Chapter 5

THE ACOUSTIC WAVE EQUATION AND SIMPLE SOLUTIONS

5.1 INTRODUCTION

Acoustic waves constitute one kind of pressure fluctuation that can exist in a compressible fluid. In addition to the audible pressure fields of moderate intensity, the most familiar, there are also *ultrasonic* and *infrasonic* waves whose frequencies lie beyond the limits of hearing, *high-intensity* waves (such as those near jet engines and missiles) that may produce a sensation of pain rather than sound, *nonlinear* waves of still higher intensities, and *shock* waves generated by explosions and supersonic aircraft.

Inviscid fluids exhibit fewer constraints to deformations than do solids. The restoring forces responsible for propagating a wave are the pressure changes that occur when the fluid is compressed or expanded. Individual elements of the fluid move back and forth in the direction of the forces, producing adjacent regions of compression and rarefaction similar to those produced by longitudinal waves in a bar.

The following terminology and symbols will be used:

$\vec{r}$ = equilibrium position of a fluid element

$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z} \tag{5.1.1}$$

($\hat{x}$, $\hat{y}$, and $\hat{z}$ are the unit vectors in the x, y, and z directions, respectively)

$\vec{\xi}$ = *particle displacement* of a fluid element from its equilibrium position

$$\vec{\xi} = \xi_x\hat{x} + \xi_y\hat{y} + \xi_z\hat{z} \tag{5.1.2}$$

$\vec{u}$ = *particle velocity* of a fluid element

$$\vec{u} = \frac{\partial\vec{\xi}}{\partial t} = u_x\hat{x} + u_y\hat{y} + u_z\hat{z} \tag{5.1.3}$$

ρ = *instantaneous density* at (x, y, z)

ρ_0 = *equilibrium density* at (x, y, z)

s = *condensation* at (x, y, z)

$$s = (\rho - \rho_0)/\rho_0 \tag{5.1.4}$$

$\rho - \rho_0 = \rho_0 s =$ *acoustic density* at (x, y, z)
$\mathcal{P} =$ *instantaneous pressure* at (x, y, z)
$\mathcal{P}_0 =$ *equilibrium pressure* at (x, y, z)
$p =$ *acoustic pressure* at (x, y, z)

$$p = \mathcal{P} - \mathcal{P}_0 \tag{5.1.5}$$

$c =$ *thermodynamic speed of sound* of the fluid
$\Phi =$ *velocity potential* of the wave

$$\vec{u} = \nabla\Phi \tag{5.1.6}$$

$T_K =$ temperature in kelvins (K)
$T =$ temperature in degrees Celsius (or centigrade) (°C)

$$T + 273.15 = T_K \tag{5.1.7}$$

The terms *fluid element* and *particle* mean an infinitesimal volume of the fluid large enough to contain millions of molecules so that the fluid may be thought of as a continuous medium, yet small enough that all acoustic variables are uniform throughout.

The molecules of a fluid do not have fixed mean positions in the medium. Even without the presence of an acoustic wave, they are in constant random motion with average velocities far in excess of any particle velocity associated with the wave motion. However, a small volume may be treated as an unchanging unit since those molecules leaving its confines are replaced (on the average) by an equal number with identical properties. The macroscopic properties of the element remain unchanged. As a consequence, it is possible to speak of particle displacements and velocities when discussing acoustic waves in fluids, as was done for elastic waves in solids. The fluid is assumed to be lossless so there are no dissipative effects such as those arising from viscosity or heat conduction. The analysis will be limited to waves of relatively small amplitude, so changes in the density of the medium will be small compared with its equilibrium value. These assumptions are necessary to arrive at the simplest equations for sound in fluids. It is fortunate that experiments show these simplifications are successful and lead to an adequate description of most common acoustic phenomena. However, there are situations where these assumptions are violated and the theory must be modified.

5.2 THE EQUATION OF STATE

For fluid media, the equation of state must relate three physical quantities describing the thermodynamic behavior of the fluid. For example, the *equation of state for a perfect gas*

$$\mathcal{P} = \rho r T_K \tag{5.2.1}$$

gives the general relationship between the total pressure $\mathcal{P}$ in pascals (Pa), the density ρ in kilograms per cubic meter (kg/m^3), and the absolute temperature T_K in kelvins (K) for a large number of gases under equilibrium conditions. The quantity r is the *specific gas constant* and depends on the *universal gas constant* $\mathcal{R}$ and the *molecular weight* M of the particular gas. See Appendix A9. For air, $r \approx 287$ J/(kg·K).

Greater simplification can be achieved if the thermodynamic process is restricted. For example, if the fluid is contained within a vessel whose walls are highly thermally conductive, then slow variations in the volume of the vessel will result in thermal energy being transferred between the walls and the fluid. If the walls have sufficient thermal capacity, they and the fluid will remain at a constant temperature. In this case, the perfect gas is described by the *isotherm*

$$\mathcal{P}/\mathcal{P}_0 = \rho/\rho_0 \qquad \text{(perfect gas isotherm)} \tag{5.2.2}$$

In contrast, acoustic processes are nearly *isentropic* (adiabatic and reversible). The thermal conductivity of the fluid and the temperature gradients of the disturbance are small enough that no appreciable thermal energy transfer occurs between adjacent fluid elements. Under these conditions, the *entropy* of the fluid remains nearly constant. The acoustic behavior of the perfect gas under these conditions is described by the *adiabat*

$$\mathcal{P}/\mathcal{P}_0 = (\rho/\rho_0)^\gamma \qquad \text{(perfect gas adiabat)} \tag{5.2.3}$$

where γ is the *ratio of specific heats* (or *ratio of heat capacities*). Finite thermal conductivity results in a conversion of acoustic energy into random thermal energy so that the acoustic disturbance attenuates slowly with time or distance. This and other dissipative effects will be considered in Chapter 8.

For fluids other than a perfect gas, the adiabat is more complicated. In these cases it is preferable to determine experimentally the isentropic relationship between pressure and density fluctuations. This relationship can be represented by a Taylor's expansion

$$\mathcal{P} = \mathcal{P}_0 + \left(\frac{\partial \mathcal{P}}{\partial \rho}\right)_{\rho_0} (\rho - \rho_0) + \frac{1}{2}\left(\frac{\partial^2 \mathcal{P}}{\partial \rho^2}\right)_{\rho_0} (\rho - \rho_0)^2 + \cdots \tag{5.2.4}$$

wherein the partial derivatives are determined for the isentropic compression and expansion of the fluid about its equilibrium density. If the fluctuations are small, only the lowest order term in $(\rho - \rho_0)$ need be retained. This gives a linear relationship between the pressure fluctuation and the change in density

$$\mathcal{P} - \mathcal{P}_0 \approx \mathcal{B}(\rho - \rho_0)/\rho_0 \tag{5.2.5}$$

with $\mathcal{B} = \rho_0(\partial\mathcal{P}/\partial\rho)_{\rho_0}$ the *adiabatic bulk modulus* discussed in Appendix A11. In terms of acoustic pressure p and condensation s, (5.2.5) can be rewritten as

$$\boxed{p \approx \mathcal{B}s} \tag{5.2.6}$$

The essential restriction is that the condensation is small.

Another approach in expressing the adiabat of any fluid is to model it on the adiabat of the perfect gas. This is done by generalizing $\mathscr{P}_0$ and γ to be empirically determined coefficients for the fluid in question. Expanding (5.2.3) in a Taylor's series in s and rearranging to isolate the acoustic pressure $p = \mathscr{P} - \mathscr{P}_0$ yields

$$p = \mathscr{P}_0\left[\gamma s + \tfrac{1}{2}\gamma(\gamma - 1)s^2 + \cdots\right] \tag{5.2.7}$$

Comparing this with (5.2.4) and equating the coefficients through second order in s reveals that $\mathscr{P}_0$ and γ can be expressed thermodynamically in general as

$$\gamma\mathscr{P}_0 = \mathscr{B} \tag{5.2.8}$$

$$\gamma - 1 \equiv \frac{B}{A} = \frac{\rho_0}{\mathscr{B}}\left(\frac{\partial \mathscr{B}}{\partial \rho}\right)_{\rho_0} \tag{5.2.9}$$

[Both $\mathscr{B}$ and $(\partial\mathscr{B}/\partial\rho)_{\rho_0}$ are evaluated under adiabatic conditions.] The quantity B/A is the *parameter of nonlinearity* of the fluid. Thus, knowing $\mathscr{B}$ and its derivative, we can determine $\mathscr{P}_0$ and γ. The equality of coefficients fails for terms of third order and above in s, but it has been demonstrated that these higher order terms are completely negligible for situations of practical importance.[1] Use of standard thermodynamic relationships allows the right sides of the above two equations to be expressed in terms of other thermodynamic properties of the fluid that are much more easily determined experimentally.

For liquids like water, simple alcohols, liquid metals, and many organic compounds, γ lies between about 4 and 12 and $\mathscr{P}_0$ between about 1×10^3 and 5×10^3 atm. The constant $\mathscr{P}_0$ suggests a fictitious *adiabatic internal pressure,* as if the liquid in its acoustic behavior were a gas under this hydrostatic pressure. The coefficient γ is an empirical constant whose difference from unity measures the nonlinear relationship between acoustic pressure and condensation. (Elsewhere, unless explicitly stated otherwise, it is the ratio of specific heats.)

5.3 THE EQUATION OF CONTINUITY

To connect the motion of the fluid with its compression or expansion, we need a functional relationship between the particle velocity $\vec{u}$ and the instantaneous density ρ. Consider a small rectangular parallelepiped volume element $dV = dx\,dy\,dz$, which is *fixed in space* and through which elements of the fluid travel. The net rate with which mass flows into the volume through its surface must equal the rate with which the mass within the volume increases. Referring to Fig. 5.3.1, we see that the net influx of mass into this spatially fixed volume resulting from flow in the x direction is

$$\left[\rho u_x - \left(\rho u_x + \frac{\partial(\rho u_x)}{\partial x}dx\right)\right]dy\,dz = -\frac{\partial(\rho u_x)}{\partial x}\,dV \tag{5.3.1}$$

[1] Beyer, *Nonlinear Acoustics,* Naval Ship Systems Command (1974).

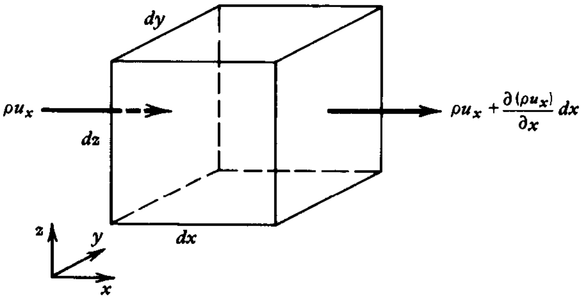

Figure 5.3.1 An elemental spatially fixed volume of fluid showing the rate of mass flow into and out of the volume resulting from fluid flowing in the x direction. A similar diagram can be drawn for fluid flowing in the y and z directions.

Similar expressions give the net influx for the y and z directions, so that the total influx must be

$$-\left(\frac{\partial(\rho u_x)}{\partial x}+\frac{\partial(\rho u_y)}{\partial y}+\frac{\partial(\rho u_z)}{\partial z}\right)dV = -\nabla\cdot(\rho\vec{u})\,dV \tag{5.3.2}$$

The rate with which the mass increases in the volume is $(\partial\rho/\partial t)\,dV$. The net influx must equal the rate of increase,

$$\boxed{\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\vec{u}) = 0} \tag{5.3.3}$$

This is the *exact continuity equation*. The second term on the left involves the product of particle velocity and instantaneous density, both of which are acoustic variables. However, if we write $\rho = \rho_0(1+s)$, require ρ_0 to be a sufficiently weak function of time, and assume that s is very small, (5.3.3) becomes

$$\rho_0\frac{\partial s}{\partial t}+\nabla\cdot(\rho_0\vec{u}) = 0 \tag{5.3.4}$$

the *linear continuity equation*. Furthermore, if ρ_0 is only a weak function of space

$$\boxed{\frac{\partial s}{\partial t}+\nabla\cdot\vec{u} = 0} \tag{5.3.5}$$

5.4 THE SIMPLE FORCE EQUATION: EULER'S EQUATION

In real fluids, the existence of viscosity and the failure of acoustic processes to be perfectly adiabatic introduce dissipative terms. As mentioned earlier, these effects will be investigated in Chapter 8.

Consider a fluid element $dV = dx\,dy\,dz$, which *moves with the fluid* and contains a mass dm of fluid. The net force $d\vec{f}$ on the element will accelerate it according to Newton's second law $d\vec{f} = \vec{a}\,dm$. In the absence of viscosity, the net force experienced by the element in the x direction is

$$df_x = \left[\mathcal{P} - \left(\mathcal{P} + \frac{\partial \mathcal{P}}{\partial x}dx\right)\right]dy\,dz = -\frac{\partial \mathcal{P}}{\partial x}\,dV \tag{5.4.1}$$

There are analogous expressions for df_y and df_z. The presence of the gravitational field introduces an additional force in the vertical direction of $\vec{g}\rho\,dV$, where $|\vec{g}| \approx 9.8\ \text{m/s}^2$ is the acceleration of gravity. Combination of these terms results in

$$d\vec{f} = -\nabla\mathcal{P}\,dV + \vec{g}\rho\,dV \tag{5.4.2}$$

The expression for the acceleration of the fluid element is a little more complicated. The particle velocity $\vec{u}$ is a function of both time and space. When the fluid element with velocity $\vec{u}(x, y, z, t)$ at position (x, y, z) and time t moves to a new location $(x + dx, y + dy, z + dz)$ at a later time $t + dt$, its new velocity is expressed by the leading terms of its Taylor expansion

$$\begin{aligned}\vec{u}(x + u_x\,dt, y + u_y\,dt, z + u_z\,dt, t + dt)\\ = \vec{u}(x, y, z, t) + \frac{\partial \vec{u}}{\partial x}u_x\,dt + \frac{\partial \vec{u}}{\partial y}u_y\,dt + \frac{\partial \vec{u}}{\partial z}u_z\,dt + \frac{\partial \vec{u}}{\partial t}\,dt\end{aligned} \tag{5.4.3}$$

Thus the acceleration of the chosen element is

$$\vec{a} = \lim_{dt\to 0}\frac{\vec{u}(x + u_x\,dt, y + u_y\,dt, z + u_z\,dt, t + dt) - \vec{u}(x, y, z, t)}{dt} \tag{5.4.4}$$

or

$$\vec{a} = \frac{\partial \vec{u}}{\partial t} + u_x\frac{\partial \vec{u}}{\partial x} + u_y\frac{\partial \vec{u}}{\partial y} + u_z\frac{\partial \vec{u}}{\partial z} \tag{5.4.5}$$

If we define the vector operator $(\vec{u}\cdot\nabla)$ as

$$(\vec{u}\cdot\nabla) \equiv u_x\frac{\partial}{\partial x} + u_y\frac{\partial}{\partial y} + u_z\frac{\partial}{\partial z} \tag{5.4.6}$$

then $\vec{a}$ can be written more conveniently as

$$\vec{a} = \frac{\partial \vec{u}}{\partial t} + (\vec{u}\cdot\nabla)\vec{u} \tag{5.4.7}$$

Since the mass dm of the element is $\rho\,dV$, substitution into $d\vec{f} = \vec{a}\,dm$ gives

$$-\nabla\mathcal{P} + \vec{g}\rho = \rho\left(\frac{\partial \vec{u}}{\partial t} + (\vec{u}\cdot\nabla)\vec{u}\right) \tag{5.4.8}$$

This nonlinear, inviscid force equation is *Euler's equation* with gravity. In the case of no acoustic excitation, $\vec{g}\rho_0 = \nabla\mathcal{P}_0$, and thus $\nabla\mathcal{P} = \nabla p + \vec{g}\rho_0$ so that (5.4.8) becomes

$$-\frac{1}{\rho_0}\nabla p + \vec{g}s = (1+s)\left(\frac{\partial \vec{u}}{\partial t} + (\vec{u}\cdot\nabla)\vec{u}\right) \tag{5.4.9}$$

If we now make the assumptions that $|\vec{g}s| \ll |\nabla p|/\rho_0$, that $|s| \ll 1$, and that $|(\vec{u}\cdot\nabla)\vec{u}| \ll |\partial\vec{u}/\partial t|$, then

$$\boxed{\rho_0\frac{\partial \vec{u}}{\partial t} = -\nabla p} \tag{5.4.10}$$

This is the *linear Euler's equation*, valid for acoustic processes of small amplitude.

5.5 THE LINEAR WAVE EQUATION

The linearized equations (5.2.6), (5.3.4), and (5.4.10) can be combined to yield a single differential equation with one dependent variable. First, take the divergence of (5.4.10),

$$\nabla\cdot\left(\rho_0\frac{\partial \vec{u}}{\partial t}\right) = -\nabla^2 p \tag{5.5.1}$$

where $\nabla\cdot\nabla = \nabla^2$ is the three-dimensional Laplacian. Next, take the time derivative of (5.3.4) and use the facts that space and time are independent and ρ_0 is no more than a weak function of time,

$$\rho_0\frac{\partial^2 s}{\partial t^2} + \nabla\cdot\left(\rho_0\frac{\partial \vec{u}}{\partial t}\right) = 0 \tag{5.5.2}$$

Elimination of the divergence term between these two equations gives

$$\nabla^2 p = \rho_0\frac{\partial^2 s}{\partial t^2} \tag{5.5.3}$$

Equation (5.2.6) allows the condensation to be expressed as $s = p/\mathcal{B}$, and with $\mathcal{B}$ no more than a weak function of time,

$$\boxed{\nabla^2 p = \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2}} \tag{5.5.4}$$

where c is the *thermodynamic speed of sound* defined by

$$c^2 = \mathcal{B}/\rho_0 \tag{5.5.5}$$

Equation (5.5.4) is the *linear, lossless wave equation* for the propagation of sound in fluids with phase speed c. Since the above derivation never required a restriction on $\mathcal{B}$ or ρ_0 with respect to *space,* (5.5.4) is valid for propagation in media with sound speeds that are functions of space, such as found in the atmosphere or the ocean.

Use of (5.5.5) shows that the adiabat can be written as

$$p = \rho_0 c^2 s \tag{5.5.6}$$

If ρ_0 and c are only weak functions of space, then p and s are essentially proportional and the condensation satisfies the wave equation.

Since the curl of the gradient of a function must vanish, $\nabla \times \nabla f = 0$, (5.4.10) shows that the particle velocity is irrotational, $\nabla \times \vec{u} = 0$. This means that it can be expressed as the gradient of a scalar function Φ,

$$\vec{u} = \nabla\Phi \tag{5.5.7}$$

which was previously identified as the velocity potential. The physical meaning of this useful result is that the acoustic excitation of an *inviscid* fluid involves no rotational flow. A real fluid has finite viscosity and the particle velocity is not curl-free everywhere. For most acoustic processes, rotational effects are small and confined to the vicinity of boundaries. They exert little influence on the propagation of sound, so that (5.5.7) can be assumed true to very high accuracy in acoustic propagation.

Substitution of (5.5.7) into (5.4.10) and requiring ρ_0 to be no more than a gradual function of space gives

$$\nabla\left(\rho_0 \frac{\partial \Phi}{\partial t} + p\right) = 0 \tag{5.5.8}$$

The quantity in parentheses can be chosen to vanish identically if there is no acoustic excitation so that

$$p = -\rho_0 \frac{\partial \Phi}{\partial t} \tag{5.5.9}$$

Thus, Φ satisfies the wave equation within the same approximations.

5.6 SPEED OF SOUND IN FLUIDS

By combining (5.2.5) and (5.5.5), we get an expression for the thermodynamic speed of sound

$$c^2 = \left(\frac{\partial \mathcal{P}}{\partial \rho}\right)_{adiabat} \tag{5.6.1}$$

This is a characteristic property of the fluid and depends on the equilibrium conditions.

When a sound wave propagates through a perfect gas, the adiabat may be utilized to derive an important special form of (5.6.1). Direct differentiation of (5.2.3) leads to

$$\left(\frac{\partial \mathcal{P}}{\partial \rho}\right)_{adiabat} = \gamma \frac{\mathcal{P}}{\rho} \tag{5.6.2}$$

Evaluating this expression at ρ_0 and substituting into (5.6.1), we obtain

$$c^2 = \gamma \mathcal{P}_0 / \rho_0 \tag{5.6.3}$$

Substitution of the appropriate values for air from Appendix A10 gives

$$c_0 = (1.402 \times 1.01325 \times 10^5 / 1.293)^{1/2} = 331.5 \text{ m/s} \tag{5.6.4}$$

as the theoretical value for the speed of sound in air at 0°C and 1 atm pressure. This is in excellent agreement with measured values and supports the assumption that acoustic processes in a fluid are adiabatic. For most real gases at constant temperature, the ratio $\mathcal{P}_0 / \rho_0$ is nearly independent of pressure so that the speed of sound is a function only of temperature. An alternate expression for the speed of sound in a perfect gas is found from (5.2.1) and (5.6.3) to be

$$c^2 = \gamma r T_K \tag{5.6.5}$$

The speed is proportional to the square root of the absolute temperature. In terms of the speed c_0 at 0°C, this becomes

$$c = c_0 (T_K / 273)^{1/2} = c_0 (1 + T/273)^{1/2} \tag{5.6.6}$$

Theoretical prediction of the speed of sound for liquids is considerably more difficult than for gases. However, it is possible to show theoretically that $\mathcal{B} = \gamma \mathcal{B}_T$, where $\mathcal{B}_T$ is the isothermal bulk modulus. Since $\mathcal{B}_T$ is much easier to measure experimentally than $\mathcal{B}$, a convenient expression for the speed of sound in liquids is obtained from (5.5.5) and $\mathcal{B}_T$,

$$c^2 = \gamma \mathcal{B}_T / \rho_0 \tag{5.6.7}$$

where γ, $\mathcal{B}_T$, and ρ_0 all vary with the equilibrium temperature and pressure of the liquid. Since no simple theory is available for predicting these variations, they must be measured experimentally and the resulting speed of sound expressed as a numerical formula. For example, in distilled water a simplified formula for c in m/s is

$$c(\mathcal{P}, t) = 1402.7 + 488t - 482t^2 + 135t^3 + (15.9 + 2.8t + 2.4t^2)(\mathcal{P}_G / 100) \tag{5.6.8}$$

where $\mathcal{P}_G$ is the gauge pressure in bar (1 bar $= 10^5$ Pa) and $t = T/100$, with T in degrees Celsius. A gauge pressure $\mathcal{P}_G$ of zero means an equilibrium pressure $\mathcal{P}_0$ of 1 atm (1.01325 bar). This equation is accurate to within 0.05% for $0 < T < 100$°C and $0 \leq \mathcal{P}_G \leq 200$ bar.

5.7 HARMONIC PLANE WAVES

In this and the next few sections, discussion will be restricted to homogeneous, isotropic fluids in which the speed of sound c is a constant throughout. Propagation in fluids having spatially dependent sound speeds will be deferred until Section 5.14.

The characteristic property of a *plane wave* is that each acoustic variable has constant amplitude and phase on any plane perpendicular to the direction of propagation. Since the surfaces of constant phase for any diverging wave become nearly planar far from their source, we may expect that the properties of diverging waves will, at large distances, become very similar to those of plane waves.

If the coordinate system is chosen so that the plane wave propagates along the x axis, the wave equation reduces to

$$\frac{\partial^2 p}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} \tag{5.7.1}$$

where $p = p(x,t)$. Direct comparison with (2.3.6) shows that the mathematical development of the solutions for transverse waves in Sections 2.4 and 2.5 can be applied here and need not be repeated. Let us therefore proceed directly to harmonic plane waves and the relationships among the acoustic variables.

The complex form of the harmonic solution for the acoustic pressure of a plane wave is

$$\mathbf{p} = \mathbf{A}e^{j(\omega t - kx)} + \mathbf{B}e^{j(\omega t + kx)} \tag{5.7.2}$$

and the associated particle velocity, from (5.4.10),

$$\vec{\mathbf{u}} = \mathbf{u}\hat{x} = [(\mathbf{A}/\rho_0 c)e^{j(\omega t - kx)} - (\mathbf{B}/\rho_0 c)e^{j(\omega t + kx)}]\hat{x} \tag{5.7.3}$$

is parallel to the direction of propagation.

If we use a subscript "+" to designate a wave traveling in the $+x$ direction and a subscript "−" for a wave traveling in the $-x$ direction, then

$$\mathbf{p}_+ = \mathbf{A}e^{j(\omega t - kx)} \quad \text{and} \quad \mathbf{p}_- = \mathbf{B}e^{j(\omega t + kx)} \tag{5.7.4}$$

$$\mathbf{u}_\pm = \pm\mathbf{p}_\pm/\rho_0 c \tag{5.7.5}$$

$$\mathbf{s}_\pm = \mathbf{p}_\pm/\rho_0 c^2 \tag{5.7.6}$$

$$\mathbf{\Phi}_\pm = -\mathbf{p}_\pm/j\omega\rho_0 \tag{5.7.7}$$

For a plane wave traveling in some *arbitrary* direction, it is plausible to try a solution of the form

$$\mathbf{p} = \mathbf{A}e^{j(\omega t - k_x x - k_y y - k_z z)} \tag{5.7.8}$$

Substitution into (5.5.4) shows that this is acceptable if

$$(\omega/c)^2 = k_x^2 + k_y^2 + k_z^2 \tag{5.7.9}$$

Definition of the *propagation vector* $\vec{k}$,

$$\vec{k} = k_x\hat{x} + k_y\hat{y} + k_z\hat{z} \tag{5.7.10}$$

which has magnitude ω/c, and a position vector $\vec{r}$,

$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z} \tag{5.7.11}$$

that gives the location of the point (x, y, z) with respect to the origin of the coordinate system, allows the trial solution (5.7.8) to be expressed as

$$\mathbf{p} = \mathbf{A}e^{j(\omega t - \vec{k}\cdot\vec{r})} \tag{5.7.12}$$

The surfaces of constant phase are given by $\vec{k}\cdot\vec{r} = constant$. Since, from the definition of the gradient, $\vec{k} = \nabla(\vec{k}\cdot\vec{r})$ is a vector perpendicular to the surfaces of constant phase, $\vec{k}$ points in the direction of propagation. The magnitude of $\vec{k}$ is the *wave number* (or *propagation constant*) k and $k_x/k, k_y/k$, and k_z/k are the direction cosines of $\vec{k}$ with respect to the x, y, and z axes.

As a special case, let us examine a plane wave whose surfaces of constant phase are parallel to the z axis. Equation (5.7.8) reduces to

$$\mathbf{p} = \mathbf{A}e^{j(\omega t - k_x x - k_y y)} \tag{5.7.13}$$

The surfaces of constant phase are given by

$$y = -(k_x/k_y)x + constant \tag{5.7.14}$$

which describes plane surfaces parallel to the z axis with a slope of $-(k_x/k_y)$ in the x-y plane. If we examine $\mathbf{p}$ as a function of x and t for $y = 0$, we have

$$\mathbf{p}(x, 0, t) = \mathbf{A}e^{j(\omega t - k_x x)} \tag{5.7.15}$$

This oblique "slice" of the wave has an apparent wavelength $\lambda_x = 2\pi/k_x$ measured in the x direction. From Fig. 5.7.1 we see that $\lambda/\lambda_x = \cos\phi$ so that $k_x = k\cos\phi$. The same argument applies in the y direction for fixed x and yields $k_y = k\sin\phi$. Thus,

$$\vec{k} = k\cos\phi\,\hat{x} + k\sin\phi\,\hat{y} \tag{5.7.16}$$

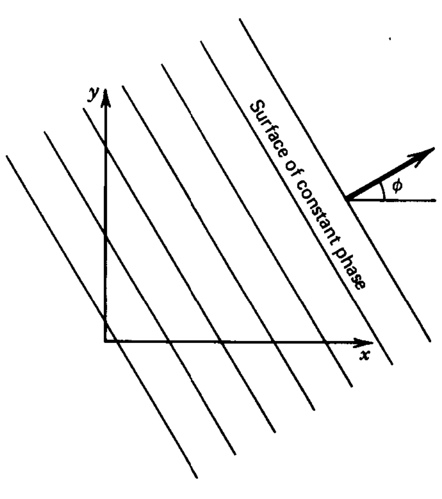

Figure 5.7.1 Surfaces of constant phase for a plane wave with wave number k traveling perpendicular to the z axis in a direction ϕ from the x axis.

and $\vec{k}$ is perpendicular to the z axis, pointing into the first quadrant of the x-y plane with an angle ϕ measured counterclockwise from the x axis. Substitution of $\vec{k}$ into (5.7.12) yields the convenient form

$$\mathbf{p} = \mathbf{A}e^{j(\omega t - kx\cos\phi - ky\sin\phi)} \tag{5.7.17}$$

5.8 ENERGY DENSITY

The energy transported by acoustic waves through a fluid medium is of two forms: (1) the *kinetic energy* of the moving elements and (2) the *potential energy* of the compressed fluid. Consider a small fluid element that moves with the fluid and occupies volume V_0 of the undisturbed fluid. The mass of the element is $\rho_0 V_0$ and its kinetic energy is

$$E_k = \tfrac{1}{2}\rho_0 V_0 u^2 \tag{5.8.1}$$

The change in potential energy associated with a volume change from V_0 to V is

$$E_p = -\int_{V_0}^{V} p\,dV \tag{5.8.2}$$

The negative sign indicates that the potential energy will increase (work is done *on* the element) when its volume is decreased by a positive acoustic pressure p. To carry this out, it is necessary to express all variables under the integral sign in terms of one variable—p, for example. From conservation of mass we have $\rho V = \rho_0 V_0$ so that

$$dV = -(V/\rho)\,d\rho \tag{5.8.3}$$

Now, with the use of $dp/d\rho = c^2$,

$$dV = (V/\rho c^2)\,dp \tag{5.8.4}$$

Substitution into (5.8.2) and integration of the acoustic pressure from 0 to p gives

$$E_p = \tfrac{1}{2}(p^2/\rho_0 c^2)V_0 \tag{5.8.5}$$

within the linear approximations. The total acoustic energy of the volume element is then

$$E = E_k + E_p = \tfrac{1}{2}\rho_0 V_0[u^2 + (p/\rho_0 c)^2] \tag{5.8.6}$$

and the *instantaneous energy density* $\mathscr{E}_i = E/V_0$ in joules per cubic meter (J/m^3) is

$$\mathscr{E}_i = \tfrac{1}{2}\rho_0[u^2 + (p/\rho_0 c)^2] \tag{5.8.7}$$

Both the pressure p and the particle speed u must be the *real* quantities obtained from the superposition of all acoustic waves present.

The instantaneous particle speed and acoustic pressure are functions of both position and time, and consequently the instantaneous energy density $\mathscr{E}_i$ is not necessarily constant throughout the fluid. The time average of $\mathscr{E}_i$ gives the *energy density* $\mathscr{E}$ at any point in the fluid:

$$\mathscr{E} = \langle \mathscr{E}_i \rangle_T = \frac{1}{T}\int_0^T \mathscr{E}_i \, dt \tag{5.8.8}$$

where the time interval T is one period of a harmonic wave.

The above expressions apply to any linear acoustic wave. To proceed further, it is necessary to know the relationship between p and u. For a plane harmonic wave traveling in the $\pm x$ direction, reference to (5.7.5) shows that $p = \pm\rho_0 cu$ so that (5.8.7) gives

$$\mathscr{E}_i = \rho_0 u^2 = p^2/\rho_0 c^2 \tag{5.8.9}$$

and if P and U are the amplitudes of the acoustic pressure and particle speed,

$$\mathscr{E} = PU/2c = P^2/2\rho_0 c^2 = \rho_0 U^2/2 \tag{5.8.10}$$

In more complicated cases, there is no guarantee that $p = \pm\rho_0 cu$ nor that the energy density is given by $\mathscr{E} = PU/2c$. However, (5.8.10) is approximately correct for progressive waves when the radii of curvature of the surfaces of constant phase are much greater than a wavelength. This occurs, for example, for spherical or cylindrical waves at distances of many wavelengths from their sources.

5.9 ACOUSTIC INTENSITY

The *instantaneous intensity* $I(t)$ of a sound wave is the instantaneous rate per unit area at which work is done by one element of fluid on an adjacent element. It is given by $I(t) = pu$ in watts per square meter (W/m^2). The *intensity* I is the *time average* of $I(t)$, the time-averaged rate of energy transmission through a unit area normal to the direction of propagation,

$$I = \langle I(t) \rangle_T = \langle pu \rangle_T = \frac{1}{T}\int_0^T pu \, dt \tag{5.9.1}$$

where for a monofrequency wave T is the period.

For a plane harmonic wave traveling in the $\pm x$ direction, $p = \pm\rho_0 cu$, so that

$$I = \pm P^2/2\rho_0 c \tag{5.9.2}$$

There is a similarity between (5.9.2) and corresponding equations for electromagnetic waves and voltage waves on transmission lines. First, reexpress (5.9.2) in terms of effective (root-mean-square) amplitudes. If we define F_e as the *effective amplitude* of a periodic quantity $f(t)$, then

$$F_e = \left(\frac{1}{T}\int_0^T f^2(t)\, dt\right)^{1/2} \tag{5.9.3}$$

where T is the period of the motion. For harmonic waves this yields

$$P_e = P/\sqrt{2} \quad \text{and} \quad U_e = U/\sqrt{2} \tag{5.9.4}$$

so that

$$I_\pm = \pm P_e U_e = \pm P_e^2/\rho_0 c \tag{5.9.5}$$

for a plane wave traveling in either the $+x$ or $-x$ direction. It must be emphasized that, while (5.9.1) is completely general, $I_\pm = \pm P_e U_e$ is exact only for plane harmonic waves and is approximately true for diverging waves at great distances from their sources.

5.10 SPECIFIC ACOUSTIC IMPEDANCE

The ratio of acoustic pressure to the associated particle speed in a medium is the *specific acoustic impedance*

$$\mathbf{z} = \mathbf{p}/\mathbf{u} \tag{5.10.1}$$

For plane waves this ratio is

$$\mathbf{z} = \pm\rho_0 c \tag{5.10.2}$$

The choice of sign depends on whether propagation is in the plus or minus x direction. The MKS unit of specific acoustic impedance is the Pa·s/m, often called the *rayl* (1 MKS rayl = 1 Pa· s/m) in honor of John William Strutt, Baron Rayleigh (1842–1919). The product $\rho_0 c$ often has greater acoustical significance as a characteristic property of the medium than does either ρ_0 or c individually. For this reason $\rho_0 c$ is called the *characteristic impedance* of the medium.

Although the specific acoustic impedance of the medium is a real quantity for progressive plane waves, this is not true for standing plane waves or for diverging waves. In general, $\mathbf{z}$ will be complex

$$\mathbf{z} = r + jx \tag{5.10.3}$$

where r is the *specific acoustic resistance* and x the *specific acoustic reactance* of the medium for the particular wave being considered.

The characteristic impedance of a medium for acoustic waves is analogous to the wave impedance $\sqrt{\mu/\varepsilon}$ of a dielectric medium for electromagnetic waves and to the characteristic impedance Z_0 of an electric transmission line. Numerical values of $\rho_0 c$ for some fluids and solids are given in Appendix A10.

For air at a temperature of 20°C and atmospheric pressure, the density is 1.21 kg/m^3 and the speed of sound is 343 m/s, giving

$$\rho_0 c = 415 \text{ Pa·s/m} \qquad \text{(air at 20°C)} \tag{5.10.4}$$

In distilled water at 20°C and 1 atm, the speed of sound is 1482.1 m/s and its density is 998.2 kg/m^3, resulting in a characteristic impedance of

$$\rho_0 c = 1.48 \times 10^6 \text{ Pa·s/m} \qquad \text{(water at 20°C)} \tag{5.10.5}$$

5.11 SPHERICAL WAVES

Expressed in spherical coordinates, the Laplacian operator is

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2}{\partial \phi^2} \tag{5.11.1}$$

where $x = r\sin\theta\cos\phi, y = r\sin\theta\sin\phi$, and $z = r\cos\theta$ (see Appendix A7). If the waves have spherical symmetry, the acoustic pressure p is a function of radial distance and time but not of the angular coordinates, and this equation simplifies to

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} \tag{5.11.2}$$

The wave equation for spherically symmetric pressure fields is then

$$\frac{\partial^2 p}{\partial r^2} + \frac{2}{r}\frac{\partial p}{\partial r} = \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} \tag{5.11.3}$$

Conservation of energy and the relationship $I = P^2/2\rho_0 c$ lead us to expect that the pressure amplitude might fall off as $1/r$, so that the quantity rp would have amplitude independent of r. Rewriting (5.11.3) with rp treated as the dependent variable results in

$$\frac{\partial^2(rp)}{\partial r^2} = \frac{1}{c^2}\frac{\partial^2(rp)}{\partial t^2} \tag{5.11.4}$$

If the product rp in this equation is considered as a single variable, the equation is the same as the plane wave equation with the general solution

$$p = \frac{1}{r}f_1(ct - r) + \frac{1}{r}f_2(ct + r) \tag{5.11.5}$$

for all $r > 0$. The solution fails at $r = 0$. The first term represents a spherical wave diverging from the origin with speed c; the second term represents a wave converging on the origin. For the outgoing wave, the solution fails at the origin because some source of sound is required to supply the energy carried away, and our wave equation does not contain any term representing this energy source. (See Sections 5.15 and 5.16.) In practice, this means that the medium must be excluded from some volume of space including the origin, and this volume must be occupied by whatever vibrating body serves as the sound source. For the incoming waves, energy is being focused at the origin and the small-amplitude approximations will fail. This failure will manifest itself in a *nonlinear* wave equation and strong acoustic losses limiting the attainable amplitudes.

The most important diverging spherical waves are harmonic. Such waves are represented in complex form by

$$\mathbf{p} = \frac{\mathbf{A}}{r}e^{j(\omega t - kr)} \tag{5.11.6}$$

Use of the relationships developed in Section 5.5 for a general wave allows the other acoustic variables to be expressed in terms of the pressure

$$\mathbf{\Phi} = -\mathbf{p}/j\omega\rho_0 \tag{5.11.7}$$

$$\vec{\mathbf{u}} = \nabla\mathbf{\Phi} = \hat{r}(1 - j/kr)\mathbf{p}/\rho_0 c \tag{5.11.8}$$

The observed acoustic variables are obtained by taking the real parts of (5.11.6)–(5.11.8).

It is apparent from (5.11.8) that, in contrast with plane waves, the particle speed is *not* in phase with the pressure. The specific acoustic impedance is not $\rho_0 c$, but rather

$$\mathbf{z} = \rho_0 c \frac{kr}{\left[1 + (kr)^2\right]^{1/2}} e^{j\theta} \tag{5.11.9}$$

or

$$\mathbf{z} = \rho_0 c \cos\theta \, e^{j\theta} \tag{5.11.10}$$

$$\cot\theta = kr \tag{5.11.11}$$

A geometric representation of θ is given in Fig. 5.11.1. As is true with many other acoustic phenomena, the product kr is the determining factor, rather than k or r separately. Since $kr = 2\pi r/\lambda$, the angle θ is a function of the ratio of the source distance to the wavelength. When the distance from the source is only a small fraction of a wavelength, the phase difference between the complex pressure and particle speed is large. At distances corresponding to a considerable number of wavelengths, **p** and **u** are very nearly in phase and the spherical wave assumes the characteristics of a plane wave. This is to be expected, since the wave fronts become essentially planar at great distances from the source.

Separating (5.11.9) into real and imaginary parts, we have

$$\mathbf{z} = \rho_0 c \frac{(kr)^2}{1 + (kr)^2} + j\rho_0 c \frac{kr}{1 + (kr)^2} \tag{5.11.12}$$

The first term is the specific acoustic resistance and the second term the specific acoustic reactance. Both approach zero for very small values of kr, but for very large values of kr the resistive term approaches $\rho_0 c$ and the reactive term approaches zero.

The absolute magnitude z of the specific acoustic impedance is equal to the ratio of the pressure amplitude P of the wave to its speed amplitude U,

$$z = P/U = \rho_0 c \cos\theta \tag{5.11.13}$$

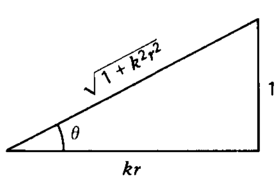

Figure 5.11.1 The relationship between θ and kr at a distance r from the source of a spherical wave of wave number k.

and the relationship between pressure and speed amplitude may be written as

$$P = \rho_0 c U \cos\theta \tag{5.11.14}$$

For large values of kr, $\cos\theta$ approaches unity and the relationship between pressure and speed is that for a plane wave. As the distance from the source of a spherical acoustic wave to the point of observation is decreased, both kr and $\cos\theta$ decrease, so that larger and larger particle speeds are associated with a given pressure amplitude. For very small distances from a point source, the particle speed corresponding to even very low acoustic pressures becomes impossibly large: a source small compared to a wavelength is inherently incapable of generating waves of large intensity.

Let us rewrite (5.11.6) as

$$\mathbf{p} = \frac{A}{r} e^{j(\omega t - kr)} \tag{5.11.15}$$

where we have chosen a new origin of time so that $\mathbf{A}$ is a real constant A. Then A/r is the *pressure amplitude* of the wave. The pressure amplitude in a spherical wave is not constant, as it is for a plane wave, but decreases inversely with the distance from the source. The actual pressure is the real part of (5.11.15),

$$p = \frac{A}{r} \cos(\omega t - kr) \tag{5.11.16}$$

Since $\mathbf{u} = \mathbf{p}/\mathbf{z}$, the corresponding complex expression for the particle speed is

$$\mathbf{u} = \frac{A}{r\mathbf{z}} e^{j(\omega t - kr)} \tag{5.11.17}$$

Replacing $\mathbf{z}$ by (5.11.10) and then taking the real part of the resulting expression gives the actual particle speed,

$$u = \frac{1}{\rho_0 c} \frac{A}{r} \frac{1}{\cos\theta} \cos(\omega t - kr - \theta) \tag{5.11.18}$$

It is apparent that, since θ is a function of kr, the speed amplitude

$$U = \frac{1}{\rho_0 c} \frac{A}{r} \frac{1}{\cos\theta} \tag{5.11.19}$$

is not inversely proportional to the distance from the source.

For a harmonic spherical wave (5.9.1) yields

$$I = \frac{1}{T} \int_0^T P \cos(\omega t - kr)\, U \cos(\omega t - kr - \theta)\, dt = \frac{PU\cos\theta}{2} = \frac{P^2}{2\rho_0 c} \tag{5.11.20}$$

where the factor $\cos\theta$ is analogous to the power factor of an alternating-current circuit. Note that the formula $I = P^2/2\rho_0 c$ is *exactly* true for both plane and spherical waves.

The average rate at which energy flows through a closed spherical surface of radius r surrounding a source of symmetric spherical waves is

$$\Pi = 4\pi r^2 I = 4\pi r^2 P^2/2\rho_0 c \tag{5.11.21}$$

or since $p = A/r$

$$\Pi = 2\pi A^2/\rho_0 c \tag{5.11.22}$$

The average rate of energy flow through any spherical surface surrounding the origin is independent of the radius of the surface, a statement of energy conservation in a lossless medium.

5.12 DECIBEL SCALES

It is customary to describe sound pressures and intensities using logarithmic scales known as *sound levels*. One reason for this is the very wide range of sound pressures and intensities encountered in the acoustic environment; audible intensities range from approximately 10^{-12} to 10 W/m^2. Using a logarithmic scale compresses the range of numbers required to describe this wide range of intensities and is also consistent with the fact that humans judge the relative loudness of two sounds by the ratio of their intensities.

The most generally used logarithmic scale for describing sound levels is the *decibel* (dB) scale. The *intensity level* IL of a sound of intensity I is defined by

$$IL = 10\log(I/I_{ref}) \tag{5.12.1}$$

where I_{ref} is a reference intensity, IL is expressed in *decibels referenced to* I_{ref} (dB *re* I_{ref}), and "log" represents the logarithm to base 10.

We have shown in Sections 5.9 and 5.11 that intensity and effective pressure of progressive plane and spherical waves are related by $I = P_e^2/\rho_0 c$. Consequently, the intensities in (5.12.1) may be replaced by expressions for pressure, leading to the sound pressure level

$$SPL = 20\log(P_e/P_{ref}) \tag{5.12.2}$$

where SPL is expressed in dB *re* P_{ref} with P_e the measured effective pressure amplitude of the sound wave and P_{ref} the reference effective pressure amplitude. If we choose $I_{ref} = P_{ref}^2/\rho_0 c$, then IL *re* $I_{ref} = SPL$ *re* P_{ref}.

Throughout the scientific disciplines a number of units are used to specify pressures, and many of these are found in acoustics. In addition, reference levels of various degrees of antiquity are encountered. Let us first catalog a few units:

CGS units

1 dyne/cm^2, also called the microbar (μbar). (The microbar was originally 10^{-6} atm but is now *defined* as 1 dyne/cm^2.)

MKS units

1 pascal (Pa), *defined* as 1 N/m^2 in the SI system of units

Others

1 atmosphere (atm) $\equiv 1.01325 \times 10^5$ Pa $= 1.01325 \times 10^6\,\mu$bar
1 kilogram/cm^2 (kgf/cm^2) $\equiv 0.980665 \times 10^5$ Pa $= 0.967841$ atm

Equivalents

1 μbar $\equiv 0.1$ N/m^2 $\equiv 10^5\ \mu$Pa

The reference standard for airborne sounds is 10^{-12} W/m^2, which is approximately the intensity of a 1 kHz pure tone that is just barely audible to a person with unimpaired hearing. Substitution of this intensity into (5.9.2) shows that it corresponds to a peak pressure amplitude of

$$P = (2\rho_0 cI)^{1/2} = 2.89 \times 10^{-5}\ \text{Pa} \tag{5.12.3}$$

or a corresponding effective (root-mean-square) pressure of

$$P_e = P/\sqrt{2} = 20.4\ \mu\text{Pa} \tag{5.12.4}$$

This latter pressure, rounded to 20 μPa, is the reference for sound pressure levels in air. Essentially identical numerical results are obtained in air using either 10^{-12} W/m^2 in (5.12.1) or 20 μPa in (5.12.2) for plane or spherical progressive waves. However, in certain more complex sound fields, such as standing waves, intensity and pressure are no longer simply related by (5.9.5) and (5.11.20) and consequently (5.12.1) and (5.12.2) will not yield identical results. Since the voltage outputs of microphones and hydrophones commonly used in acoustic measurements are proportional to pressure, sound pressure levels are used more widely than intensity levels.

Three different pressures are encountered as reference pressures in underwater acoustics. One is an effective pressure of 20 μPa (the same as the reference pressure in air). The second reference pressure is 1 μbar and the third is 1 μPa. The last is now the standard.

This abundance of reference pressures can lead to confusion unless care is taken to always specify the reference pressure being used: *SPL re* 20 μPa, *re* 1 μPa, or *re* 1 μbar. Table 5.12.1 summarizes the various conventions.

From the above discussion, note that a given acoustic pressure in air corresponds to a much higher intensity than does the same acoustic pressure in water. Since (5.9.5) or (5.11.20) shows that, for a given pressure amplitude, intensity is inversely

Table 5.12.1 References and conversions for sound pressure levels

Medium	*Reference*	*Nearly equivalent to*
Air	10^{-12} W/m^2	20 μPa
	20 μPa = 0.0002 μbar	10^{-12} W/m^2
Water	1 μbar = $10^5\mu$Pa	6.76×10^{-9} W/m^2
	0.0002 μbar = 20μPa	2.70×10^{-16} W/m^2
	1 μPa	6.76×10^{-19} W/m^2

SPL re 1 μbar + 100 = *SPL re* 1 μPa
SPL re 0.0002 μbar − 74 = *SPL re* 1 μbar
SPL re 0.0002 μbar + 26 = *SPL re* 1 μPa

proportional to the characteristic impedance of the medium, the ratio of the intensity in air to that in water for the *same acoustic pressure* is $(1.48 \times 10^6)/415 = 3570$. On the other hand, if we compare two acoustic waves of the *same frequency and particle displacement*, the ratio of the intensity in air to that in water is 1/3570.

Because of the conveniences afforded by decibel scales, electrical quantities are often specified in terms of levels. For example, the voltage level VL is defined as

$$VL(re\ V_{ref}) = 20\log(V/V_{ref}) \tag{5.12.5}$$

where V is the effective voltage and V_{ref} is some convenient reference effective voltage.

By convention, the subscript "e" and the adjective "effective" are omitted when specifying effective amplitudes of electrical quantities. Two common reference voltages are 1 V and 0.775 V. (This latter stems from an old reference, the voltage required to dissipate 1 mW of electrical power in a 600 ohm resistor.) Comparison of voltage levels referenced to the two common reference voltages reveals that

$$VL(re\ 0.775\ \text{V}) = VL(re\ 1\ \text{V}) + 2.21 \tag{5.12.6}$$

The abilities of electroacoustic sources and receivers to convert between electrical and acoustic quantities can be expressed in terms of *sensitivities*. For example, the *open circuit receiving sensitivity* $\mathcal{M}_o$ of a microphone is defined as

$$\mathcal{M}_o = (V/P_e)_{I=0} \tag{5.12.7}$$

where V is the output voltage produced (with negligible output current I) when the microphone is placed at a point where the effective pressure amplitude was P_e in the absence of the microphone. This is one of a number of sensitivities that can be defined for a microphone; more detail will be found in Chapter 14. A sensitivity $\mathcal{M}$ is usually expressed in terms of the associated *sensitivity level* $\mathcal{ML}$

$$\mathcal{ML}(re\ \mathcal{M}_{ref}) = 20\log(\mathcal{M}/\mathcal{M}_{ref}) \tag{5.12.8}$$

where $\mathcal{M}_{ref}$ is a reference sensitivity such as 1 V/μbar or 1 V/Pa.

Relationships among P, V, and $\mathcal{M}_o$ can be expressed in terms of either the fundamental quantities or the associated levels. For example, assume that a microphone of known sensitivity level $\mathcal{ML}$ dB *re* M_{ref} gives an output level VL dB *re* V_{ref}, and we wish to know the sound pressure level SPL dB *re* P_{ref} of the sound field. Algebraic manipulation reveals

$$SPL(re\ P_{ref}) = VL(re\ V_{ref}) - \mathcal{ML}(re\ \mathcal{M}_{ref}) + 20\log\left(\frac{V_{ref}/P_{ref}}{\mathcal{M}_{ref}}\right) \tag{5.12.9}$$

In complete analogy, an acoustic source is characterized by a source sensitivity $\mathcal{S} = P_e/V$ and a source sensitivity level $\mathcal{SL}$

$$\mathcal{SL}\left(re\ \mathcal{S}_{ref}\right) = 20\log\left(\frac{P_e/V}{\mathcal{S}_{ref}}\right) \tag{5.12.10}$$

where V is the voltage applied to the electrical input of the source, P_e is the effective pressure at some specified location (usually on the acoustic axis of the source extrapolated back from large distances to 1 m from the face of the source), and $\mathcal{S}_{ref}$ is a reference sensitivity such as 1 μPa/V or 1 μbar/V.

*5.13 CYLINDRICAL WAVES

Three-dimensional cylindrical waves have significant applications in atmospheric and underwater propagation. The wave equation for cylindrical propagation is (5.5.4) with the Laplacian expressed in cylindrical coordinates,

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}\right)p = \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} \tag{5.13.1}$$

Recall that the physical interpretation of r depends on the coordinates being used. In spherical coordinates, r denotes the *radial distance from the origin* to the field point in any direction. In cylindrical coordinates it refers to the *perpendicular distance from the z axis* to the field point.

Assuming harmonic solutions and separation of variables,

$$\mathbf{p}(r,\theta,z,t) = \mathbf{R}(r)\mathbf{\Theta}(\theta)\mathbf{Z}(z)e^{j\omega t} \tag{5.13.2}$$

allows (5.13.1) to be decomposed into three differential equations and provides a relationship for the separation constants,

$$\begin{aligned} \frac{d^2\mathbf{R}}{dr^2} + \frac{1}{r}\frac{d\mathbf{R}}{dr} + \left(k_r^2 - \frac{m^2}{r^2}\right)\mathbf{R} &= 0 \\ \frac{d^2\mathbf{Z}}{dz^2} + k_z^2\mathbf{Z} &= 0 \\ \frac{d^2\mathbf{\Theta}}{d\theta^2} + m^2\mathbf{\Theta} &= 0 \\ (\omega/c)^2 = k^2 &= k_r^2 + k_z^2 \end{aligned} \tag{5.13.3}$$

The equation for $\mathbf{\Theta}$ is the same as for the circular membrane. If we assume azimuthal symmetry, then $m = 0$. The equation for $\mathbf{Z}$ is solved by sinusoidal or complex exponential functions and corresponds to oblique waves whose propagation vectors have a projection on the z axis of k_z. The simplest case is $k_z = 0$, which describes waves whose surfaces of constant phase are cylinders concentric with the z axis. These two simplifications leave us with the z-independent, cylindrically symmetric solutions of the radial wave equation

$$\frac{d^2\mathbf{R}}{dr^2} + \frac{1}{r}\frac{d\mathbf{R}}{dr} + k^2\mathbf{R} = 0 \tag{5.13.4}$$

Reference to Section 4.4 and use of $m = 0$ gives the general solution

$$\mathbf{p}(r,t) = [\mathbf{A}J_0(kr) + \mathbf{B}Y_0(kr)]e^{j\omega t} \tag{5.13.5}$$

Since Y_0 diverges as $r \to 0$, (5.13.5) fails at $r = 0$ unless $\mathbf{B} = 0$. The reasons for this failure are identical with those discussed for spherical waves in Section 5.11, so when $\mathbf{B} \neq 0$ the z-axis must be excluded from the volume within which (5.13.5) can be applied.

Examination of (5.13.5) reveals that if $\mathbf{p}$ is to be a traveling wave, it must be a complex function of space. Furthermore, assuming that $I = P^2/2\rho_0 c$ is at least approximately true at large distances and using conservation of energy suggests that the pressure $\mathbf{p}(r,t)$ should be proportional to

$$\frac{1}{\sqrt{r}} e^{j(\omega t \pm kr)} \tag{5.13.6}$$

The $\pm$ sign in the exponent gives incoming or outgoing waves. The combinations of **A** and **B** that will produce (5.13.6) in the limit $r \to \infty$ can be found from the large-argument asymptotic forms of J_0 and Y_0,

$$\begin{aligned} J_0(kr) &\to (2/\pi kr)^{1/2} \cos(kr - \pi/4) \\ Y_0(kr) &\to (2/\pi kr)^{1/2} \sin(kr - \pi/4) \end{aligned} \tag{5.13.7}$$

Equation (5.13.5) will take on the form (5.13.6) if $\mathbf{B} = \pm j\mathbf{A}$. These combinations are the Bessel functions of the third kind, or *Hankel functions*,

$$\begin{aligned} H_0^{(1)}(kr) &= J_0(kr) + jY_0(kr) \\ H_0^{(2)}(kr) &= J_0(kr) - jY_0(kr) \end{aligned} \tag{5.13.8}$$

For an outgoing harmonic cylindrical wave with azimuthal symmetry and independent of z, the appropriate solution of (5.13.4) is

$$\mathbf{p}(r,t) = \mathbf{A} H_0^{(2)}(kr) e^{j\omega t} \tag{5.13.9}$$

While (5.13.9) was developed by imposing the asymptotic behavior (5.13.6) and using the asymptotic form of the Hankel function for large kr, it is an exact solution of (5.13.4) for all $r > 0$. (This is often referred to as imposing a *radiation boundary condition at infinity*.) For large kr this solution has asymptotic behavior

$$\mathbf{p}(r,t) \to \mathbf{A}(2/\pi kr)^{1/2} e^{j(\omega t - kr + \pi/4)} \tag{5.13.10}$$

Generating the velocity potential $\mathbf{\Phi}$ with (5.5.9), and then using (5.5.7) gives the particle speed

$$\mathbf{u}(r,t) = -j(\mathbf{A}/\rho_0 c) H_1^{(2)}(kr) e^{j\omega t} \tag{5.13.11}$$

with the help of Appendix A4. The specific acoustic impedance **z** follows at once:

$$\mathbf{z} = j\rho_0 c H_0^{(2)}(kr)/H_1^{(2)}(kr) \tag{5.13.12}$$

In the limit $kr \gg 1$, the asymptotic approximations of the Hankel functions show that $\mathbf{z} \to \rho_0 c$ at large distances. This is to be expected, since as kr increases beyond unity, the radii of curvature of the surfaces of constant phase become much larger than a wavelength and the waveform looks locally more and more like a plane wave.

Calculation of the acoustic intensity is a little more complicated. The instantaneous intensity is $I(r,t) = pu$. This yields

$$I(r,t) = (A^2/\rho_0 c)\left[J_0(kr)\cos\omega t + Y_0(kr)\sin\omega t\right]\left[J_1(kr)\sin\omega t - Y_1(kr)\cos\omega t\right] \tag{5.13.13}$$

where for ease we have chosen time so that $\mathbf{A} = A$. Taking the time average leaves us with the intensity

$$I(r) = (A^2/2\rho_0 c)\left[J_1(kr)Y_0(kr) - J_0(kr)Y_1(kr)\right] \tag{5.13.14}$$

The quantity in square brackets is the *Wronskian* of $J_0(kr)$ and $Y_0(kr)$ and has the known value $2/\pi kr$. Substitution gives us the result

$$I(r) = \frac{2A^2/\pi kr}{2\rho_0 c} = \frac{P_{as}^2}{2\rho_0 c} \tag{5.13.15}$$

where P_{as} is the *asymptotic* amplitude

$$P_{as} = A(2/\pi kr)^{1/2} \tag{5.13.16}$$

of $\mathbf{p}(r,t)$. The intensity falls off as $1/r$, as conservation of energy in a lossless fluid says it must for a cylindrically diverging wave, but the intensity is *not* simply $P^2/2\rho_0 c$ everywhere, as it was for plane and spherical waves.

*5.14 RAYS AND WAVES

Up to this point, we have considered the propagation of sound in a homogeneous medium having a constant speed of sound. The speed of sound is often a function of space and instead of plane, spherical, and cylindrical waves of infinite spatial extent we find waves whose directions of propagation change as they traverse the medium. One technique for studying this effect is based on the assumption that the energy is carried along reasonably well-defined paths through the medium, so that it is useful to think of *rays* rather than waves. In many cases, description in terms of rays is much easier than in terms of waves. However, rays are not exact replacements for waves, but only approximations that are valid under certain rather restrictive conditions.

(a) The Eikonal and Transport Equations

The wave equation with spatially dependent sound speed is

$$\left(\nabla^2 - \frac{1}{c^2(x,y,z)}\frac{\partial^2}{\partial t^2}\right)\mathbf{p}(x,y,z,t) = 0 \tag{5.14.1}$$

For sound traversing such a fluid, the amplitude varies with position and the surfaces of constant phase can be complicated. Assume a trial solution

$$\mathbf{p}(x,y,z,t) = A(x,y,z)e^{j\omega[t-\Gamma(x,y,z)/c_0]} \tag{5.14.2}$$

where Γ has units of length and c_0 is a reference speed to be defined later. The quantity Γ/c_0 is the *eikonal*.The values of (x,y,z) for which Γ is constant define the surfaces of constant phase. From the basic definition of the gradient, $\nabla\Gamma$ is everywhere perpendicular to these surfaces.

Substituting the trial solution into (5.14.1) and collecting real and imaginary parts gives

$$\begin{aligned} -\frac{\nabla^2 A}{A} + \left(\frac{\omega}{c_0}\right)^2 \nabla\Gamma\cdot\nabla\Gamma &= \left(\frac{\omega}{c}\right)^2 \\ 2\frac{\nabla A}{A}\cdot\nabla\Gamma + \nabla^2\Gamma &= 0 \end{aligned} \tag{5.14.3}$$

These equations are difficult to solve because they are coupled and nonlinear. However, if we require

$$\left|\frac{\nabla^2 A}{A}\right| \ll \left(\frac{\omega}{c}\right)^2 \tag{5.14.4}$$

then the first of (5.14.3) assumes the simpler approximate form

$$\boxed{\nabla\Gamma \cdot \nabla\Gamma = (c_0/c)^2 = n^2} \tag{5.14.5}$$

where $n = c_0/c$ is the *index of refraction*. Equation (5.14.5) is the *eikonal equation*. It is immediately clear that

$$\nabla\Gamma = n\hat{s} \tag{5.14.6}$$

where the unit vector $\hat{s}$ gives the local direction of propagation. Given $\hat{s}$ at a point in the sound field and then tracing how that specific $\hat{s}$ changes direction as it is advanced point to point within the fluid defines a *ray path*, the trajectory followed by the particular ray. Since according to (5.14.6) the local direction of propagation of the ray is perpendicular to the eikonal, in this approximation each ray is always perpendicular to the local surface of constant phase. Sufficient conditions for satisfying (5.14.4) are (1) the amplitude of the wave and (2) the speed of sound do not change significantly over distances comparable to a wavelength. If we consider a beam of sound with transverse dimensions much greater than a wavelength traveling through a fluid, (5.14.4) states that the eikonal equation may be applied over the central portion of the beam where A is not rapidly varying. At the edges of the beam, however, A may rapidly reduce to zero over distances on the order of a wavelength and the restriction (5.14.4) fails. The failure manifests itself in the *diffraction* of sound at the edges of the beam—analogous to the diffraction of light through a slit or pinhole. This means that (5.14.5) is accurate only in the limit of high frequencies—how high depends on the spatial variations of c and A. More rigorous *necessary* conditions can be stated, but their physical meanings are less direct. Indeed, there are propagating waves (Problem 5.14.10) that do not satisfy the sufficient conditions, but for which (5.14.5) is valid.

Analysis of the *transport equation*, the second of (5.14.3), will provide further justification for the concept of rays. Substitution of (5.14.6) into this equation and a little manipulation (Problem 5.14.4*a*) gives

$$\frac{d}{ds}\ln(nA^2) = -\nabla \cdot \hat{s} \tag{5.14.7}$$

For distances more than a few wavelengths away from the source, the intensity is

$$I = P^2/2\rho_0 c = nA^2/2\rho_0 c_0 \tag{5.14.8}$$

so that (5.14.7) becomes

$$\boxed{\frac{1}{I}\frac{dI}{ds} = -\nabla \cdot \hat{s}} \tag{5.14.9}$$

The left side is the fractional change of intensity per unit distance *along a ray path* and $\nabla \cdot \hat{s}$ describes how the rays converge or diverge. Now apply Gauss's theorem to the volume defined by the bundle of rays shown in Fig. 5.14.1. The volume is chosen so that the rays pass only through the end caps. Integrate (5.14.9) over the volume $S\,\Delta h$. On the left side the volume integral becomes $(1/I)(dI/ds)S\,\Delta h = S[d(\ln I)/ds]\,\Delta h$. On the right side, use of Gauss's theorem converts the volume integral into a surface integral of $\hat{s} \cdot \hat{n}$. Since the rays enter and leave the volume only through the end caps, this integral yields the incremental change $-\Delta S$ in the cross-sectional area of the bundle of rays. Finally, recognize that ΔS is obtained along the ray path, so that $\Delta S = (dS/ds)\,\Delta h$. This gives us $d(\ln I)/ds = -d(\ln S)/ds$

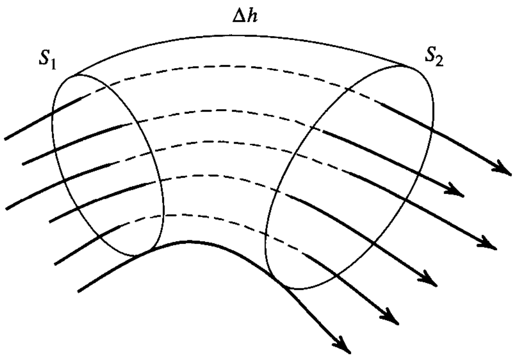

Figure 5.14.1 An elemental volume of a ray bundle with end caps of areas S_1 and S_2 separated by a distance Δh along the rays.

and the result

$$IS = constant \tag{5.14.10}$$

Thus, within the limitations of the eikonal equation, the energy within a ray bundle remains constant. This is the mathematical justification for the intuitive concept that energy in a sound wave travels along rays. Any mathematical or geometrical technique that allows a bundle of rays to be traced through space will allow calculation of the intensity throughout space.

(b) The Equations for the Ray Path

Solution of the eikonal equation (5.14.6) gives the direction $\hat{s}$ for each ray at every point along its path. The problem of obtaining the ray paths is equivalent to solving for the successive locations of $\hat{s}$. First, express $\hat{s}$ in terms of its direction cosines,

$$\begin{aligned} \hat{s} &= \alpha\hat{x} + \beta\hat{y} + \gamma\hat{z} \\ \alpha^2 + \beta^2 + \gamma^2 &= 1 \end{aligned} \tag{5.14.11}$$

where the direction cosines are $\alpha = dx/ds, \beta = dy/ds$, and $\gamma = dz/ds$ with dx, dy, and dz the coordinate changes resulting from a step ds in the $\hat{s}$ direction along the ray path. If the change in any scalar along the ray

$$\frac{d}{ds} = \alpha\frac{\partial}{\partial x} + \beta\frac{\partial}{\partial y} + \gamma\frac{\partial}{\partial z} \tag{5.14.12}$$

is applied to both sides of the first of (5.14.11), the components become

$$\begin{aligned} \frac{d}{ds}(n\alpha) &= \frac{\partial n}{\partial x} \\ \frac{d}{ds}(n\beta) &= \frac{\partial n}{\partial y} \\ \frac{d}{ds}(n\gamma) &= \frac{\partial n}{\partial z} \end{aligned} \tag{5.14.13}$$

(See Problem 5.14.4*b* for details.) The eikonal equation relates the changes in the direction of propagation of a ray to the gradient of the local index of refraction. Given $n(x, y, z)$, it is possible to trace the trajectories of every element of a wave front through the medium. A simple example follows.

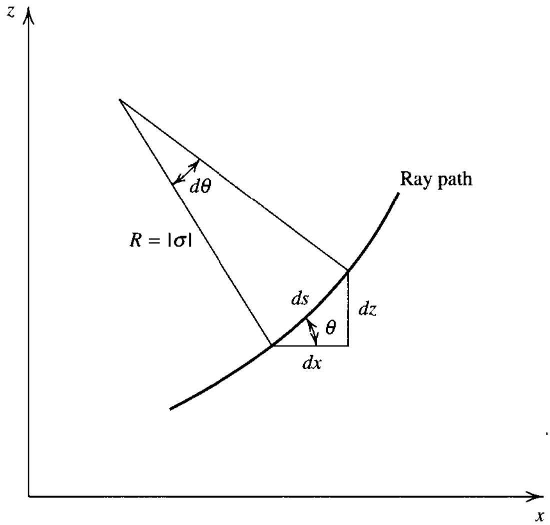

Figure 5.14.2 An element of a ray path in the x-z plane of length ds making an angle θ with the x axis will have a radius of curvature $R = |c/(g\cos\theta)|$, where c is the speed of sound and g is the sound speed gradient.

(c) The One-Dimensional Gradient

The speed of sound can often be considered a function of only one spatial dimension. In both the ocean and the atmosphere, for example, variations of the speed of sound with horizontal range are generally much weaker than the variations with depth or height.

Let the index of refraction be a function of z alone, where z is the vertical coordinate. Then (5.14.13) becomes

$$\begin{aligned} \frac{d}{ds}(n\alpha) &= 0 \\ \frac{d}{ds}(n\beta) &= 0 \\ \frac{d}{ds}(n\gamma) &= \frac{dn}{dz} \end{aligned} \tag{5.14.14}$$

If the coordinate axes are oriented so that a ray starts off in the x-z plane and makes an angle θ with the x axis (see Fig. 5.14.2), the initial value of β is zero and, according to the second of the above equations, β will remain zero and the ray path will stay in the x-z plane. We can then identify $\alpha = \cos\theta$ and $\gamma = \sin\theta$, and the remaining equations in (5.14.14) become

$$\begin{aligned} \frac{d}{ds}(n\cos\theta) &= 0 \\ \frac{d}{ds}(n\sin\theta) &= \frac{dn}{dz} \end{aligned} \tag{5.14.15}$$

The first of (5.14.15) reveals that $n\cos\theta$ must have the same value at every point along a particular ray path. If we specify the angle of elevation θ_0 where the ray path encounters the reference speed c_0, we then have a statement of *Snell's law,*

$$\boxed{\frac{\cos\theta}{c} = \frac{\cos\theta_0}{c_0}} \tag{5.14.16}$$

From the definition of $n = c_0/c$ we see that dc/dz has the opposite sign from dn/dz. Then, the second of (5.14.15) shows that when the sound speed increases in the z direction, θ must decrease along the ray—the ray turns toward the lower sound speed. When the sound speed decreases in the z direction, θ increases along the ray—the ray still turns toward the lower sound speed. A *ray always bends toward the neighboring region of lower sound speed.* While this equation cannot be solved without knowing the dependence of c on z, it can be put into a geometrical form. With reference to Fig. 5.14.2, $dz = \sin\theta\, ds$ and $ds = \sigma\, d\theta$, where σ is a measure of the amount and orientation of the curvature of the ray path. For Fig. 5.14.2, $d\theta$ increases along the ray, so σ is *positive*. If the ray path were to curve the other way (with negative second derivative), σ would be *negative*. The magnitude of σ is the *radius of curvature* R. Use of these geometrical relationships along with (5.14.15) and (5.14.16) gives

$$\begin{aligned} \sigma &= -\frac{1}{g}\frac{c_0}{\cos\theta_0} \\ g &= \frac{dc}{dz} \end{aligned} \qquad (5.14.17)$$

where g is the *gradient of the sound speed*. The radius of curvature R of the ray is inversely proportional to $|g|$ at each point along the path. Each ray path must be computed separately since each has its own value of the Snell's law constant $(\cos\theta_0)/c_0$. See Chapter 15 for examples of ray tracing when g is piecewise constant.

(d) Phase and Intensity Considerations

Let I_1 be the acoustic intensity referred to a distance 1 meter from a source along a bundle of rays with initial angle of elevation θ_0. It is desired to know the intensity I of this bundle at some range x as shown in Fig. 5.14.3. For a lossless medium, the intensity multiplied by the cross-sectional area of the bundle must be constant. Let S_1 be the cross section of the bundle at 1 meter from the source and S the cross section at range x, where the intensity is I. Examination of the geometry of the figure reveals $S = x\,\Delta\phi\,\sin\theta\,dx$ and $S_1 = \Delta\phi\,d\theta_0\cos\theta_0$

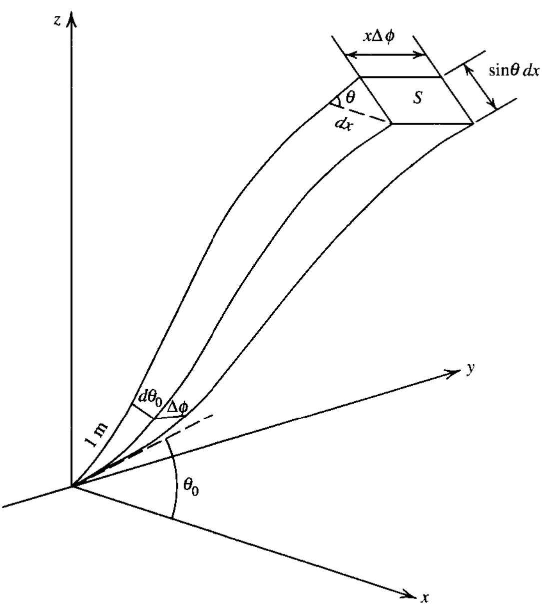

Figure 5.14.3 A ray bundle in the x-z plane that is used to determine intensity from conservation of energy. At $x = 1$ m the cross-sectional area of the bundle is $\Delta\phi\, d\theta_0 \cos\theta_0$, where $\Delta\phi$ is the horizontal angular width of the bundle, $d\theta_0$ its initial vertical angular width, and θ_0 the initial angle of elevation. The area at range x where the ray makes an angle θ with the horizontal is $x\,\Delta\phi \sin\theta\, dx$, where $dx = (\partial x/\partial\theta_0)_z\, d\theta_0$.

so that conservation of energy provides $Ix \sin\theta\, dx = I_1 \cos\theta_0\, d\theta_0$. The element dx can be expressed by $dx = (\partial x/\partial\theta_0)_z\, d\theta_0$, where the range x must be written as a function of θ_0 and z. Combination of the above equations results in

$$\frac{I}{I_1} = \frac{1}{x}\frac{\cos\theta_0}{\sin\theta}\frac{1}{(\partial x/\partial\theta_0)_z} \tag{5.14.18}$$

When neighboring rays from the source intersect at some field point, the partial derivative vanishes and the intensity becomes infinite. The *locus* of such neighboring points may form a surface of infinite intensity, called a *caustic*. The intensity does not really become infinite on a caustic, of course, because the conditions necessary for the validity of the eikonal equation fail. Caustics do, however, identify regions of high intensity where there is strong focusing of acoustic energy.

A different situation can occur when *nonadjacent* ray paths intersect at some point away from the source. An example would be reflection from a boundary, for which the direct and reflected ray paths intersect. For a continuous monofrequency signal generated by the source, there are two different approaches to this combination:

1. **Incoherent Summation.** If spatial irregularities and fluctuations in the boundary or the speed of sound profile are sufficient to randomize the relative phases of the signals propagating over intersecting ray paths, then we can make a *random phase approximation*. Under this approximation, a reasonable estimate of the average acoustic intensity where the different paths intersect is the sum of the intensities for the individual rays. The acoustic pressure amplitude is then the square root of the sum of the squares of the pressure amplitudes of the signals where they intersect.
2. **Coherent Summation.** If, however, the irregularities in propagation do not appreciably affect the phases of the signals, *phase coherence* is retained and it is necessary to calculate the travel time Δt of each signal along its path so that the relative phases can be obtained. The total pressure and phase of the combination is then obtained by adding the phasors with proper regard for the phases.

A typical case of continuous wave propagation may lie somewhere between these two idealizations. Coherence is favored by short-range, low-frequency, smooth boundaries, few boundary reflections, and a stable and smooth speed of sound profile. Random phasing is favored by the converse conditions. The travel time can be calculated in a number of ways, each simple to derive:

$$\Delta t = \int_0^s \frac{1}{c}\, ds = \int_{x_0}^x \frac{1}{c\cos\theta}\, dx = \int_{z_0}^z \frac{1}{c\sin\theta}\, dz = \int_{\theta_0}^\theta \frac{1}{g\cos\theta}\, d\theta \tag{5.14.19}$$

where each integrand must be expressed as a function of the variable of integration.

For very short transient acoustic signals, the travel times along the various ray paths may be so different that the individual arrivals do not overlap each other. This would then yield a combined signal in which each of the arrivals along a different ray path would be separate and distinct. As the transients become longer, however, partial overlapping would generate a complicated combination.

*5.15 THE INHOMOGENEOUS WAVE EQUATION

In previous sections we developed a wave equation that applied to regions of space devoid of any sources of acoustic energy. However, a source must be present to generate an acoustic field. Certain sources internal to the region of interest can be taken into account by introducing time-dependent boundary conditions, as described for strings, bars, and membranes. In Chapter 7, this is the procedure that will be used to relate the motion of the surface of a source to the sound field created by the source. However, there are times when it is more convenient to adopt an approach that builds the sources into the wave equation by modifying the fundamental equations to include source terms.

1. If mass is injected (or appears to be) into the space at a rate per unit volume $G(\vec{r},t)$, the linearized equation of continuity becomes

$$\rho_0 \frac{\partial s}{\partial t} + \nabla \cdot \left(\rho_0 \vec{u}\right) = G(\vec{r},t) \tag{5.15.1}$$

This $G(\vec{r},t)$ is generated by a closed surface that changes volume, such as the outer surface of an explosion, an imploding evacuated glass sphere, or a loudspeaker in an enclosed cabinet.

2. If there are *body forces* present in the fluid, a body force per unit volume $\vec{F}(\vec{r},t)$ must be included in Euler's equation. The linearized equation of motion becomes

$$\rho_0 \frac{\partial \vec{u}}{\partial t} + \nabla p = \vec{F}(\vec{r},t) \tag{5.15.2}$$

Examples of this kind of force are those produced by a source that moves through the fluid without any change in volume, such as the cone of an unbaffled loudspeaker or a vibrating sphere of constant volume.

If these two modifications are combined with the linearized equation of state, an inhomogeneous wave equation is obtained,

$$\nabla^2 p - \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} = -\frac{\partial G}{\partial t} + \nabla \cdot \vec{F} \tag{5.15.3}$$

3. A third type of sound source was first described by Lighthill[2] in 1952. Lighthill's result includes the effects of shear and bulk viscosity and its derivation is beyond the scope of this text. However, in virtually all cases of practical interest, the contributions from viscous forces are completely negligible and a simplified derivation can be made. The source of acoustic excitation lies in the convective term $(\vec{u} \cdot \nabla)\vec{u}$ of the acceleration. Retaining this term and discarding the terms involving viscosity and gravity in (5.4.8) gives

$$-\nabla p = \rho\left(\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\vec{u}\right) = \frac{\partial(\rho \vec{u})}{\partial t} - \vec{u}\frac{\partial \rho}{\partial t} + \rho(\vec{u} \cdot \nabla)\vec{u} \tag{5.15.4}$$

Use the nonlinear continuity equation (5.3.3) to replace $\vec{u}(\partial\rho/\partial t)$ with $-\vec{u}\nabla \cdot (\rho\vec{u})$, take the time derivative of (5.3.3) and the divergence of (5.15.4), eliminate the common term, and use (5.5.6) to express ρ in terms of p in the linear term. The result is an inhomogeneous wave equation

$$\nabla^2 p - \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} = -\nabla \cdot [\vec{u}\nabla \cdot (\rho\vec{u}) + \rho(\vec{u} \cdot \nabla)\vec{u}] \tag{5.15.5}$$

The source term can be given direct physical meaning if it is rewritten

$$\nabla^2 p - \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} = -\frac{\partial^2 \left(\rho u_i u_j\right)}{\partial x_i \, \partial x_j} \tag{5.15.6}$$

Tensor notation has been used for economy of notation. The subscripts i and j take on the values 1, 2, and 3 and represent the x, y, and z directions. A *summation convention* is

[2]Lighthill, *Proc. R. Soc. (London) A*, **211**, 564 (1952).

used, wherein if any subscript appears more than once, it is assumed that subscript is summed over all its values. For example, $\partial u_i/\partial x_i$ is equivalent to $\nabla \cdot \vec{u}$ and $u_j(\partial u_i/\partial x_j)$ is equivalent to $(\vec{u} \cdot \nabla)\vec{u}$. Thus, there are nine quantities in the source term. This source term describes the spatial rates of change of momentum flux *within* the fluid, and Lighthill showed that it is responsible for the sounds produced by regions of *turbulence*, as in the exhaust of a jet engine. [See Problem 5.15.3 to show that the source term in (5.15.5) is equivalent to that in (5.15.6).]

It can be seen that each of these three source terms [described separately in (1), (2), and (3) above] arises independently, so that the complete inhomogeneous lossless wave equation accounting for mass injection, body forces, and turbulence is

$$\nabla^2 p - \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} = -\frac{\partial G}{\partial t} + \nabla \cdot \vec{F} - \frac{\partial^2 (\rho u_i u_j)}{\partial x_i \, \partial x_j} \tag{5.15.7}$$

The effects of gravity could be included by adding a term $\nabla \cdot (\rho_0 \vec{g} s)$ to the left side of (5.15.7) and speed of sound profiles by considering c a function of position. The sources on the right side of (5.15.7) will be related to *monopole, dipole,* and *quadrupole* radiation in Section 7.10.

*5.16 THE POINT SOURCE

The monofrequency spherical wave given by (5.11.15) is a solution to the homogeneous wave equation (5.5.4) everywhere except at $r = 0$. (This is consistent with the fact that there must be a source at $r = 0$ to generate the wave.) However, (5.11.15) does satisfy the inhomogeneous wave equation

$$\nabla^2 \mathbf{p} - \frac{1}{c^2}\frac{\partial^2 \mathbf{p}}{\partial t^2} = -4\pi A \delta(\vec{r}) e^{j\omega t} \tag{5.16.1}$$

for all $\vec{r}$. The three-dimensional delta function $\delta(\vec{r})$ is defined by

$$\int_V \delta(\vec{r})\, dV = \begin{cases} 1 & \vec{r} = 0 \in V \\ 0 & \vec{r} = 0 \notin V \end{cases} \tag{5.16.2}$$

To prove this, multiply both sides of (5.16.1) by dV, integrate over a volume V that includes $\vec{r} = 0$, and use (5.16.2) to evaluate the delta function integral and Gauss's theorem to reduce the volume integral to a surface integral. This gives

$$\int_S \nabla \mathbf{p} \cdot \hat{n}\, dS - \frac{1}{c^2}\int_V \frac{\partial^2 \mathbf{p}}{\partial t^2}\, dV = -4\pi A e^{j\omega t} \tag{5.16.3}$$

where $\hat{n}$ is the unit outward normal to the surface S of V. Now, substitute (5.11.15) for p and carry out the surface integration over a sphere centered on $\vec{r} = 0$. See Problem 5.16.1.

To generalize to a point source located at $\vec{r} = \vec{r}_0$, make the appropriate change of variable in (5.11.15):

$$\mathbf{p} = \frac{A}{|\vec{r} - \vec{r}_0|} \exp\left[j\left(\omega t - k|\vec{r} - \vec{r}_0|\right)\right] \tag{5.16.4}$$

This is a solution of

$$\nabla^2 \mathbf{p} - \frac{1}{c^2}\frac{\partial^2 \mathbf{p}}{\partial t^2} = -4\pi A \delta(\vec{r} - \vec{r}_0)\, e^{j\omega t} \tag{5.16.5}$$

In the proper circumstances, incorporation of a point source directly into the wave equation provides considerable mathematical simplification. (See Sections 7.10 and 9.7–9.9 as examples.) We will, however, use this formalism only when necessary, utilizing in most cases methods more closely related to elementary physical intuition.

PROBLEMS

5.2.1. (*a*) Linearize (5.2.3) by assuming $s \ll 1$. Then, by comparing this result with (5.2.5), obtain the adiabatic bulk modulus of a perfect gas in terms of $\mathcal{P}_0$ and γ. (*b*) With the help of (5.2.1) applied to equilibrium conditions, obtain the temperature dependence of $\mathcal{B}$ at constant volume.

5.2.2. Another form of the perfect gas law is $\mathcal{P}V = n\mathcal{R}T_K$, where n is the number of moles and $\mathcal{R} = 8.3143$ J/(mol·K) is the universal gas constant (the mole is the molecular weight M in grams). Obtain a relationship between r and $\mathcal{R}$. Evaluate $\mathcal{R}$ in J/(kmol·K) (the kilomole is the molecular weight in kilograms).

5.2.3. If the adiabat for a fluid is presented in the form $\mathcal{P} = \mathcal{P}_0 + A[(\rho - \rho_0)/\rho_0] + \frac{1}{2}B[(\rho - \rho_0)/\rho_0]^2$ and is to be written as $(\mathcal{P}/\mathcal{P}_0) = (\rho/\rho_0)^a$, find an approximate expression for the exponent a. *Hint:* Expand $(\rho/\rho_0)^a$ about ρ_0 through second order and equate coefficients. Relate the results to (5.2.9) and (5.2.7).

5.2.4. The major constituents in *standard air* and the percentage and molecular weight in grams of each are: nitrogen (N_2), 78.084, 28.0134; oxygen (O_2), 20.948, 31.9988; argon (Ar), 0.934, 39.948; carbon dioxide (CO_2), 0.031, 44.010. (*a*) Calculate the effective molecular weight of air. (*b*) Obtain the specific gas constant $r = \mathcal{R}/M$ for air and compare with the value listed in Appendix A1.

5.3.1. From the linear continuity equation (5.3.4), show that the condensation and particle displacement are related by $s = -\nabla \cdot \vec{\xi}$. *Hint:* Assume ρ_0 is independent of time. The integral of (5.3.4) over time must yield a constant independent of the forms of s and $\vec{u}$. Evaluate the constant when there is no sound.

5.4.1. Show that the change in the density of a particular fluid element moving with velocity $\vec{u}$ is given by $(\partial \rho/\partial t) + \vec{u} \cdot \nabla \rho$.

5.4.2. A flow is incompressible if a fluid element does not change its density as the element moves. From Problem 5.4.1, this means $(\partial \rho/\partial t) + \vec{u} \cdot \nabla \rho = 0$. (*a*) Show that for an incompressible fluid the equation of continuity reduces to $\nabla \cdot \vec{u} = 0$. (*b*) Write Euler's equation for the flow of an incompressible fluid. (*c*) What is c for an incompressible fluid?

5.5.1. Use the adiabat and the linearized equations of continuity and motion to show that all the scalar acoustic variables obey the wave equation $\nabla^2 - (1/c^2)\,\partial^2/\partial t^2 = 0$ within the accuracy of the linearizing approximations and the near constancy of ρ_0 and c.

5.5.2. (*a*) Use the adiabat and the linearized equations of continuity and motion (and the near constancy of ρ_0 and c) to show that $\nabla(\nabla \cdot \vec{u}) = (1/c^2)\,\partial^2\vec{u}/\partial t^2$. (*b*) Show that, since $\vec{u}$ is irrotational, this is equivalent to $\nabla^2\vec{u} = (1/c^2)\,\partial^2\vec{u}/\partial t^2$. (*c*) Write the latter equation in spherical coordinates with spherical symmetry and compare it with the wave equation for the pressure in the same coordinates. (See Appendix A7 for $\nabla^2\vec{u}$.)

5.6.1. (*a*) Find the speed of sound in hydrogen at 1 atm and 0°C from its values of $\mathcal{P}_0$, ρ_0, and γ. (*b*) Compare with the result given in Appendix A10. Is your agreement within the round-off of the tabulated values? (*c*) What error in temperature would give the same disagreement?

5.6.2. (*a*) By means of (5.6.8), determine the speed of sound in distilled water at atmospheric pressure and a temperature of 30°C. (*b*) What is the rate of change of the speed of sound in water with respect to temperature at this temperature?

5.6.3. (*a*) For a perfect gas, does c vary with the equilibrium pressure? With the instantaneous pressure in an acoustic process? (*b*) Find c for a perfect gas that obeys the isotherm (5.2.2). (*c*) Compare the value of c from (*b*) to that for air at 20°C.

5.7.1. If $\vec{\mathbf{u}} = \hat{x}U\exp[j(\omega t - kx)]$, show that $|(\vec{\mathbf{u}} \cdot \nabla)\vec{\mathbf{u}}|/|\partial\vec{\mathbf{u}}/\partial t| = U/c$, the acoustic Mach number. Relate this to the relevant assumption made to obtain the linear Euler's equation (5.4.10).

5.7.2. For an acoustic wave with propagation constant k, show that the mathematical assumption made to obtain (5.5.8) is equivalent to requiring $|(1/\rho_0)\nabla\rho_0| \ll k$. Physically, what does this mean?

5.7.3. For a plane wave $\mathbf{u} = U\exp[j(\omega t - kx)]$, find expressions for the acoustic Mach number U/c (*a*) in terms of P, ρ_0, and c and (*b*) in terms of s.

5.7.4. Using (5.7.8) for an oblique wave, obtain the velocity potential and then the acoustic particle velocity, and show that the velocity is parallel to the propagation vector.

5.7.5. (*a*) Show that if the density is not approximated by ρ_0 in the gravity term in Euler's equation, the wave equation for acoustic pressure contains a term $\nabla \cdot (\vec{g}\rho_0 s)$. (*b*) Show for a plane wave that this term is negligible as long as $\omega \gg |\vec{g}|/c$. Evaluate $|\vec{g}|/c$ for water and air.

5.7.6. For an acoustic wave of angular frequency ω, find a condition justifying ignoring any time dependence in ρ_0 in the linear equation of continuity.

5.8.1. Two parallel traveling plane waves have different angular frequencies ω_1 and ω_2 and pressure amplitudes P_1 and P_2. (*a*) Show that the instantaneous energy density $\mathscr{E}_i$ at a point in space varies between $(P_1 + P_2)^2/\rho_0 c^2$ and $(P_1 - P_2)^2/\rho_0 c^2$. (*b*) Show that the total energy density $\mathscr{E}$ at the point averages to the sum of the individual energy densities of each wave alone. *Hint:* Let the averaging time be much greater than $2\pi/|\omega_1 - \omega_2|$.

5.9.1. If $\mathbf{p} = P\exp[j(\omega t - kx)]$, find (*a*) the acoustic density, (*b*) the particle speed, (*c*) the velocity potential, (*d*) the energy density, and (*e*) the intensity.

5.9.2. (*a*) Derive an equation expressing the adiabatic temperature rise ΔT produced in a gas by an acoustic pressure p. (*b*) What is the amplitude of the temperature fluctuation produced by a sound of intensity 10 W/m^2 in air at 20°C and standard atmospheric pressure?

5.9.3. Repeat Problem 5.9.1 for the standing wave $p = P\cos(\omega t)\cos(kx)$.

5.10.1. For a wave consisting of two waves traveling in the $+x$ direction but with different frequencies, show that the specific acoustic impedance is $\rho_0 c$.

5.10.2. Show that for any plane wave traveling in the $+x$ direction, the specific acoustic impedance is $\rho_0 c$. *Hint:* Let $\mathbf{\Phi} = \mathbf{f}(ct - x)$ and generate $\mathbf{p}$ and $\mathbf{u}$ from $\mathbf{\Phi}$.

5.10.3. Find the specific acoustic impedance for a standing wave $\mathbf{p} = P\sin kx\exp(j\omega t)$.

5.11.1C. For values of kr between 0.1 and 10, plot the specific acoustic resistance and reactance. In what range of kr do these quantities make transitions between low- and high-frequency behaviors? What are their maximum values?

5.11.2. Given a small source of spherical waves in air, at a radial distance of 10 cm compute the difference in phase angle between pressure and particle velocity for 10 Hz, 100 Hz, 1 kHz, and 10 kHz. Compute the magnitude of the specific acoustic impedance for each frequency at this location.

5.11.3. For a spherical wave $\mathbf{p} = (A/r)\cos(kr)\exp(j\omega t)$, find (*a*) the particle speed, (*b*) the specific acoustic impedance, (*c*) the instantaneous intensity, and (*d*) the intensity.

5.11.4. Show that the specific acoustic reactance of a spherical wave is a maximum for $kr = 1$.

5.11.5. At some location, the pressure amplitude and particle speed of a 100 Hz sound wave in air are measured to be 2 Pa and 0.0100 m/s. Assuming that this is a spherical wave, find the distance from the source. What additional measurements could be made at this same place to determine the direction of the source?

5.11.6C. Plot the magnitude and the phase of the specific acoustic impedance (normalized by dividing by $\rho_0 c$) of a spherical wave as a function of kr. Above what value of kr does the spherical wave approximate the behavior of a plane wave within about 10%?

5.12.1C. A spherical wave in air has a sound pressure amplitude of 100 dB *re* 20 μPa at 1 m from the origin. (*a*) Plot the ratio of amplitudes of the pressure P and the particle speed U as a function of r for various frequencies. (*b*) Is the distance at which the ratio P/U comes to within 10% of $\rho_0 c$ independent of frequency? (*c*) If not, plot this distance as a function of frequency.

5.12.2. For a 171 Hz plane traveling wave in air with a sound pressure level of 40 dB *re* 20 μPa, find (*a*) the acoustic pressure amplitude, (*b*) the intensity, (*c*) the acoustic particle speed amplitude, (*d*) the acoustic density amplitude, (*e*) the particle displacement amplitude, and (*f*) the condensation amplitude.

5.12.3. A plane sound wave in air of 100 Hz has a peak acoustic pressure amplitude of 2 Pa. (*a*) What is its intensity and its intensity level? (*b*) What is its peak particle displacement amplitude? (*c*) What is its peak particle speed amplitude? (*d*) What is its effective or rms pressure? (*e*) What is its sound pressure level *re* 20 μPa?

5.12.4. An acoustic wave has a sound pressure level of 80 dB *re* 1 μbar. Find (*a*) the sound pressure level *re* 1 μPa and (*b*) the sound pressure level *re* 20 μPa.

5.12.5. (*a*) Show that a plane wave having an effective acoustic pressure of 1 μbar in air has an intensity level of 74 dB *re* 10^{-12} W/m^2. (*b*) Find the intensity (W/m^2) produced by an acoustic plane wave in water of $SPL(1\ \mu\text{bar}) = 120$ dB. (*c*) What is the ratio of the acoustic pressure in water for a plane wave to that of a similar wave in air of equal intensity?

5.12.6. (*a*) Determine the energy density and effective pressure amplitude of a plane wave in air of intensity level 70 dB *re* 10^{-12} W/m^2. (*b*) Determine the energy density and effective pressure amplitude of a plane wave in water if its sound pressure level is 70 dB *re* 1 μbar.

5.12.7. (*a*) Show that at constant $\mathcal{P}_0$ the characteristic impedance of a gas is inversely proportional to the square root of its absolute temperature T_K. (*b*) What is the characteristic impedance of air at 0°C? At 80°C? (*c*) If the pressure amplitude of a sound wave remains constant, what is its percent change in intensity as the temperature increases from 0°C to 80°C? (*d*) What would be the corresponding change in intensity level? In pressure level?

5.12.8. Cavitation may take place at the face of a sonar transducer when the sound pressure amplitude being produced exceeds the hydrostatic pressure in the water. (*a*) For a hydrostatic pressure of 200,000 Pa, what is the highest intensity that may be radiated without producing cavitation? (*b*) What is the sound pressure level of this sound *re* 1 μbar? (*c*) What is the condensation amplitude? (*d*) At what depth in the ocean would this hydrostatic pressure be found?

5.12.9. A transmitter generates a sound pressure level at 1 m of 100 dB *re* 1 μbar for a driving voltage of 100 V (rms). Find the sensitivity level in dB *re* 1 μbar/V.

5.12.10. A transmitter has a sensitivity level of 60 dB *re* 1 μbar/V. Find its sensitivity level *re* 1 μPa/V and *re* 20 μPa/V.

5.12.11. The receiving sensitivity level of a hydrophone is -80 dB *re* 1 V/μbar. (*a*) Express this level *re* 1 V/μPa. (*b*) What will be the (rms) output voltage if the pressure field is 80 dB *re* 1 μbar?

5.12.12. A microphone reads 1 mV for an incident effective pressure level of 120 dB *re* 20 μPa. Find the sensitivity level of the microphone *re* 1 V/μbar and 1 V/20 μPa.

5.13.1C. Compare the magnitude of $H_0^{(2)}(kr)$ with its asymptotic expression and find the value of kr beyond which the disagreement is within 10%.

5.13.2C. Find the value of kr beyond which $|\mathbf{z}|$ in (5.13.12) is within 10% of $\rho_0 c$.

5.13.3. For various z, test the assertion that the Wronskian of $J_0(z)$ and $Y_0(z)$ is $2/\pi z$.

5.13.4. Find the fractional change in pressure amplitude for each doubling of the propagation distance for (*a*) spherical waves, (*b*) cylindrical waves for $kr \gg 1$, (*c*) plane waves.

5.13.5. Assume that $k_z \neq 0$ in (5.13.3). (*a*) Show that

$$\mathbf{p} = H_0^{(2)}(k_r r) \sin k_z z \, e^{j\omega t}$$

is a solution of (5.13.3). (*b*) Write $\sin k_z z$ in terms of complex exponentials and show that $\mathbf{p}$ consists of two outward-traveling waves, each having conical surfaces of constant phase. (*c*) Find the angles of elevation and depression with respect to the $z = 0$ plane of the propagation vectors.

5.14.1. (*a*) If c is a function only of z, show that $d\theta/ds = -[(\cos\theta_0)/c_0]\, dc/dz$, with θ_0 the angle of elevation of the ray where $c = c_0$. (*b*) If the gradient $g = dc/dz$ is a constant, find the radius of curvature R of the ray in terms of g, c, and θ. Is R a constant? (*c*) If the temperature of air decreases linearly with height z, verify that $c(z) = c_0 - gz$, where $g > 0$. If the temperature decreases 5 C°/km, find the radius of curvature of a ray that is horizontal at $z = 0$ (assume $c_0 = 340$ m/s). At what horizontal range will this ray have risen to a height of 10 m?

5.14.2. Assume the speed of sound is given by the quasi-parabolic profile $c(z) = c_0[1 - (\varepsilon z)^2]^{-1/2}$. Let the depth $z = 0$, which defines the axis of the sound channel, lie well below the surface of the ocean. (*a*) Find the equation $z(x)$ for rays emitted by a source at $(x, z) = (0, 0)$ with angles of elevation or depression $\pm\theta_0$. *Hint:* use Snell's law, $dz/dx = \tan\theta$, and $\int (a^2 - u^2)^{-1/2}\, du = \sin^{-1}(u/a)$. (*b*) For a given ray, find an expression for the average speed with which energy propagates out to a distance x lying on the channel axis. Explain why there is no dependence on the parameter ε. For small angles, approximate your expression through the first nonzero term in θ_0. (*c*) For $|\theta_0| \le \pi/8$, show that $c(z)$ is a good approximation of the parabolic profile $c_0[1 + \frac{1}{2}(\varepsilon z)^2]$. What is the percentage discrepancy at 22°? (*d*) A certain ocean channel with axis more than about 1 km below the ocean surface can be approximated by $c(z)$ with $c_0 = 1475$ m/s and $\varepsilon = 1.5 \times 10^{-4}$ m^{-1}. Calculate the travel speeds of (*c*) for $\theta_0 = 0$, 1, 2, 5, and 10°. (*e*) For each of the angles in (*d*), determine the greatest height above the channel axis reached by the ray and the distance between successive axis crossings. (*f*) Explain why the results of (*d*) and (*e*) are not inconsistent.

5.14.3. Assume the speed of sound is given by the quasi-linear profile $c(z) = c_0[1 - \varepsilon|z|]^{-1/2}$. Let the depth $z = 0$, which defines the axis of the sound channel, lie well below the surface of the ocean. (*a*) Find the equation $z(x)$ for rays emitted by a source at $(x, z) = (0, 0)$ with angles of elevation or depression $\pm\theta_0$. *Hint:* use Snell's law and $dz/dx = \tan\theta$. (*b*) For a ray with initial angle θ_0, find the distance Δx between x-axis intercepts and the maximum distance Δz it attains above or below $z = 0$. (*c*) For a given ray, find an expression for the average speed with which energy propagates out to a distance x lying on the channel axis. Explain why there is no dependence on the parameter ε. For small angles, approximate your expression through the first nonzero term in θ_0. (*d*) For $|\theta_0| \leq \pi/8$, show that $c(z)$ is a good approximation of the linear profile $c_0(1 + \frac{1}{2}\varepsilon|z|)$. What is the percentage discrepancy at 22°? (*d*) A certain ocean channel with axis more than about 1 km below the ocean surface can be approximated by a quasi-bilinear profile with $c_0 = 1467$ m/s and $\varepsilon_1 = 4.0 \times 10^{-5}$ m^{-1} above the axis and $\varepsilon_2 = 2.0 \times 10^{-5}$ m^{-1} below. Calculate the travel speeds of (*c*) for $\theta_0 = 0$, 1, 2, 5, and 10°. (*e*) For each of the angles in (*d*), determine the greatest distances above and below the channel axis reached by the ray and the two distances between successive axis crossings. (*f*) Explain why the results of (*d*) and (*e*) are not inconsistent.

5.14.4. (*a*) Verify (5.14.7). *Hint:* Substitute (5.14.6) into (5.14.3) and note that $\nabla A \cdot (n\hat{s}) = n(dA/ds)$ and $\nabla \cdot (n\hat{s}) = dn/ds + n\nabla \cdot \hat{s}$. (*b*) Obtain (5.14.13) from (5.14.12). Deal with this component by component. Show that the x component of $d(\nabla\Gamma)/ds$ can be written as $d(n\alpha)/ds$ from (5.14.6) and as $(\alpha\, \partial/\partial x + \beta\, \partial/\partial y + \gamma\, \partial/\partial z)(\partial\Gamma/\partial x)$ from (5.14.12). In the latter expression, exchange orders of differentiation, use (5.14.6), expand the derivatives, and regroup using $\alpha^2 + \beta^2 + \gamma^2 = 1$.

5.14.5. If the speed of sound in water is 1500 m/s at the surface and increases linearly with depth at a rate of 0.017/s, find the range at which a ray emitted horizontally from a source at 100 m depth will reach the surface.

5.14.6. For the conditions of Problem 5.14.5, calculate the ratio of the intensity when the ray reaches the surface to that at 1 meter from the source. Compare this to the intensity if the spreading were spherical.

5.14.7. Plot, as a function of time, the phase coherent sum of two sinusoidal signals of equal frequency and amplitude for phase differences from 0° to 360° in steps of 45°. For each case, calculate the ratio of the intensity of the summed signal to the intensity of the individual signals and compare to results obtained from the plots.

5.14.8. Sound from a single source arrives at some point over two separate paths. Let the two signals have pressures $p_1 = P_1 \cos(\omega t)$ and $p_2 = P_2 \cos(\omega t + \phi)$ at the point. (*a*) Under the assumption that the waves are plane and essentially parallel, show that the intensity at that point is $I = [(P_1 + P_2 \cos\phi)^2 + (P_2 \sin\phi)^2]/2\rho_0 c$. (*b*) If incoherence effects now cause ϕ to be a slowly varying function of time compared to the period of the waves, show that the total intensity at the point is the sum of the individual intensities of the two waves. *Hint:* Let the accumulated values of ϕ be distributed with equal probability over the interval $0 \leq \phi < 2\pi$.

5.14.9C. Plot, as a function of time, the sum of two quasi-random signals of about equal intensity. Verify that the intensity of the summed signals is the sum of the intensities of the individual signals. *Hint:* Construct the signals from sinusoids, each with its phase independently randomized at each time step.

5.14.10. Show that a spherically symmetric outward-traveling wave in an isospeed medium satisfies (5.14.5) identically for all r.

5.14.11C. The sound speed in deep water can be approximated by two layers: an upper layer in which the sound speed decreases linearly from 1500 m/s at the surface to 1475 m/s at 1000 m, and an infinitely deep layer in which the sound speed increases at a constant rate of 0.017 s^{-1}. For a source at the surface, (*a*) plot the distance at which a ray returns to the surface for depression angles between 0°

and 10°. (*b*) Find the depression angle and range of the ray that reaches the surface closest to the source. (*c*) The region where rays with different source angles reach the surface at the same range is the *resweep zone*. Find the width of this region. (*d*) Find the greatest depth attained by a ray that contributes to this region.

5.14.12. The sound speed in air varies linearly from 343 m/s at the ground to 353 m/s at 100 m altitude and then decreases above this. For a source at ground level, find (*a*) the maximum elevation angle for a ray that returns to ground level, and (*b*) the range at which this ray returns to ground level.

5.14.13C. At the range found in Problem 5.14.12, find the difference in the times of arrival of the ray that leaves the source horizontally and the ray that leaves at the maximum elevation angle.

5.14.14. (*a*) Show that, within the approximations yielding the eikonal equation, $\nabla p = pk\nabla\Gamma$. (*b*) The intensity, written explicitly as a vector, is $\vec{I} = \langle p\vec{u}\rangle_T$. Using the relationship between p and $\vec{u}$, show that $\vec{I}$ is parallel to the ray path.

5.15.1. Express in vector notation: (*a*) $u_i v_i$, (*b*) $\partial u_j/\partial x_j$, (*c*) $u_i\,\partial f/\partial x_i$, (*d*) $f\,\partial u_j/\partial x_j + u_i\,\partial f/\partial x_i$.

5.15.2. Express in subscript (tensor) notation: (*a*) $(\vec{u}\cdot\nabla)f$, (*b*) $\nabla\cdot[(\vec{u}\cdot\nabla)\vec{u}]$, (*c*) $\nabla\cdot[\vec{u}(\nabla\cdot\vec{u})]$.

5.15.3. Show that the right sides of (5.15.5) and (5.15.6) are equivalent. *Hint:* Take $(\partial/\partial x_j)$ of $(\rho u_j)u_i$ and convert to vector notation.

5.16.1. Prove the equality given by (5.16.3). *Hint:* Use the indefinite integral relation $\int x\exp(-jx)\,dx = \exp(-jx) + jx\exp(-jx)$.

5.16.2. Show that in spherical coordinates with spherical symmetry the three-dimensional delta function can be written as $\delta(\vec{r}) = (4\pi r^2)^{-1}\delta(r)$, where $\delta(r)$ is the one-dimensional delta function.

5.16.3. Show that in cylindrical coordinates with radial symmetry for a source lying on the z axis at $z = z_0$, the three-dimensional delta function $\delta(\vec{r} - \vec{r}_0)$ can be written as $\delta(\vec{r} - \vec{r}_0) = (2\pi r)^{-1}\delta(r)\delta(z - z_0)$.

5.16.4. (*a*) Show that $p = (A/r)f(t - r/c)$ is a solution of the inhomogeneous wave equation $\nabla^2 p - (1/c^2)\partial^2 p/\partial t^2 = -4\pi A\delta(\vec{r})f(t)$. (*b*) Show that $p = (1/r)\delta(t - r/c)$ is a solution of this equation when $f(t) = \delta(t)$.