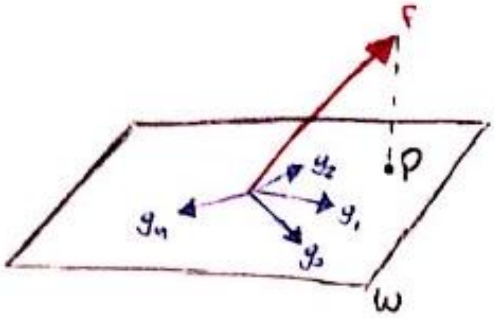
Minimizando distância



- W = espaço gerado por $\{g_1, g_2, \dots, g_n\}$
- $V \supset W$ e $f \in V$
- V espaço vetorial com produto interno $\langle \cdot, \cdot \rangle$

- Podemos fazer duas coisas:
  - ① Projetar ortogonalmente f em W
  - ② Encontrar o vetor de W que está mais próximo de f

- Projetar ortogonalmente é $A^T A x = A^T b$ dado por:

$$\begin{bmatrix} \langle g_0, g_0 \rangle & (\dots) & \langle g_0, g_n \rangle \\ (\vdots) & & (\vdots) \\ \langle g_n, g_0 \rangle & (\dots) & \langle g_n, g_n \rangle \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ (\vdots) \\ c_n \end{bmatrix} = \begin{bmatrix} \langle f, g_0 \rangle \\ \langle f, g_1 \rangle \\ (\vdots) \\ \langle f, g_n \rangle \end{bmatrix}$$

Projeção é $p = c_0 g_0 + c_1 g_1 + (\dots) + c_n g_n$

①=② O que é a distância de f a um vetor $g \in W$? $\|f - g\|$

② Queremos minimizar, entre todos os $g \in W$, a $(\text{distância})^2$ acima

$$\|f-g\|^2 = \|f - (\textstyle\sum c_i g_i)\|^2 = \langle (f - \sum c_i g_i), (f - \sum c_i g_i) \rangle$$

$$E^2(c_0, c_1, \dots, c_n) = \|f - \textstyle\sum c_i g_i\|^2$$

- Queremos minimizar E^2 para encontrar o mínimo, procuramos um ponto crítico, isto é, procuramos $(\bar{c}_0, \dots, \bar{c}_n)$, tal que $\dfrac{\partial E^2}{\partial c_j}(\bar{c}_0, \dots, \bar{c}_n) = 0$

$$E^2(c_0,\dots,c_n) = \langle f,f\rangle - 2\langle f, \sum c_j g_j\rangle + \langle \sum c_j g_j , \sum c_k g_k \rangle$$

$$= \langle f,f\rangle - 2\sum_{j} \langle f, c_j g_j\rangle + \sum_{j,k} c_j c_k \langle g_j, g_k\rangle$$

$$\frac{\partial E^2}{\partial c_i} = -2\langle f, g_i\rangle + 2\sum_{j} c_j \langle g_j, g_j\rangle$$

$$\boxed{\frac{\partial E^2}{\partial c_i} = 0 \iff \sum_{j=0}^{n} c_j \langle g_j, g_j\rangle = \langle f, g_i\rangle, \ \forall i = 1,\dots,n}$$

Exemplo

1) $f(x) = e^x$

$g_0(x) = 1 ; \ g_1(x) = x ; \ g_2(x) = x^2$

$$\langle f, g\rangle = \int_0^1 f(x)g(x)\,dx$$

$$\langle g_i, g_i\rangle = \int_0^1 x^{i+j}\,dx = \left.\frac{x^{i+j+1}}{i+j+1}\right|_0^1 = \frac{1}{i+j+1}$$

$$\begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} e-1 \\ 1 \\ e-2 \end{bmatrix}$$

$p(x) = 1{,}013 + 0{,}8511x + 0{,}8391x^2$

E se quiséssemos projetar $f(x) = e^x$ no espaço de polinômio de grau ≤ 10?

$$\begin{bmatrix} 1 & 1/2 & 1/3 & (\dots) & 1/11 \\ (\vdots) & \ddots & & & (\vdots) \\ 1/11 & (\dots) & & & 1/21 \end{bmatrix} \begin{bmatrix} c_0 \\ (\vdots) \\ c_{10} \end{bmatrix} = \begin{bmatrix} e-1 \\ 1 \\ e-2 \\ (\vdots) \end{bmatrix}$$

- Usando o Método de Gram-Schimidt para enconcontrar uma base ortogonal para o espaço de polinômios no qual estamos interessados

Exemplo: $\{\text{Polinômios de grau} \leq 2\} = \omega$

$$\begin{cases} g_0(x) = 1 \\ g_1(x) = x \\ g_2(x) = x^2 \end{cases} \rightsquigarrow \begin{array}{l} q_0(x) = g_0(x) = 1 \\ q_1(x) = x - 1/2 \\ q_2(x) = x^3 - x + 1/6 \end{array}$$

$$q_1(x) = g_1(x) - \frac{\langle g_1, q_0 \rangle}{\langle q_0, q_0 \rangle} q_0 = x - \frac{\int_0^1 x\,dx}{\int_0^1 1\,dx} \cdot 1 = x - \frac{1}{2}$$

$$q_2(x) = g_2(x) - \left(\begin{array}{l} \text{Projeção de } g_2 \text{ no espaço gerado} \\ \text{por } \{g_0, g_1\} \Leftrightarrow \text{espaço gerado por } \{q_0, q_1\} \end{array} \right)$$

$$= g_2(x) - \frac{\langle g_2, q_0 \rangle}{\langle q_0, q_0 \rangle} \cdot q_0(x) - \frac{\langle g_2, q_1 \rangle}{\langle q_1, q_1 \rangle} q_1(x)$$

$$= x^2 - \frac{\int_0^1 x\,dx}{\int_0^1 1\,dx} \cdot 1 - \frac{\int_0^1 x^2(x - 1/2)\,dx}{\int_0^1 (x - 1/2)^2\,dx} \cdot (x - 1/2)$$

$$= x^2 - \frac{1}{3} - \frac{1/4 - 1/6}{1/12}(x - 1/2) = x^3 - x + \frac{1}{6}$$

• Projetar $f(x)=e^x$ no espaço de polinômios de grau ≤ 2

Usemos a base ortogonal $\{1, x-\frac{1}{2}, x^2-x+\frac{1}{6}\}$

Sistema normal: $$\left[\langle q_i, q_j\rangle\right]\begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \langle f, q_0\rangle \\ \langle f, q_1\rangle \\ \langle f, q_2\rangle \end{bmatrix}$$

obs.: $$\left[\langle q_i, q_j\rangle\right] = \begin{bmatrix} \langle q_0,q_0\rangle & & 0 \\ & \langle q_1,q_1\rangle & \\ 0 & & \langle q_2,q_2\rangle \end{bmatrix}$$

$$c_0 = \frac{\langle f, q_0\rangle}{\langle q_0,q_0\rangle} \qquad c_1 = \frac{\langle f, q_1\rangle}{\langle q_1,q_1\rangle} \qquad c_2 = \frac{\langle f, q_2\rangle}{\langle q_2,q_2\rangle}$$

$$p(x) = c_0\, q_0(x) + c_1\, q_1(x) + c_2\, q_2(x)$$

$$\boxed{p(x) = \frac{\langle f, q_0\rangle}{\langle q_0,q_0\rangle} q_0(x) + \frac{\langle f, q_1\rangle}{\langle q_1,q_1\rangle} q_1(x) + \frac{\langle f, q_2\rangle}{\langle q_2,q_2\rangle} q_2(x)}$$

$$\langle q_0,q_0\rangle = \int_0^1 1\, dx = 1 \qquad \langle q_1,q_1\rangle = \int_0^1 (x-\tfrac{1}{2})^2 dx = \frac{1}{12} \qquad \langle q_2,q_2\rangle = \int_0^1 (x^2-x+\tfrac{1}{6})^2 dx = \frac{1}{180}$$

$$\langle f, q_0\rangle = \int_0^1 e^x\, dx = e-1 \qquad \langle f, q_1\rangle = \int_0^1 e^x (x-\tfrac{1}{2}) dx = \frac{3}{2} - \frac{e}{2}$$

$$\langle f, q_2\rangle = \int_0^1 e^x \left(x^2 - x + \frac{1}{6}\right) dx = \frac{7e-19}{6}$$

$$\begin{aligned} p(x) &= (e-1) + (18-6e)\left(x-\frac{1}{2}\right) + (210e-570)\left(x^2-x+\frac{1}{6}\right) \\ &= (210e-570)x^2 + (-216e+588)x + 39e-105 \\ &= 0{,}839184\, x^2 + 0{,}851125\, x + 1{,}01299 \end{aligned}$$