More Applications

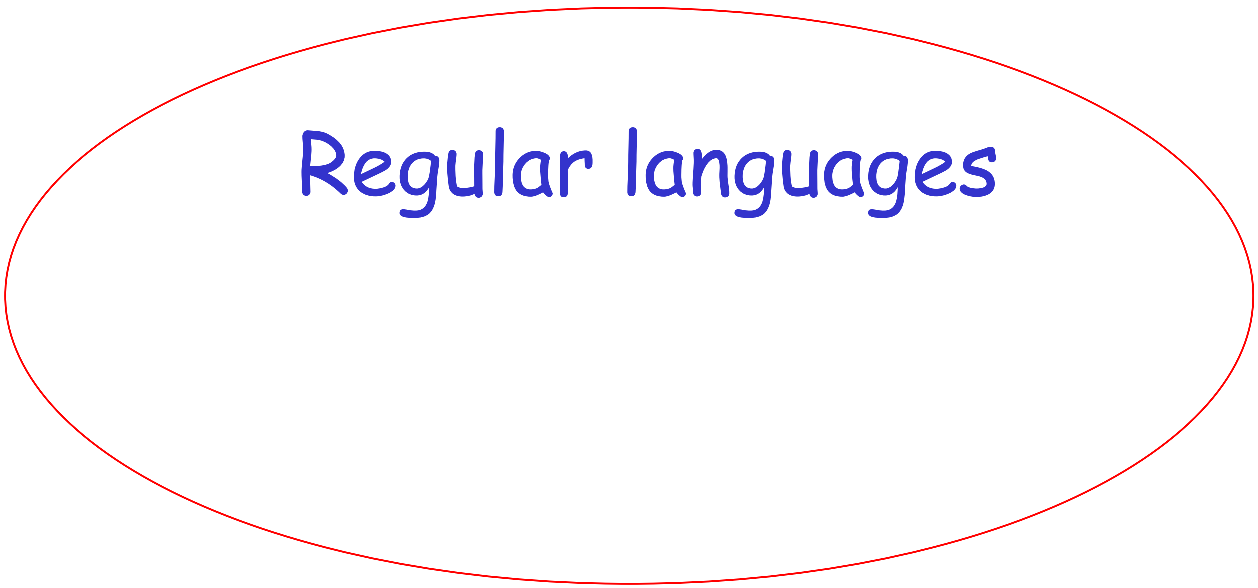
of

the Pumping Lemma

The Pumping Lemma:

- Given a infinite regular language L
- there exists an integer p (critical length)
- for any string $w \in L$ with length $|w| \geq p$
- we can write $w = x\,y\,z$
- with $|x\,y| \leq p$ and $|y| \geq 1$
- such that: $x\,y^i\,z \in L \qquad i = 0, 1, 2, \ldots$

Non-regular languages $L=\{vv^R : v \in \Sigma^*\}$



Theorem: The language

$$L = \{vv^R : v \in \Sigma^*\} \qquad \Sigma = \{a, b\}$$

is not regular

Proof: Use the Pumping Lemma

$$L = \{vv^R : v \in \Sigma^*\}$$

Assume for contradiction that L is a regular language

Since L is infinite we can apply the Pumping Lemma

$$L = \{vv^R : v \in \Sigma^*\}$$

Let p be the critical length for L

Pick a string w such that: $w \in L$

and length $|w| \geq p$

We pick $w = a^p b^p b^p a^p$

From the Pumping Lemma:

we can write: $w = a^p b^p b^p a^p = x\, y\, z$

with lengths: $|\, x\, y \,| \leq p, \quad |\, y \,| \geq 1$

$$w = xyz = \overbrace{a\ldots aa\ldots a\ldots a}^{p}\overbrace{b\ldots b}^{p}\overbrace{b\ldots b}^{p}\overbrace{a\ldots a}^{p}$$

$$w = xyz = \underbrace{a\ldots a}_{x}\underbrace{a\ldots a}_{y}\underbrace{\ldots ab\ldots bb\ldots ba\ldots a}_{z}$$

Thus: $y = a^k, \quad 1 \leq k \leq p$

$$x\,y\,z = a^p b^p b^p a^p \qquad y = a^k, \;\; 1 \le k \le p$$

From the Pumping Lemma: $x\,y^i\,z \in L$

$$i = 0, 1, 2, \ldots$$

Thus: $x\,y^2\,z \in L$

$$x\ y\ z = a^p b^p b^p a^p \qquad y = a^k, \quad 1 \le k \le p$$

From the Pumping Lemma: $x\ y^2\ z \ \in\ L$

$$xy^2z = \overbrace{a...aa...aa...a...a}^{p+k}\overbrace{b...b}^{p}\overbrace{b...b}^{p}\overbrace{a...a}^{p} \in L$$

$$\underbrace{a...a}_{x}\underbrace{a...a}_{y}\underbrace{a...a}_{y}\underbrace{a...ab...bb...ba...a}_{z}$$

Thus: $a^{p+k} b^p b^p a^p \quad \in L$

$$a^{p+k}b^p b^p a^p \quad \in L \qquad\qquad k \geq 1$$

BUT: $L = \{vv^R : v \in \Sigma^*\}$

$\Downarrow$

$$a^{p+k}b^p b^p a^p \quad \notin L$$

CONTRADICTION!!!

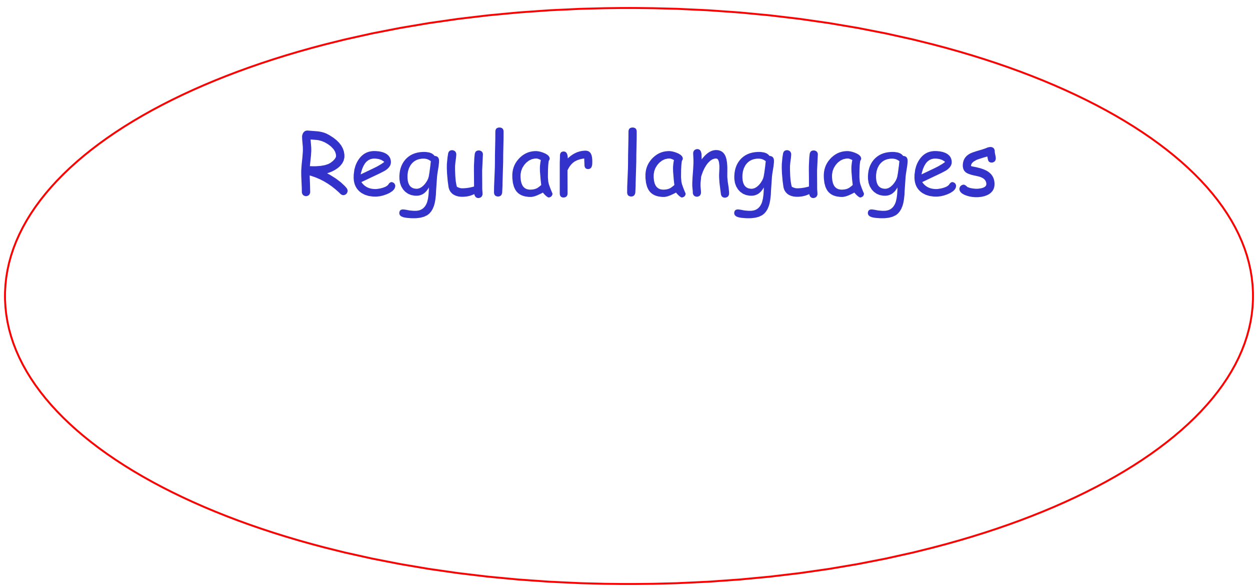
Therefore: Our assumption that L is a regular language is not true

Conclusion: L is not a regular language

END OF PROOF

Non-regular languages

$$L = \{a^n b^l c^{n+l} : \ n, l \geq 0\}$$



Theorem: The language

$$L = \{a^n b^l c^{n+l} : \ n, l \geq 0\}$$

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Proof: Use the Pumping Lemma

$$L = \{a^n b^l c^{n+l} : n, l \geq 0\}$$

Assume for contradiction
that L is a regular language

Since L is infinite
we can apply the Pumping Lemma

$$L=\{a^n b^l c^{n+l}: \; n,l \geq 0\}$$

Let p be the critical length of L

Pick a string w such that: $w \in L$ and

length $|w| \geq p$

We pick $w = a^p b^p c^{2p}$

From the Pumping Lemma:

We can write $w = a^p b^p c^{2p} = x\,y\,z$

With lengths $|x\,y| \le p, \quad |y| \ge 1$

$$w = xyz = \overbrace{a...aa...aa...a}^{p}\overbrace{b...b}^{p}\overbrace{c...cc...c}^{2p}$$

$$w = xyz = \underbrace{a...a}_{x}\underbrace{a...a}_{y}\underbrace{a...ab...bc...cc...c}_{z}$$

Thus: $y = a^k, \quad 1 \le k \le p$

$$x\,y\,z = a^p b^p c^{2p} \qquad y = a^k, \quad 1 \le k \le p$$

From the Pumping Lemma: $x\,y^i\,z \in L$

$$i = 0, 1, 2, \ldots$$

Thus: $x\,y^0\,z = xz \in L$

$$x\,y\,z = a^p b^p c^{2p} \qquad y = a^k, \quad 1 \le k \le p$$

From the Pumping Lemma: $xz \in L$

$$xz = \overbrace{a...aa...a}^{p-k}\overbrace{b...b}^{p}\overbrace{c...cc...c}^{2p} \in L$$

$$\underbrace{a...a}_{x}\underbrace{a...ab...bc...cc...c}_{z}$$

Thus: $a^{p-k} b^p c^{2p} \in L$

$$a^{p-k}b^p c^{2p} \quad \in L \qquad\qquad k \geq 1$$

BUT: $L = \{a^n b^l c^{n+l} : \ n, l \geq 0\}$

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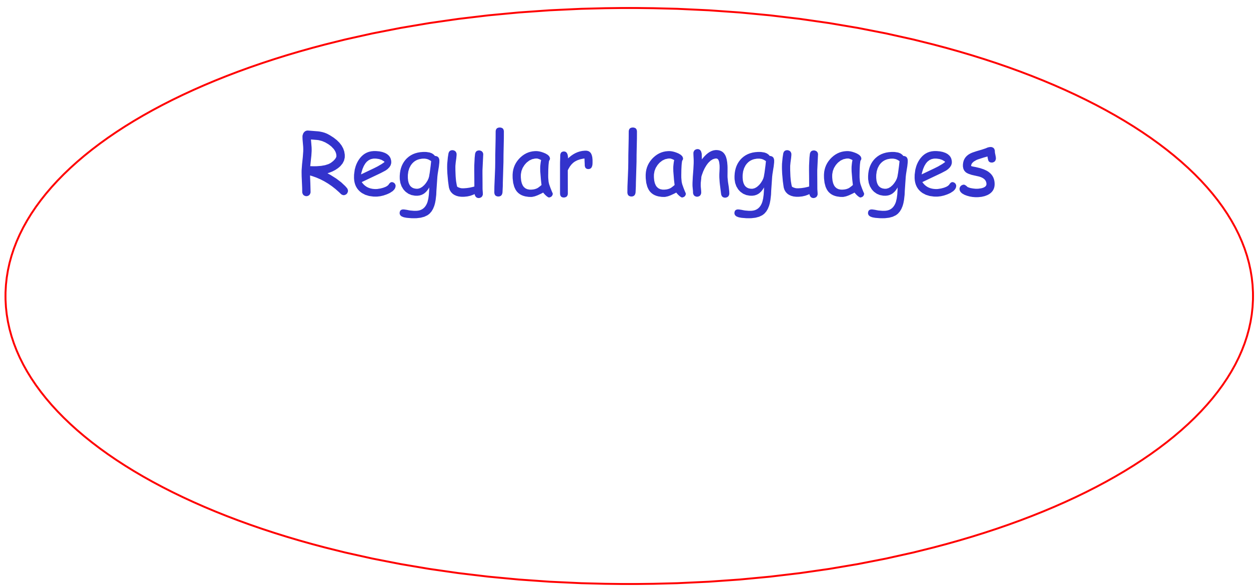
Non-regular languages $L = \{a^{n!} : \ n \geq 0\}$



Theorem: The language $L = \{a^{n!} : \; n \geq 0\}$

is not regular

$$n! \; = \; 1 \cdot 2 \Lambda \; (n-1) \cdot n$$

Proof: Use the Pumping Lemma

$$L = \{a^{n!} : \; n \geq 0\}$$

Assume for contradiction
that L is a regular language

Since L is infinite
we can apply the Pumping Lemma

$$L = \{a^{n!} : n \geq 0\}$$

Let p be the critical length of L

Pick a string w such that: $w \in L$

length $|w| \geq p$

We pick $w = a^{p!}$

From the Pumping Lemma:

We can write $w = a^{p!} = x\ y\ z$

With lengths $|x\ y| \leq p, \quad |y| \geq 1$

$$w = xyz = a^{p!} = \overbrace{\underbrace{a\ldots a}_{x}\underbrace{a\ldots a}_{y}a\ldots a}^{p}\overbrace{a\ldots aa\ldots a}^{p!-p}$$

(where z is $a\ldots aa\ldots aa\ldots a$, the part after y)

Thus: $y = a^k, \quad 1 \leq k \leq p$

$$x\,y\,z = a^{p!} \qquad y = a^k, \quad 1 \le k \le p$$

From the Pumping Lemma: $x\,y^i\,z \in L$

$$i = 0, 1, 2, \ldots$$

Thus: $x\,y^2\,z \in L$

$$x\ y\ z = a^{p!} \qquad y = a^k, \;\; 1 \le k \le p$$

From the Pumping Lemma: $x\ y^2\ z \ \in\ L$

$$xy^2z = \overbrace{\underbrace{a...a}_{x}\underbrace{a...a}_{y}\underbrace{a...a}_{y}a...a}^{p+k}\overbrace{a...aa...a}^{p!-p} \ \in\ L$$

where $a...aa...aa...aa...a$ after the second y together form z.

Thus: $a^{p!+k} \ \in L$

$$a^{p!+k} \in L \qquad 1 \le k \le p$$

Since: $L = \{a^{n!} : \ n \ge 0\}$

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There must exist z such that:

$$p!+k = z!$$

However: $p!+k \leq p!+p$ for $p>1$

$$\leq p!+p!$$

$$< p!\,p + p!$$

$$= p!(p+1)$$

$$= (p+1)!$$

⇩

$$p!+k < (p+1)!$$

⇩

$p!+k \neq z!$ for any z

for $p = 1$

we could pick string $w = a^{p'!}$

where $p' > p$

and we would obtain the same conclusion:

$p'! + k \neq z!$ for any z

$$a^{p!+k} \in L \qquad 1 \leq k \leq p$$

BUT: $L = \{a^{n!} : n \geq 0\}$

$\Downarrow$

$$a^{p!+k} \notin L$$

CONTRADICTION!!!

Therefore: Our assumption that L is a regular language is not true

Conclusion: L is not a regular language

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