MAP 2320 – MÉTODOS NUMÉRICOS EM EQUAÇÕES DIFERENCIAIS II

2º Semestre - 2019

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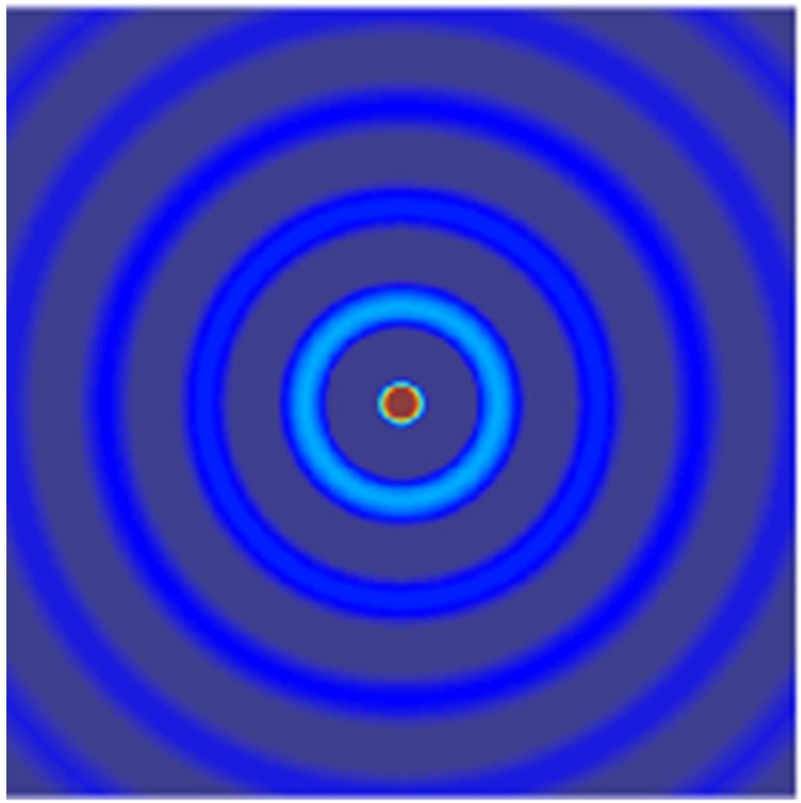
$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u$$



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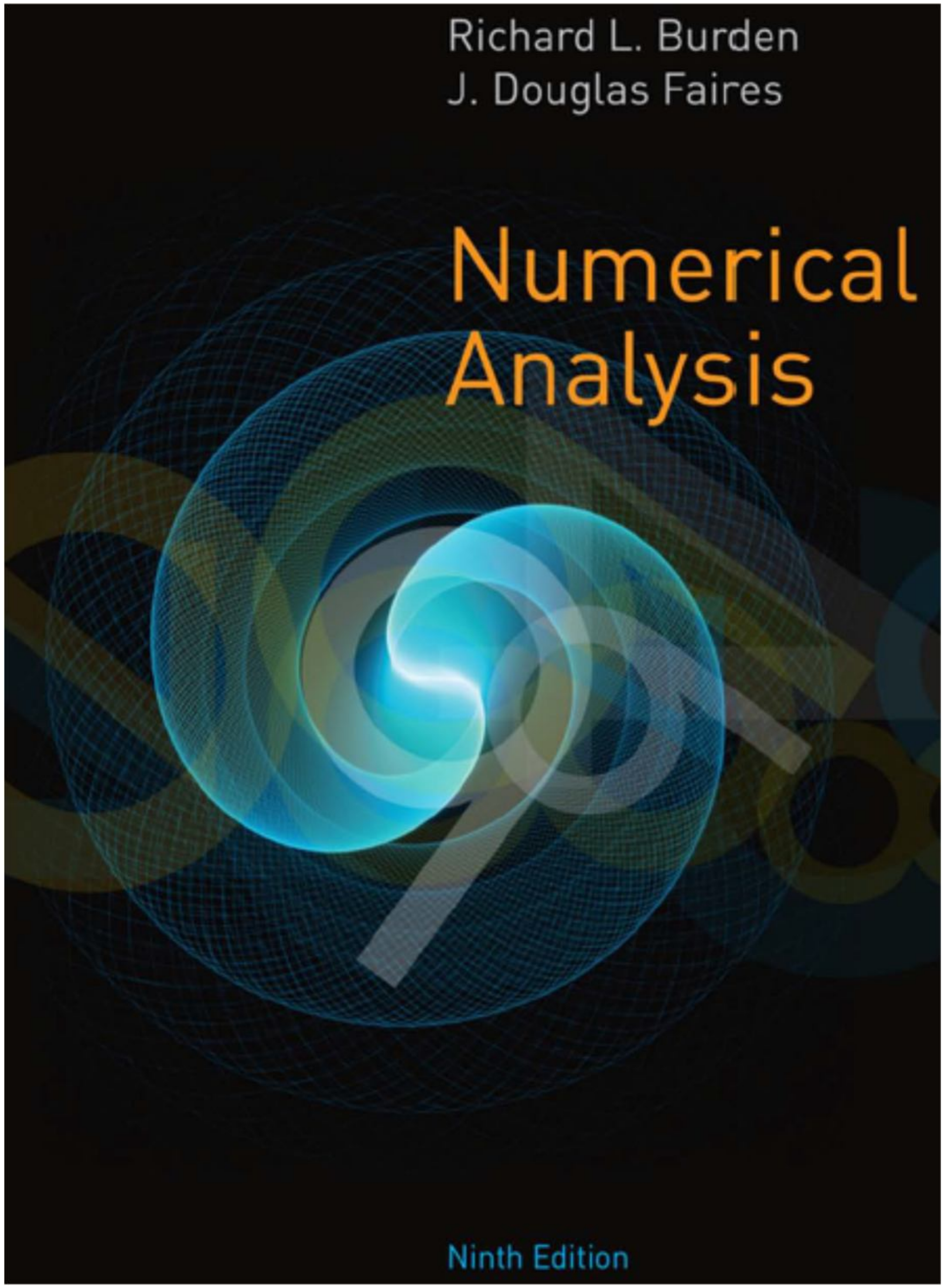
Richard L. Burden
J. Douglas Faires
Numerical Analysis
Ninth Edition

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Numerical Analysis

NINTH EDITION

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Youngstown State University

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12.3 Hyperbolic Partial Differential Equations

In this section, we consider the numerical solution to the **wave equation**, an example of a *hyperbolic* partial differential equation. The wave equation is given by the differential equation

$$\frac{\partial^2 u}{\partial t^2}(x,t) - \alpha^2 \frac{\partial^2 u}{\partial x^2}(x,t) = 0, \quad 0 < x < l, \quad t > 0, \tag{12.16}$$

subject to the conditions

$$u(0,t) = u(l,t) = 0, \quad \text{for} \quad t > 0,$$

$$u(x,0) = f(x), \quad \text{and} \quad \frac{\partial u}{\partial t}(x,0) = g(x), \quad \text{for} \quad 0 \le x \le l,$$

where α is a constant dependent on the physical conditions of the problem.

Select an integer $m > 0$ to define the x-axis grid points using $h = l/m$. In addition, select a time-step size $k > 0$. The mesh points (x_i, t_j) are defined by

$$x_i = ih \quad \text{and} \quad t_j = jk,$$

for each $i = 0, 1, \ldots, m$ and $j = 0, 1, \ldots$.

At any interior mesh point (x_i, t_j), the wave equation becomes

$$\frac{\partial^2 u}{\partial t^2}(x_i, t_j) - \alpha^2 \frac{\partial^2 u}{\partial x^2}(x_i, t_j) = 0. \tag{12.17}$$

$$h = \Delta x, \qquad k = \Delta t$$

$$D^2u(\bar{x}) \;=\; \frac{1}{h^2}[u(\bar{x}-h) - 2u(\bar{x}) + u(\bar{x}+h)]$$

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The difference method is obtained using the centered-difference quotient for the second partial derivatives given by

$$\frac{\partial^2 u}{\partial t^2}(x_i, t_j) = \frac{u(x_i, t_{j+1}) - 2u(x_i, t_j) + u(x_i, t_{j-1})}{k^2} - \frac{k^2}{12}\frac{\partial^4 u}{\partial t^4}(x_i, \mu_j),$$

where $\mu_j \in (t_{j-1}, t_{j+1})$, and

$$\frac{\partial^2 u}{\partial x^2}(x_i, t_j) = \frac{u(x_{i+1}, t_j) - 2u(x_i, t_j) + u(x_{i-1}, t_j)}{h^2} - \frac{h^2}{12}\frac{\partial^4 u}{\partial x^4}(\xi_i, t_j),$$

where $\xi_i \in (x_{i-1}, x_{i+1})$. Substituting these into Eq. (12.17) gives

$$\frac{u(x_i, t_{j+1}) - 2u(x_i, t_j) + u(x_i, t_{j-1})}{k^2} - \alpha^2 \frac{u(x_{i+1}, t_j) - 2u(x_i, t_j) + u(x_{i-1}, t_j)}{h^2}$$
$$= \frac{1}{12}\left[k^2 \frac{\partial^4 u}{\partial t^4}(x_i, \mu_j) - \alpha^2 h^2 \frac{\partial^4 u}{\partial x^4}(\xi_i, t_j)\right].$$

Neglecting the error term

$$\tau_{i,j} = \frac{1}{12}\left[k^2 \frac{\partial^4 u}{\partial t^4}(x_i, \mu_j) - \alpha^2 h^2 \frac{\partial^4 u}{\partial x^4}(\xi_i, t_j)\right], \qquad (12.18)$$

leads to the difference equation

$$\frac{w_{i,j+1} - 2w_{i,j} + w_{i,j-1}}{k^2} - \alpha^2 \frac{w_{i+1,j} - 2w_{i,j} + w_{i-1,j}}{h^2} = 0.$$

Define $\lambda = \alpha k/h$. Then we can write the difference equation as

$$w_{i,j+1} - 2w_{i,j} + w_{i,j-1} - \lambda^2 w_{i+1,j} + 2\lambda^2 w_{i,j} - \lambda^2 w_{i-1,j} = 0$$

and solve for $w_{i,j+1}$, the most advanced time-step approximation, to obtain

$$w_{i,j+1} = 2(1-\lambda^2)w_{i,j} + \lambda^2(w_{i+1,j} + w_{i-1,j}) - w_{i,j-1}. \tag{12.19}$$

This equation holds for each $i = 1, 2, \ldots, m-1$ and $j = 1, 2, \ldots$. The boundary conditions give

$$w_{0,j} = w_{m,j} = 0, \quad \text{for each } j = 1, 2, 3, \ldots, \tag{12.20}$$

and the initial condition implies that

$$w_{i,0} = f(x_i), \quad \text{for each } i = 1, 2, \ldots, m-1. \tag{12.21}$$

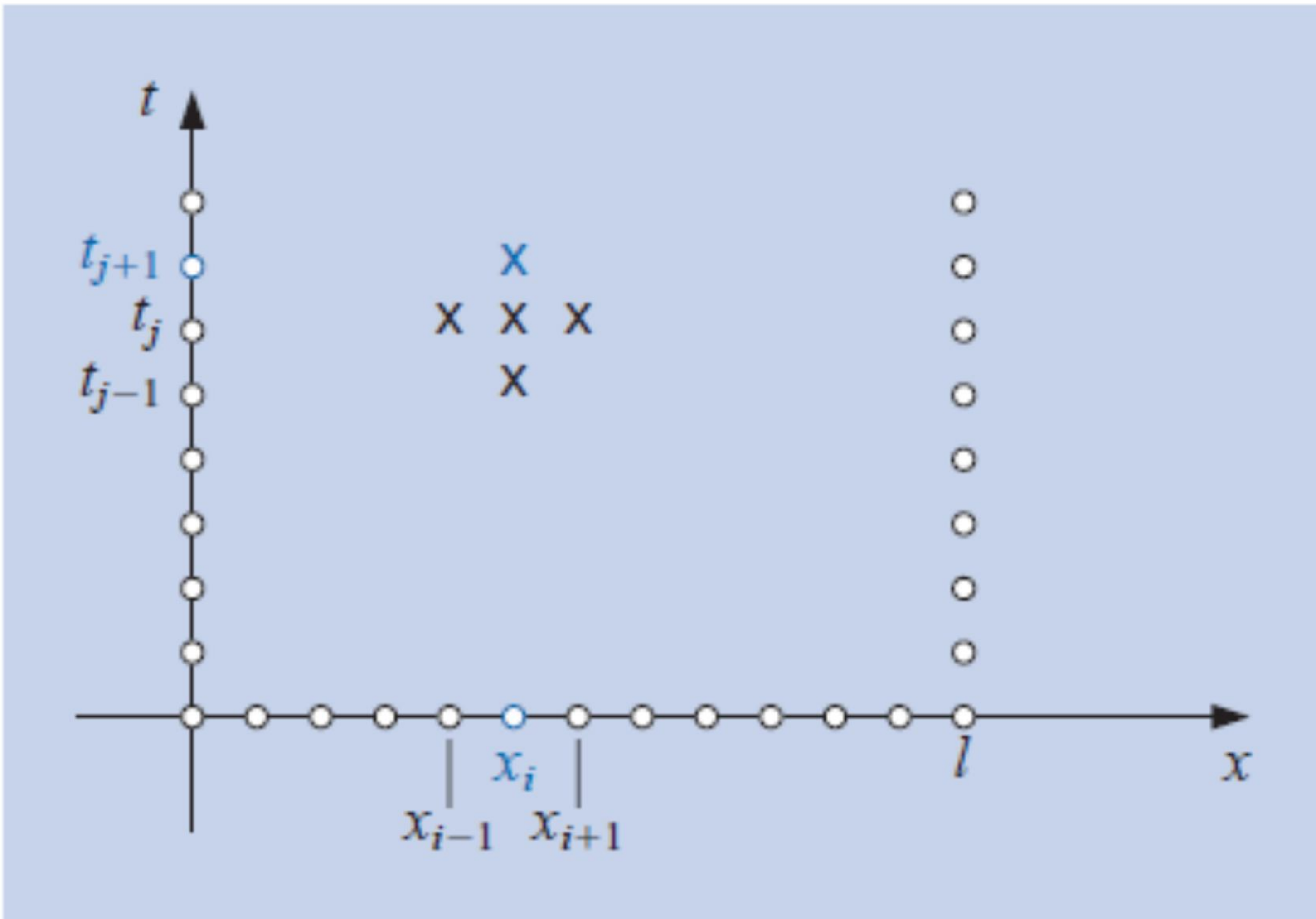
Writing this set of equations in matrix form gives

$$\begin{bmatrix} w_{1,j+1} \\ w_{2,j+1} \\ \vdots \\ w_{m-1,j+1} \end{bmatrix} = \begin{bmatrix} 2(1-\lambda^2) & \lambda^2 & 0 & \cdots & 0 \\ \lambda^2 & 2(1-\lambda^2) & \lambda^2 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \lambda^2 \\ 0 & \cdots & 0 & \lambda^2 & 2(1-\lambda^2) \end{bmatrix} \begin{bmatrix} w_{1,j} \\ w_{2,j} \\ \vdots \\ w_{m-1,j} \end{bmatrix} - \begin{bmatrix} w_{1,j-1} \\ w_{2,j-1} \\ \vdots \\ w_{m-1,j-1} \end{bmatrix}. \tag{12.22}$$

Equations (12.18) and (12.19) imply that the $(j+1)$st time step requires values from the jth and $(j-1)$st time steps. (See Figure 12.12.) This produces a minor starting problem because values for $j=0$ are given by Eq. (12.20), but values for $j=1$, which are needed in Eq. (12.18) to compute $w_{i,2}$, must be obtained from the initial-velocity condition

$$\frac{\partial u}{\partial t}(x,0) = g(x), \quad 0 \le x \le l.$$

Figure 12.12



One approach is to replace $\partial u/\partial t$ by a forward-difference approximation,

$$\frac{\partial u}{\partial t}(x_i, 0) = \frac{u(x_i, t_1) - u(x_i, 0)}{k} - \frac{k}{2}\frac{\partial^2 u}{\partial t^2}(x_i, \tilde{\mu}_i), \tag{12.23}$$

for some $\tilde{\mu}_i$ in $(0, t_1)$. Solving for $u(x_i, t_1)$ in the equation gives

$$\begin{aligned} u(x_i, t_1) &= u(x_i, 0) + k\frac{\partial u}{\partial t}(x_i, 0) + \frac{k^2}{2}\frac{\partial^2 u}{\partial t^2}(x_i, \tilde{\mu}_i) \\ &= u(x_i, 0) + kg(x_i) + \frac{k^2}{2}\frac{\partial^2 u}{\partial t^2}(x_i, \tilde{\mu}_i). \end{aligned}$$

Deleting the truncation term gives the approximation,

$$w_{i,1} = w_{i,0} + kg(x_i), \quad \text{for each } i = 1, \ldots, m-1. \tag{12.24}$$

However, this approximation has truncation error of only $O(k)$ whereas the truncation error in Eq. (12.19) is $O(k^2)$.

To obtain a better approximation to $u(x_i, 0)$, expand $u(x_i, t_1)$ in a second Maclaurin polynomial in t. Then

$$u(x_i, t_1) = u(x_i, 0) + k\frac{\partial u}{\partial t}(x_i, 0) + \frac{k^2}{2}\frac{\partial^2 u}{\partial t^2}(x_i, 0) + \frac{k^3}{6}\frac{\partial^3 u}{\partial t^3}(x_i, \hat{\mu}_i),$$

for some $\hat{\mu}_i$ in $(0, t_1)$. If f'' exists, then

$$\frac{\partial^2 u}{\partial t^2}(x_i, 0) = \alpha^2\frac{\partial^2 u}{\partial x^2}(x_i, 0) = \alpha^2\frac{d^2 f}{dx^2}(x_i) = \alpha^2 f''(x_i)$$

and

$$u(x_i, t_1) = u(x_i, 0) + kg(x_i) + \frac{\alpha^2 k^2}{2}f''(x_i) + \frac{k^3}{6}\frac{\partial^3 u}{\partial t^3}(x_i, \hat{\mu}_i).$$

This produces an approximation with error $O(k^3)$:

$$w_{i1} = w_{i0} + kg(x_i) + \frac{\alpha^2 k^2}{2}f''(x_i).$$

If $f \in C^4[0,1]$ but $f''(x_i)$ is not readily available, we can use the difference equation in Eq. (4.9) to write

$$f''(x_i) = \frac{f(x_{i+1}) - 2f(x_i) + f(x_{i-1})}{h^2} - \frac{h^2}{12} f^{(4)}(\tilde{\xi}_i),$$

for some $\tilde{\xi}_i$ in (x_{i-1}, x_{i+1}). This implies that

$$u(x_i, t_1) = u(x_i, 0) + kg(x_i) + \frac{k^2\alpha^2}{2h^2}[f(x_{i+1}) - 2f(x_i) + f(x_{i-1})] + O(k^3 + h^2k^2).$$

Because $\lambda = k\alpha/h$, we can write this as

$$\begin{aligned} u(x_i, t_1) &= u(x_i, 0) + kg(x_i) + \frac{\lambda^2}{2}[f(x_{i+1}) - 2f(x_i) + f(x_{i-1})] + O(k^3 + h^2k^2) \\ &= (1 - \lambda^2) f(x_i) + \frac{\lambda^2}{2} f(x_{i+1}) + \frac{\lambda^2}{2} f(x_{i-1}) + kg(x_i) + O(k^3 + h^2k^2). \end{aligned}$$

Thus, the difference equation,

$$w_{i,1} = (1 - \lambda^2) f(x_i) + \frac{\lambda^2}{2} f(x_{i+1}) + \frac{\lambda^2}{2} f(x_{i-1}) + kg(x_i), \tag{12.25}$$

can be used to find $w_{i,1}$, for each $i = 1, 2, \ldots, m-1$. To determine subsequent approximates we use the system in (12.22).

Algorithm 12.4 uses Eq. (12.25) to approximate $w_{i,1}$, although Eq. (12.24) could also be used. It is assumed that there is an upper bound for the value of t to be used in the stopping technique, and that $k = T/N$, where N is also given.

ALGORITHM
12.4

Wave Equation Finite-Difference

To approximate the solution to the wave equation

$$\frac{\partial^2 u}{\partial t^2}(x,t) - \alpha^2 \frac{\partial^2 u}{\partial x^2}(x,t) = 0, \quad 0 < x < l, \quad 0 < t < T,$$

subject to the boundary conditions

$$u(0,t) = u(l,t) = 0, \quad 0 < t < T,$$

and the initial conditions

$$u(x,0) = f(x), \quad \text{and} \quad \frac{\partial u}{\partial t}(x,0) = g(x), \quad \text{for} \quad 0 \le x \le l,$$

ALGORITHM 12.4

INPUT endpoint l; maximum time T; constant α; integers $m \geq 2$, $N \geq 2$.

OUTPUT approximations $w_{i,j}$ to $u(x_i, t_j)$ for each $i = 0, \ldots, m$ and $j = 0, \ldots, N$.

Step 1 Set $h = l/m$;
$k = T/N$;
$\lambda = k\alpha/h$.

Step 2 For $j = 1, \ldots, N$ set $w_{0,j} = 0$;
$w_{m,j} = 0$;

Step 3 Set $w_{0,0} = f(0)$;
$w_{m,0} = f(l)$.

Step 4 For $i = 1, \ldots, m-1$ (*Initialize for $t = 0$ and $t = k$.*)
set $w_{i,0} = f(ih)$;

$$w_{i,1} = (1 - \lambda^2) f(ih) + \frac{\lambda^2}{2}[f((i+1)h) + f((i-1)h)] + kg(ih).$$

Step 5 For $j = 1, \ldots, N-1$ (*Perform matrix multiplication.*)
for $i = 1, \ldots, m-1$
set $w_{i,j+1} = 2(1 - \lambda^2)w_{i,j} + \lambda^2(w_{i+1,j} + w_{i-1,j}) - w_{i,j-1}$.

Step 6 For $j = 0, \ldots, N$
set $t = jk$;
for $i = 0, \ldots, m$
set $x = ih$;
OUTPUT $(x, t, w_{i,j})$.

Step 7 STOP. (*The procedure is complete.*) ■

Example 1

Approximate the solution to the hyperbolic problem

$$\frac{\partial^2 u}{\partial t^2}(x,t) - 4\frac{\partial^2 u}{\partial x^2}(x,t) = 0, \quad 0 < x < 1, \quad 0 < t,$$

with boundary conditions

$$u(0,t) = u(1,t) = 0, \quad \text{for} \quad 0 < t,$$

and initial conditions

$$u(x,0) = \sin(\pi x), \quad 0 \le x \le 1, \quad \text{and} \quad \frac{\partial u}{\partial t}(x,0) = 0, \quad 0 \le x \le 1,$$

using $h = 0.1$ and $k = 0.05$. Compare the results with the exact solution

$$u(x,t) = \sin \pi x \cos 2\pi t.$$

Example 1

Solution Choosing $h = 0.1$ and $k = 0.05$ gives $\lambda = 1$, $m = 10$, and $N = 20$. We will choose a maximum time $T = 1$ and apply the Finite-Difference Algorithm 12.4. This produces the approximations $w_{i,N}$ to $u(0.1i, 1)$ for $i = 0, 1, \ldots, 10$. These results are shown in Table 12.6 and are correct to the places given. ■

Console

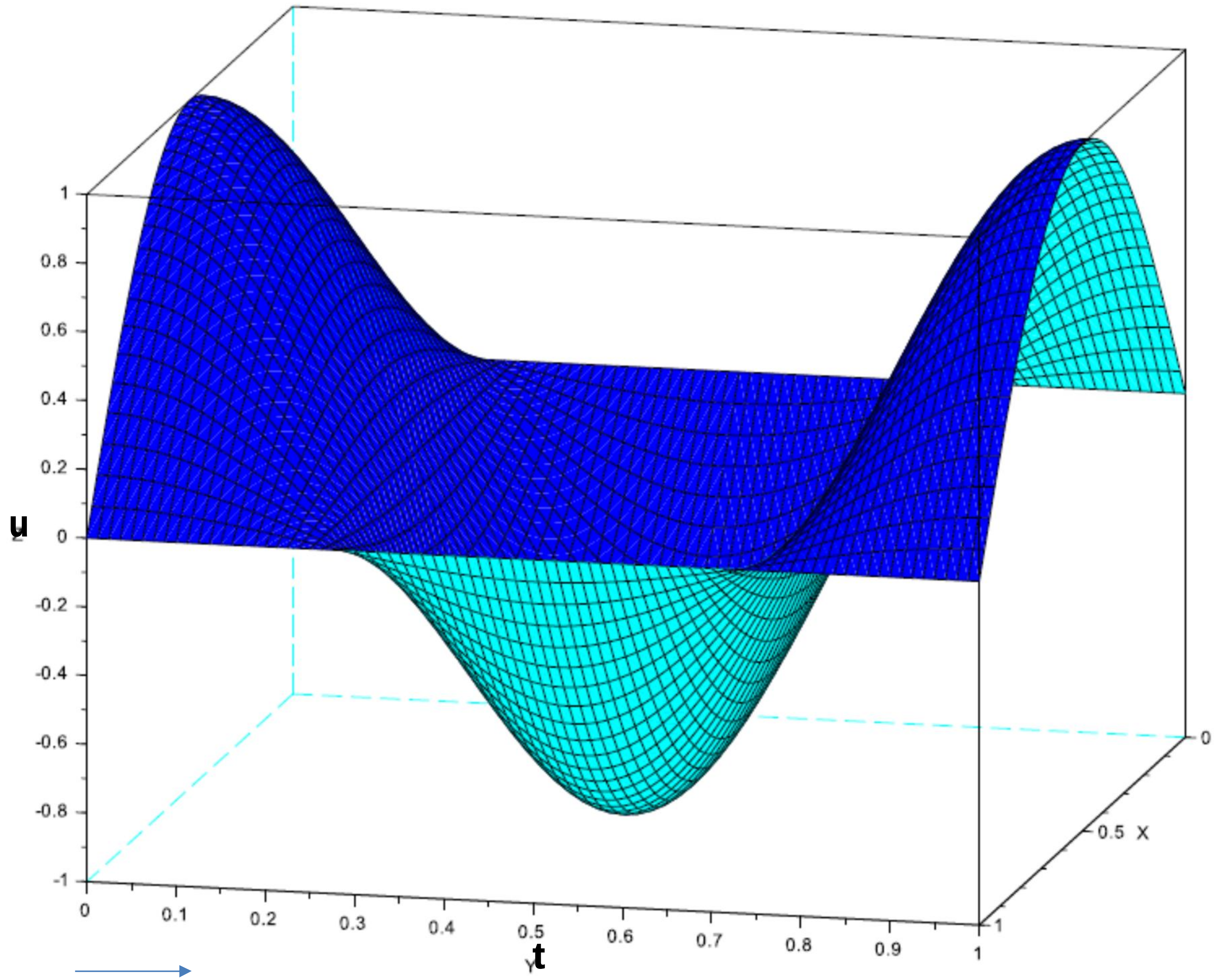
```
-->u(:,21)
 ans  =

      0.
    0.3090170
    0.5877853
    0.8090170
    0.9510565
      1.
    0.9510565
    0.8090170
    0.5877853
    0.3090170
      0.
```

Minha implementação em Scilab

Table 12.6

x_i	$w_{i,20}$
0.0	0.0000000000
0.1	0.3090169944
0.2	0.5877852523
0.3	0.8090169944
0.4	0.9510565163
0.5	1.0000000000
0.6	0.9510565163
0.7	0.8090169944
0.8	0.5877852523
0.9	0.3090169944
1.0	0.0000000000



Minha implementação em Scilab, m = 40

The results of the example were very accurate, more so than the truncation error $O(k^2 + h^2)$ would lead us to believe. This is because the true solution to the equation is infinitely differentiable. When this is the case, Taylor series gives

$$\frac{u(x_{i+1},t_j) - 2u(x_i,t_j) + u(x_{i-1},t_j)}{h^2}$$
$$= \frac{\partial^2 u}{\partial x^2}(x_i,t_j) + 2\left[\frac{h^2}{4!}\frac{\partial^4 u}{\partial x^4}(x_i,t_j) + \frac{h^4}{6!}\frac{\partial^6 u}{\partial x^6}(x_i,t_j) + \cdots\right]$$

and

$$\frac{u(x_i,t_{j+1}) - 2u(x_i,t_j) + u(x_i,t_{j-1})}{k^2}$$
$$= \frac{\partial^2 u}{\partial t^2}(x_i,t_j) + 2\left[\frac{k^2}{4!}\frac{\partial^4 u}{\partial t^4}(x_i,t_j) + \frac{h^4}{6!}\frac{\partial^6 u}{\partial t^6}(x_i,t_j) + \cdots\right].$$

Since $u(x,t)$ satisfies the partial differential equation,

$$\frac{u(x_i,t_{j+1}) - 2u(x_i,t_j) + u(x_i,t_{j-1})}{k^2} - \alpha^2\frac{u(x_{i+1},t_j) - 2u(x_i,t_j) + u(x_{i-1},t_j)}{h^2}$$
$$= 2\Bigg[\frac{1}{4!}\left(k^2\frac{\partial^4 u}{\partial t^4}(x_i,t_j) - \alpha^2 h^2\frac{\partial^4 u}{\partial x^4}(x_i,t_j)\right)$$
$$+\frac{1}{6!}\left(k^4\frac{\partial^6 u}{\partial t^6}(x_i,t_j) - \alpha^2 h^4\frac{\partial^6 u}{\partial x^6}(x_i,t_j)\right) + \cdots\Bigg]. \tag{12.26}$$

However, differentiating the wave equation gives

$$k^2 \frac{\partial^4 u}{\partial t^4}(x_i, t_j) = k^2 \frac{\partial^2}{\partial t^2}\left[\alpha^2 \frac{\partial^2 u}{\partial x^2}(x_i, t_j)\right] = \alpha^2 k^2 \frac{\partial^2}{\partial x^2}\left[\frac{\partial^2 u}{\partial t^2}(x_i, t_j)\right]$$
$$= \alpha^2 k^2 \frac{\partial^2}{\partial x^2}\left[\alpha^2 \frac{\partial^2 u}{\partial x^2}(x_i, t_j)\right] = \alpha^4 k^2 \frac{\partial^4 u}{\partial x^4}(x_i, t_j),$$

and we see that since $\lambda^2 = (\alpha^2 k^2 / h^2) = 1$, we have

$$\frac{1}{4!}\left[k^2 \frac{\partial^4 u}{\partial t^4}(x_i, t_j) - \alpha^2 h^2 \frac{\partial^4 u}{\partial x^4}(x_i, t_j)\right] = \frac{\alpha^2}{4!}[\alpha^2 k^2 - h^2]\frac{\partial^4 u}{\partial x^4}(x_i, t_j) = 0.$$

Continuing in this manner, all the terms on the right-hand side of (12.26) are 0, implying that the local truncation error is 0. The only errors in Example 1 are those due to the approximation of $w_{i,1}$ and to round-off.

As in the case of the Forward-Difference method for the heat equation, the Explicit Finite-Difference method for the wave equation has stability problems. In fact, it is necessary that $\lambda = \alpha k / h \le 1$ for the method to be stable. (See [IK], p. 489.) The explicit method given in Algorithm 12.4, with $\lambda \le 1$, is $O(h^2 + k^2)$ convergent if f and g are sufficiently differentiable. For verification of this, see [IK], p. 491.

Although we will not discuss them, there are implicit methods that are unconditionally stable. A discussion of these methods can be found in [Am], p. 199, [Mi], or [Sm,G].

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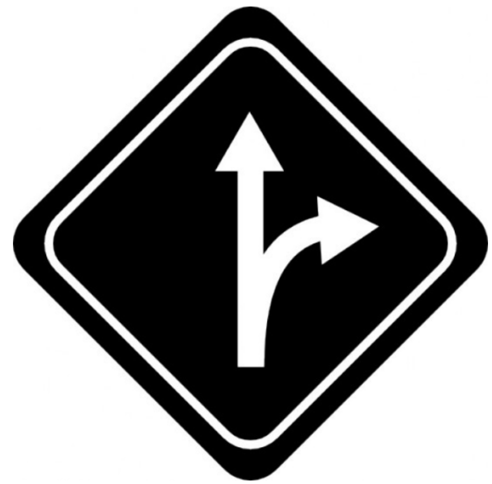


4.1.2 *Numerical Treatment*

4.1.2.1 A Simple Explicit Method

The first idea is just to use central differences for both time and space derivatives, i.e.,

$$\frac{u_i^{j+1} - 2u_i^j + u_i^{j-1}}{\triangle t^2} = c^2 \frac{u_{i+1}^j - 2u_i^j + u_{i-1}^j}{\triangle x^2}, \tag{4.8}$$

or, with $\alpha = c\triangle t/\triangle x$

$$\boxed{u_i^{j+1} = -u_i^{j-1} + 2(1-\alpha^2)u_i^j + \alpha^2(u_{i+1}^j + u_{i-1}^j)}. \tag{4.9}$$

Schematical representation of the scheme (4.9) is shown on Fig. 4.1.

Note that one should also implement initial conditions (4.6). In order to implement the second initial condition one needs the virtual point u_i^{-1},

$$u_t(x_i, 0) = g(x_i) = \frac{u_i^1 - u_i^{-1}}{2\triangle t} + \mathcal{O}(\triangle t^2).$$

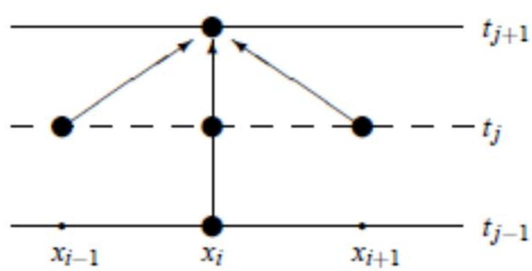


Fig. 4.1 Schematical visualization of the numerical scheme (4.9) for (4.2).

http://pauli.uni-muenster.de/tp/fileadmin/lehre/NumMethoden/WS0910/ScriptPDE/Wave.pdf

With $g_i := g(x_i)$ one can rewrite the last expression as

$$u_i^{-1} = u_i^1 - 2\triangle t g_i + \mathcal{O}(\triangle t^2),$$

and the second time row can be calculated as

$$\boxed{u_i^1 = \triangle t g_i + (1-\alpha^2) f_i + \frac{1}{2}\alpha^2 (f_{i-1} + f_{i+1}),} \tag{4.10}$$

where $u(x_i, 0) = u_i^0 = f(x_i) = f_i$.

von Neumann Stability Analysis

In order to investigate the stability of the explicit scheme (4.9) we start with the usual ansatz (1.21)

$$\varepsilon_i^{j+1} = g^j e^{ikx_i},$$

which leads to the following expression for the amplification factor $g(k)$

$$g^2 = 2(1-\alpha^2)g - 1 + 2\alpha^2 g \cos(k\triangle x).$$

After several transformations the last expression becomes just a quadratic equation for g, namely

$$g^2 - 2\beta g + 1 = 0\,, \tag{4.11}$$

where

$$\beta = 1 - 2\alpha^2 \sin^2\left(\frac{k\triangle x}{2}\right).$$

Solutions of the equation for $g(k)$ read

$$g_{1,2} = \beta \pm \sqrt{\beta^2 - 1}\,.$$

Notice that if $\beta > 1$ then at least one of absolute value of $g_{1,2}$ is bigger that one. Therefor one should desire for $\beta < 1$, i.e.,

$$g_{1,2} = \beta \pm i\sqrt{\beta^2 - 1}$$

and

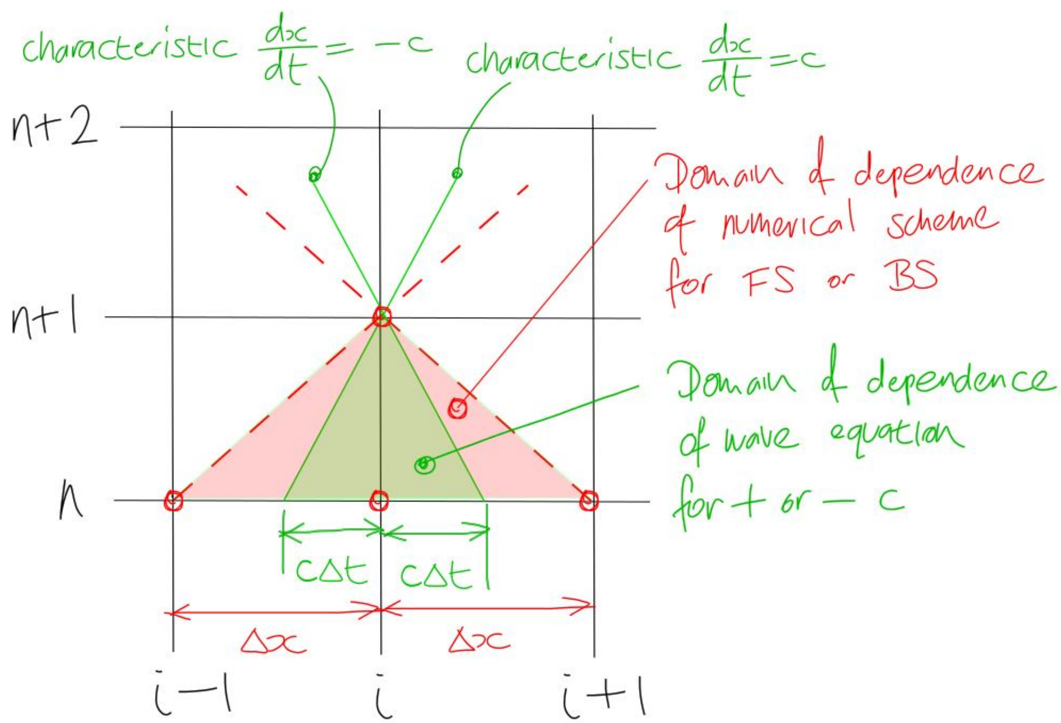
$$|g|^2 = \beta^2 + 1 - \beta^2 = 1\,.$$

That is, the scheme (4.9) is conditional stable. The stability condition reads

$$-1 \leq 1 - 2\alpha^2 \sin^2\left(\frac{k\triangle x}{2}\right) \leq 1,$$

what is equivalent to the standart CFL condition (2.7)

$$\alpha = \frac{c\triangle t}{\triangle x} \leq 1\,.$$



Domínio de dependência da equação contido no estêncil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3o Trabalho Computacional

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Considere a Equação da Onda na corda, $0 < x < 1$, presa pelas extremidades, com velocidade da onda, a = 1, e condição inicial de velocidade nula.

Para os seguintes deslocamentos iniciais analise a precisão (norma 2 do erro) do algoritmo de diferenças finitas, após um período, com a redução do espaçamento , respeitando o no. de CFL ($\lambda = \frac{ak}{h} \leq 1$).

(i) $$f(x) = \begin{cases} 0, & 0 \leq x < 1/3 \\ 1/2, & 1/3 \leq x \leq 2/3 \\ 0, & 2/3 < x \leq 1 \end{cases}$$

(i) $f(x) = \begin{cases} 0, & 0 \le x < 1/3 \\ 1/2, & 1/3 \le x \le 2/3 \\ 0, & 2/3 < x \le 1 \end{cases}$

(iii) $f(x) = \frac{1}{2} EXP\left[-50(x - 1/2)^2\right]$

O que você pode observar?

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A Guide to Numerical Methods for Transport Equations

2010

Dmitri Kuzmin

Friedrich-Alexander-Universität Erlangen-Nürnberg

http://www.mathematik.uni-dortmund.de/~kuzmin/cfdbook.html

Mathematics of Transport Phenomena

$$\frac{\partial}{\partial t}\int_V u(\mathbf{x},t)\,\mathrm{d}\mathbf{x} + \int_S \mathbf{f}\cdot\mathbf{n}\,\mathrm{d}s = \int_V s(\mathbf{x},t)\,\mathrm{d}\mathbf{x}.$$

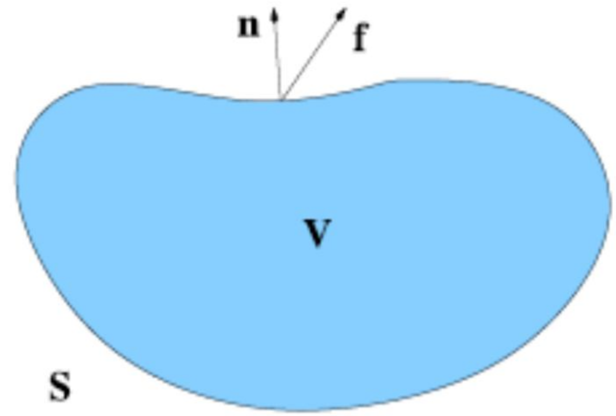

Fig. 1.1 A fixed control volume V bounded by the control surface S.

Quantidade conservada

Fonte (ou sumidouro) da
quantidade conservada

Fluxo da quantidade conservada

Massa – Quantidade de Movimento - Energia

If the functions $u(\mathbf{x},t)$ and $\mathbf{f}(\mathbf{x},t)$ are differentiable, then the divergence theorem, as applied to the surface integral in (1.2), yields the identity

$$\int_V \left[\frac{\partial u(\mathbf{x},t)}{\partial t} + \nabla\cdot\mathbf{f}(\mathbf{x},t) - s(\mathbf{x},t)\right] \mathrm{d}\mathbf{x} = 0.$$

Since the choice of V is arbitrary, the expression in the square brackets must vanish, so the evolution of $u(\mathbf{x},t)$ is governed by the partial differential equation (PDE)

$$\frac{\partial u(\mathbf{x},t)}{\partial t} + \nabla\cdot\mathbf{f}(\mathbf{x},t) = s(\mathbf{x},t). \tag{1.3}$$

massa transportada

Gradiente de concentração

In general, both convective and diffusive effects must be taken into account, so

$$\mathbf{f}(\mathbf{x},t) = \mathbf{v}(\mathbf{x},t)u - \mathscr{D}(\mathbf{x},t)\rho\nabla c. \tag{1.7}$$

However, the rates of convective and diffusive transport may be quite different. For example, the transport of pollutants in a river is dominated by convection, whereas the spreading of pollutants in a lake is dominated by diffusion (dispersion).

Velocidade de transporte

Coeficiente de difusão

Convecção
Ou
Advecção

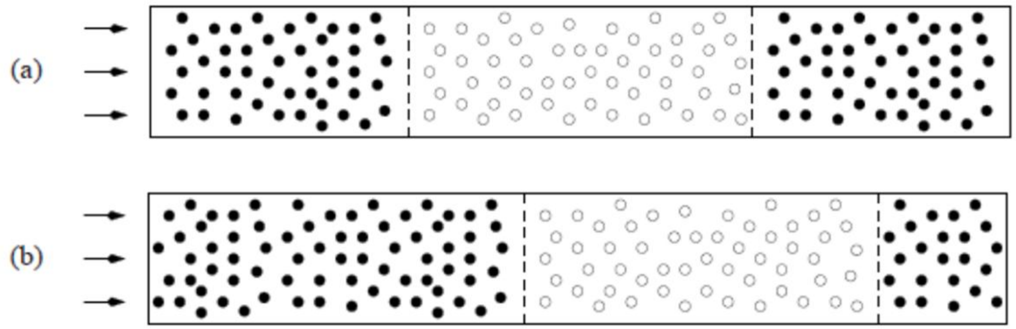

Fig. 1.2 Transport of tracer particles in a pipe filled with moving water.

Difusão

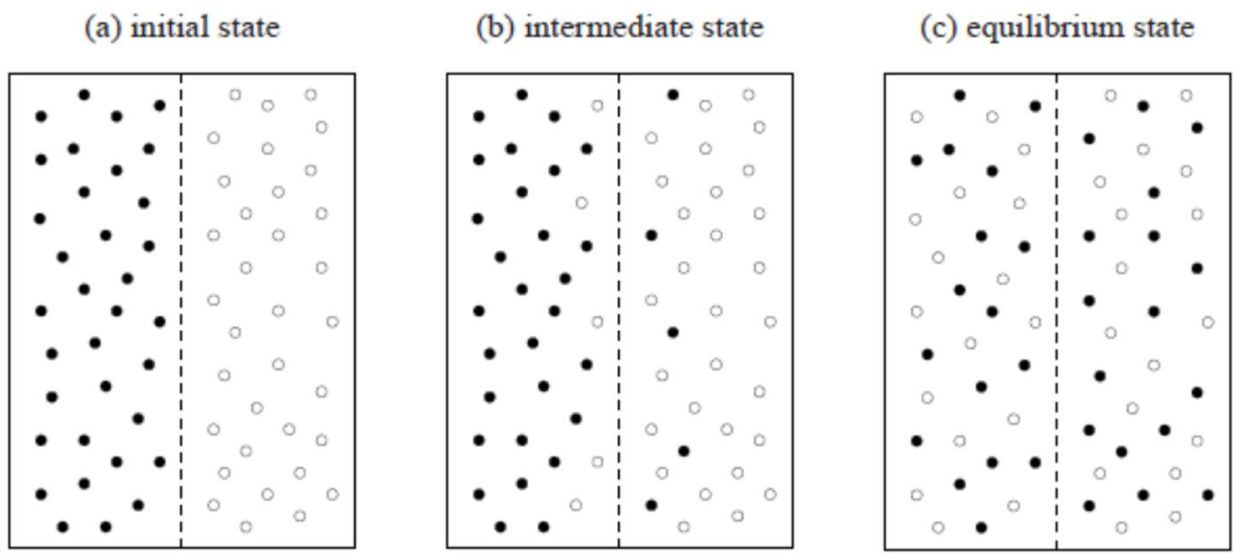

Fig. 1.3 Random motion of molecules across an interface in a stationary liquid.

The Generic Transport Equation

$$\frac{\partial \rho c}{\partial t} + \nabla \cdot (\mathbf{v}\rho c) - \nabla \cdot (\mathscr{D}\rho \nabla c) = s.$$

- the rate-of-change term $\frac{\partial \rho c}{\partial t}$ is the net gain/loss of mass per unit volume and time;
- the convective term $\nabla \cdot (\mathbf{v}\rho c)$ is due to the downstream transport with velocity $\mathbf{v}$;
- the diffusive term $-\nabla \cdot (\mathscr{D}\rho \nabla c)$ is due to a nonuniform spatial distribution of c;
- the source or sink term s combines all other effects that create or destroy ρc.

$$\frac{\partial u}{\partial t} + \nabla \cdot (\mathbf{v}u) - \nabla \cdot (\mathscr{D}\nabla u) = s.$$

convection-diffusion-reaction (CDR) equation

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Summary of Model Problems

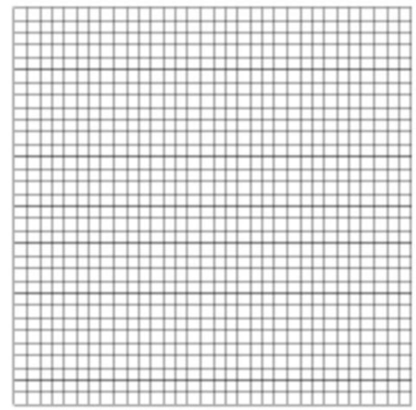
Table 1.1 Summary of models for convection, diffusion, and reaction processes.

PDE type	multidimensional	one-dimensional
elliptic	$\nabla \cdot (\mathbf{v}u - \mathscr{D}\nabla u) = s$ $-\nabla \cdot (\mathscr{D}\nabla u) = s$	$v\frac{\partial u}{\partial x} - d\frac{\partial^2 u}{\partial x^2} = s$ $-d\frac{\partial^2 u}{\partial x^2} = s$
hyperbolic	$\nabla \cdot (\mathbf{v}u) = s$ $\frac{\partial u}{\partial t} + \nabla \cdot (\mathbf{v}u) = s$	$v\frac{\partial u}{\partial x} = s$ $\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial x} = s$
parabolic	$\frac{\partial u}{\partial t} - \nabla \cdot (\mathscr{D}\nabla u) = s$ $\frac{\partial u}{\partial t} + \nabla \cdot (\mathbf{v}u - \mathscr{D}\nabla u) = s$	$\frac{\partial u}{\partial t} - d\frac{\partial^2 u}{\partial x^2} = s$ $\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial x} - d\frac{\partial^2 u}{\partial x^2} = s$

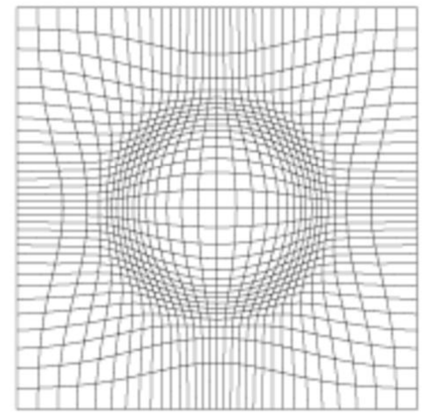
Space Discretization Techniques

Computational Meshes

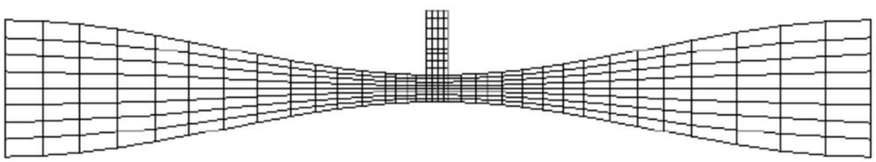
(a) structured, uniform



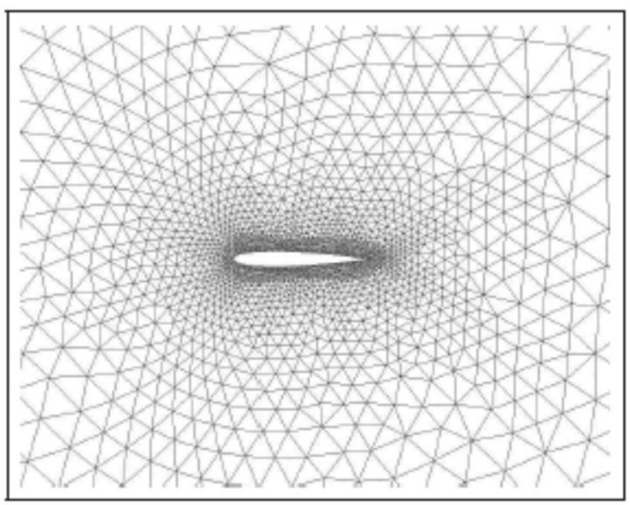
(b) structured, deformed



(d) block-structured, 2 subdomains



(e) unstructured, triangular



(f) unstructured, quadrilateral

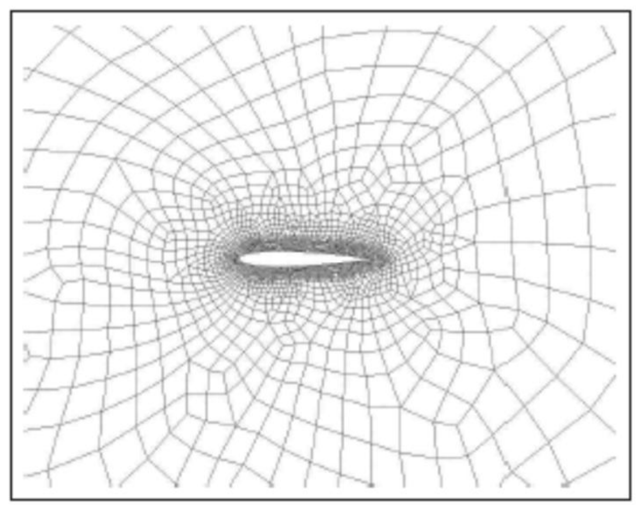


Fig. 1.4 Examples of computational meshes for two-dimensional domains.

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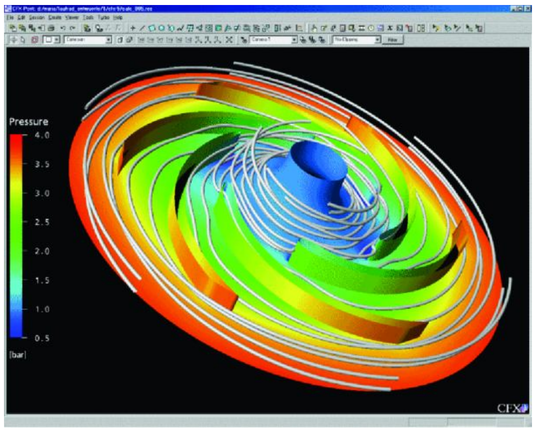
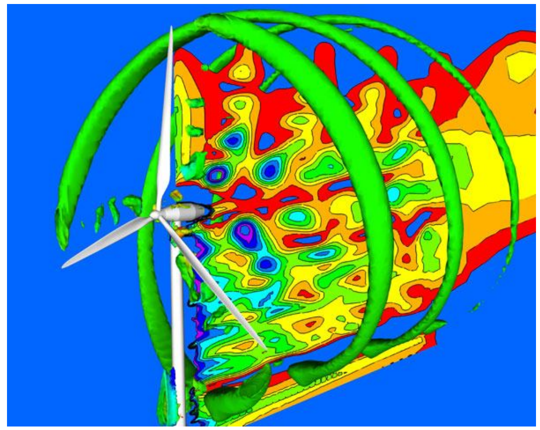
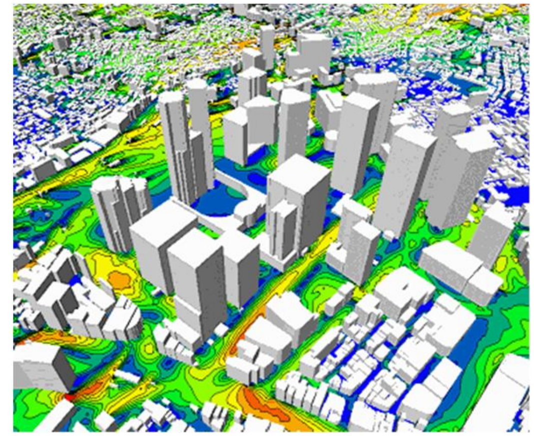
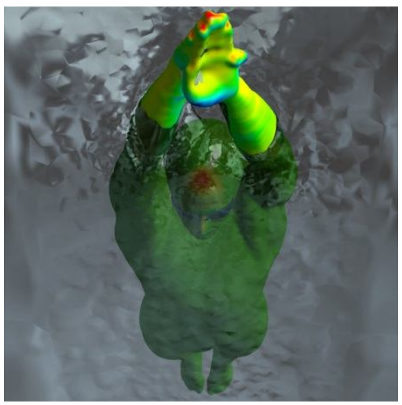
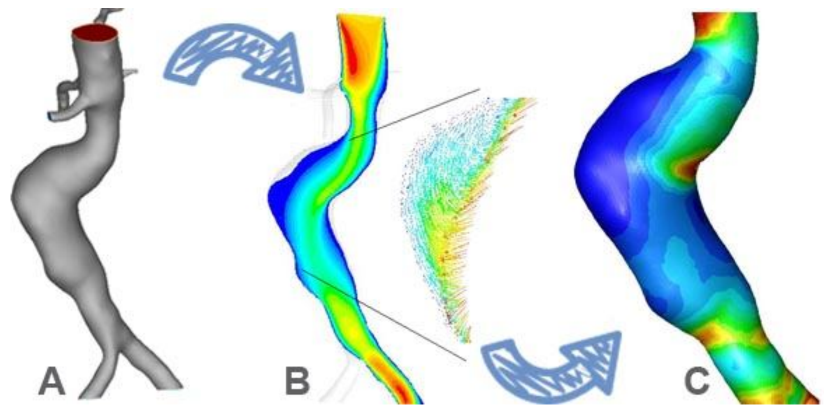
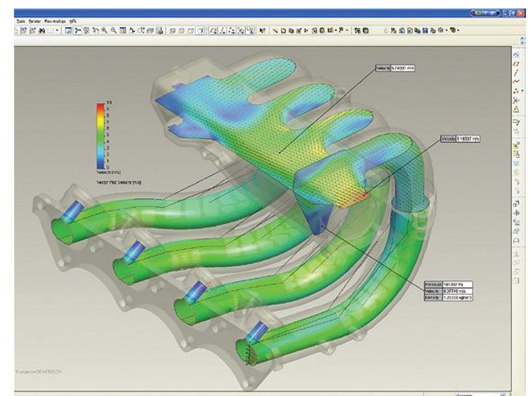

Pressure
4.0
3.5
3.0
2.5
2.0
1.5
1.0
0.5
[bar]
CFX

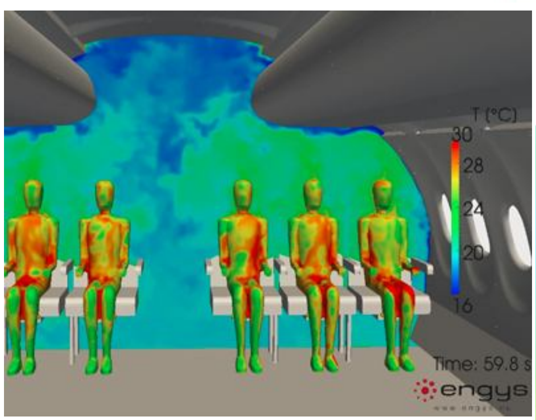


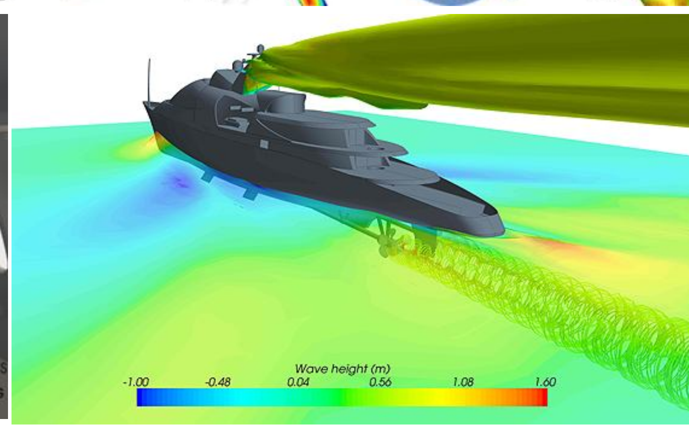


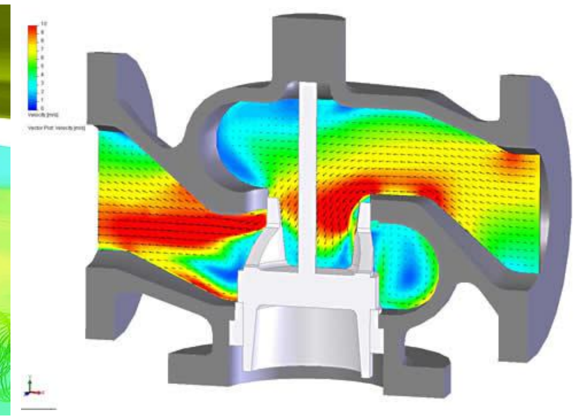



A
B
C




T (°C)
30
28
24
20
16
Time: 59.8 s
engys


Wave height (m)
-1.00
-0.48
0.04
0.56
1.08
1.60



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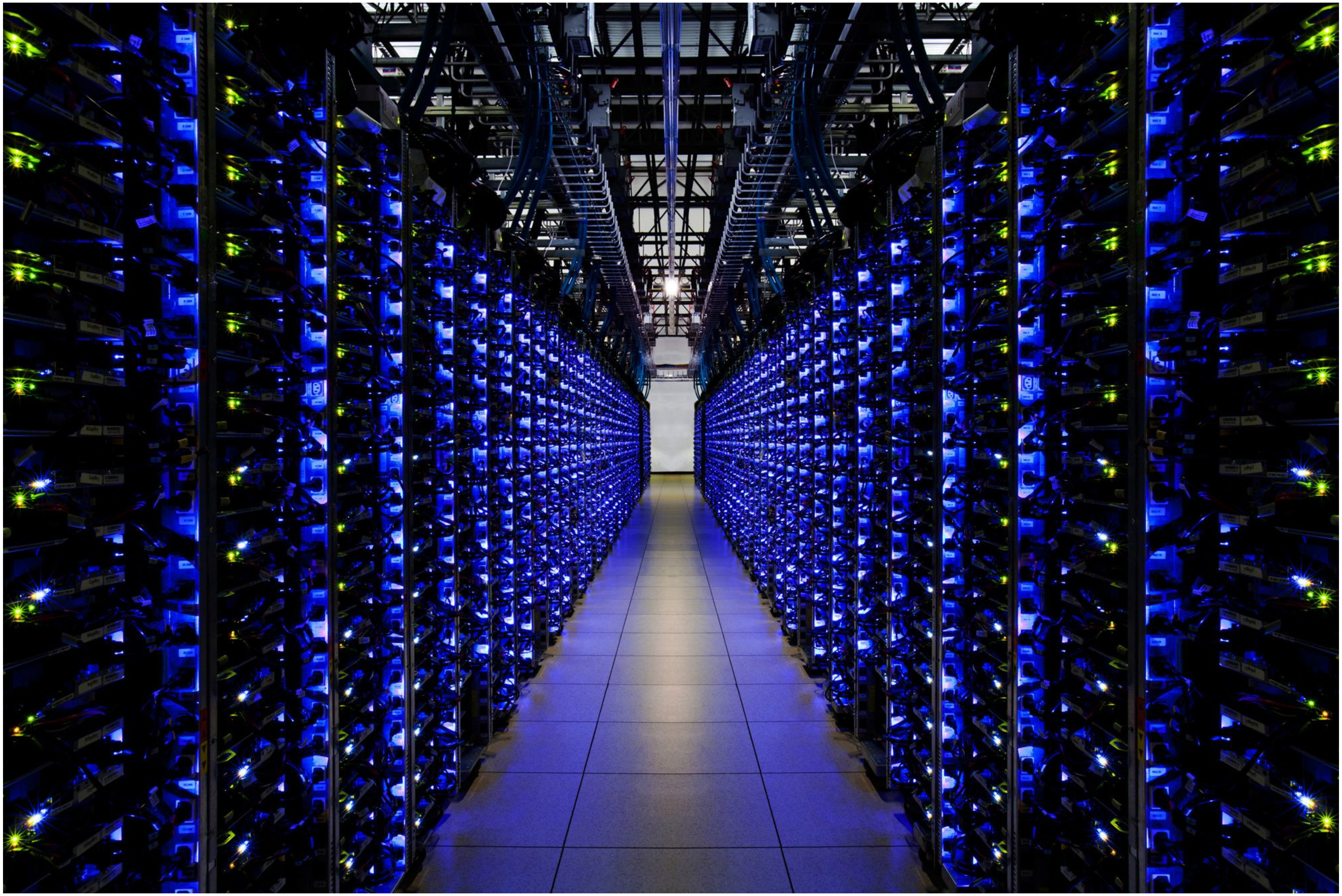

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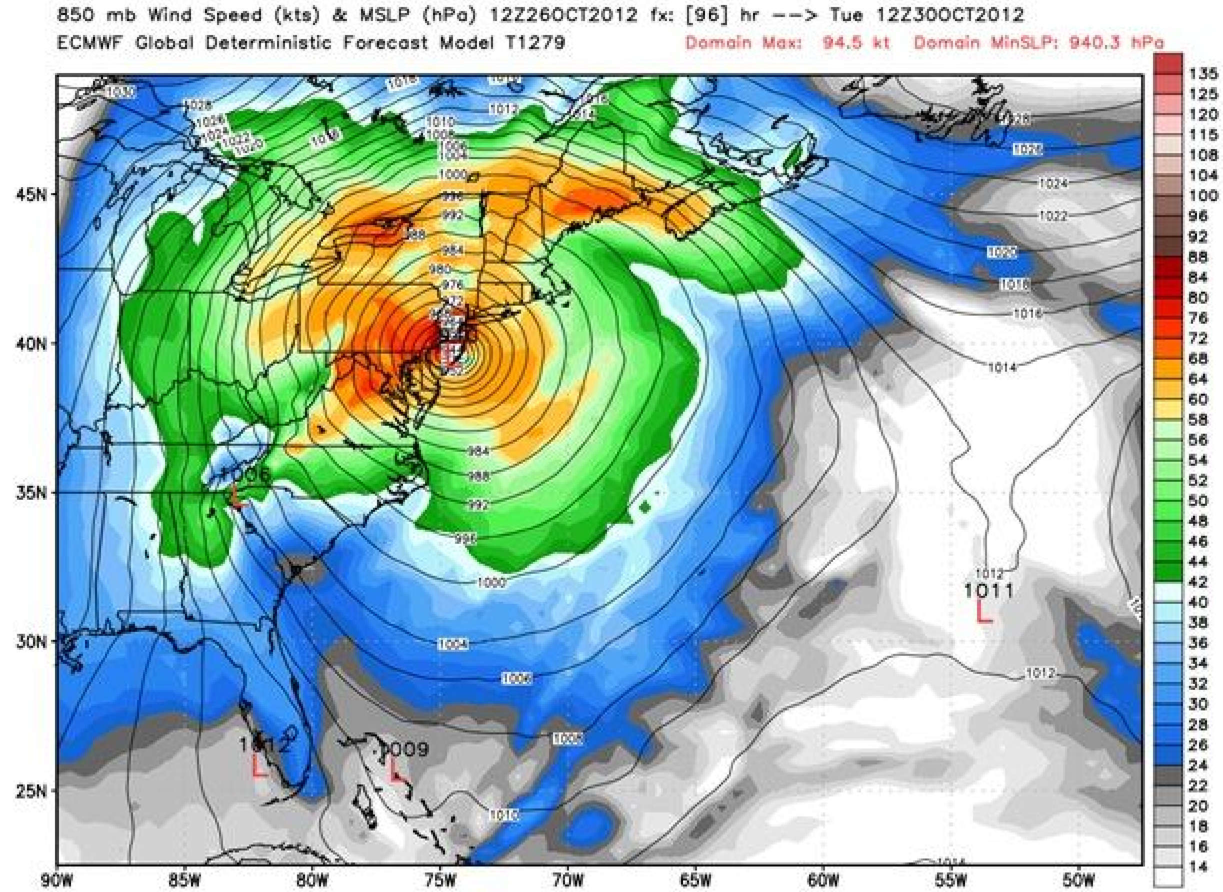

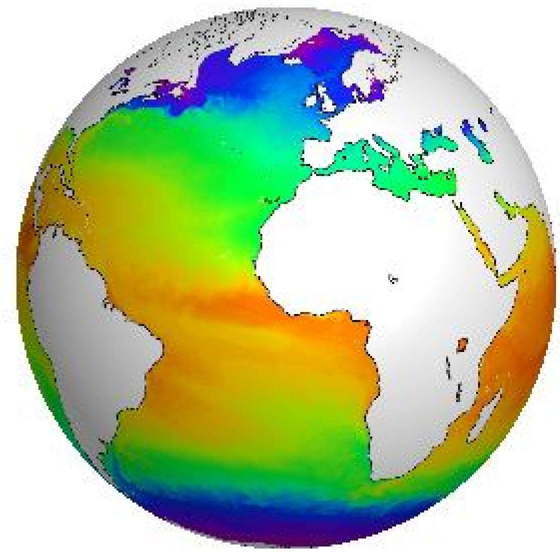

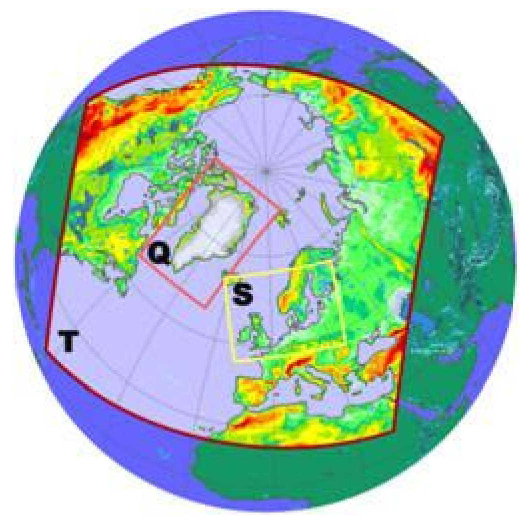

$$\frac{\partial u}{\partial t}+\dot{\sigma}\,\frac{\partial u}{\partial \sigma}+u\,\frac{\partial u}{\partial x}+v\,\frac{\partial u}{\partial y}-fv-\frac{uv}{r}\tan\phi+g\,\frac{\partial z}{\partial x}+c_p\theta\,\frac{\partial \pi}{\partial x}+F_x=0$$

$$\frac{\partial v}{\partial t}+\dot{\sigma}\,\frac{\partial v}{\partial \sigma}+u\,\frac{\partial v}{\partial x}+v\,\frac{\partial v}{\partial y}+fu+\frac{u^2}{r}\tan\phi+g\,\frac{\partial z}{\partial x}+c_p\theta\,\frac{\partial \pi}{\partial y}+F_y=0$$

$$\frac{\partial (gz)}{\partial \sigma}+c_p\theta\,\frac{\partial \pi}{\partial \sigma}=0,$$

$$\frac{\partial \theta}{\partial t}+\dot{\sigma}\,\frac{\partial \theta}{\partial \sigma}+u\,\frac{\partial \theta}{\partial x}+v\,\frac{\partial \theta}{\partial y}+H=0,$$

$$\frac{\partial p_\sigma}{\partial t}+\frac{\partial}{\partial \sigma}\,(\dot{\sigma}p_\sigma)+\frac{\partial}{\partial x}\,(up_\sigma)+\frac{\partial}{\partial y}\,(vp_\sigma)-\frac{vp_\sigma}{r}\tan\phi=0, \qquad \pi=\left(\frac{p}{P}\right)^{\kappa}.$$

MAP 2320 – MÉTODOS NUMÉRICOS EM EQUAÇÕES DIFERENCIAIS II

2º Semestre - 2019

Roteiro do curso

- Introdução
- Séries de Fourier
- **Método de Diferenças Finitas**
- Equação do calor transiente (parabólica)
- Equação de Poisson (elíptica)
- **Equação da onda (hiperbólica)**