# Introduction to Embedded Systems

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## Module *2a*: Modeling Physical Dynamics

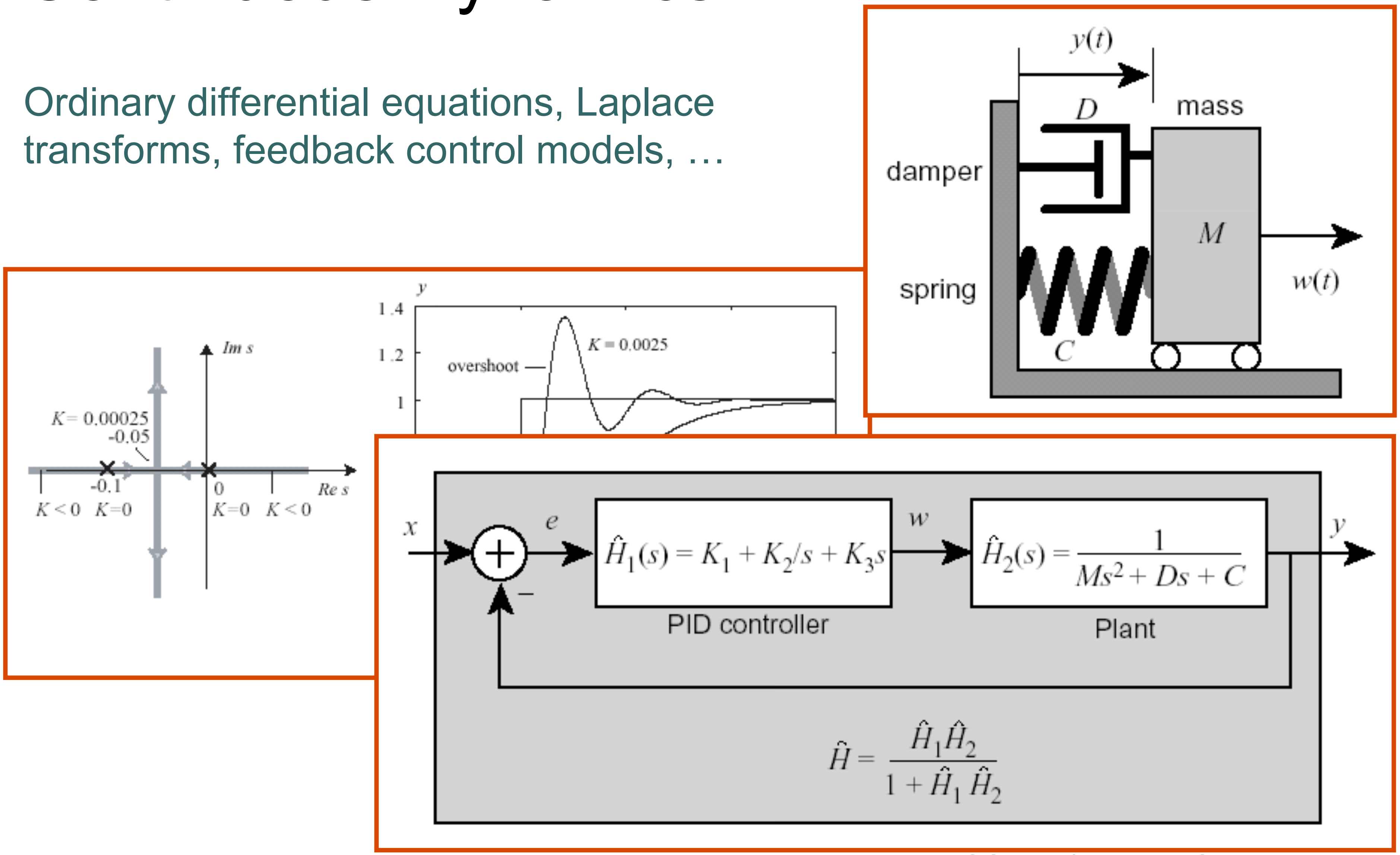
# Modeling Techniques in this Course

Models that are abstractions of **system dynamics** (how system behavior changes over time)

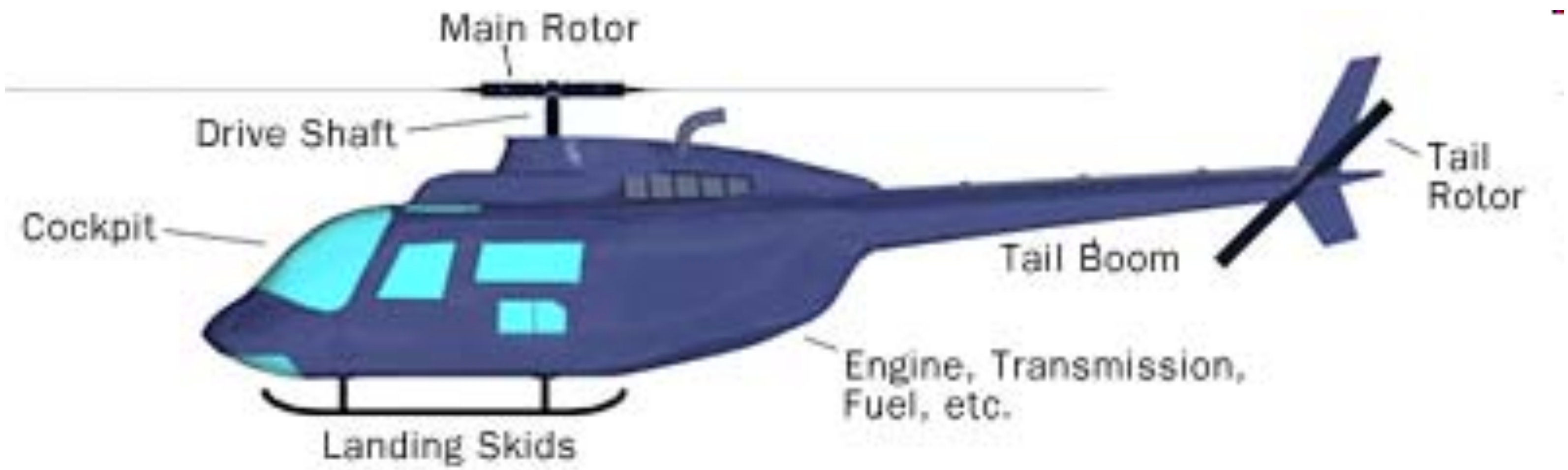
- Modeling physical phenomena – differential equations
- Feedback control systems – time-domain modeling
- Modeling modal behavior – FSMs, hybrid automata, …
- Modeling sensors and actuators –calibration, noise, …
- Hardware and software – concurrency, timing, power, …
- Networks – latencies, error rates, packet losses, …

# Today's Lecture: Modeling of Continuous Dynamics

Ordinary differential equations, Laplace transforms, feedback control models, …



# An Example: Helicopter Dynamics

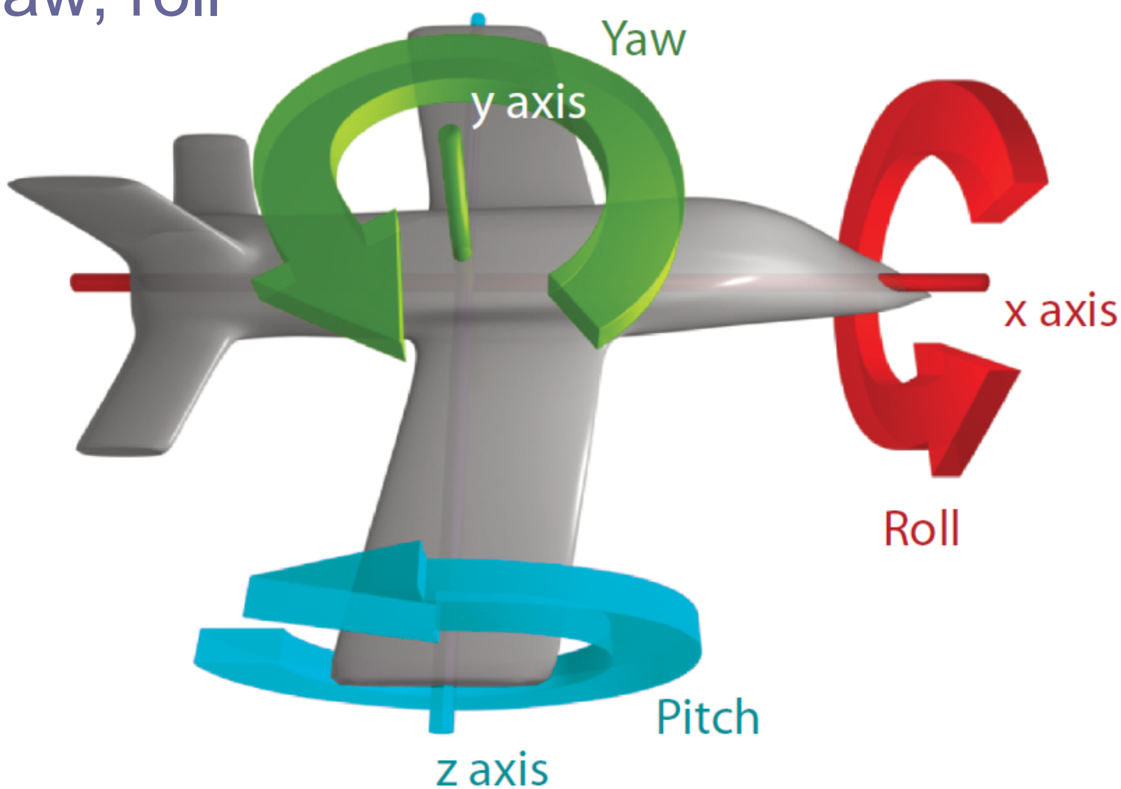



**The Fundamental Parts of any Helicopter**

# Modeling Physical Motion

Six degrees of freedom:

- Position: x, y, z
- Orientation: pitch, yaw, roll



# Notation

Position is given by three functions:

$$x : \mathbb{R} \to \mathbb{R}$$
$$y : \mathbb{R} \to \mathbb{R}$$
$$z : \mathbb{R} \to \mathbb{R}$$

where the domain $\mathbb{R}$ represents time and the co-domain (range) $\mathbb{R}$ represents position along the axis. Collecting into a vector:

$$\mathbf{x} : \mathbb{R} \to \mathbb{R}^3$$

Position at time $t \in \mathbb{R}$ is $\mathbf{x}(t) \in \mathbb{R}^3$.

# Notation

Velocity

$$\dot{\mathbf{x}} \colon \mathbb{R} \to \mathbb{R}^3$$

is the derivative, $\forall\, t \in \mathbb{R}$,

$$\dot{\mathbf{x}}(t) = \frac{d}{dt}\mathbf{x}(t)$$

Acceleration $\ddot{\mathbf{x}} \colon \mathbb{R} \to \mathbb{R}^3$ is the second derivative,

$$\ddot{\mathbf{x}} = \frac{d^2}{dt^2}\mathbf{x}$$

Force on an object is $\mathbf{F} \colon \mathbb{R} \to \mathbb{R}^3$.

# Newton's Second Law

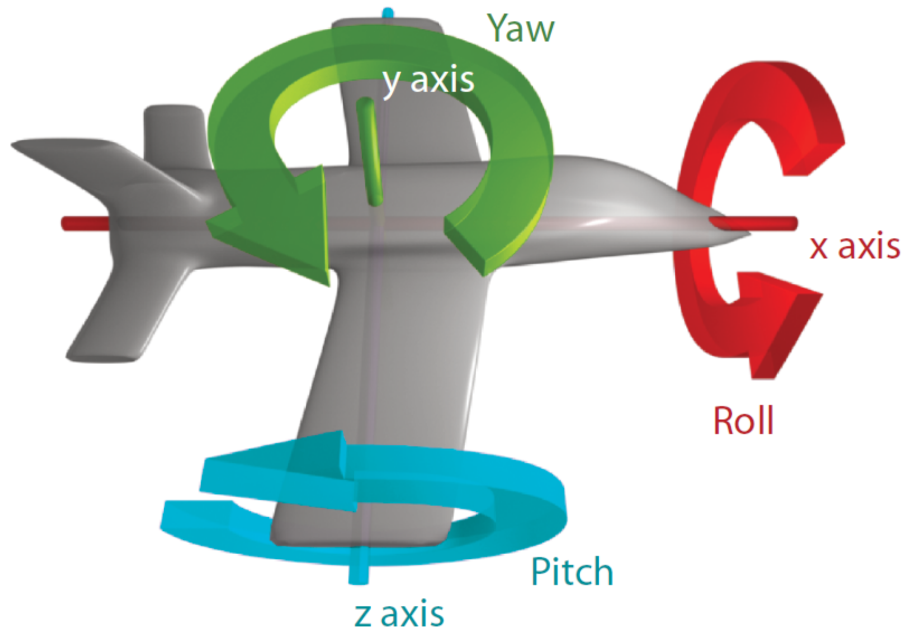
Newton's second law states $\forall\, t \in \mathbb{R}$,

$$\mathbf{F}(t) = M\ddot{\mathbf{x}}(t)$$

where $M$ is the mass. To account for initial position and velocity, convert this to an integral equation

$$\begin{aligned} \mathbf{x}(t) &= \mathbf{x}(0) + \int_0^t \dot{\mathbf{x}}(\tau)d\tau \\ &= \mathbf{x}(0) + t\dot{\mathbf{x}}(0) + \frac{1}{M}\int_0^t \int_0^\tau \mathbf{F}(\alpha)d\alpha d\tau, \end{aligned}$$

# Orientation

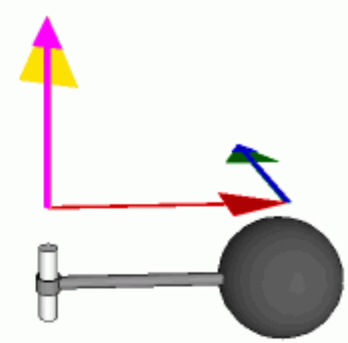



- Orientation: $\theta \colon \mathbb{R} \to \mathbb{R}^3$
- Angular velocity: $\dot{\theta} \colon \mathbb{R} \to \mathbb{R}^3$
- Angular acceleration: $\ddot{\theta} \colon \mathbb{R} \to \mathbb{R}^3$
- Torque: $\mathbf{T} \colon \mathbb{R} \to \mathbb{R}^3$

$$\theta(t) = \begin{bmatrix} \dot{\theta}_x(t) \\ \dot{\theta}_y(t) \\ \dot{\theta}_z(t) \end{bmatrix} = \begin{bmatrix} \text{roll} \\ \text{yaw} \\ \text{pitch} \end{bmatrix}$$

# Angular version of force is torque.
# For a point mass rotating around a fixed axis:

- radius of the arm: $r \in \mathbb{R}$
- force orthogonal to arm: $f \in \mathbb{R}$
- mass of the object: $m \in \mathbb{R}$



$$T_y(t) = r f(t)$$

angular momentum, momentum

Just as force is a push or a pull, a torque is a twist.

Units: newton-meters/radian, Joules/radian

Note that radians are meters/meter ($2\pi$ meters of circumference per 1 meter of radius), so as units, are optional.

# Rotational Version of Newton's Second Law

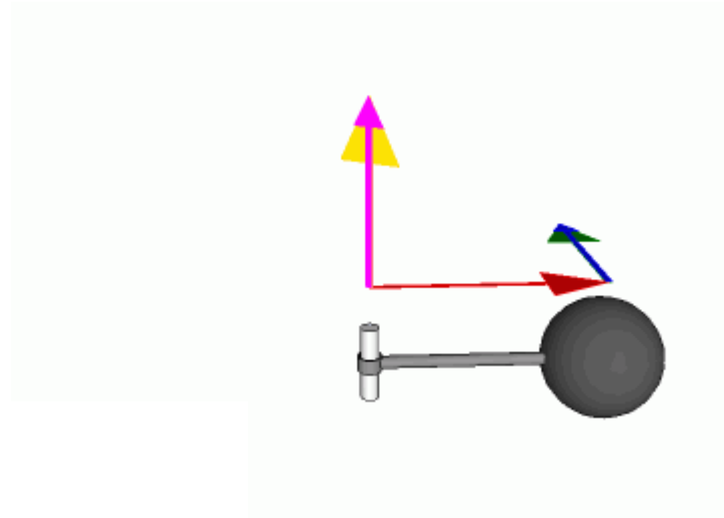
$$\mathbf{T}(t) = \frac{d}{dt}\left(I(t)\dot{\theta}(t)\right),$$

where $I(t)$ is a $3 \times 3$ matrix called the moment of inertia tensor.

$$\begin{bmatrix} T_x(t) \\ T_y(t) \\ T_z(t) \end{bmatrix} = \frac{d}{dt}\left(\begin{bmatrix} I_{xx}(t) & I_{xy}(t) & I_{xz}(t) \\ I_{yx}(t) & I_{yy}(t) & I_{yz}(t) \\ I_{zx}(t) & I_{zy}(t) & I_{zz}(t) \end{bmatrix}\begin{bmatrix} \dot{\theta}_x(t) \\ \dot{\theta}_y(t) \\ \dot{\theta}_z(t) \end{bmatrix}\right)$$

Here, for example, $T_y(t)$ is the net torque around the $y$ axis (which would cause changes in yaw), $I_{yx}(t)$ is the inertia that determines how acceleration around the $x$ axis is related to torque around the $y$ axis.

# Simplified Model



Yaw dynamics:

$$T_y(t) = I_{yy}\ddot{\theta}_y(t)$$

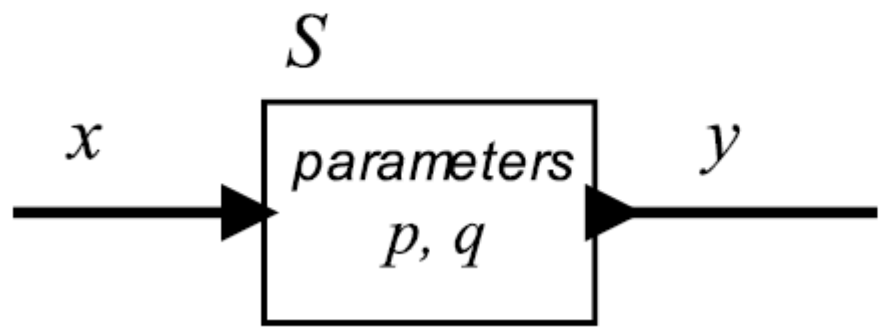
To account for initial angular velocity, write as

$$\dot{\theta}_y(t) = \dot{\theta}_y(0) + \frac{1}{I_{yy}}\int_0^t T_y(\tau)d\tau.$$

# “Plant” and Controller

# Actor Model of Systems

A *system* is a function that accepts an input *signal* and yields an output signal.




The domain and range of the system function are sets of signals, which themselves are functions.

$$x:\mathbb{R} \to \mathbb{R}, \quad y:\mathbb{R} \to \mathbb{R}$$

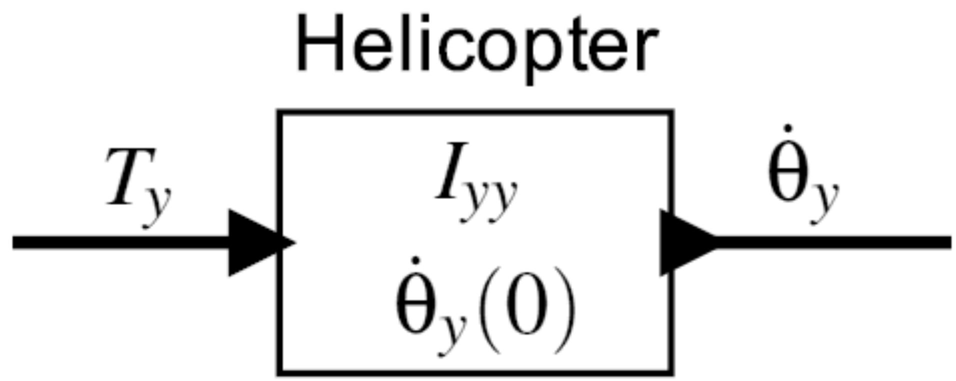
$$S: X \to Y$$

$$X = Y = (\mathbb{R} \to \mathbb{R})$$

Parameters may affect the definition of the function *S*.

# Actor Model of the Helicopter

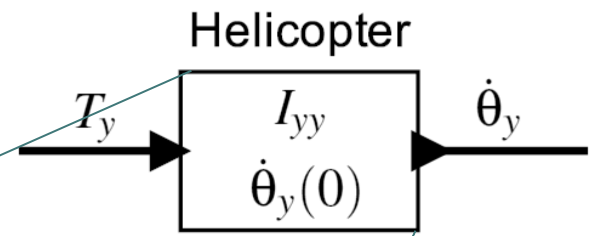
Input is the net torque of the tail rotor and the top rotor. Output is the angular velocity around the *y* axis.

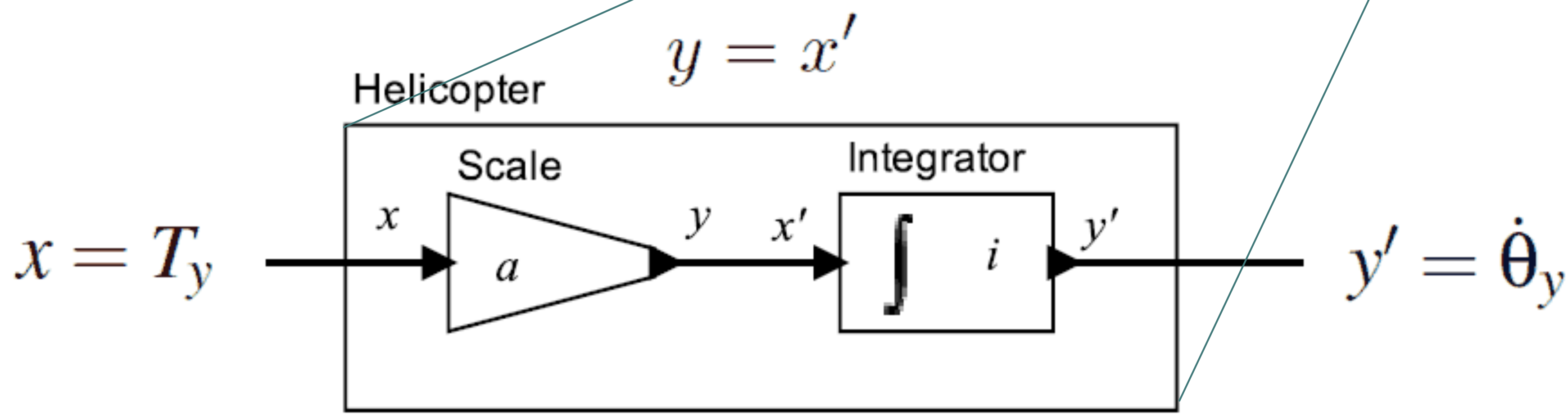



Parameters of the model are shown in the box. The input and output relation is given by the equation to the right.

$$\dot{\theta}_y(t) = \dot{\theta}_y(0) + \frac{1}{I_{yy}} \int_0^t T_y(\tau) d\tau$$

# Composition of Actor Models



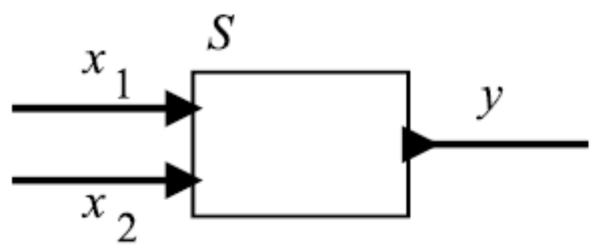




$$\forall\, t \in \mathbb{R}, \quad y(t) = ax(t) \qquad y'(t) = i + \int_0^t x'(\tau)d\tau$$

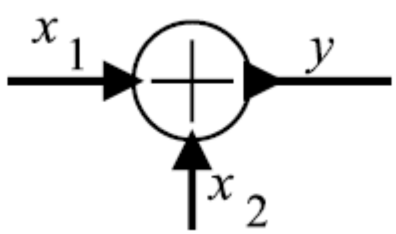
$$y = ax$$

$$a = 1/I_{yy} \qquad i = \dot{\theta}_y(0)$$

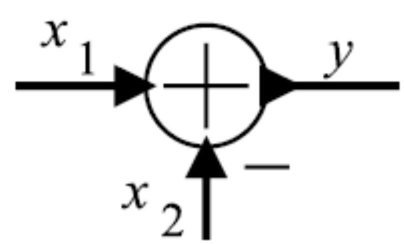
# Actor Models with Multiple Inputs




$$S\colon (\mathbb{R} \to \mathbb{R})^2 \to (\mathbb{R} \to \mathbb{R})$$

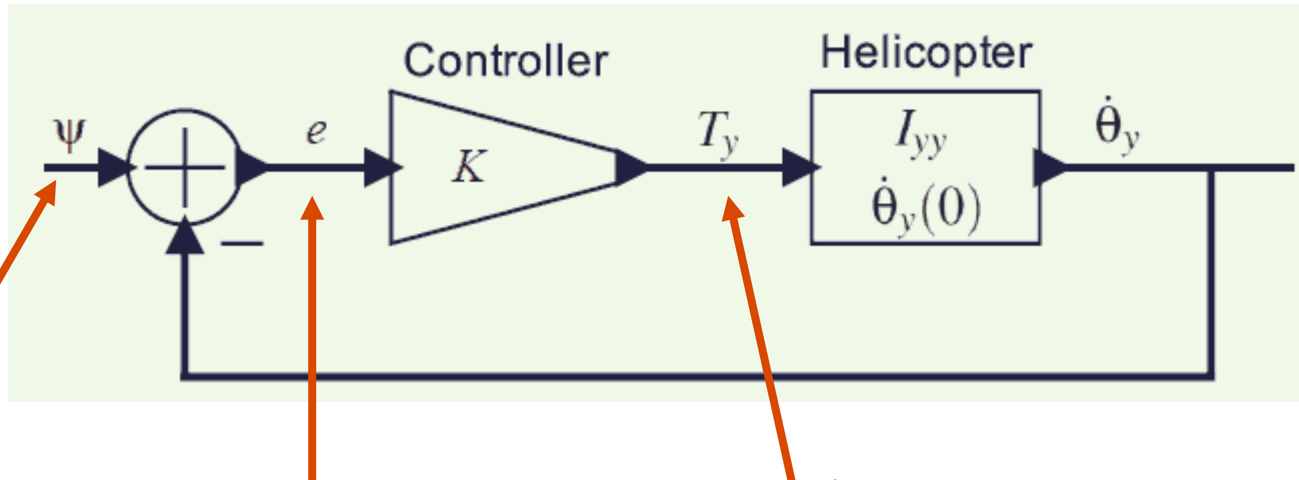



$$\forall\, t \in \mathbb{R}, \quad y(t) = x_1(t) + x_2(t)$$




$$(S(x_1, x_2))(t) = y(t) = x_1(t) - x_2(t)$$

# Proportional controller




desired angular velocity

error signal

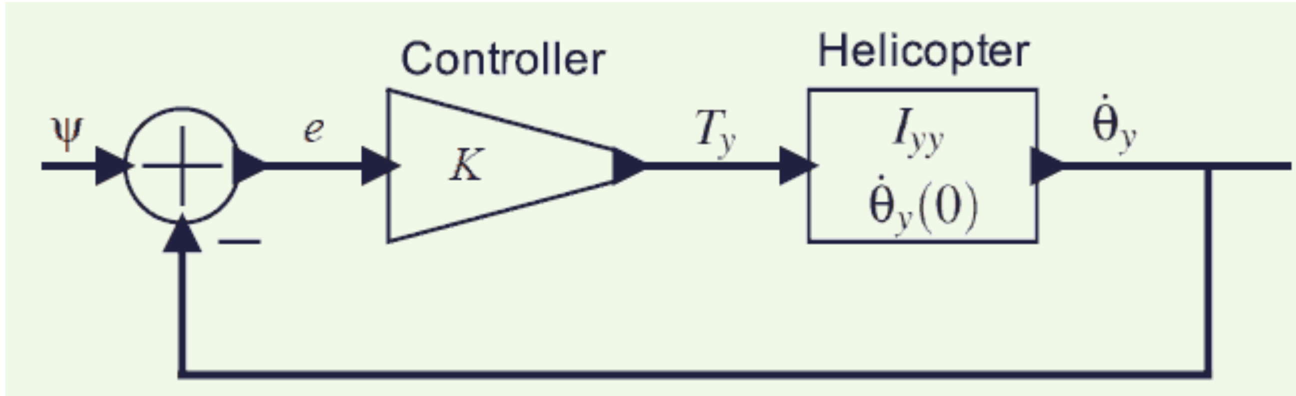
net torque

$$e(t) = \psi(t) - \dot{\theta}_y(t) \qquad T_y(t) = Ke(t)$$

$$\begin{aligned} \dot{\theta}_y(t) &= \dot{\theta}_y(0) + \frac{1}{I_{yy}} \int_0^t T_y(\tau)\, d\tau \\ &= \dot{\theta}_y(0) + \frac{K}{I_{yy}} \int_0^t (\psi(\tau) - \dot{\theta}_y(\tau))\, d\tau \end{aligned}$$

Note that the angular velocity appears on both sides, so this equation is not trivial to solve.

# Behavior of the controller




$$\dot{\theta}_y(t) = \dot{\theta}_y(0) + \frac{K}{I_{yy}} \int_0^t (\psi(\tau) - \dot{\theta}_y(\tau)) d\tau$$

Desired angular velocity: $\psi(t) = 0$

Simplifies differential equation to:

$$\dot{\theta}_y(t) = \dot{\theta}_y(0) - \frac{K}{I_{yy}} \int_0^t \dot{\theta}_y(\tau) d\tau.$$

Which can be solved as follows (see textbook):

$$\dot{\theta}_y(t) = \dot{\theta}_y(0) e^{-Kt/I_{yy}} u(t)$$

# Exercise

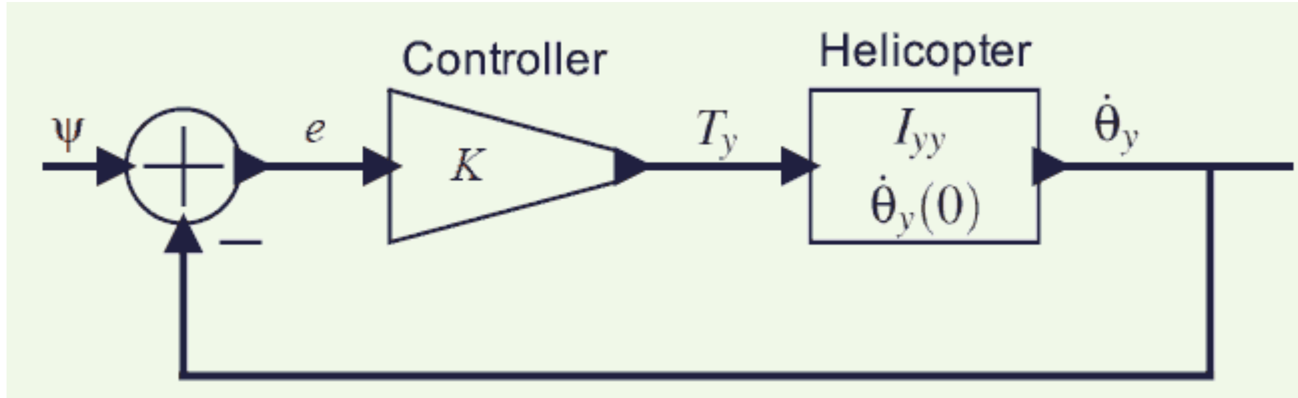
Reformulate the helicopter model so that it has two inputs, the torque of the top rotor and the torque of the tail rotor.

Show (by simulation) that if the top rotor applies a constant torque, then our controller cannot keep the helicopter from rotating. Increasing the feedback gain, however, reduces the rate of rotation.

A better controller would include an integrator in the controller. Such controllers are studied in EECS 128.

# Questions

# Behavior of the controller




$$\dot{\theta}_y(t) = \dot{\theta}_y(0) + \frac{K}{I_{yy}} \int_0^t (\psi(\tau) - \dot{\theta}_y(\tau))d\tau$$

Assume that helicopter is initially at rest,

$$\dot{\theta}(0) = 0,$$

and that the desired signal is

$$\psi(t) = au(t)$$

for some constant $a$.
By calculus (see notes), the solution is

$$\dot{\theta}_y(t) = au(t)(1 - e^{-Kt/I_{yy}})$$