A Universal Turing Machine

A limitation of Turing Machines:

Turing Machines are "hardwired"

they execute only one program

Real Computers are re-programmable

Solution: Universal Turing Machine

Attributes:

- Reprogrammable machine

- Simulates any other Turing Machine

Universal Turing Machine

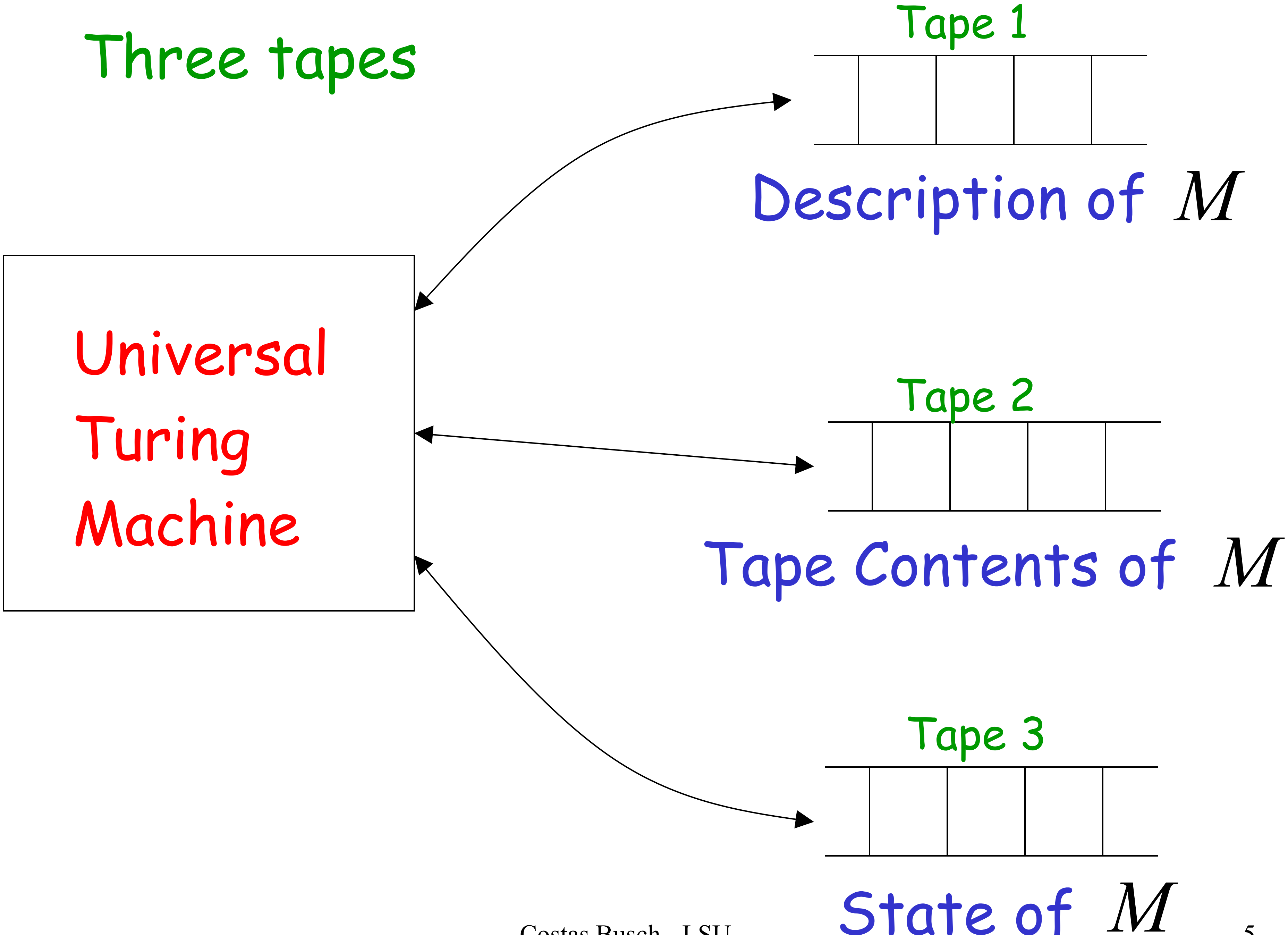
simulates any Turing Machine M

Input of Universal Turing Machine:

Description of transitions of M

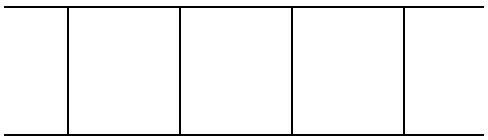
Input string of M

Three tapes



Universal
Turing
Machine
Tape 1
Description of M
Tape 2
Tape Contents of M
Tape 3
State of M

Tape 1



Description of M

We describe Turing machine M
as a string of symbols:

We encode M as a string of symbols

Alphabet Encoding

Symbols:	a	b	c	d
	↓	↓	↓	↓
Encoding:	1	11	111	1111

State Encoding

States:	q_1	q_2	q_3	q_4
	↓	↓	↓	↓
Encoding:	1	11	111	1111

Head Move Encoding

Move:	L	R
	↓	↓
Encoding:	1	11

Transition Encoding

Transition: $\delta(q_1, a) = (q_2, b, L)$

Encoding: 1 0 1 0 11 0 11 0 1

separator

Turing Machine Encoding

Transitions:

$$\delta(q_1, a) = (q_2, b, L) \qquad \delta(q_2, b) = (q_3, c, R)$$

Encoding:

10101101101 00 1101101110111011

separator

Tape 1 contents of Universal Turing Machine:

binary encoding
of the simulated machine M

Tape 1

1 0 1 0 11 0 11 0 10011 0 1 10 111 0 111 0 1100 K

A Turing Machine is described with a binary string of 0's and 1's

Therefore:

The set of Turing machines forms a language:

each string of this language is the binary encoding of a Turing Machine

Language of Turing Machines

$L = \{$ 1010110101, (Turing Machine 1)

101011101011, (Turing Machine 2)

11101011110101111, ...

...... $\}$

Countable Sets

Infinite sets are either:

Countable

or

Uncountable

Countable set:

There is a one to one correspondence (injection)
of
elements of the set
to
Positive integers (1,2,3,…)

Every element of the set is mapped to a positive number such that no two elements are mapped to same number

Example: The set of even integers is countable

Even integers: (positive) $0, \ 2, \ 4, \ 6, \ K$

Correspondence: $\downarrow \quad \downarrow \quad \downarrow \quad \downarrow$

Positive integers: $1, \ 2, \ 3, \ 4, \ K$

$2n$ corresponds to $n+1$

Example: The set of rational numbers is countable

Rational numbers: $\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \ldots$

Naïve Approach

Nominator 1

Rational numbers: $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots$

Correspondence: $\downarrow \quad \downarrow \quad \downarrow$

Positive integers: $1, 2, 3, \ldots$

Doesn't work:

we will never count numbers with nominator 2: $\frac{2}{1}, \frac{2}{2}, \frac{2}{3}, \ldots$

Better Approach

$$\frac{1}{1} \quad \frac{1}{2} \quad \frac{1}{3} \quad \frac{1}{4} \quad \cdots$$

$$\frac{2}{1} \quad \frac{2}{2} \quad \frac{2}{3} \quad \cdots$$

$$\frac{3}{1} \quad \frac{3}{2} \quad \cdots$$

$$\frac{4}{1} \quad \cdots$$

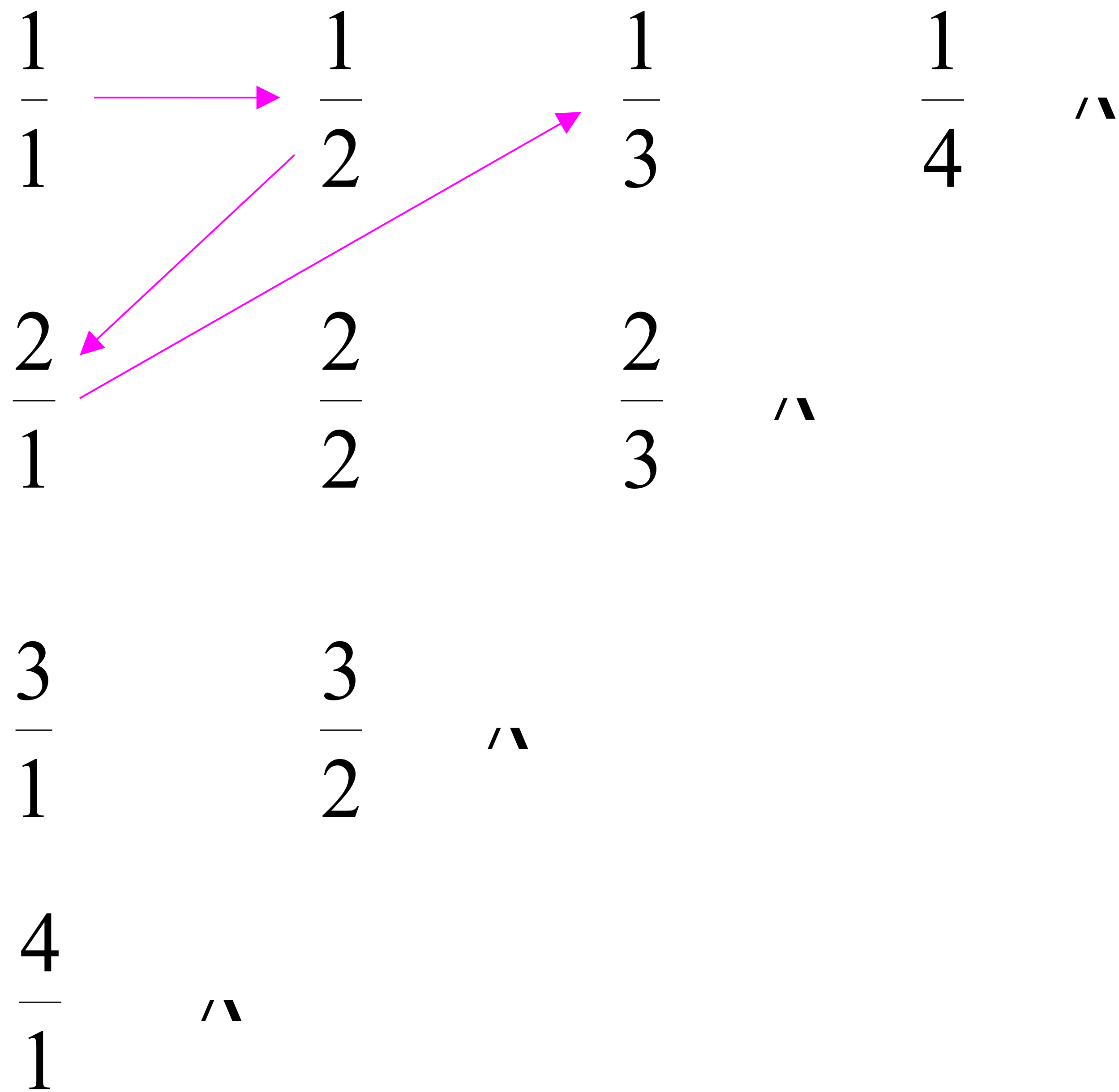
$$\frac{1}{1} \rightarrow \frac{1}{2} \quad \frac{1}{3} \quad \frac{1}{4} \quad \Lambda$$

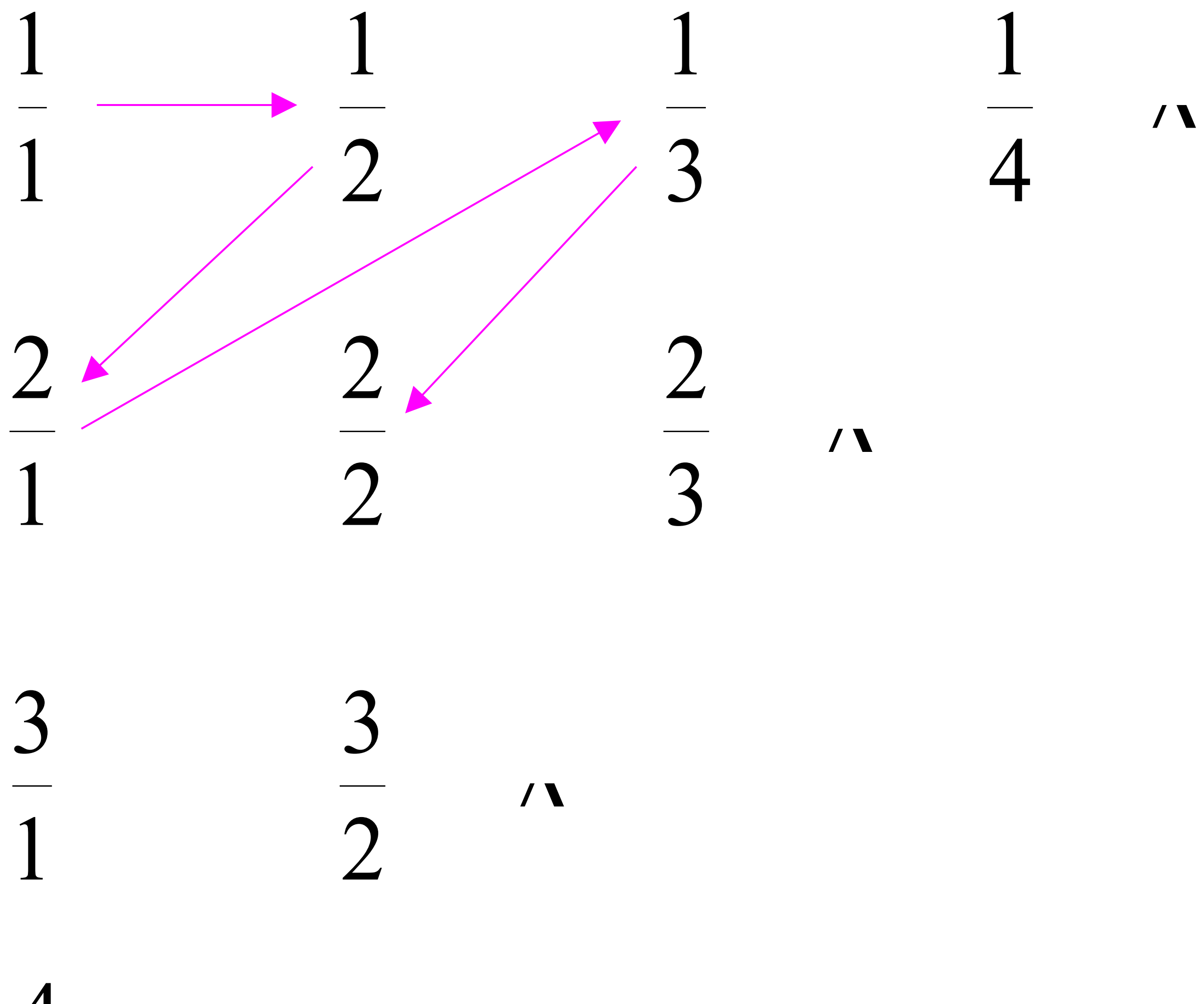
$$\frac{2}{1} \quad \frac{2}{2} \quad \frac{2}{3} \quad \Lambda$$

$$\frac{3}{1} \quad \frac{3}{2} \quad \Lambda$$

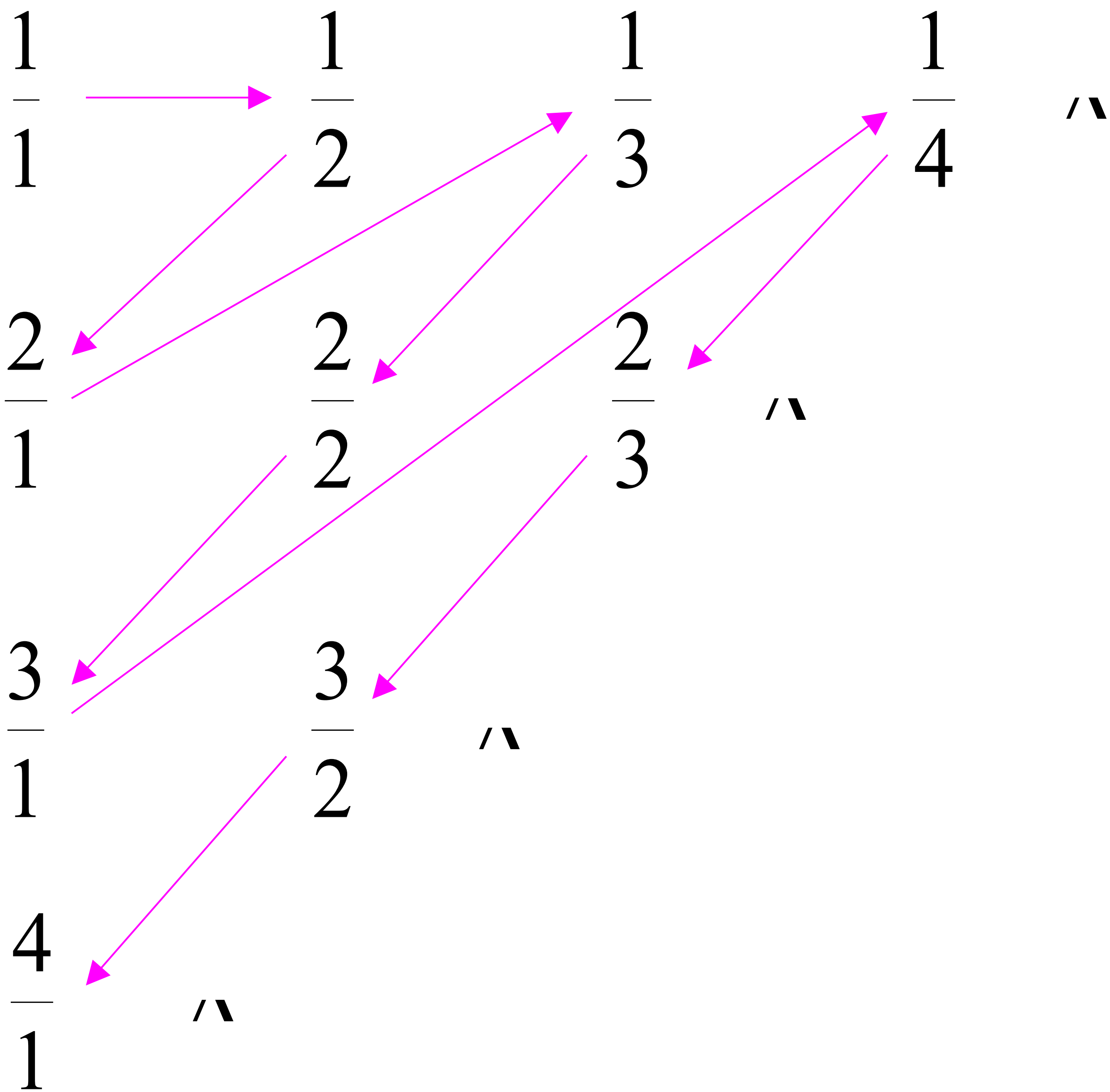
$$\frac{4}{1} \quad \Lambda$$

$\frac{1}{1}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	Λ
$\frac{2}{1}$	$\frac{2}{2}$	$\frac{2}{3}$	Λ	
$\frac{3}{1}$	$\frac{3}{2}$	Λ		
$\frac{4}{1}$	Λ			


1/1 1/2 1/3 1/4 …
2/1 2/2 2/3 …
3/1 3/2 …
4/1 …



$\frac{1}{1}$ $\frac{1}{2}$ $\frac{1}{3}$ $\frac{1}{4}$
$\frac{2}{1}$ $\frac{2}{2}$ $\frac{2}{3}$
$\frac{3}{1}$ $\frac{3}{2}$
$\frac{4}{1}$


1/1
1/2
1/3
1/4
2/1
2/2
2/3
3/1
3/2
4/1

Rational Numbers: $\frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{1}{3}, \frac{2}{2}, \ldots$

Correspondence:

Positive Integers: $1, \quad 2, \quad 3, \quad 4, \quad 5, \ldots$

We proved:

the set of rational numbers is countable
by describing an enumeration procedure
(enumerator)
for the correspondence to natural numbers

Definition

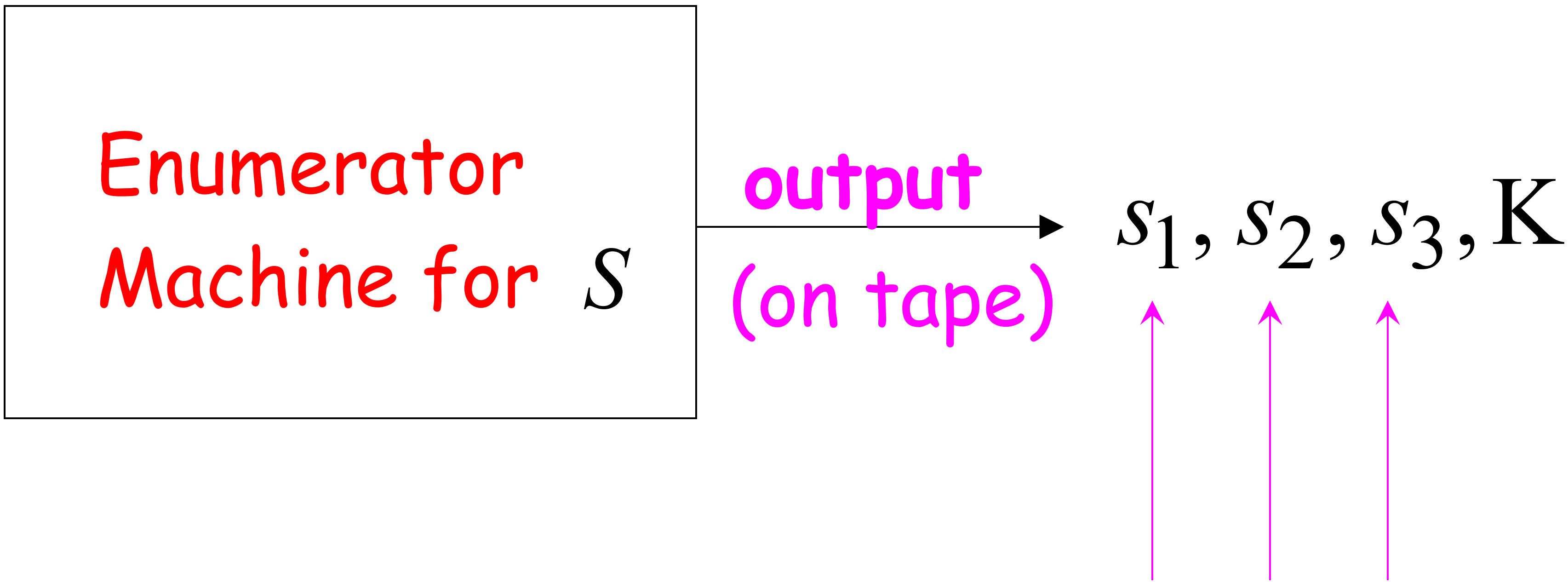
Let S be a set of strings (Language)

An **enumerator** for S is a Turing Machine
that generates (prints on tape)
all the strings of S one by one

and

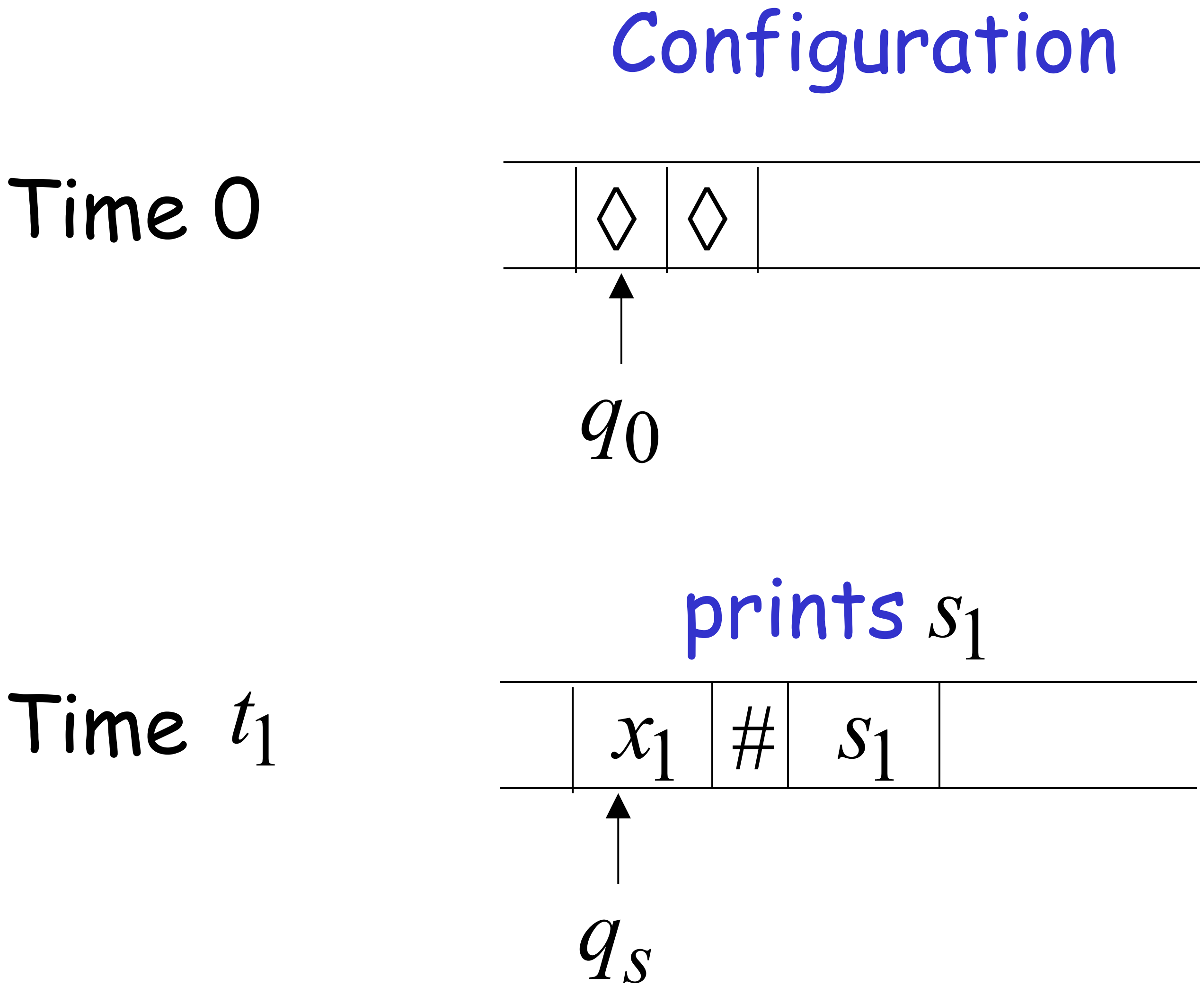
each string is generated in finite time

strings $s_1, s_2, s_3, K \in S$



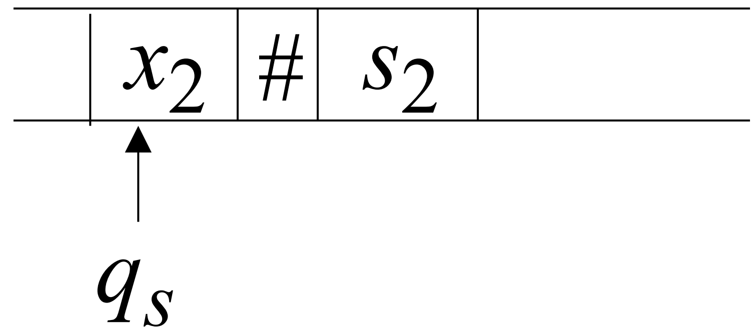
Finite time: t_1, t_2, t_3, K

Enumerator Machine



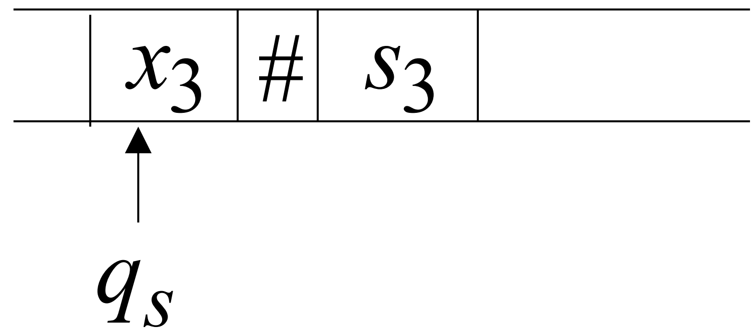
Time t_2

prints s_2



Time t_3

prints s_3



Observation:

If for a set S there is an enumerator, then the set is countable

The enumerator describes the correspondence of S to natural numbers

Example: The set of strings $S = \{a, b, c\}^+$ is countable

Approach:

We will describe an enumerator for S

Naive enumerator:

Produce the strings in lexicographic order:

$$s_1 = a$$

$$s_2 = aa$$

$$\vdots \quad aaa$$

$$aaaa$$

$$\ldots\ldots$$

Doesn't work:

strings starting with b
will never be produced

Better procedure: **Proper Order** (Canonical Order)

1. Produce all strings of length 1

2. Produce all strings of length 2

3. Produce all strings of length 3

4. Produce all strings of length 4

⋮

$s_1 = a$
$s_2 = b$
c
$\vdots$

length 1

aa
ab
ac
ba
bb
bc
ca
cb
cc

length 2

Produce strings in Proper Order:

aaa
aab
aac
......

length 3

Theorem: The set of all Turing Machines is countable

Proof: Any Turing Machine can be encoded with a binary string of 0's and 1's

Find an enumeration procedure for the set of Turing Machine strings

Enumerator:

Repeat

1. Generate the next binary string of 0's and 1's in proper order

2. Check if the string describes a Turing Machine

   if **YES**: print string on output tape

   if **NO**: ignore string

Binary strings		Turing Machines
0 ignore		
1 ignore		
00 ignore		
01		
⋮		
1 0 1 0 11 0 11 0 0		
1 0 1 0 11 0 11 0 1	$\xrightarrow{s_1}$	1 0 1 0 11 0 11 0 1
⋮		
1 0 11 0 1010010101 101	$\xrightarrow{s_2}$	1 0 11 0 1010010101 101
⋮		

End of Proof

Simpler Proof:

Each Turing machine binary string is mapped to the number representing its value

Uncountable Sets

We will prove that there is a language L which is not accepted by any Turing machine

Technique:

Turing machines are countable

Languages are uncountable

(there are more languages than Turing Machines)

Theorem:

If S is an infinite countable set, then

the powerset 2^S of S is uncountable.

The powerset 2^S contains all possible subsets of S

Example: $S = \{a, b\}$ $\quad 2^S = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$

Proof:

Since S is countable, we can list its elements in some order

$$S = \{s_1, s_2, s_3, \ldots\}$$

Elements of S

Elements of the powerset 2^S have the form:

$$\emptyset$$

$$\{s_1, s_3\}$$

$$\{s_5, s_7, s_9, s_{10}\}$$

$$\vdots$$

They are subsets of S

We encode each subset of S
with a binary string of 0's and 1's

Subset of S	Binary encoding				
	s_1	s_2	s_3	s_4	…
$\{s_1\}$	1	0	0	0	…
$\{s_2, s_3\}$	0	1	1	0	…
$\{s_1, s_3, s_4\}$	1	0	1	1	…

Every infinite binary string corresponds to a subset of S:

Example: 1 0 0 1 1 1 0 ʌ

Corresponds to: $\{s_1, s_4, s_5, s_6, \mathrm{K}\} \in 2^S$

Let's assume (for contradiction) that the powerset 2^S is countable

Then: we can list the elements of the powerset in some order

$$2^S = \{t_1, t_2, t_3, \mathrm{K}\}$$

Subsets of S

Powerset element	Binary encoding example					
t_1	1	0	0	0	0	⋯
t_2	1	1	0	0	0	⋯
t_3	1	1	0	1	0	⋯
t_4	1	1	0	0	1	⋯
⋮			⋮			

t = the binary string whose bits are the complement of the diagonal

t_1	1	0	0	0	0	Λ
t_2	1	1	0	0	0	Λ
t_3	1	1	0	1	0	Λ
t_4	1	1	0	0	1	Λ

Binary string: $t = 0011\Lambda$

(birary complement of diagonal)

The binary string $t = 0011\mathrm{K}$

corresponds to a subset of S: $t = \{s_3, s_4, \mathrm{K}\} \in 2^S$

t = the binary string whose bits are the complement of the diagonal

t_1	1	0	0	0	0	Λ
t_2	1	1	0	0	0	Λ
t_3	1	1	0	1	0	Λ
t_4	1	1	0	0	1	Λ

$$t = 0011\Lambda$$

Question: $t = t_1$? NO: differ in 1st bit

t = the binary string whose bits are the complement of the diagonal

t_1	1	0	0	0	0	Λ
t_2	1	1	0	0	0	Λ
t_3	1	1	0	1	0	Λ
t_4	1	1	0	0	1	Λ

$$t = 0011\Lambda$$

Question: $t = t_2$? **NO:** differ in 2nd bit

t = the binary string whose bits are the complement of the diagonal

t_1	1	0	0	0	0	Λ
t_2	1	1	0	0	0	Λ
t_3	1	1	0	1	0	Λ
t_4	1	1	0	0	1	Λ

$t = 0011\Lambda$

Question: $t = t_3$? NO: differ in 3rd bit

Thus: $t \neq t_i$ for every i

since they differ in the ith bit

However, $t \in 2^S \Rightarrow t = t_i$ for some i

Contradiction!!!

Therefore the powerset 2^S is uncountable

End of proof

An Application: Languages

Consider Alphabet : $A = \{a, b\}$

The set of all strings:

$$S = \{a, b\}^* = \{\varepsilon, a, b, aa, ab, ba, bb, aaa, aab, \ldots\}$$

infinite and countable

because we can enumerate
the strings in proper order

Consider Alphabet : $A = \{a, b\}$

The set of all strings:

$$S = \{a, b\}^* = \{\varepsilon, a, b, aa, ab, ba, bb, aaa, aab, \ldots\}$$

infinite and countable

Any language is a subset of S :

$$L = \{aa, ab, aab\}$$

Consider Alphabet : $A = \{a, b\}$

The set of all Strings:

$$S = A^* = \{a, b\}^* = \{\varepsilon, a, b, aa, ab, ba, bb, aaa, aab, \ldots\}$$

infinite and countable

The powerset of S contains all languages:

$$2^S = \{\emptyset, \{\varepsilon\}, \{a\}, \{a, b\}, \{aa, b\}, \ldots, \{aa, ab, aab\}, \ldots\}$$

uncountable

Consider Alphabet : $A = \{a, b\}$

countable

Turing machines: $M_1 \quad M_2 \quad M_3 \quad \Lambda$

accepts

Languages accepted
By Turing Machines: $L_1 \quad L_2 \quad L_3 \quad \Lambda$

countable

Denote: $X = \{L_1, L_2, L_3, \mathrm{K}\}$

countable

Note: $X \subseteq 2^S$

$(S = \{a, b\}^*)$

Languages accepted
by Turing machines: X countable

All possible languages: 2^S uncountable

Therefore: $X \neq 2^S$

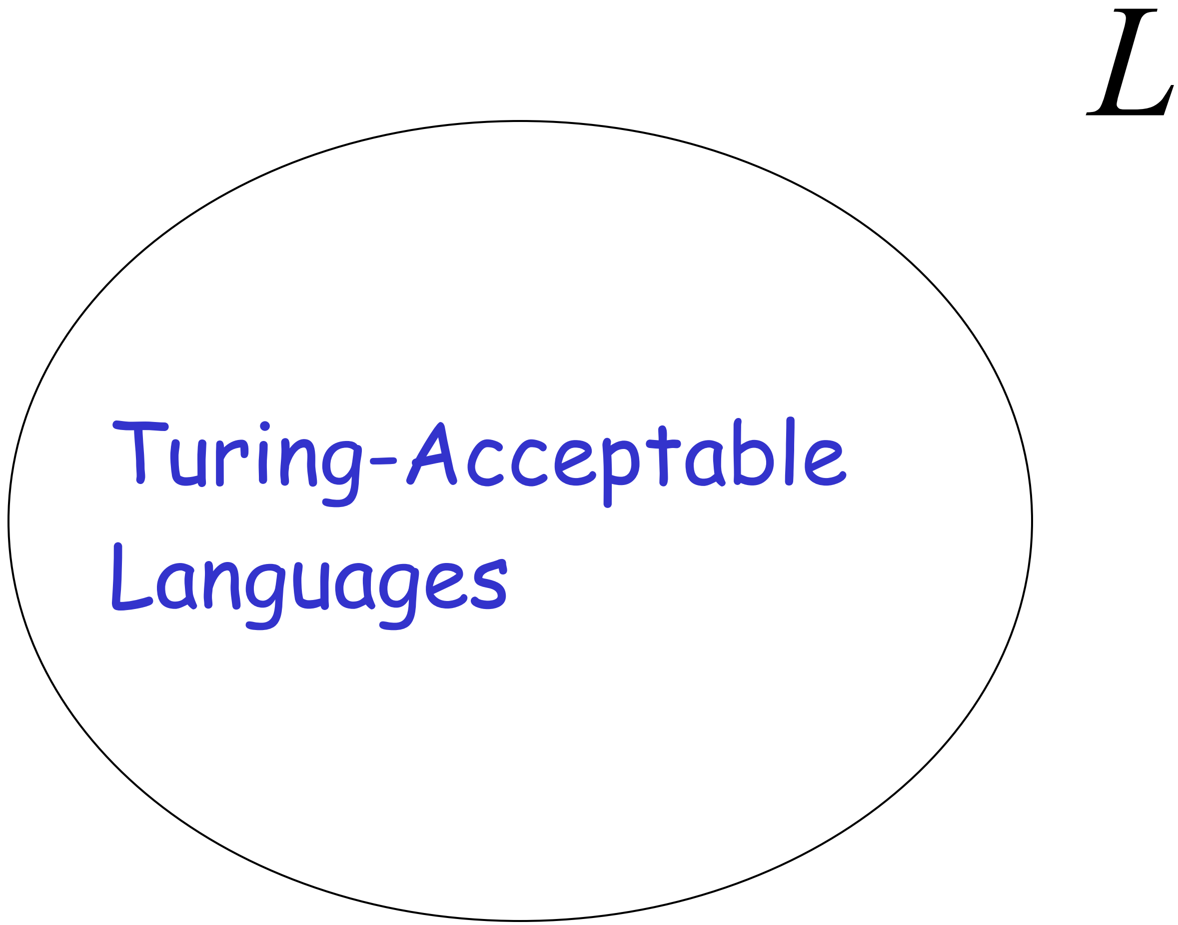
$$\left(\text{since } X \subseteq 2^S, \text{ we get } X \subset 2^S\right)$$

Conclusion:

There is a language L not accepted by any Turing Machine:

$$X \subset 2^S \quad \Longrightarrow \quad \exists L \in 2^S \text{ and } L \notin X$$

Non Turing-Acceptable Languages



Note that: $X = \{L_1, L_2, L_3, \ldots\}$

is a *multi-set* (elements may repeat) since a language may be accepted by more than one Turing machine

However, if we remove the repeated elements, the resulting set is again countable since every element still corresponds to a positive integer