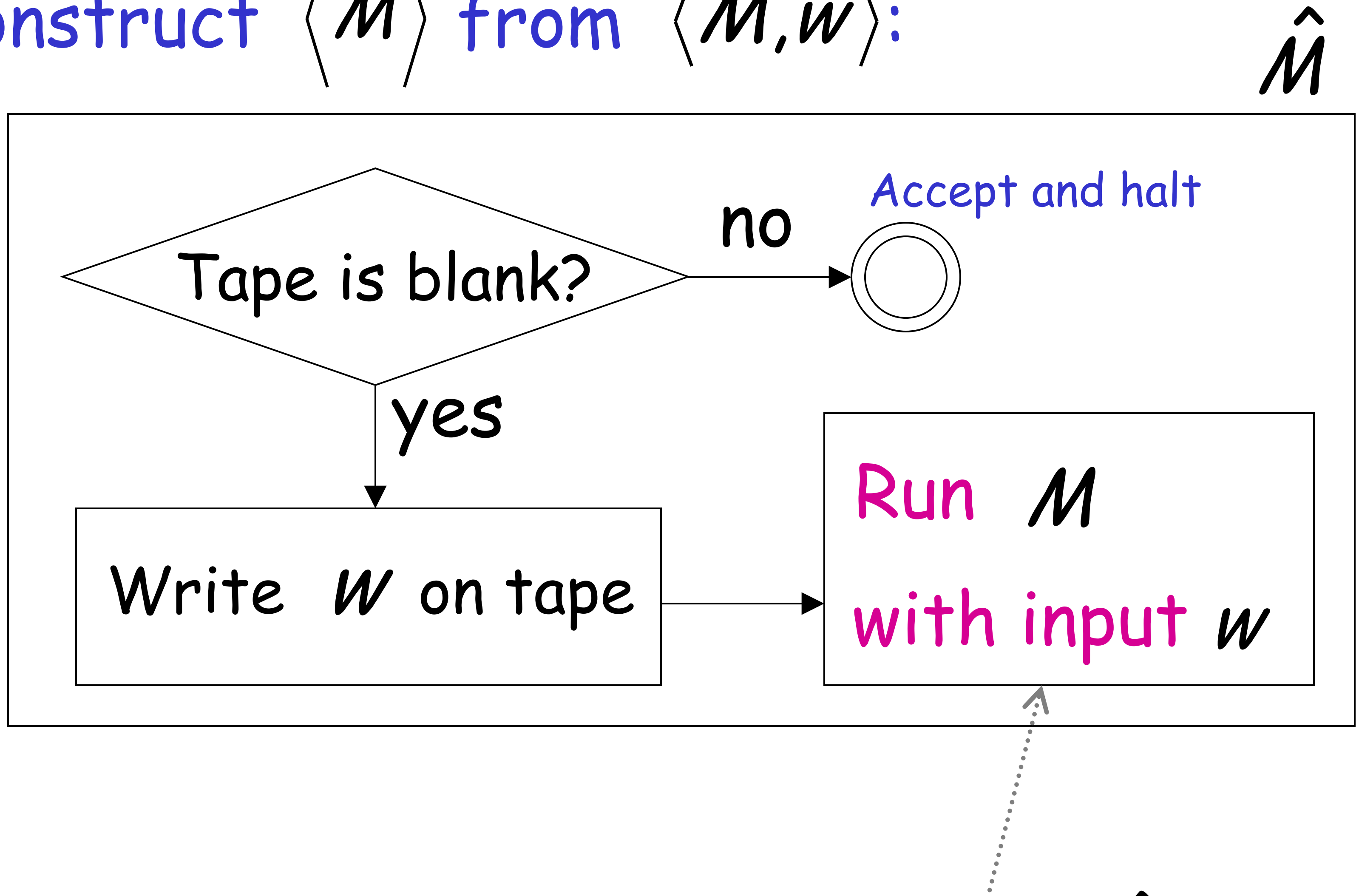
We only need to build the reduction:



So that:

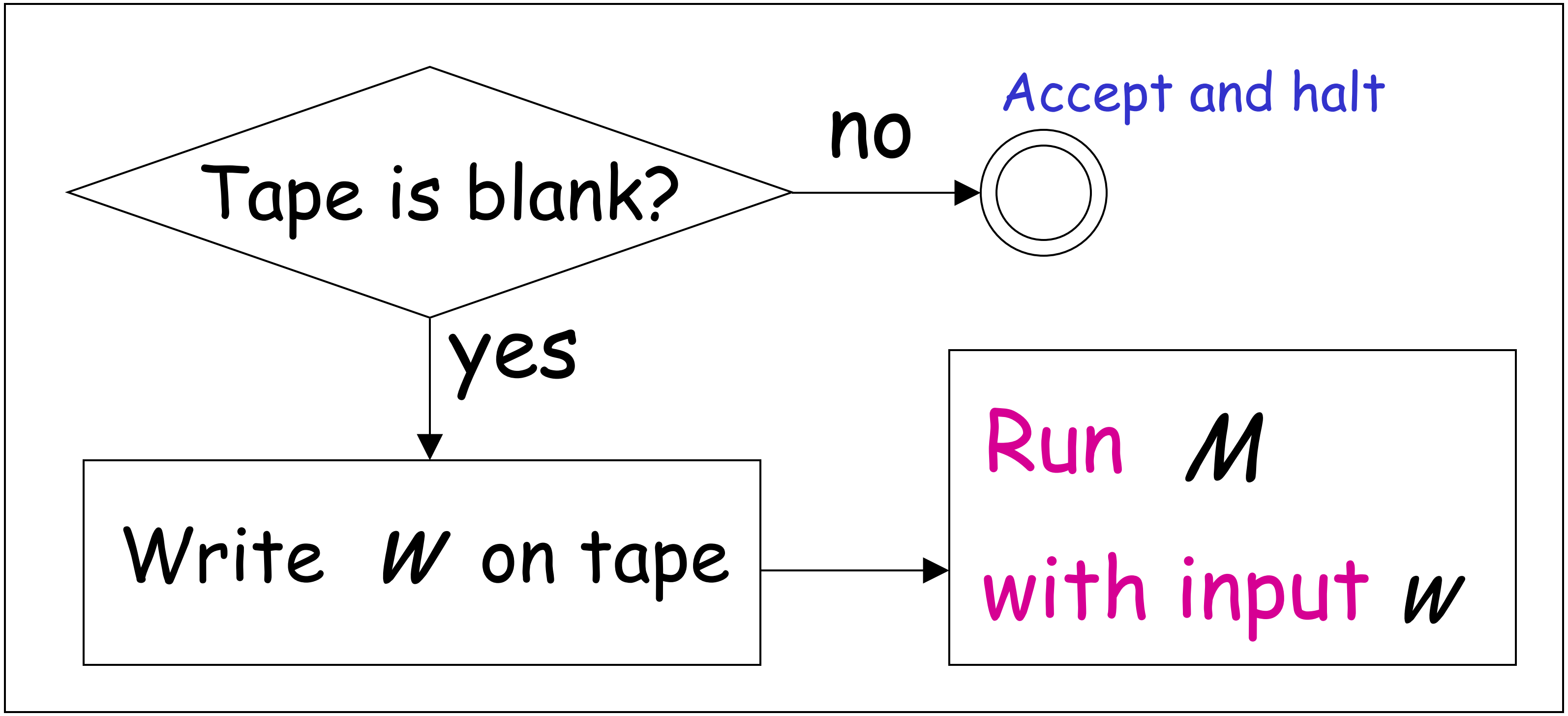
$$\langle M,w \rangle \in HALT_{TM} \iff \langle \hat{M} \rangle \in BLANK_{TM}$$

Construct $\langle \hat{M} \rangle$ from $\langle M, w \rangle$:



If M halts then $\hat{M}$ halts too

$\hat{M}$



M halts on input w

⇕

$\hat{M}$ halts when started on blank tape

M halts on input w

$\Updownarrow$

$\hat{M}$ halts when started on blank tape

Equivalently:

$$\langle M, w \rangle \in HALT_{TM} \iff \langle \hat{M} \rangle \in BLANK_{TM}$$

END OF PROOF

Theorem 3:

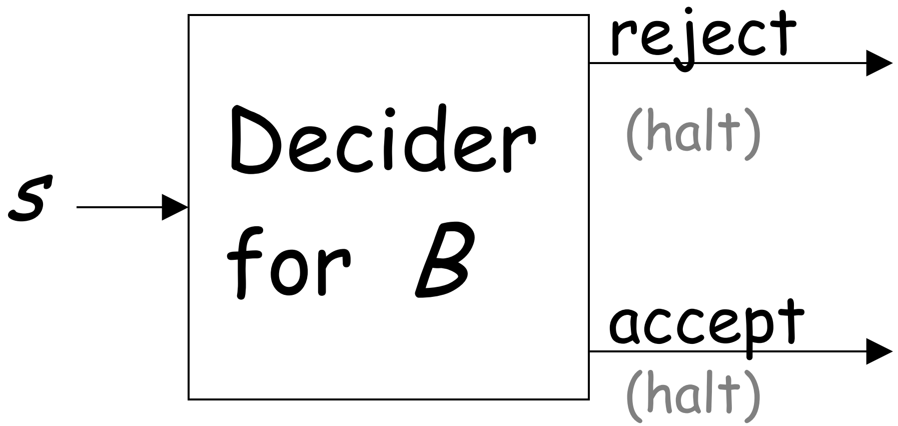
If: Language A is reduced to $\overline{B}$
and language A is undecidable

Then: B is undecidable

Proof: Suppose B is decidable
Then $\overline{B}$ is decidable
Using the decider for $\overline{B}$
build the decider for A

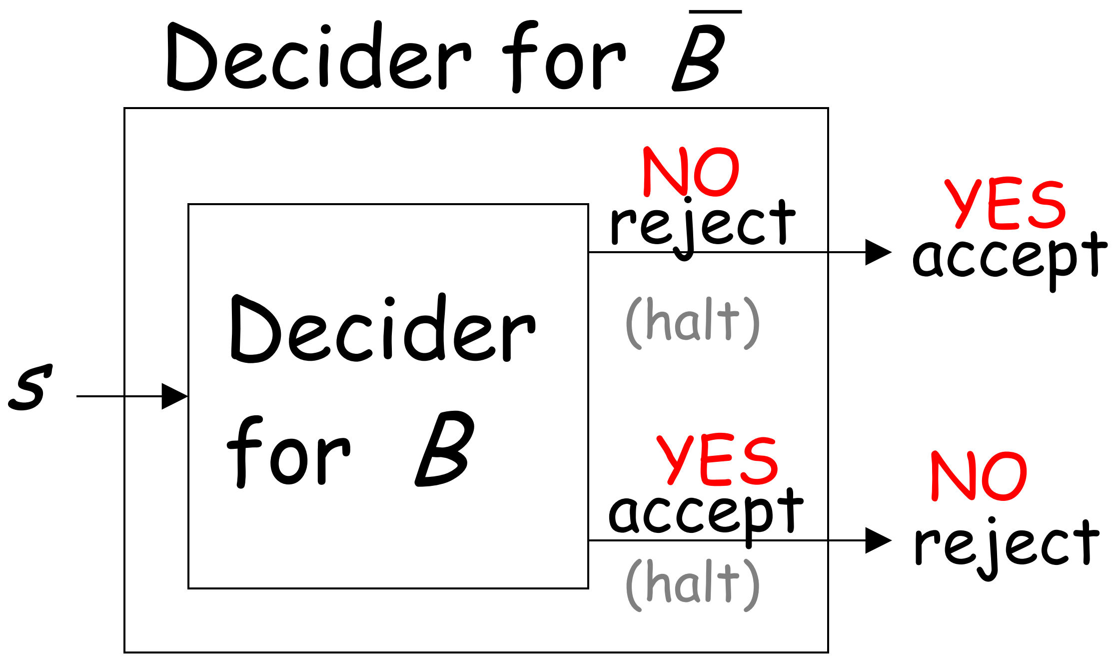
Contradiction!

Suppose B is decidable

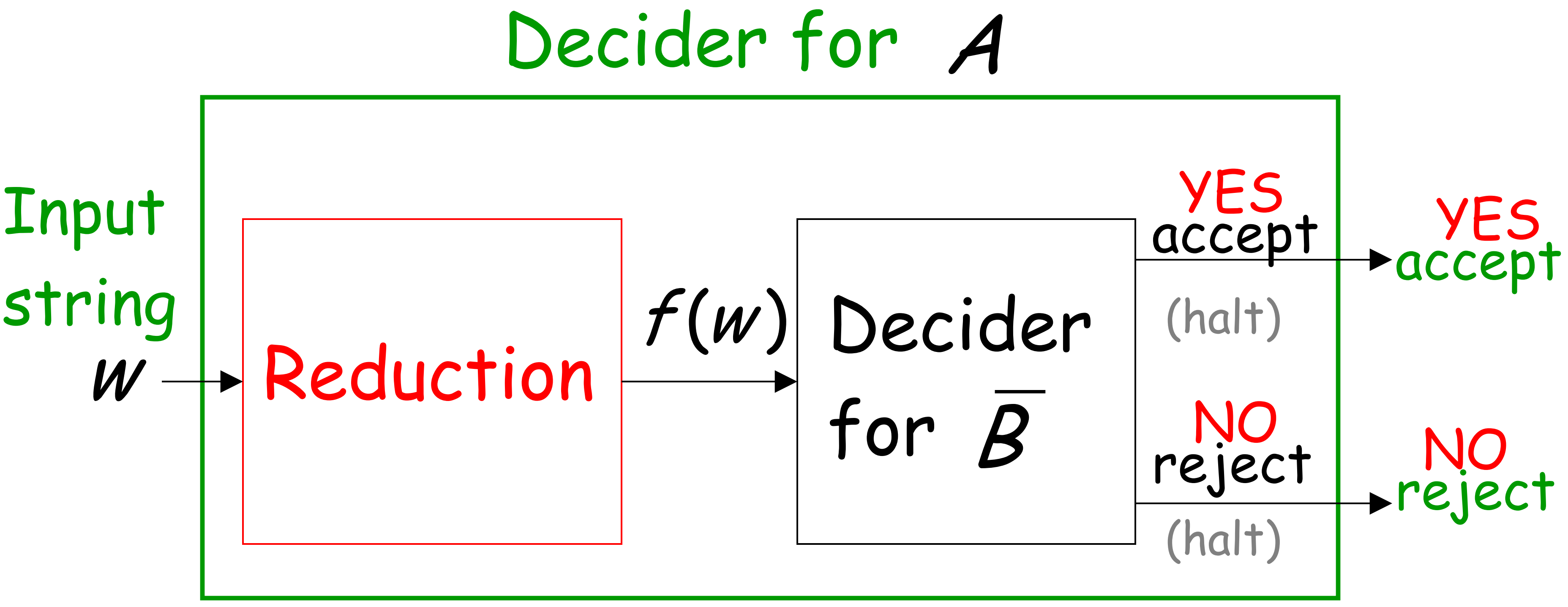


Suppose B is decidable

Then $\overline{B}$ is decidable



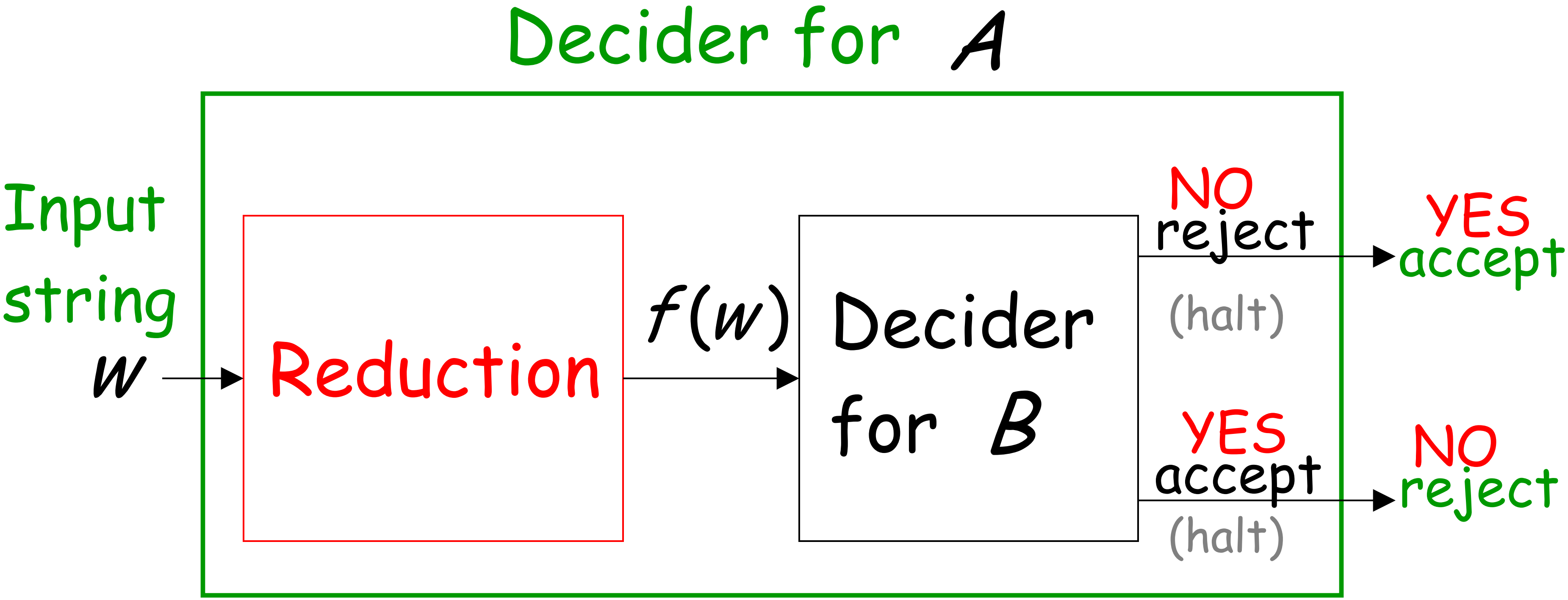
If $\overline{B}$ is decidable then we can build:



$$w \in A \iff f(w) \in \overline{B}$$

CONTRADICTION!

Alternatively:



$$w \in A \iff f(w) \notin B$$

CONTRADICTION!

END OF PROOF

Observation:

To prove that language B is undecidable
we only need to reduce
a known undecidable language A
to B (Theorem 2)
or $\overline{B}$ (Theorem 3)

Undecidable Problems for Turing Recognizable languages

Let L be a Turing-acceptable language

- L is empty?
- L is regular?
- L has size 2?

All these are undecidable problems

Let L be a Turing-acceptable language

- L is empty?
- L is regular?
- L has size 2?

Empty language problem

Input: Turing Machine M

Question: Is $L(M)$ empty? $L(M) = \emptyset$?

Corresponding language:

$$EMPTY_{TM} = \{\langle M \rangle : M \text{ is a Turing machine that accepts the empty language } \emptyset\}$$

Theorem: $EMPTY_{TM}$ is undecidable

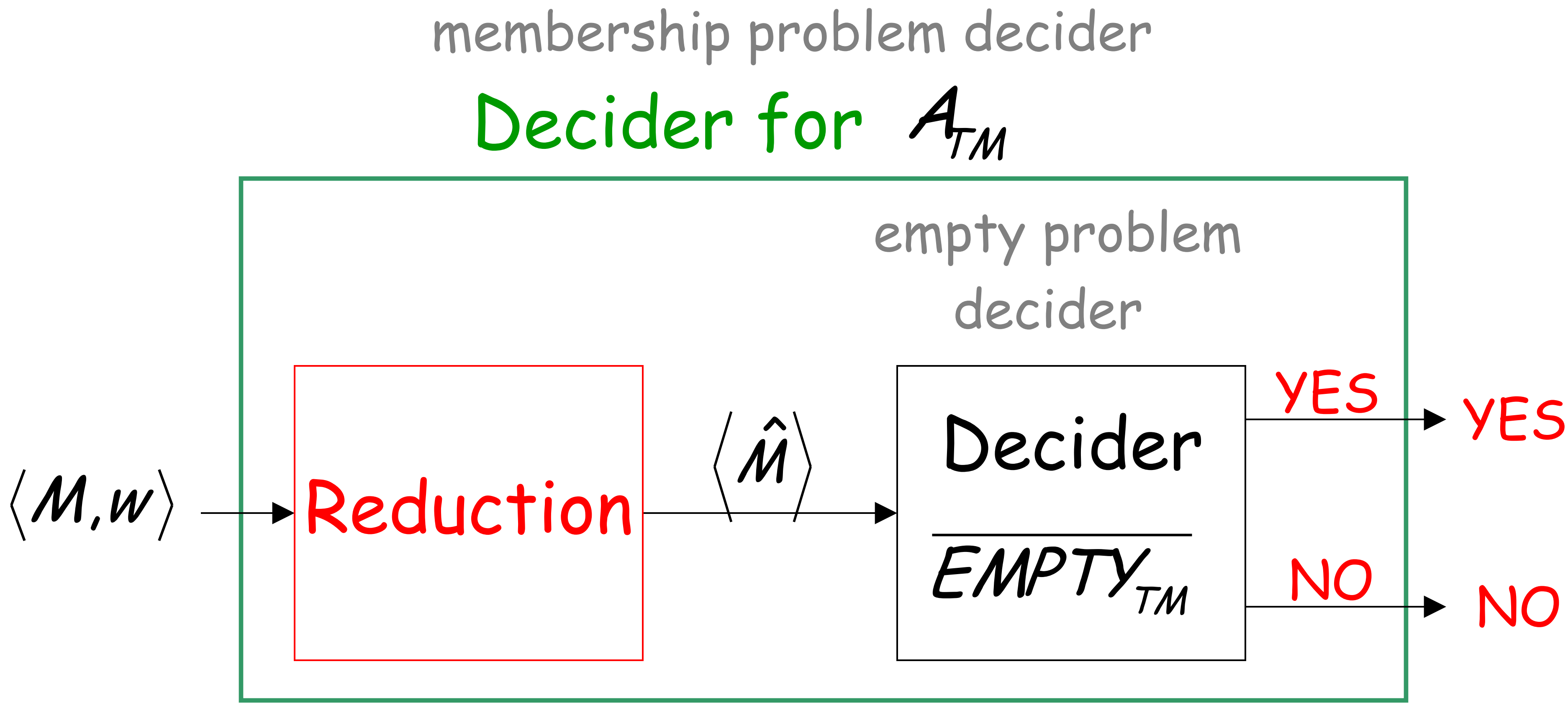
(empty-language problem is unsolvable)

Proof: Reduce

A_{TM} (membership problem)

to

$\overline{EMPTY_{TM}}$ (empty language problem)



Given the reduction,
if $\overline{EMPTY_{TM}}$ is decidable,
then A_{TM} is decidable

A contradiction!
since A_{TM}
is undecidable

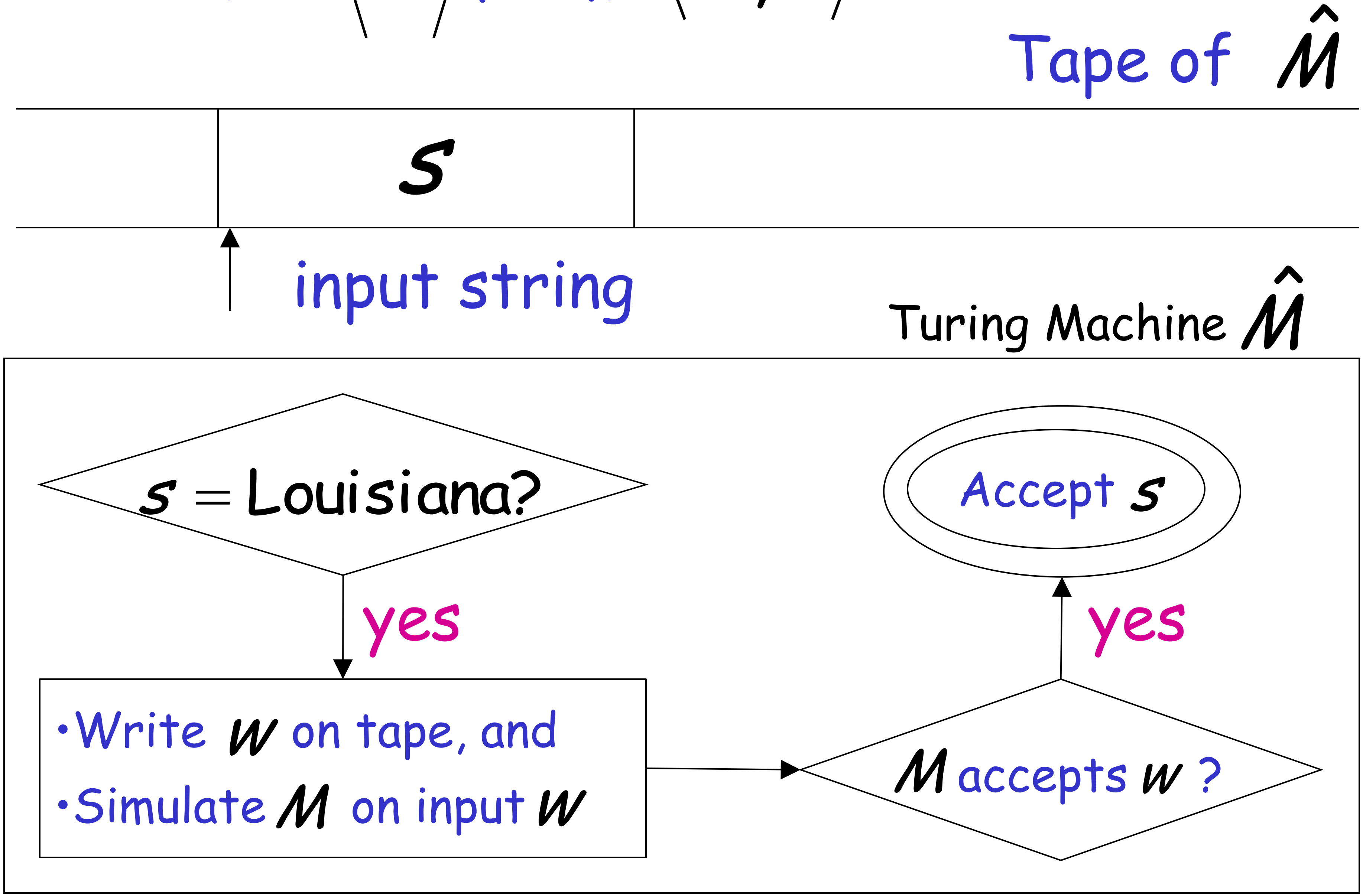
We only need to build the reduction:

$\langle M, w \rangle \longrightarrow$ **Reduction** $\longrightarrow \langle \hat{M} \rangle$

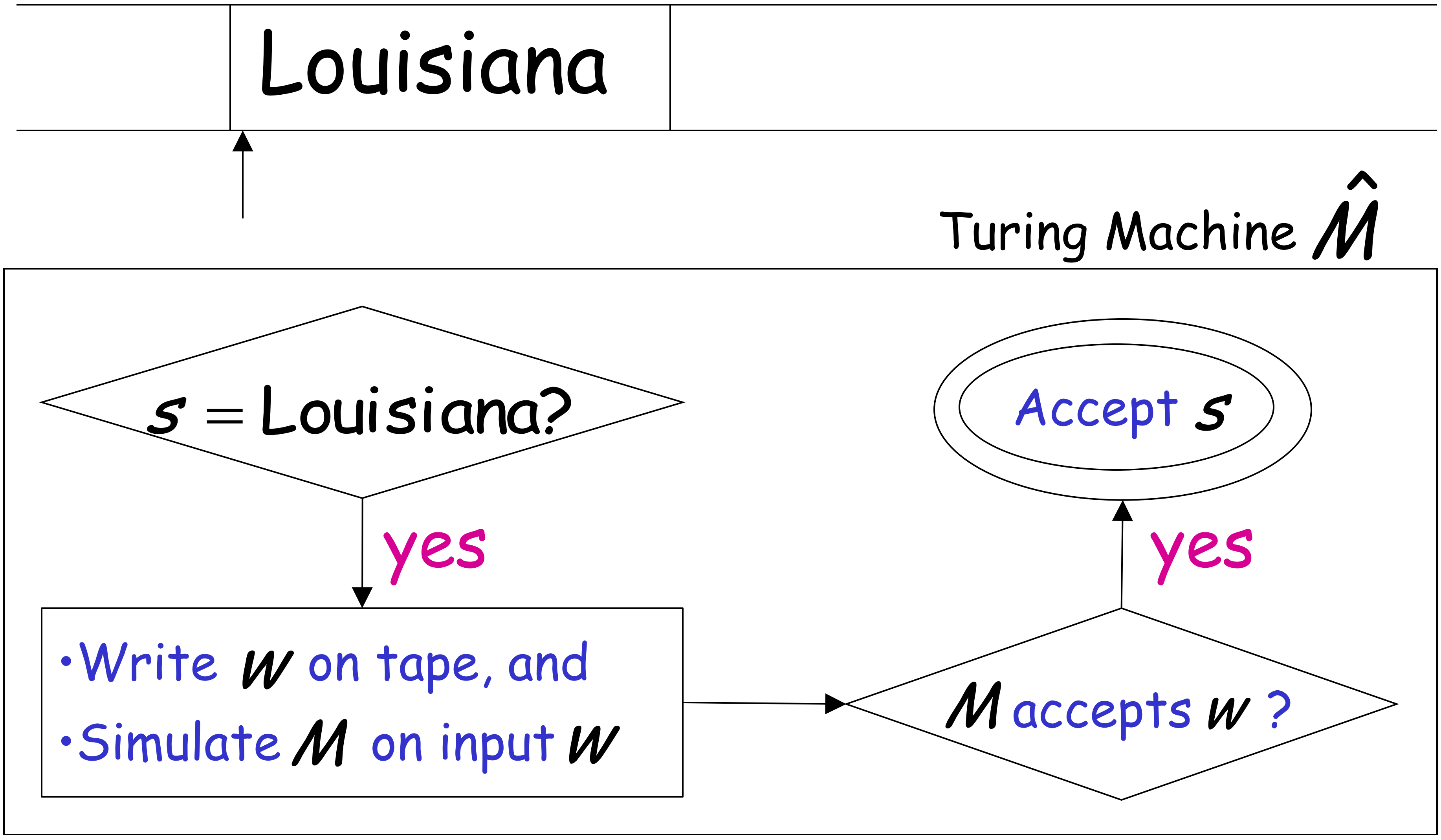
So that:

$$\langle M, w \rangle \in A_{TM} \iff \langle \hat{M} \rangle \in \overline{EMPTY_{TM}}$$

Construct $\langle \hat{M} \rangle$ from $\langle M, w \rangle$:



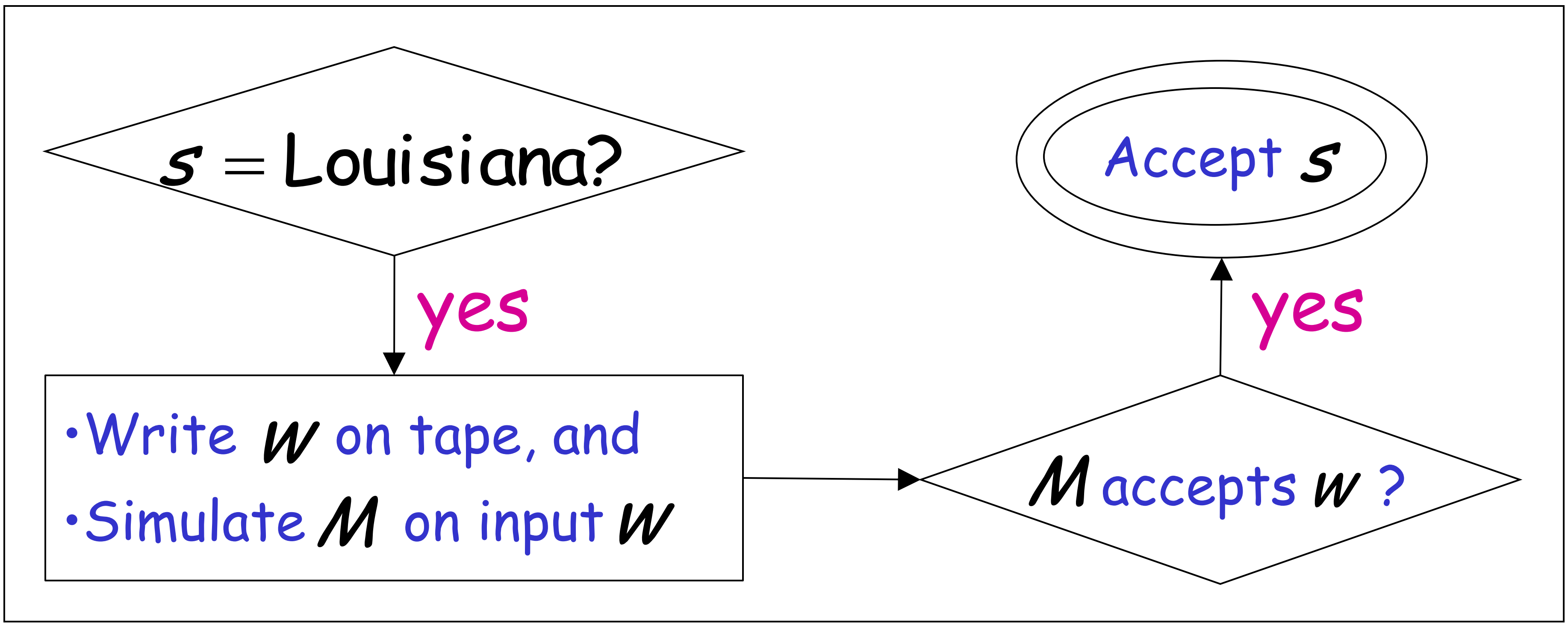
The only possible accepted string s



M accepts w ⇒ $L(\hat{M}) = \{\text{Louisiana}\} \neq \emptyset$

M does not accept w ⇒ $L(\hat{M}) = \emptyset$

Turing Machine $\hat{M}$



Therefore:

M accepts w $\Longleftrightarrow$ $L(\hat{M}) \neq \emptyset$

Equivalently:

$\langle M, w \rangle \in A_{TM}$ $\Longleftrightarrow$ $\langle \hat{M} \rangle \in \overline{EMPTY_{TM}}$

END OF PROOF

Let L be a Turing-acceptable language

- L is empty?
- L is regular?
- L has size 2?

Regular language problem

Input: Turing Machine M

Question: Is $L(M)$ a regular language?

Corresponding language:

$$REGULAR_{TM} = \{\langle M \rangle : M \text{ is a Turing machine that accepts a regular language}\}$$

Theorem: $REGULAR_{TM}$ is undecidable

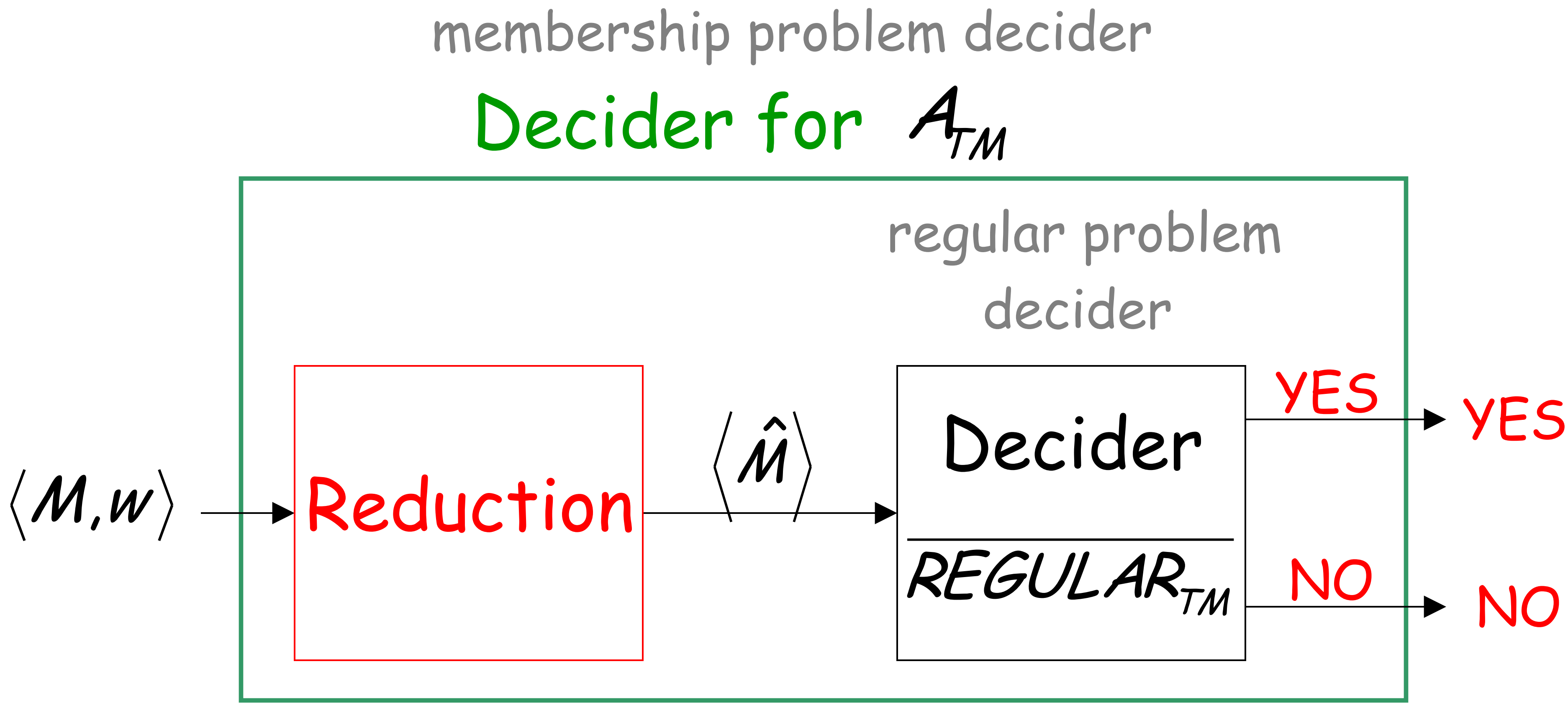
(regular language problem is unsolvable)

Proof: Reduce

A_{TM} (membership problem)

to

$\overline{REGULAR_{TM}}$ (regular language problem)



Given the reduction,
If $\overline{REGULAR_{TM}}$ is decidable,
then A_{TM} is decidable

A contradiction!
since A_{TM}
is undecidable

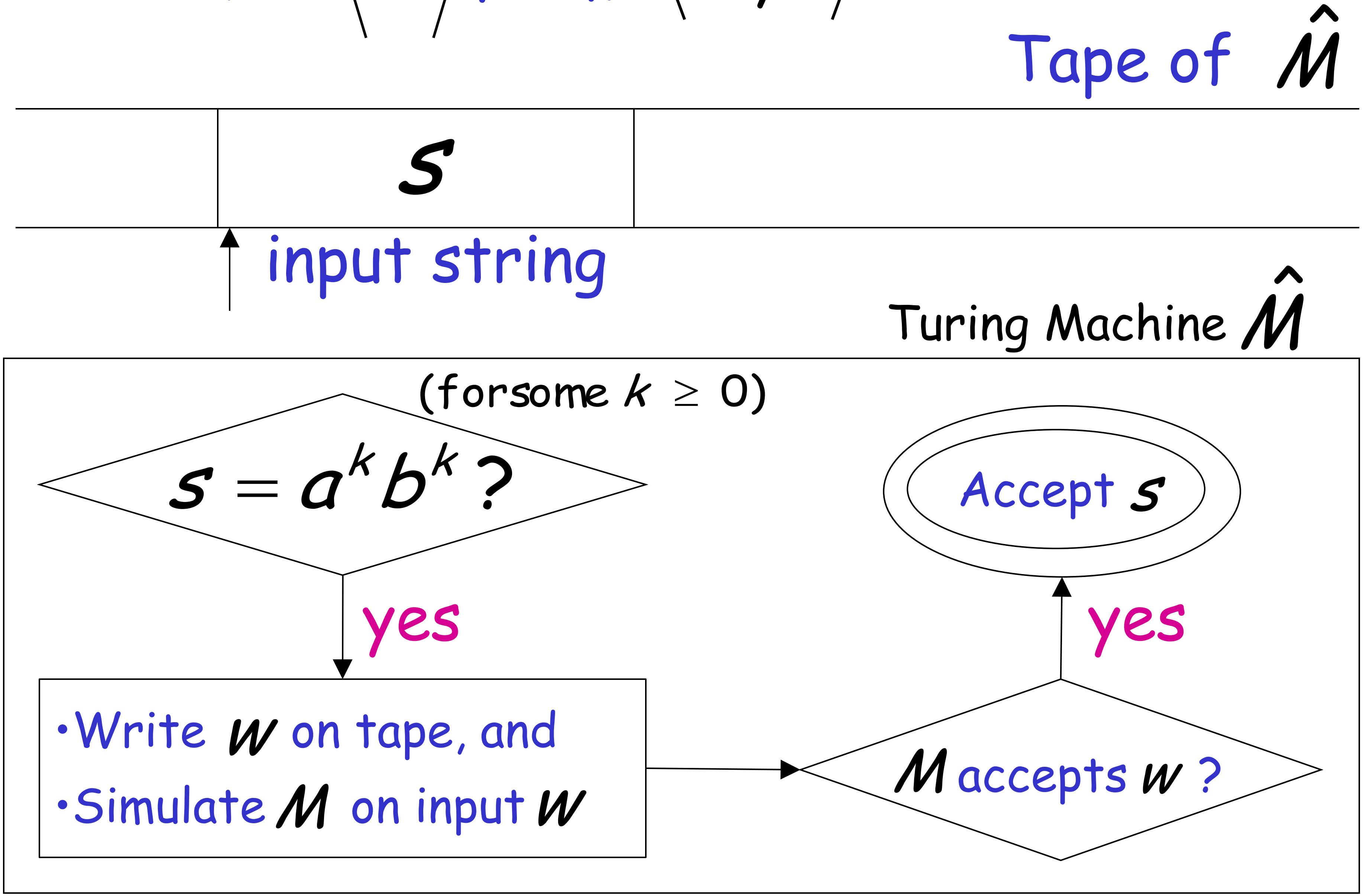
We only need to build the reduction:

$\langle M,w \rangle \longrightarrow$ **Reduction** $\longrightarrow \langle \hat{M} \rangle$

So that:

$$\langle M, w \rangle \in A_{TM} \iff \langle \hat{M} \rangle \in \overline{REGULAR_{TM}}$$

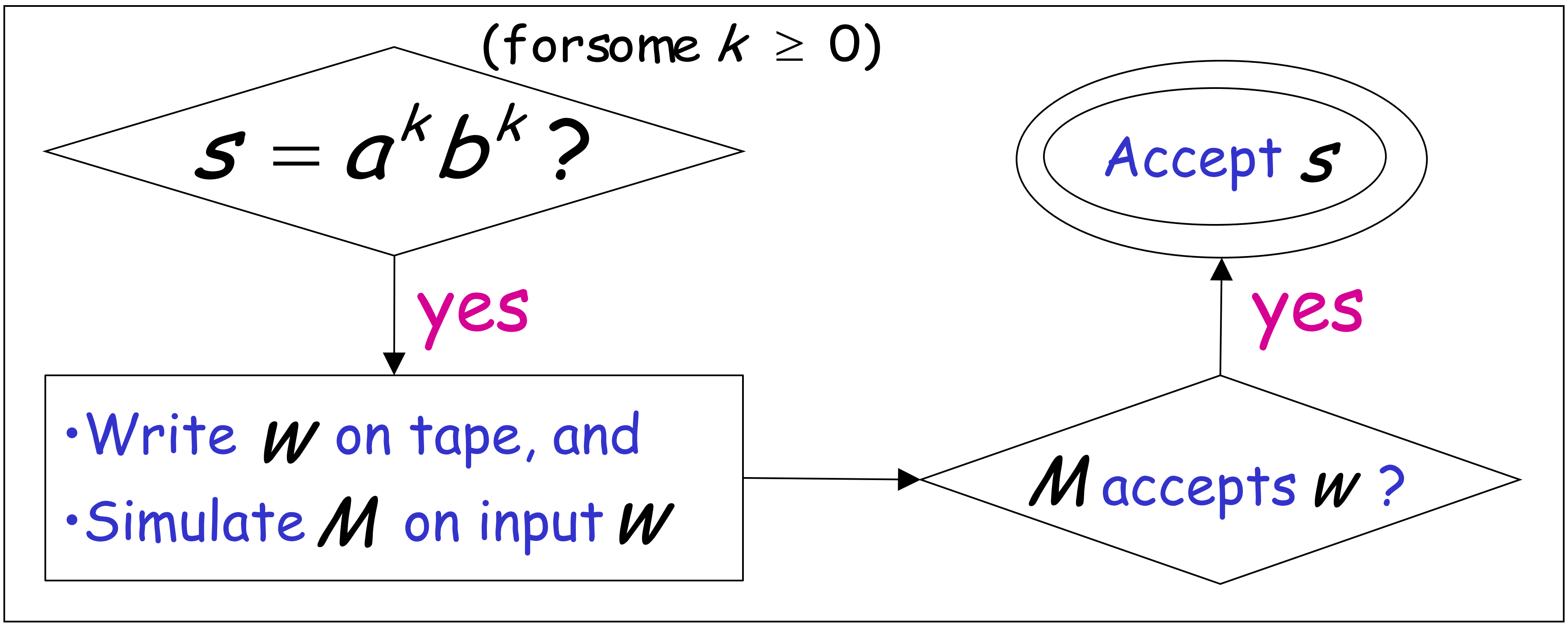
Construct $\langle \hat{M} \rangle$ from $\langle M, w \rangle$:



M accepts w ⟹ $L(\hat{M}) = \{a^n b^n : n \geq 0\}$ not regular

M does not accept w ⟹ $L(\hat{M}) = \varnothing$ regular

Turing Machine $\hat{M}$



Therefore:

M accepts w $\Longleftrightarrow$ $L(\hat{M})$ is not regular

Equivalently:

$$\langle M, w \rangle \in A_{TM} \Longleftrightarrow \langle \hat{M} \rangle \in \overline{REGULAR_{TM}}$$

END OF PROOF

Let L be a Turing-acceptable language

- L is empty?
- L is regular?
- L has size 2?

Size2 language problem

Input: Turing Machine M

Question: Does $L(M)$ have size 2 (two strings)?

$$|L(M)| = 2?$$

Corresponding language:

$$SIZE2_{TM} = \{\langle M \rangle : M \text{ is a Turing machine that accepts exactly two strings}\}$$

Theorem: $SIZE2_{TM}$ is undecidable

(size2 language problem is unsolvable)

Proof: Reduce

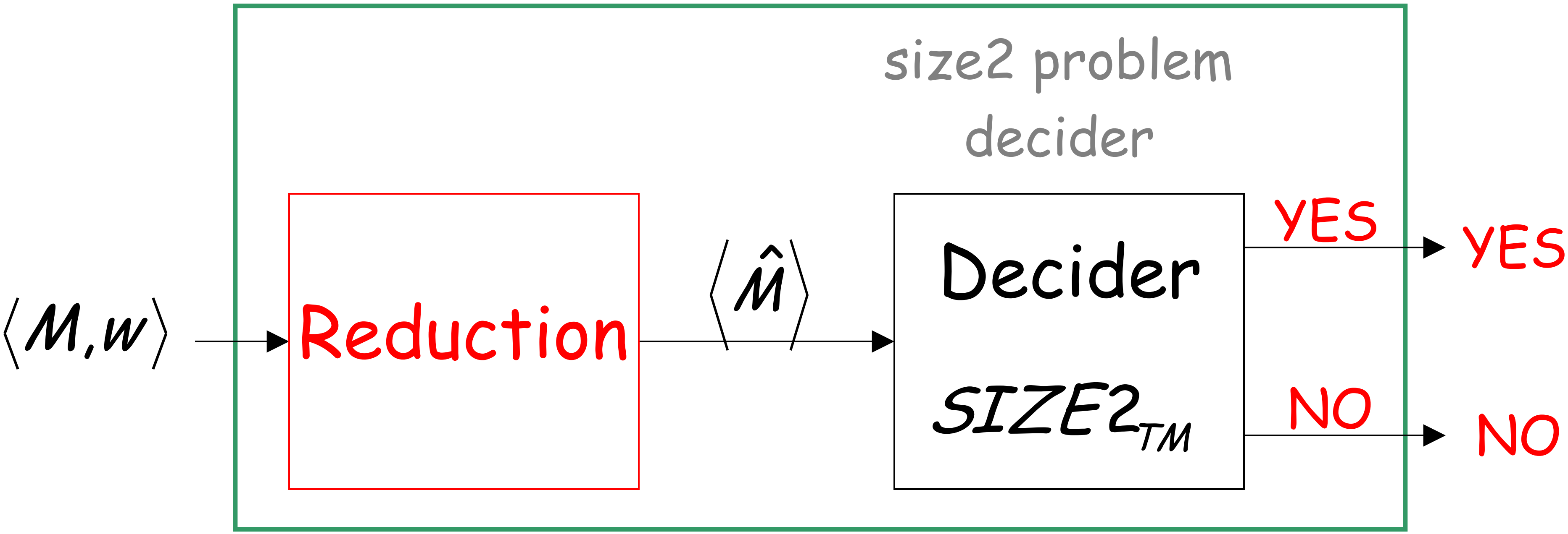
A_{TM} (membership problem)

to

$SIZE2_{TM}$ (size 2 language problem)

membership problem decider

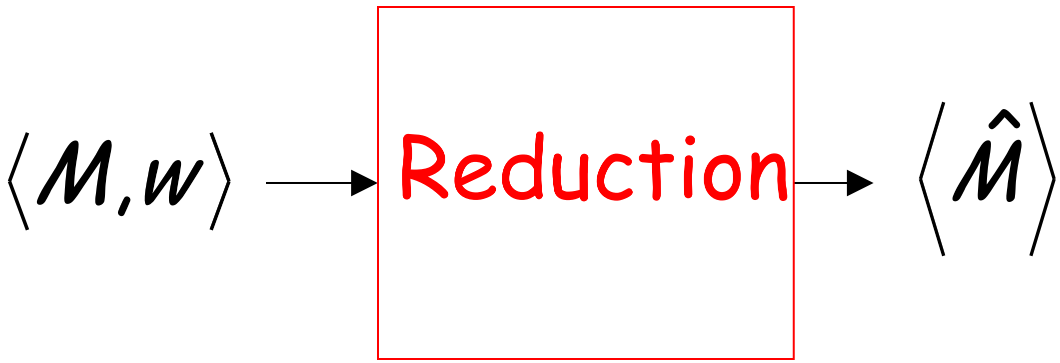
Decider for A_{TM}



Given the reduction,
If $SIZE2_{TM}$ is decidable,
then A_{TM} is decidable

A contradiction!
since A_{TM}
is undecidable

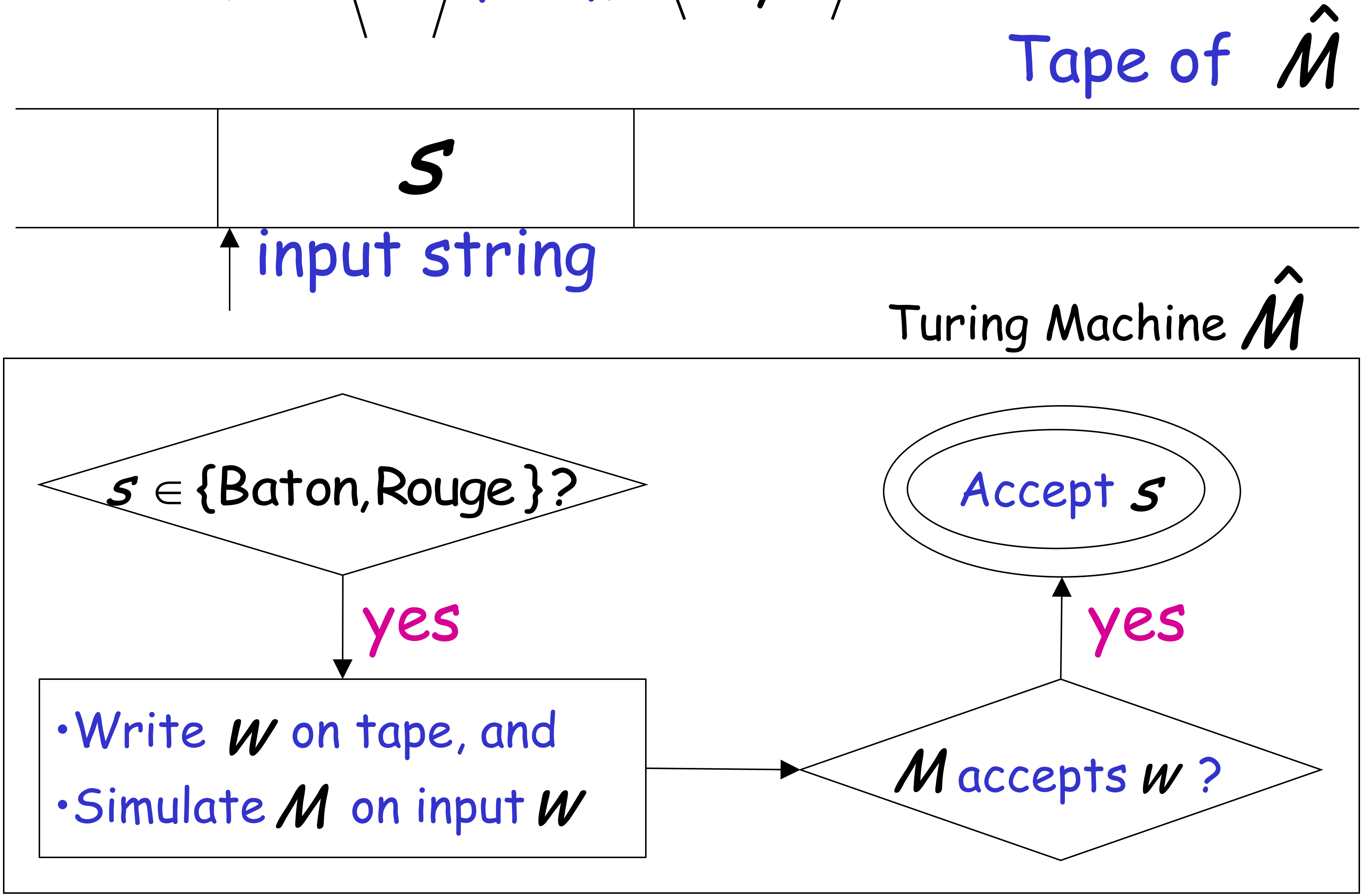
We only need to build the reduction:



So that:

$$\langle M, w \rangle \in A_{TM} \iff \langle \hat{M} \rangle \in SIZE2_{TM}$$

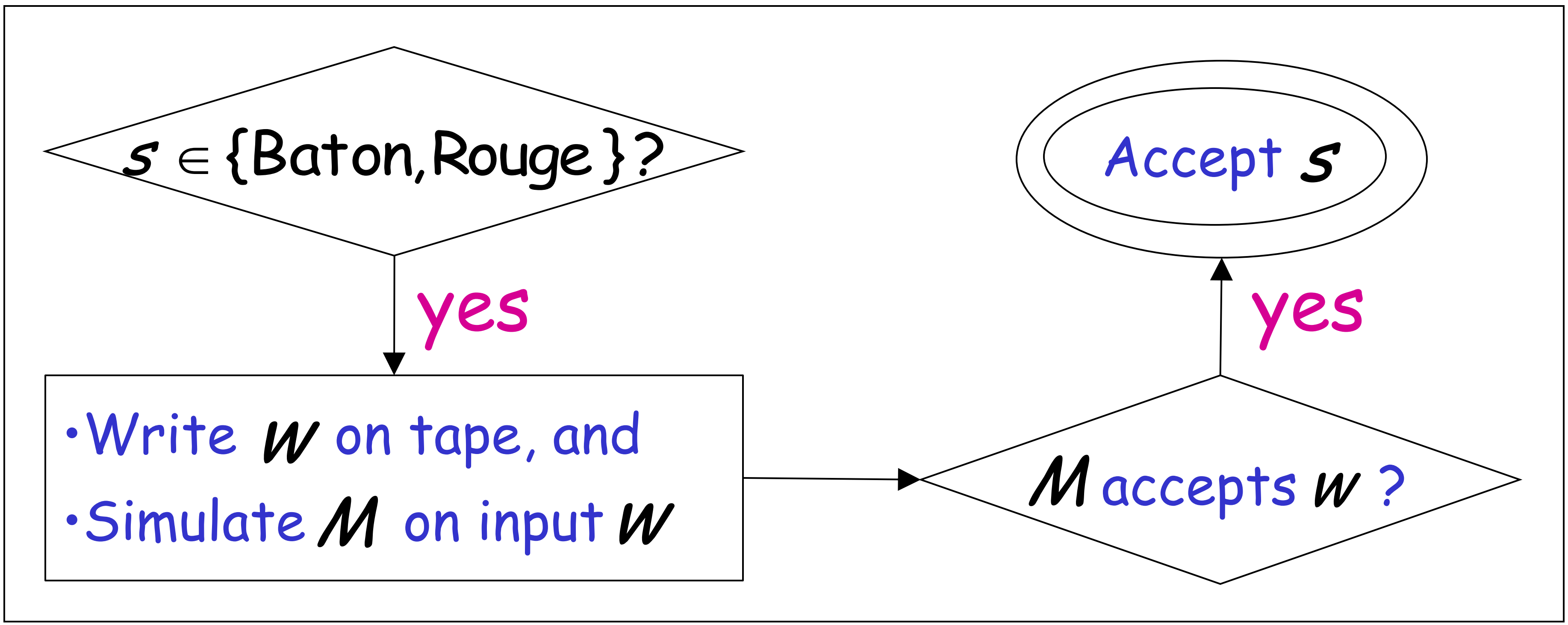
Construct $\langle \hat{M} \rangle$ from $\langle M, w \rangle$:



M accepts w ⟹ $L(\hat{M}) = \{Baton, Rouge\}$ — 2 strings

M does not accept w ⟹ $L(\hat{M}) = \emptyset$ — 0 strings

Turing Machine $\hat{M}$



Therefore:

M accepts w $\Longleftrightarrow$ $L(\hat{M})$ has size 2

Equivalently:

$$\langle M, w \rangle \in A_{TM} \Longleftrightarrow \langle \hat{M} \rangle \in SIZE2_{TM}$$

END OF PROOF