Linear Grammars

Grammars with
at most one variable at the right side
of a production

Examples:

$$S \to aSb$$
$$S \to \lambda$$

$$S \to Ab$$
$$A \to aAb$$
$$A \to \lambda$$

A Non-Linear Grammar

Grammar G :

$$S \to SS$$

$$S \to \lambda$$

$$S \to aSb$$

$$S \to bSa$$

$$L(G) = \{w: \quad n_a(w) = n_b(w)\}$$

Number of a in string w

Another Linear Grammar

Grammar G :

$$S \to A$$

$$A \to aB \mid \lambda$$

$$B \to Ab$$

$$L(G) = \{a^n b^n : n \geq 0\}$$

Right-Linear Grammars

All productions have form: $A \to xB$

or

$A \to x$

string of terminals

Example: $S \to abS$

$S \to a$

Left-Linear Grammars

All productions have form: $A \to Bx$

or

$A \to x$

string of terminals

Example: $S \to Aab$

$A \to Aab \mid B$

$B \to a$

Regular Grammars

Regular Grammars

A regular grammar is any right-linear or left-linear grammar

Examples:

G_1

$$S \to abS$$
$$S \to a$$

G_2

$$S \to Aab$$
$$A \to Aab \mid B$$
$$B \to a$$

Observation

Regular grammars generate regular languages

Examples:

G_1

$$S \to abS$$

$$S \to a$$

G_2

$$S \to Aab$$

$$A \to Aab \mid B$$

$$B \to a$$

$$L(G_1) = (ab)^* a$$

$$L(G_2) = aab(ab)^*$$

Regular Grammars
Generate
Regular Languages

Theorem

$$\left\{ \begin{array}{l} \text{Languages} \\ \text{Generated by} \\ \text{Regular Grammars} \end{array} \right\} = \left\{ \begin{array}{l} \text{Regular} \\ \text{Languages} \end{array} \right\}$$

Theorem - Part 1

$$\left\{ \begin{array}{l} \text{Languages} \\ \text{Generated by} \\ \text{Regular Grammars} \end{array} \right\} \subseteq \left\{ \begin{array}{l} \text{Regular} \\ \text{Languages} \end{array} \right\}$$

Any regular grammar generates a regular language

Theorem - Part 2

$$\left\{ \begin{array}{l} \text{Languages} \\ \text{Generated by} \\ \text{Regular Grammars} \end{array} \right\} \supseteq \left\{ \begin{array}{l} \text{Regular} \\ \text{Languages} \end{array} \right\}$$

Any regular language is generated by a regular grammar

Proof - Part 1

$$\left\{ \begin{array}{l} \text{Languages} \\ \text{Generated by} \\ \text{Regular Grammars} \end{array} \right\} \subseteq \left\{ \begin{array}{l} \text{Regular} \\ \text{Languages} \end{array} \right\}$$

The language $L(G)$ generated by any regular grammar G is regular

The case of Right-Linear Grammars

Let G be a right-linear grammar

We will prove: $L(G)$ is regular

Proof idea: We will construct NFA M with $L(M) = L(G)$

Grammar G is right-linear

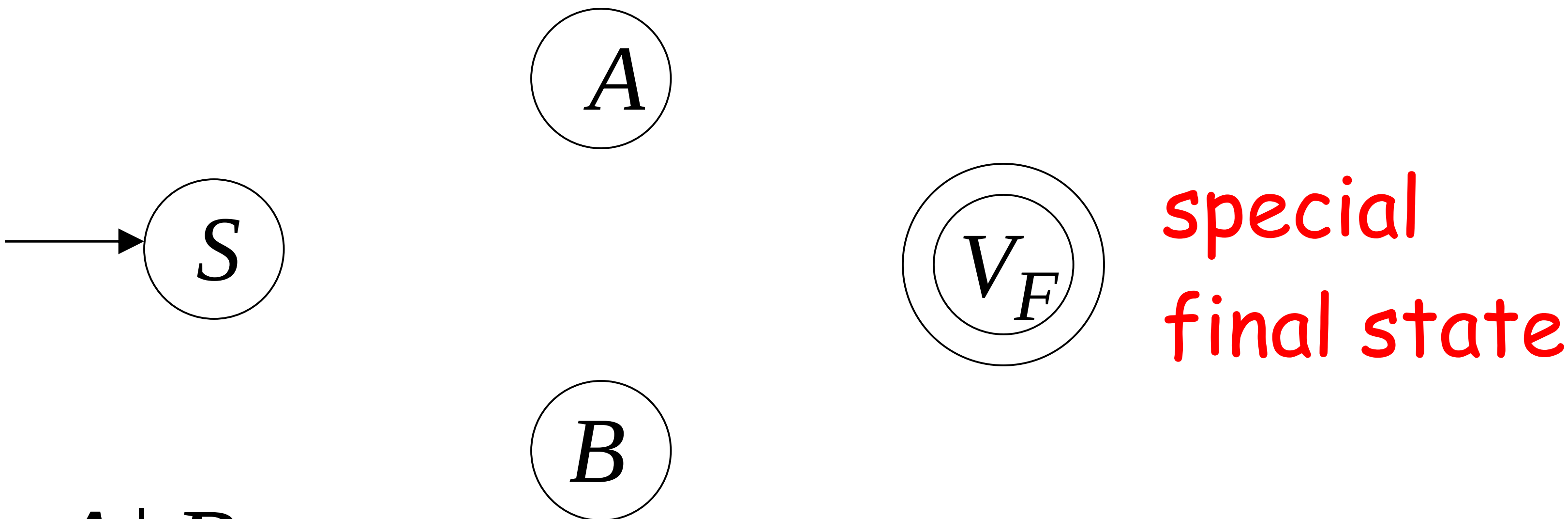
Example:

$$S \to aA \mid B$$

$$A \to aa\,B$$

$$B \to b\,B \mid a$$

Construct NFA M such that every state is a grammar variable:

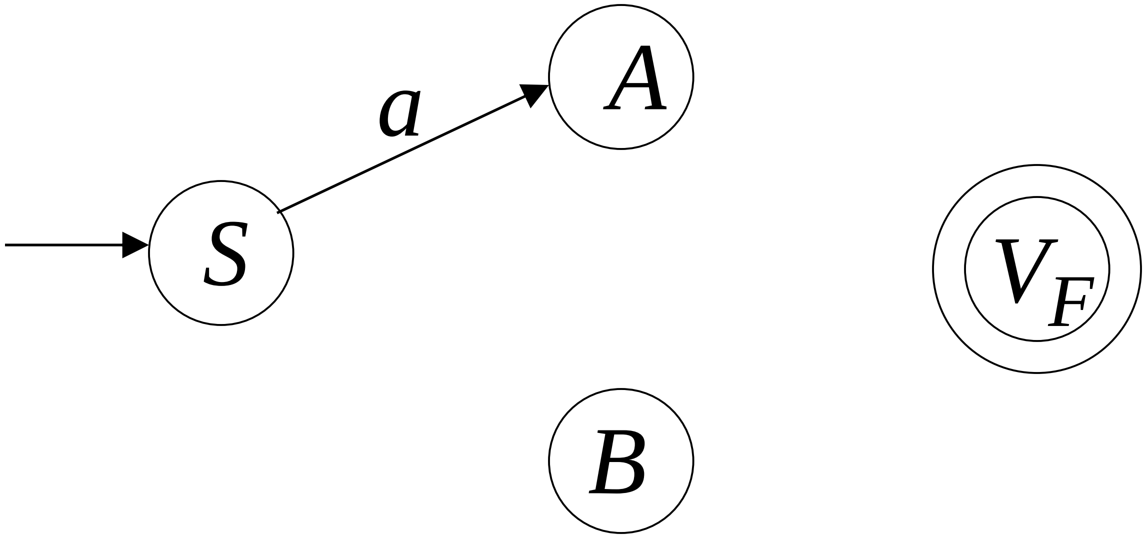


$$S \to aA \mid B$$

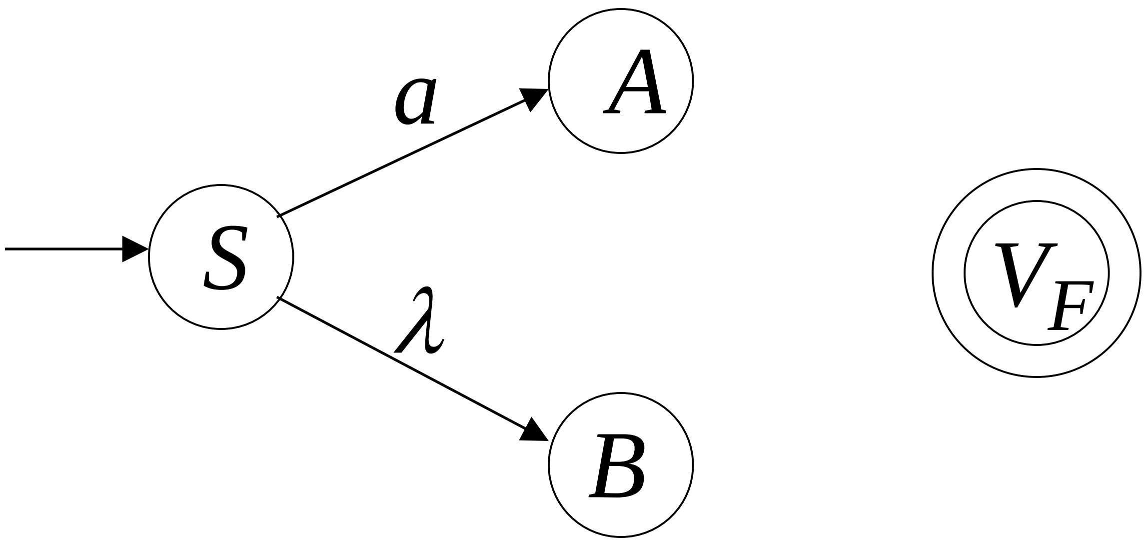
$$A \to aa\,B$$

$$B \to b\,B \mid a$$

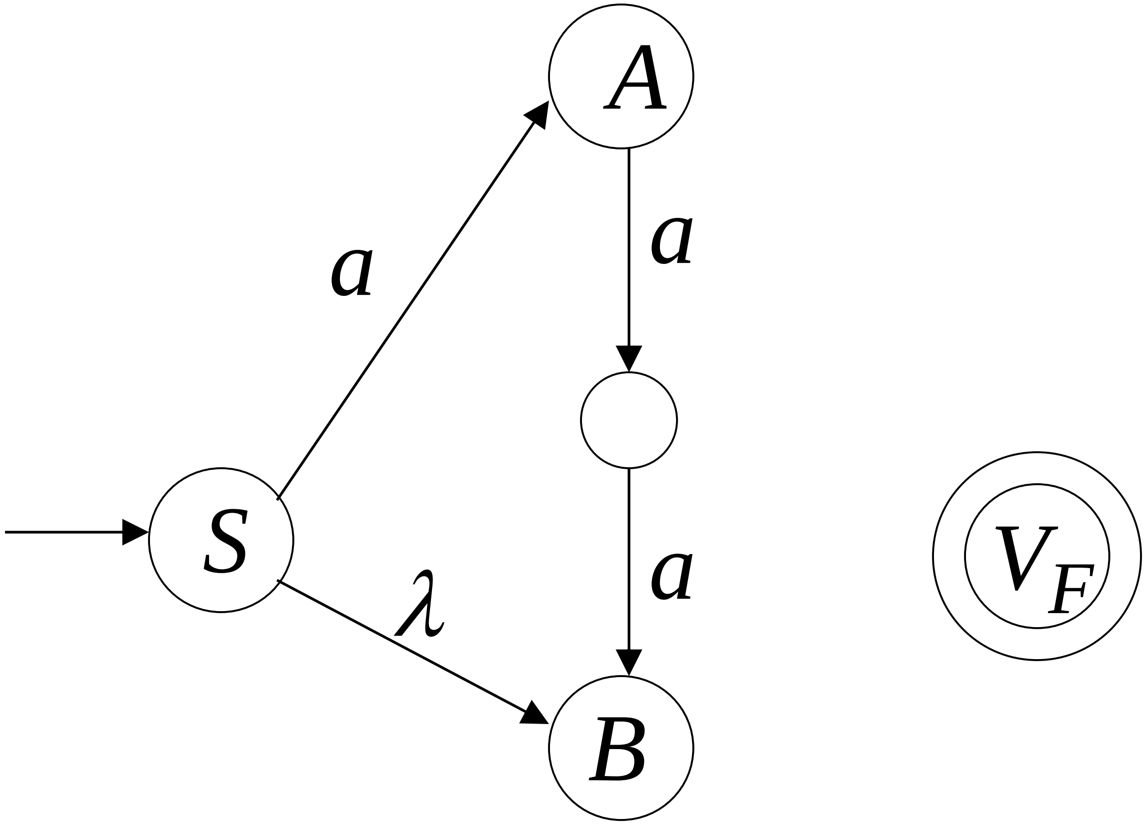
Add edges for each production:



$$S \rightarrow aA$$

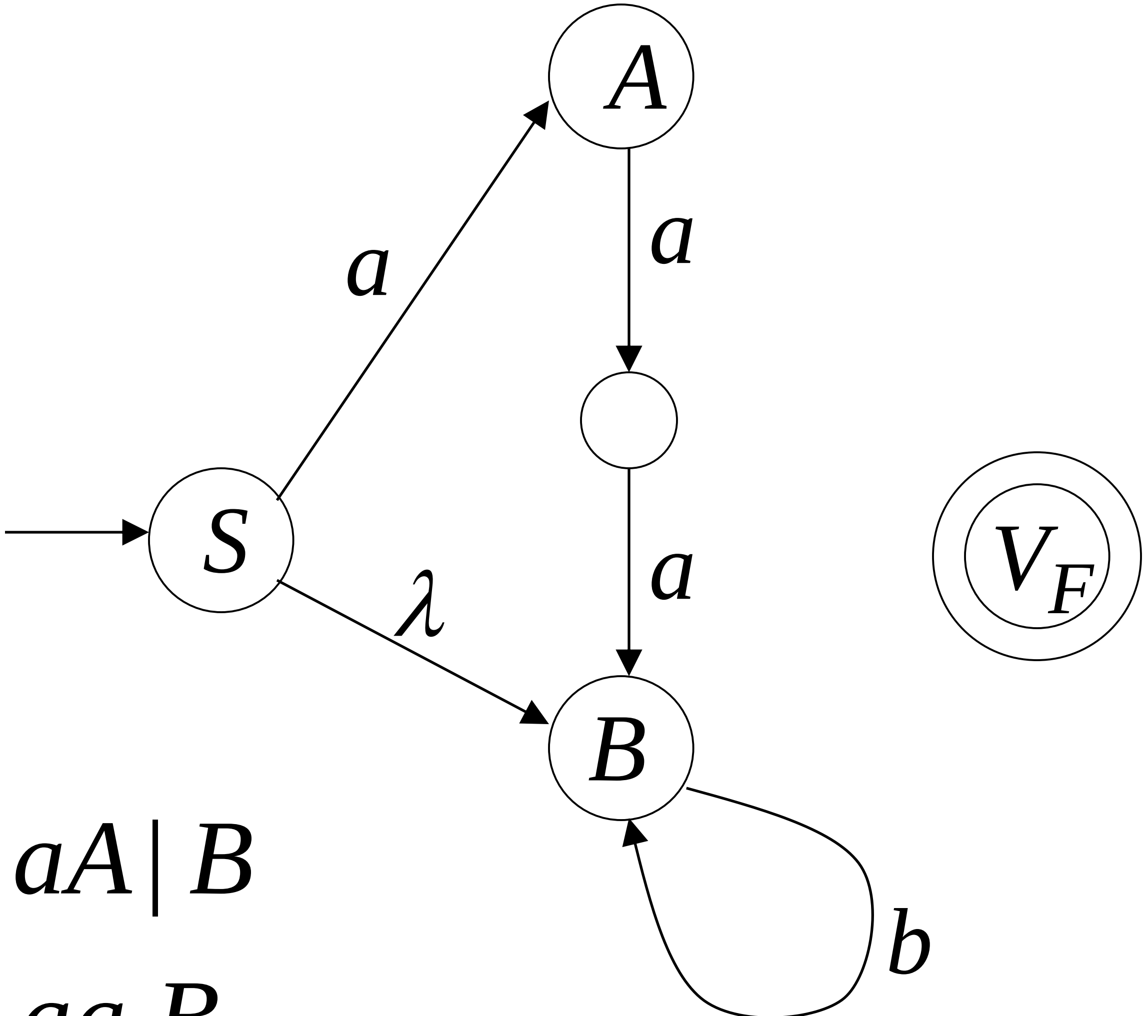


$$S \rightarrow aA \mid B$$



$$S \to aA \mid B$$

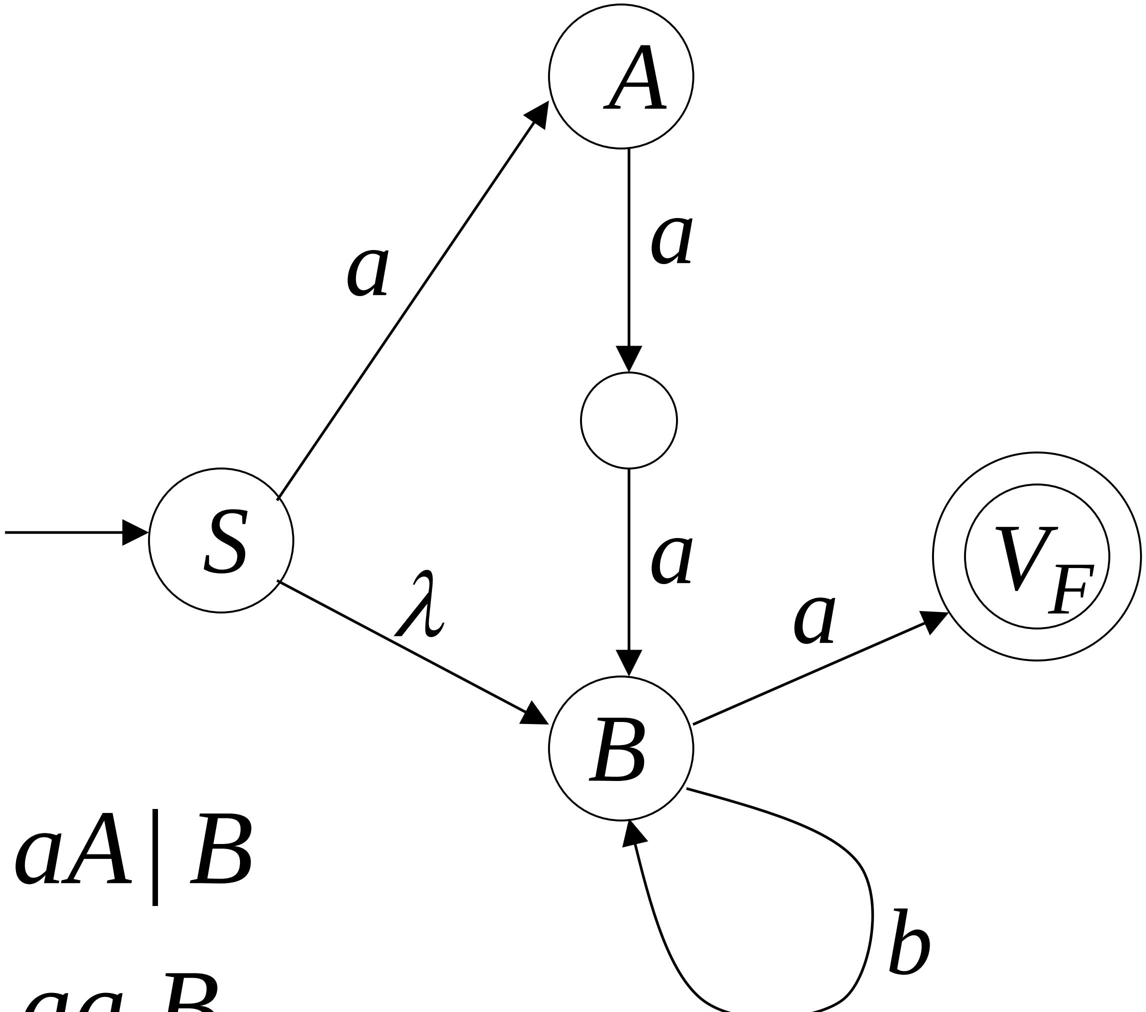
$$A \to aa\,B$$



$$S \rightarrow aA \mid B$$

$$A \rightarrow aa\ B$$

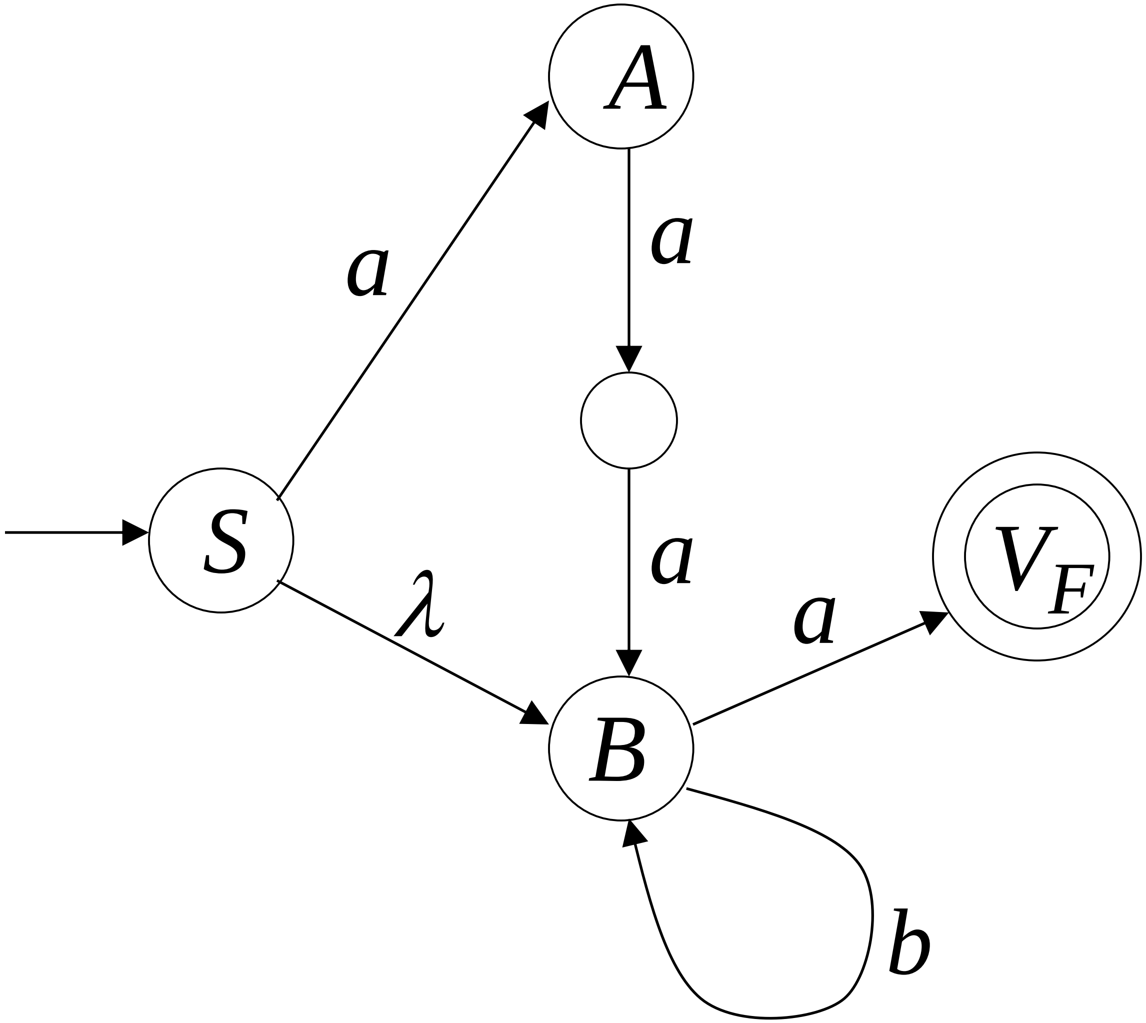
$$B \rightarrow bB$$



$$S \to aA \mid B$$

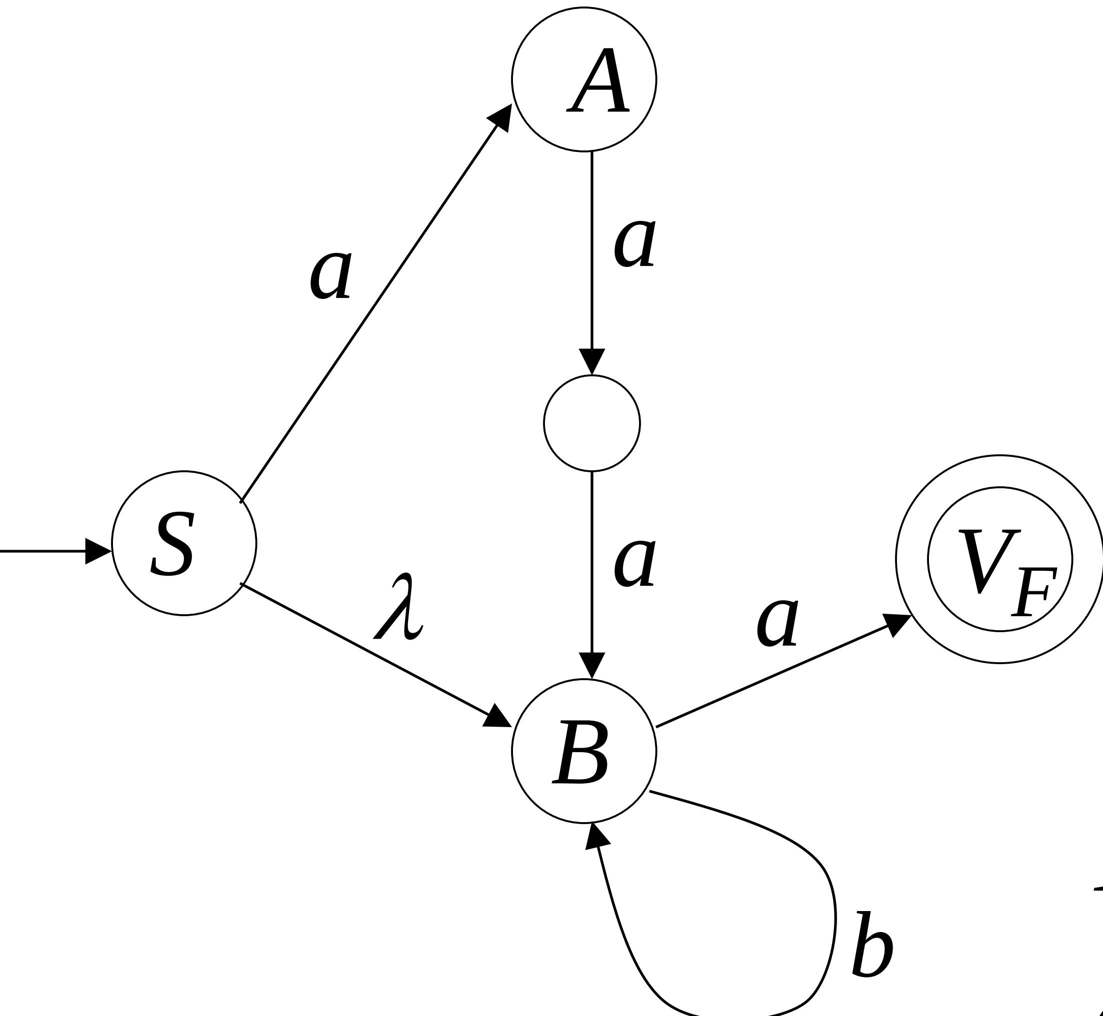
$$A \to aa\ B$$

$$B \to bB \mid a$$



$$S \Rightarrow aA \Rightarrow aaaB \Rightarrow aaabB \Rightarrow aaaba$$

NFA M



Grammar G

$$S \to aA \mid B$$

$$A \to aa\,B$$

$$B \to bB \mid a$$

$$L(M) = L(G) =$$

$$aaab^*a + b^*a$$

In General

A right-linear grammar G

has variables: $V_0, V_1, V_2, \mathrm{K}$

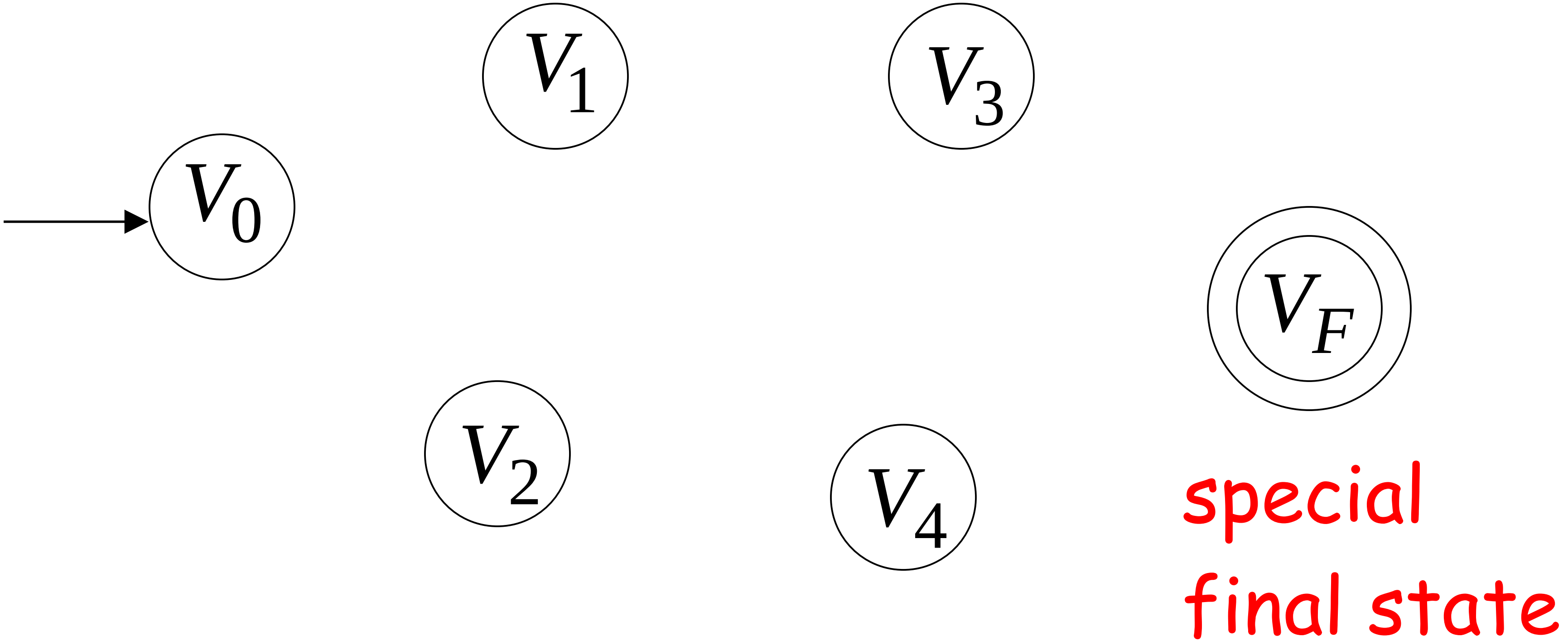
and productions: $V_i \to a_1 a_2 \Lambda\ a_m V_j$

or

$$V_i \to a_1 a_2 \Lambda\ a_m$$

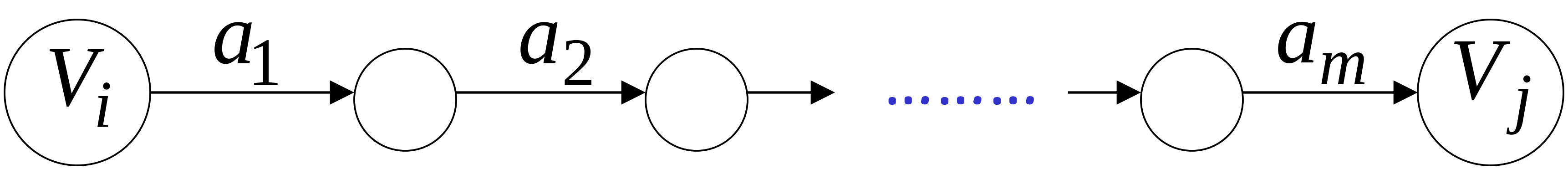
We construct the NFA M such that:

each variable V_i corresponds to a node:



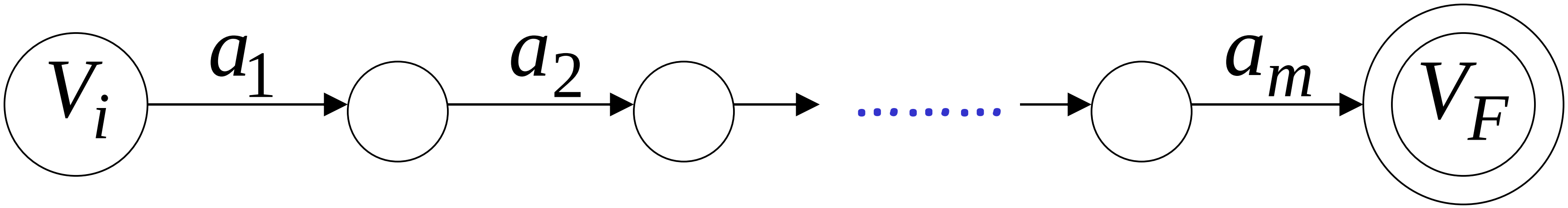
For each production: $V_i \to a_1 a_2 \Lambda \ a_m V_j$

we add transitions and intermediate nodes

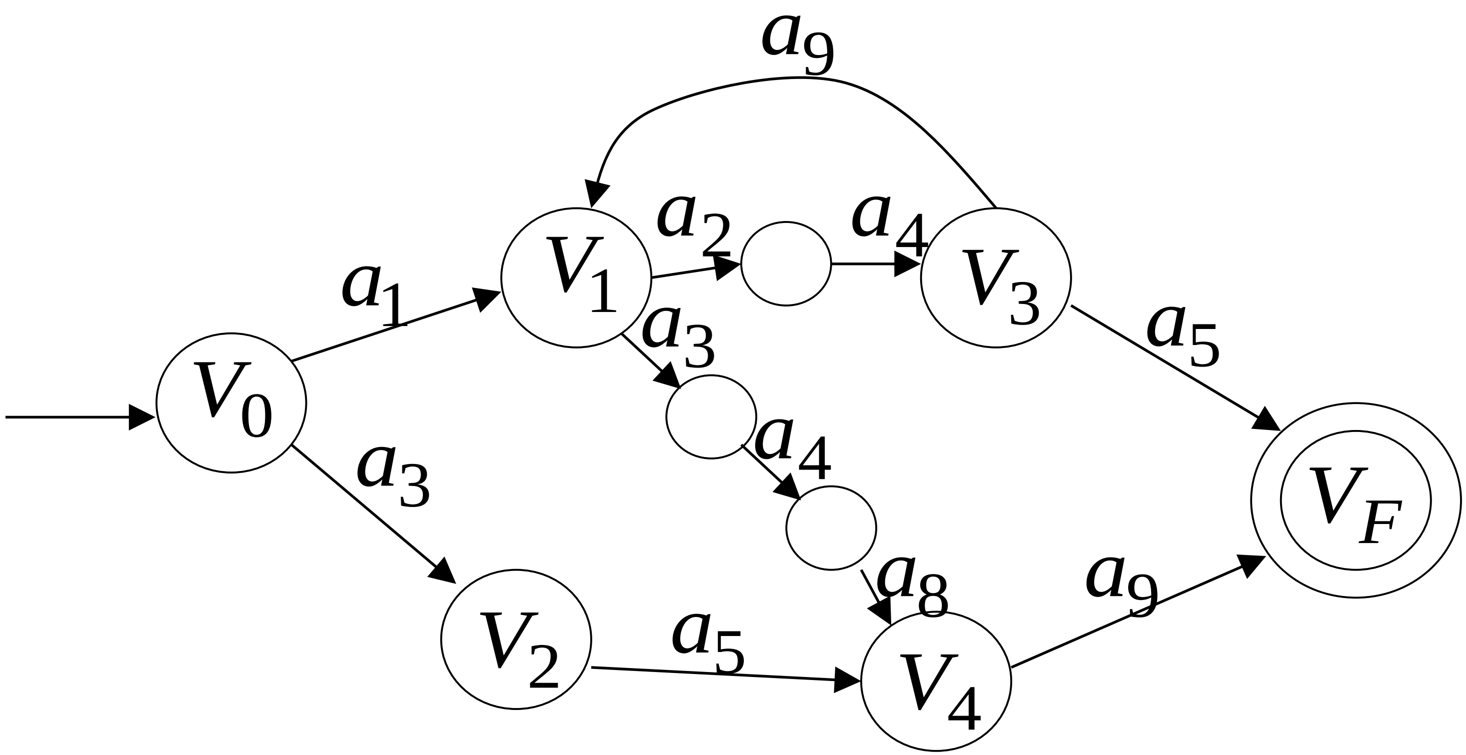


For each production: $V_i \rightarrow a_1 a_2 \Lambda\ a_m$

we add transitions and intermediate nodes



Resulting NFA M looks like this:



It holds that: $L(G) = L(M)$

The case of Left-Linear Grammars

Let G be a left-linear grammar

We will prove: $L(G)$ is regular

Proof idea:

We will construct a right-linear grammar G' with $L(G) = L(G')^R$

Since G is left-linear grammar
the productions look like:

$$A \rightarrow Ba_1 a_2 \Lambda\ a_k$$

$$A \rightarrow a_1 a_2 \Lambda\ a_k$$

Construct right-linear grammar G'

Left linear G

$$A \to Ba_1a_2\Lambda\ a_k$$

$$A \to Bv$$

Right linear G'

$$A \to a_k\Lambda\ a_2a_1B$$

$$A \to v^R B$$

Construct right-linear grammar G'

Left linear G

$$A \to a_1 a_2 \Lambda\ a_k$$

$$A \to v$$

Right linear G'

$$A \to a_k \Lambda\ a_2 a_1$$

$$A \to v^R$$

It is easy to see that: $L(G) = L(G')^R$

Since G' is right-linear, we have:

$L(G')$ → $L(G')^R$ → $L(G)$

Regular Language → Regular Language → Regular Language

Proof - Part 2

$$\left\{ \begin{array}{l} \text{Languages} \\ \text{Generated by} \\ \text{Regular Grammars} \end{array} \right\} \supseteq \left\{ \begin{array}{l} \text{Regular} \\ \text{Languages} \end{array} \right\}$$

Any regular language L is generated by some regular grammar G

Any regular language L is generated by some regular grammar G

Proof idea:

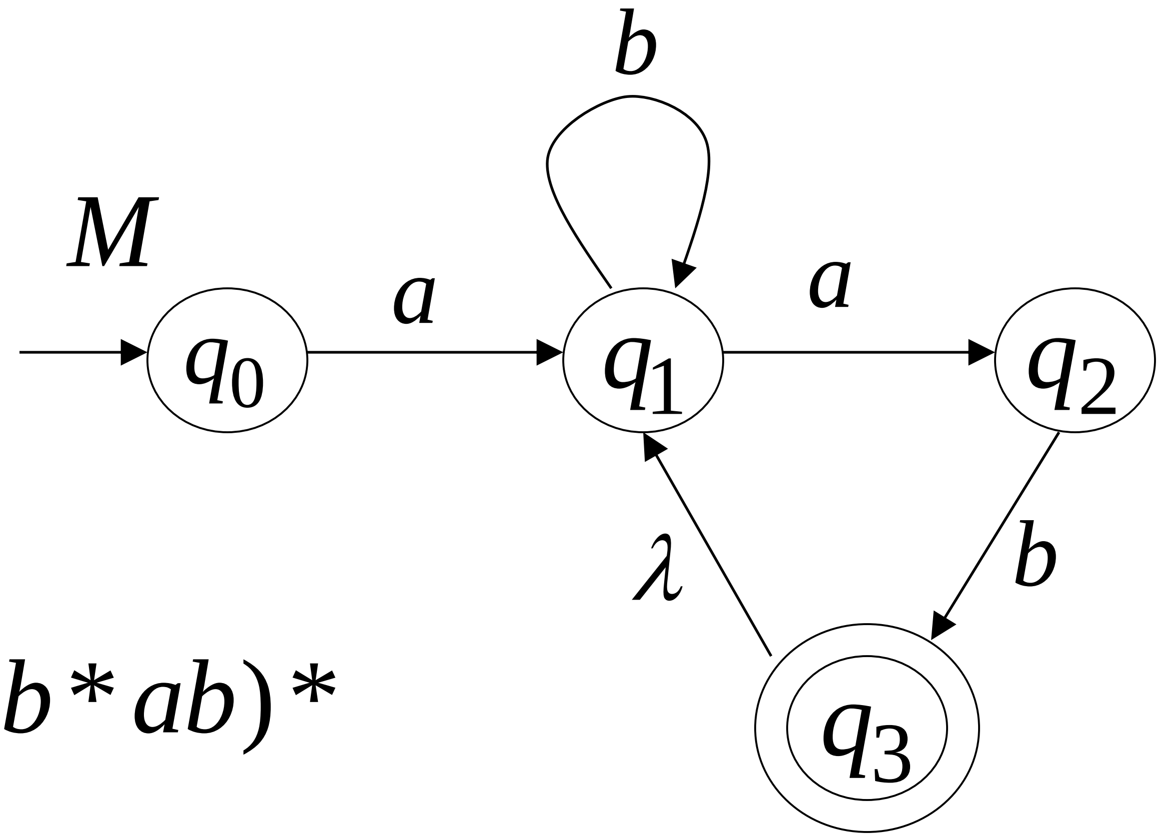
Let M be the NFA with $L = L(M)$.

Construct from M a regular grammar G such that $L(M) = L(G)$

Since L is regular

there is an NFA M such that $L = L(M)$

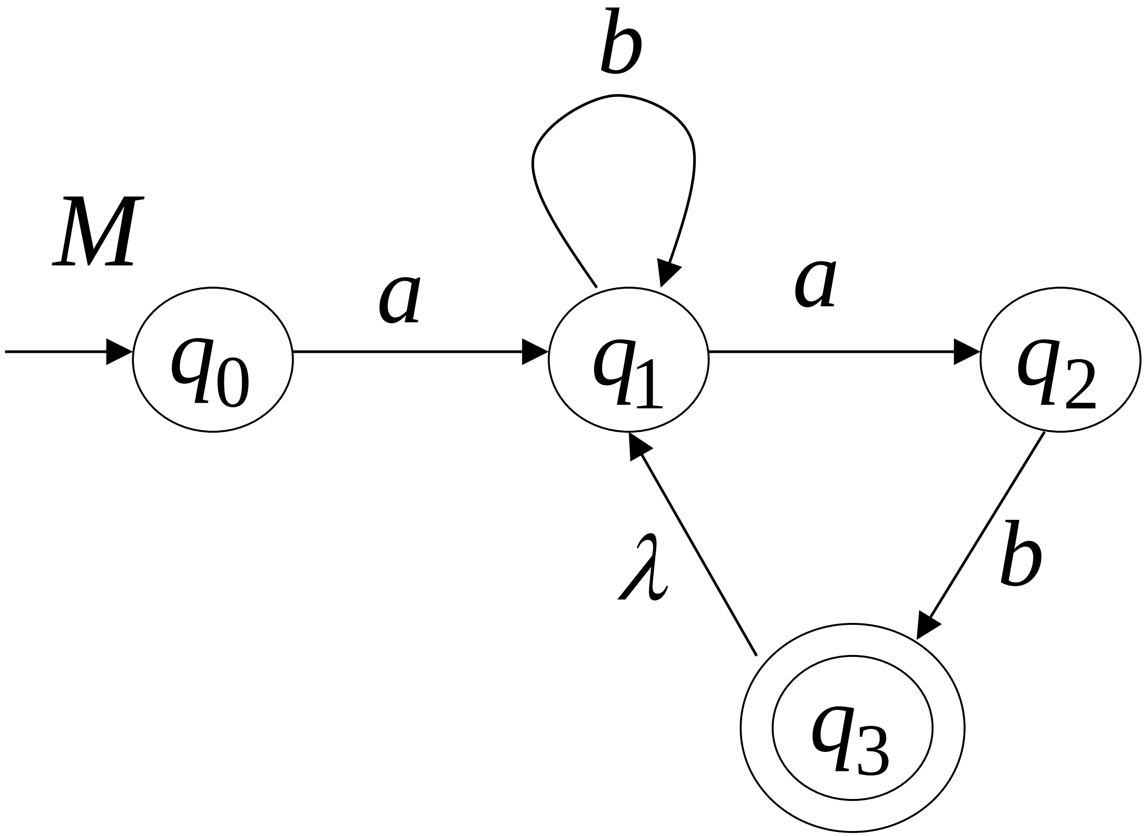
Example:



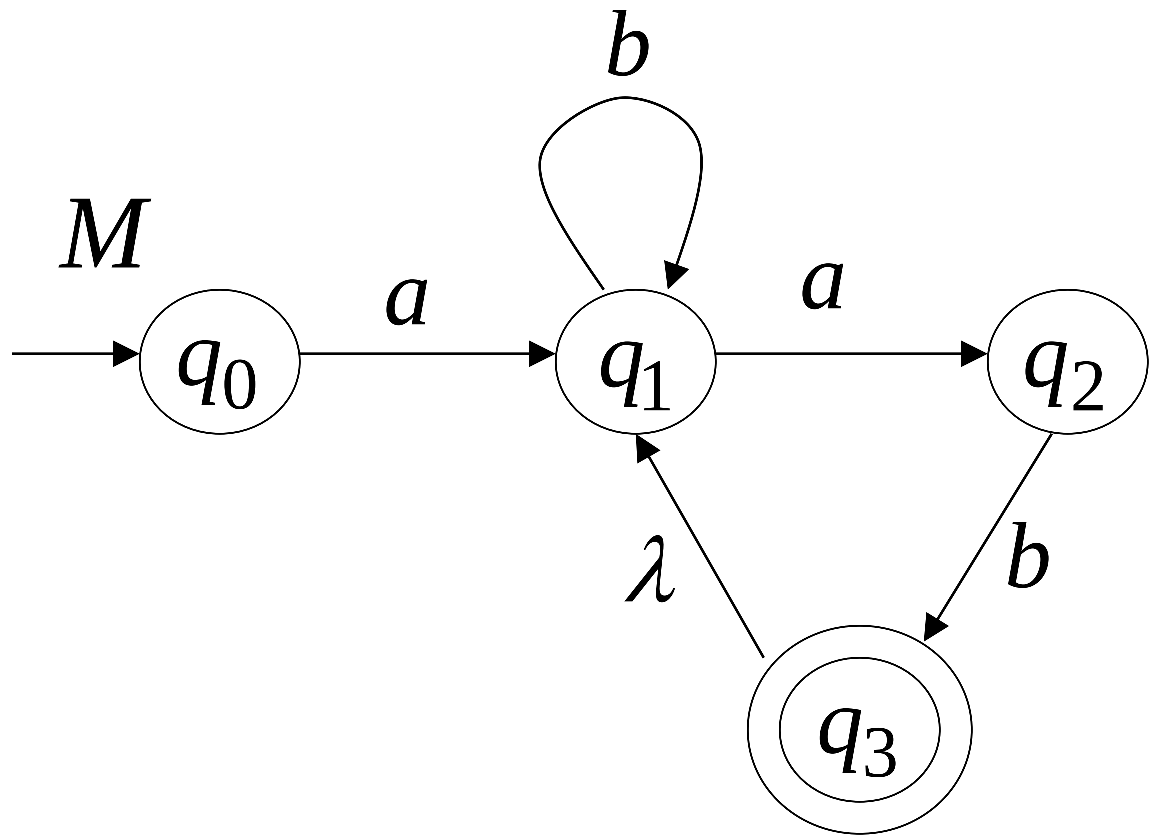
$L = ab^*ab(b^*ab)^*$

$L = L(M)$

Convert M to a right-linear grammar



$$q_0 \rightarrow aq_1$$



$$q_0 \rightarrow aq_1$$

$$q_1 \rightarrow bq_1$$

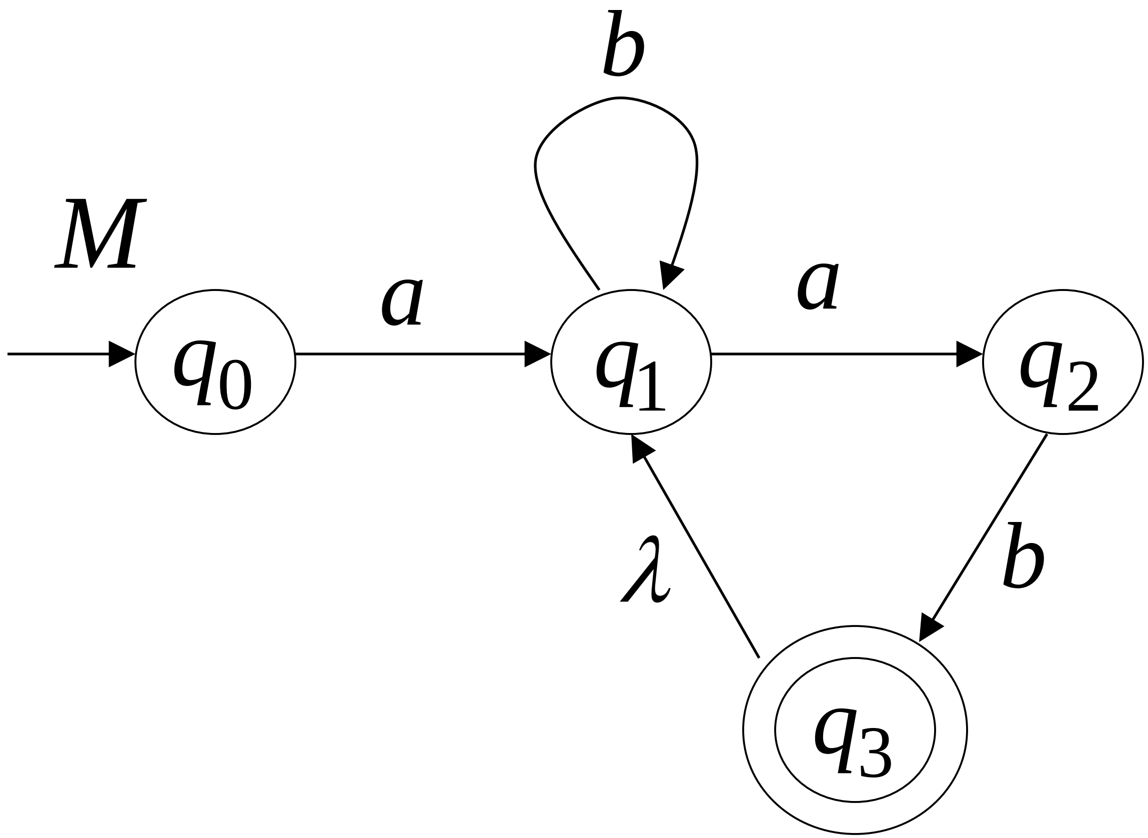
$$q_1 \rightarrow aq_2$$

$$q_0 \to a q_1$$

$$q_1 \to b q_1$$

$$q_1 \to a q_2$$

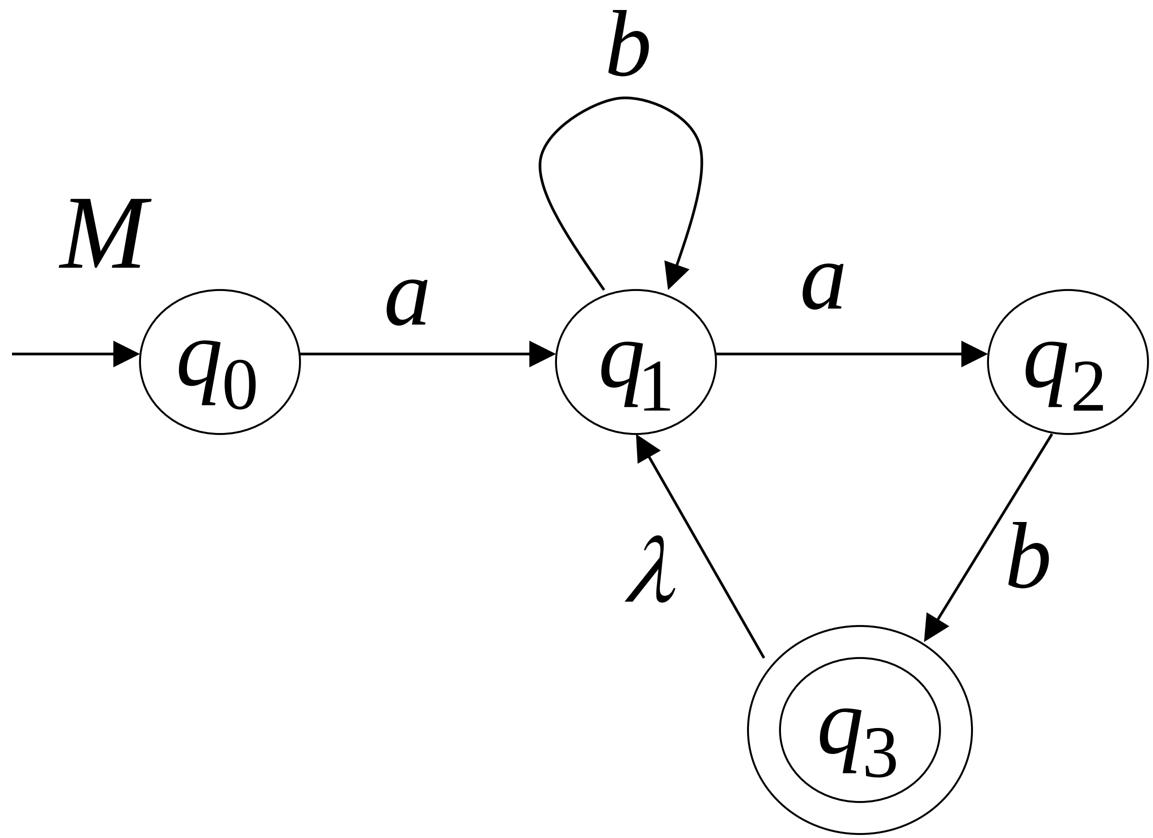
$$q_2 \to b q_3$$



$$L(G) = L(M) = L$$

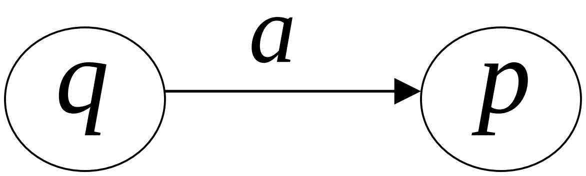
G

$$q_0 \rightarrow aq_1$$
$$q_1 \rightarrow bq_1$$
$$q_1 \rightarrow aq_2$$
$$q_2 \rightarrow bq_3$$
$$q_3 \rightarrow q_1$$
$$q_3 \rightarrow \lambda$$



In General

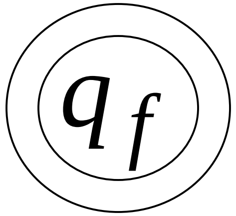
For any transition:



Add production: $q \to ap$

variable terminal variable

For any final state:



Add production: $q_f \to \lambda$

Since G is right-linear grammar

G is also a regular grammar

with $L(G) = L(M) = L$