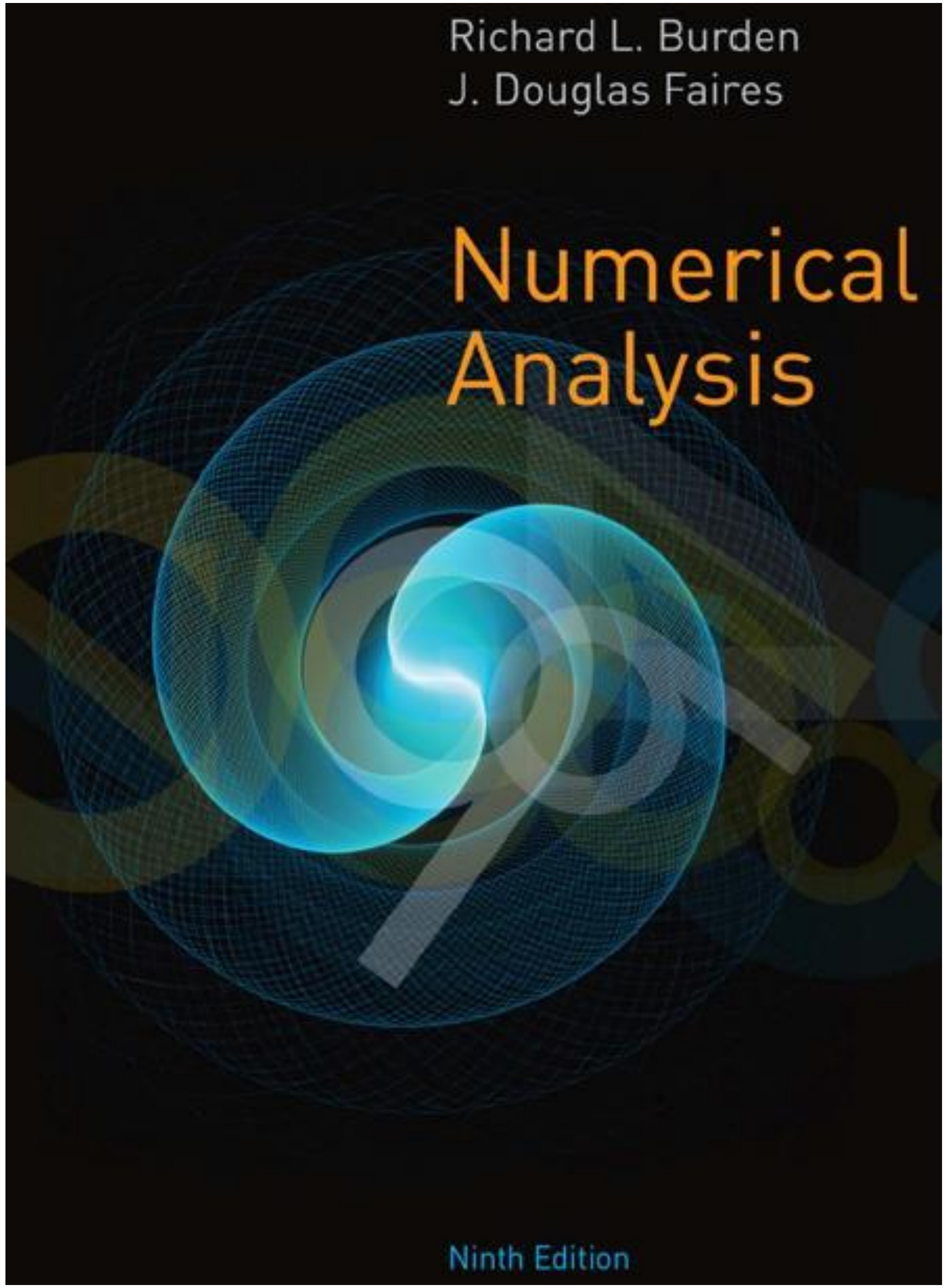
MAP 2320 – MÉTODOS NUMÉRICOS EM EQUAÇÕES DIFERENCIAIS II

2º Semestre - 2019

Prof. Dr. Luis Carlos de Castro Santos

lsantos@ime.usp.br

MAP2320


Richard L. Burden
J. Douglas Faires
Numerical Analysis
Ninth Edition

MAP2320

Numerical Analysis

NINTH EDITION

Richard L. Burden
Youngstown State University

J. Douglas Faires
Youngstown State University

12 Numerical Solutions to Partial Differential Equations 713

12.1 Elliptic Partial Differential Equations 716
12.2 Parabolic Partial Differential Equations 725
12.3 Hyperbolic Partial Differential Equations 739
12.4 An Introduction to the Finite-Element Method 746
12.5 Survey of Methods and Software 760

12.2 Parabolic Partial Differential Equations

The *parabolic* partial differential equation we consider is the heat, or diffusion, equation

$$\frac{\partial u}{\partial t}(x,t) = \alpha^2 \frac{\partial^2 u}{\partial x^2}(x,t), \quad 0 < x < l, \quad t > 0, \tag{12.6}$$

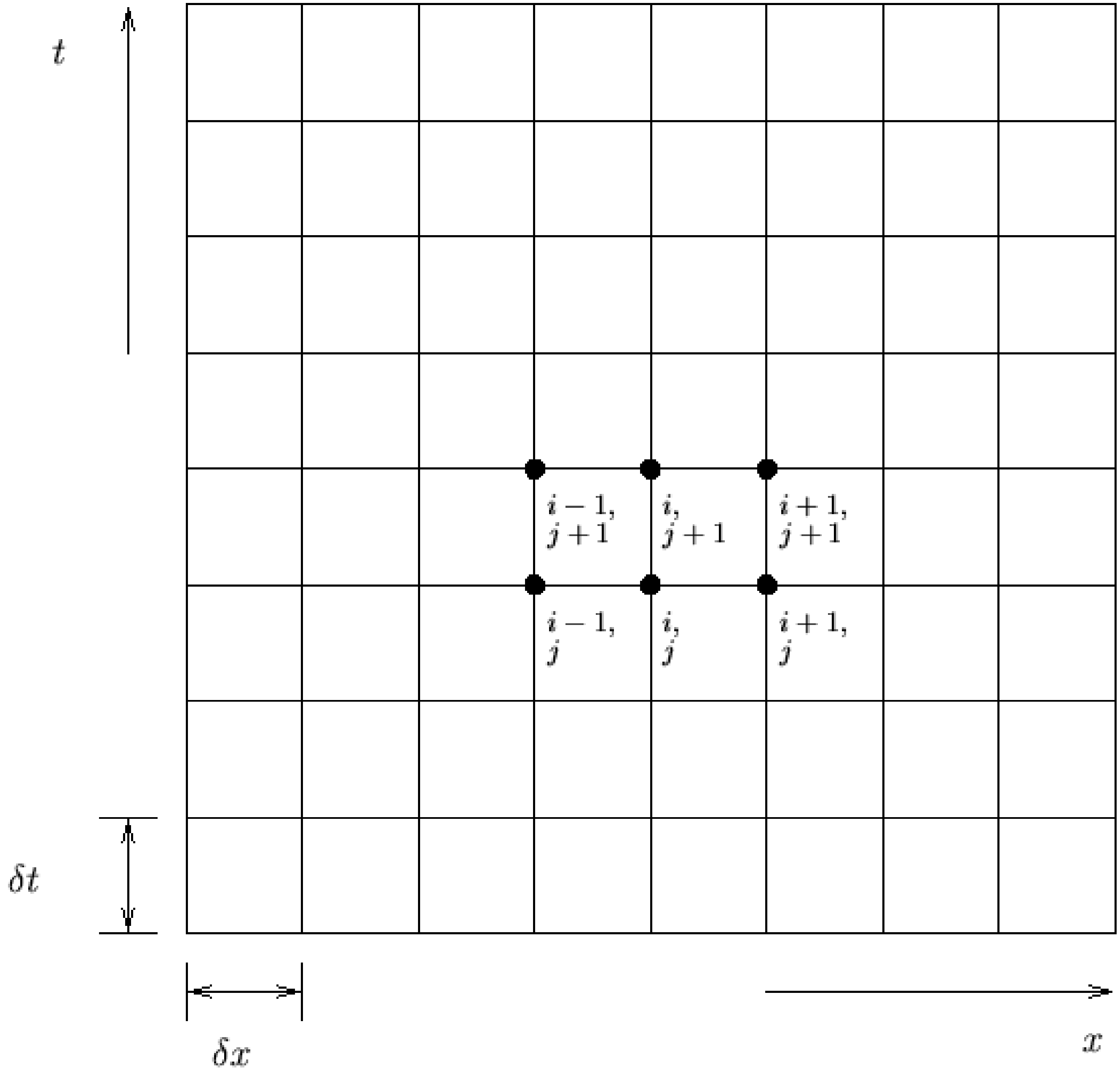
subject to the conditions

$$u(0,t) = u(l,t) = 0, \quad t > 0, \quad \text{and} \quad u(x,0) = f(x), \quad 0 \le x \le l.$$

The approach we use to approximate the solution to this problem involves finite differences and is similar to the method used in Section 12.1.

First select an integer $m > 0$ and define the x-axis step size $h = l/m$. Then select a time-step size k. The grid points for this situation are (x_i, t_j), where $x_i = ih$, for $i = 0, 1, \ldots, m$, and $t_j = jk$, for $j = 0, 1, \ldots$.

$$h = \Delta x, \qquad k = \Delta t$$



t
$i-1,$
$j+1$
$i,$
$j+1$
$i+1,$
$j+1$
$i-1,$
j
$i,$
j
$i+1,$
j
δt
δx
x

$$D_+u(\bar{x}) \equiv \frac{u(\bar{x}+h)-u(\bar{x})}{h}$$

Forward Difference Method

We obtain the difference method using the Taylor series in t to form the difference quotient

$$\frac{\partial u}{\partial t}(x_i, t_j) = \frac{u(x_i, t_j + k) - u(x_i, t_j)}{k} - \frac{k}{2}\frac{\partial^2 u}{\partial t^2}(x_i, \mu_j), \tag{12.7}$$

for some $\mu_j \in (t_j, t_{j+1})$, and the Taylor series in x to form the difference quotient

$$\frac{\partial^2 u}{\partial x^2}(x_i, t_j) = \frac{u(x_i + h, t_j) - 2u(x_i, t_j) + u(x_i - h, t_j)}{h^2} - \frac{h^2}{12}\frac{\partial^4 u}{\partial x^4}(\xi_i, t_j), \tag{12.8}$$

where $\xi_i \in (x_{i-1}, x_{i+1})$.

$$D^2u(\bar{x}) \quad = \quad \frac{1}{h^2}[u(\bar{x}-h) - 2u(\bar{x}) + u(\bar{x}+h)]$$

The parabolic partial differential equation (12.6) implies that at interior gridpoints (x_i, t_j), for each $i = 1, 2, \ldots, m-1$ and $j = 1, 2, \ldots,$ we have

$$\frac{\partial u}{\partial t}(x_i, t_j) - \alpha^2 \frac{\partial^2 u}{\partial x^2}(x_i, t_j) = 0,$$

so the difference method using the difference quotients (12.7) and (12.8) is

$$\frac{w_{i,j+1} - w_{ij}}{k} - \alpha^2 \frac{w_{i+1,j} - 2w_{ij} + w_{i-1,j}}{h^2} = 0, \tag{12.9}$$

where w_{ij} approximates $u(x_i, t_j)$.

The local truncation error for this difference equation is

$$\tau_{ij} = \frac{k}{2}\frac{\partial^2 u}{\partial t^2}(x_i, \mu_j) - \alpha^2 \frac{h^2}{12}\frac{\partial^4 u}{\partial x^4}(\xi_i, t_j). \tag{12.10}$$

$$\tau_{ij} = O(k, h^2)$$

Solving Eq. (12.9) for $w_{i,j+1}$ gives

$$w_{i,j+1} = \left(1 - \frac{2\alpha^2 k}{h^2}\right) w_{ij} + \alpha^2 \frac{k}{h^2}(w_{i+1,j} + w_{i-1,j}), \tag{12.11}$$

for each $i = 1, 2, \ldots, m-1$ and $j = 1, 2, \ldots$.

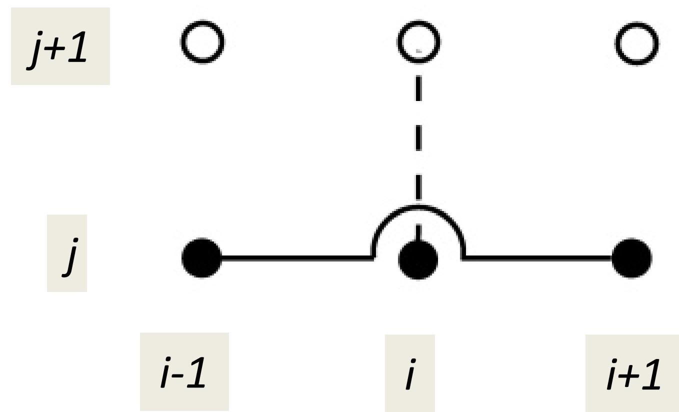
So we have

$$w_{0,0} = f(x_0), \quad w_{1,0} = f(x_1), \quad \ldots w_{m,0} = f(x_m).$$

Then we generate the next t-row by

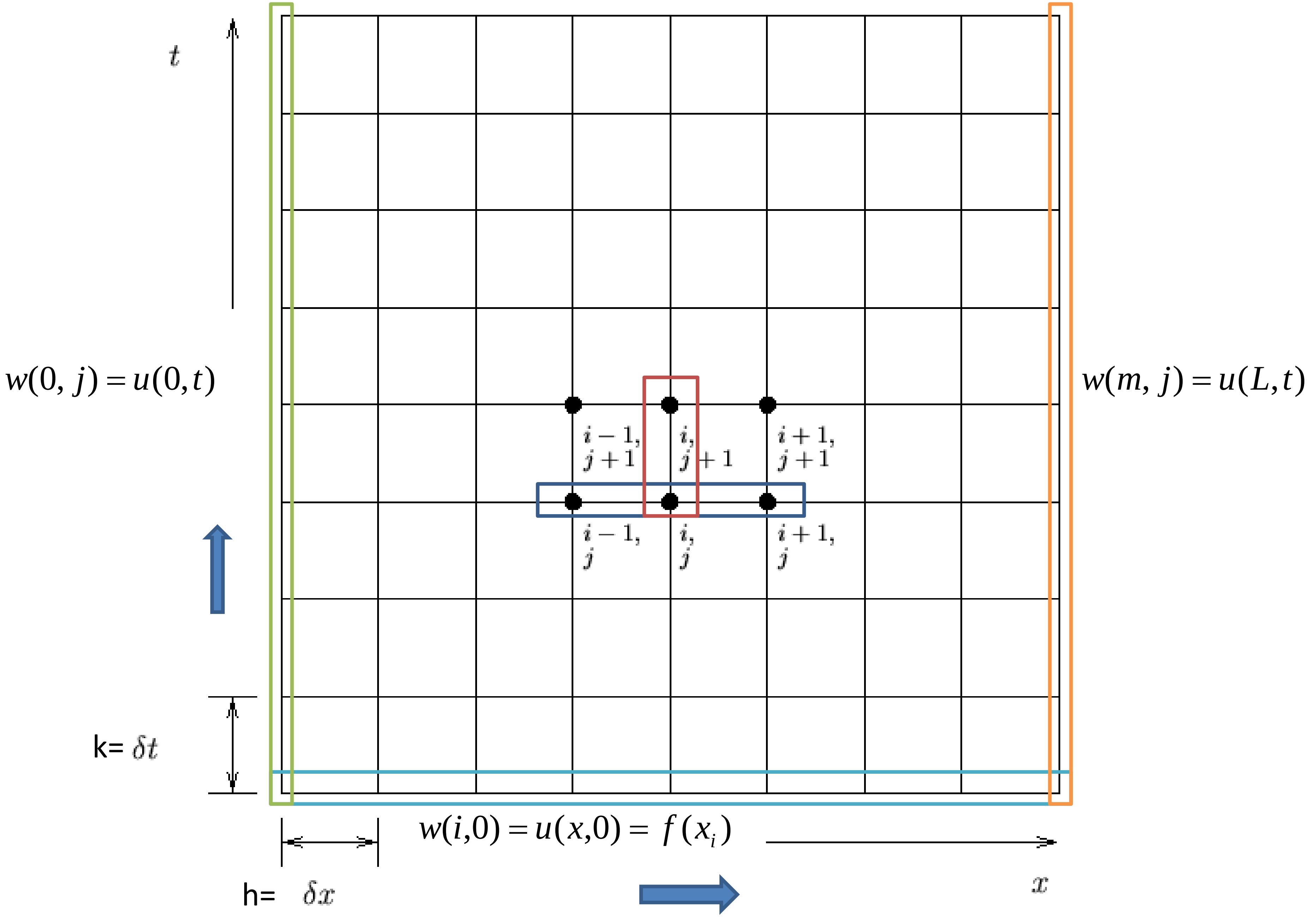
$$\begin{aligned}
w_{0,1} =& u(0, t_1) = 0;\\
w_{1,1} =& \left(1 - \frac{2\alpha^2 k}{h^2}\right) w_{1,0} + \alpha^2 \frac{k}{h^2}(w_{2,0} + w_{0,0});\\
w_{2,1} =& \left(1 - \frac{2\alpha^2 k}{h^2}\right) w_{2,0} + \alpha^2 \frac{k}{h^2}(w_{3,0} + w_{1,0});\\
&\vdots\\
w_{m-1,1} =& \left(1 - \frac{2\alpha^2 k}{h^2}\right) w_{m-1,0} + \alpha^2 \frac{k}{h^2}(w_{m,0} + w_{m-2,0});\\
w_{m,1} =& u(m, t_1) = 0.
\end{aligned}$$

Now we can use the $w_{i,1}$ values to generate all the $w_{i,2}$ values and so on.



$$w_i^{j+1} = \lambda w_{i-1}^j + (1 - 2\lambda) w_i^j + \lambda w_{i+1}^j$$

$$\lambda = \frac{\alpha^2 k}{h^2}$$



t
$w(0, j) = u(0,t)$
$w(m, j) = u(L,t)$
$i-1,$ $j+1$
$i,$ $j+1$
$i+1,$ $j+1$
$i-1,$ j
$i,$ j
$i+1,$ j
k= δt
$w(i,0) = u(x,0) = f(x_i)$
h= δx
x

The explicit nature of the difference method implies that the $(m-1)\times(m-1)$ matrix associated with this system can be written in the tridiagonal form

$$A = \begin{bmatrix} (1-2\lambda) & \lambda & 0 & \cdots & \cdots & 0 \\ \lambda & (1-2\lambda) & \lambda & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \lambda \\ 0 & \cdots & \cdots & 0 & \lambda & (1-2\lambda) \end{bmatrix},$$

where $\lambda = \alpha^2(k/h^2)$. If we let

$$\mathbf{w}^{(0)} = (f(x_1), f(x_2), \ldots, f(x_{m-1}))^t$$

and

$$\mathbf{w}^{(j)} = (w_{1j}, w_{2j}, \ldots, w_{m-1,j})^t, \quad \text{for each } j = 1, 2, \ldots,$$

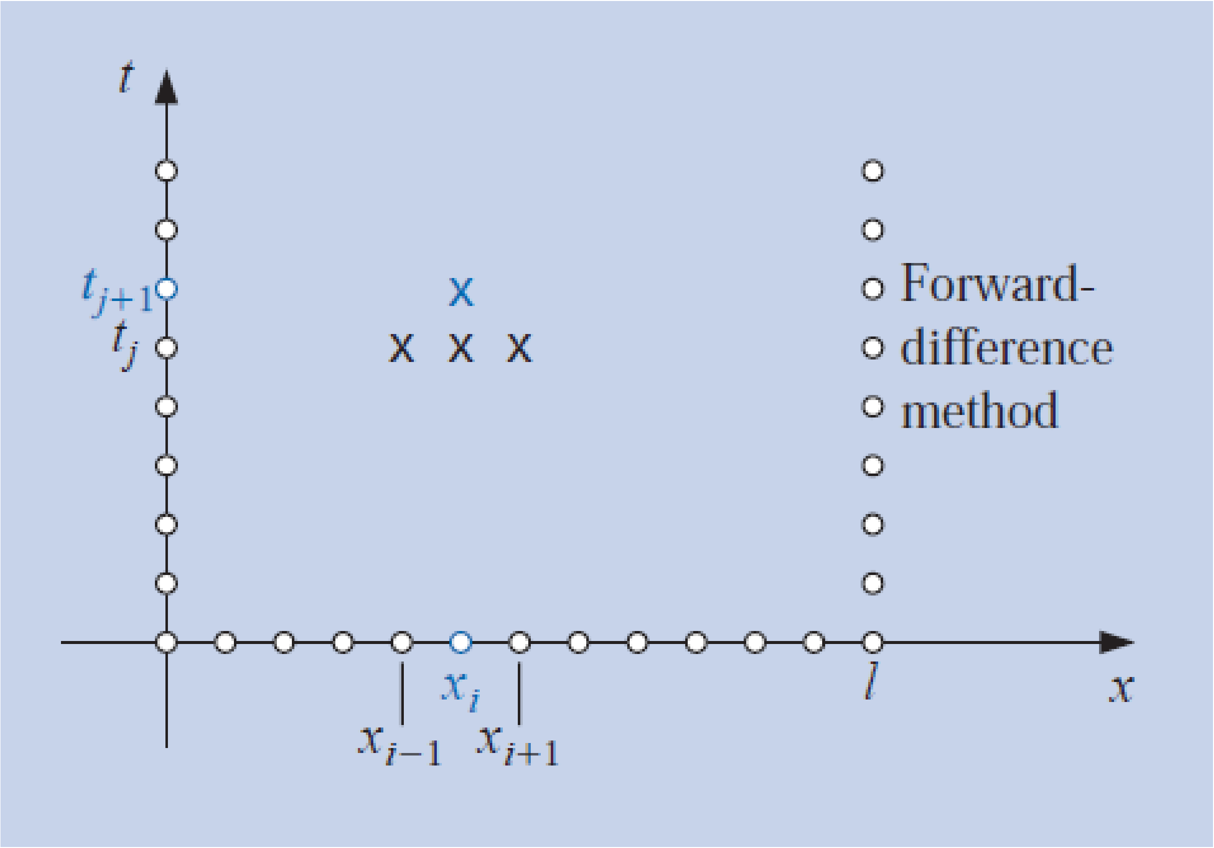
then the approximate solution is given by

$$\mathbf{w}^{(j)} = A\mathbf{w}^{(j-1)}, \quad \text{for each } j = 1, 2, \ldots,$$

so $\mathbf{w}^{(j)}$ is obtained from $\mathbf{w}^{(j-1)}$ by a simple matrix multiplication.

Apenas para efeitos de entendimento do problema discreto NÃO para implementação

This is known as the
Forward-Difference method, and the approximation at the cyan point shown in Figure 12.8 uses information from the other points marked on that figure. If the solution to the partial differential equation has four continuous partial derivatives in x and two in t, then Eq. (12.10) implies that the method is of order $O(k + h^2)$.



Example 1

Use steps sizes **(a)** $h = 0.1$ and $k = 0.0005$ and **(b)** $h = 0.1$ and $k = 0.01$ to approximate the solution to the heat equation

$$\frac{\partial u}{\partial t}(x,t) - \frac{\partial^2 u}{\partial x^2}(x,t) = 0, \quad 0 < x < 1, \quad 0 \le t,$$

with boundary conditions

$$u(0,t) = u(1,t) = 0, \quad 0 < t,$$

and initial conditions

$$u(x,0) = \sin(\pi x), \quad 0 \le x \le 1.$$

Compare the results at $t = 0.5$ to the exact solution

$$u(x,t) = e^{-\pi^2 t}\sin(\pi x).$$

Solution

(a) Forward-Difference method with $h = 0.1$, $k = 0.0005$ and $\lambda = (1)^2(0.0005/(0.1)^2) = 0.05$ gives the results in the third column of Table 12.3. As can be seen these results are quite accurate.

Table 12.3

x_i	$u(x_i, 0.5)$	$w_{i,1000}$ $k = 0.0005$	$\lvert u(x_i, 0.5) - w_{i,1000}\rvert$	$w_{i,50}$ $k = 0.01$	$\lvert u(x_i, 0.5) - w_{i,50}\rvert$
0.0	0	0		0	
0.1	0.00222241	0.00228652	6.411×10^{-5}	8.19876×10^{7}	8.199×10^{7}
0.2	0.00422728	0.00434922	1.219×10^{-4}	-1.55719×10^{8}	1.557×10^{8}
0.3	0.00581836	0.00598619	1.678×10^{-4}	2.13833×10^{8}	2.138×10^{8}
0.4	0.00683989	0.00703719	1.973×10^{-4}	-2.50642×10^{8}	2.506×10^{8}
0.5	0.00719188	0.00739934	2.075×10^{-4}	2.62685×10^{8}	2.627×10^{8}
0.6	0.00683989	0.00703719	1.973×10^{-4}	-2.49015×10^{8}	2.490×10^{8}
0.7	0.00581836	0.00598619	1.678×10^{-4}	2.11200×10^{8}	2.112×10^{8}
0.8	0.00422728	0.00434922	1.219×10^{-4}	-1.53086×10^{8}	1.531×10^{8}
0.9	0.00222241	0.00228652	6.511×10^{-5}	8.03604×10^{7}	8.036×10^{7}
1.0	0	0		0	

1000 steps

50 steps

(b) Forward-Difference method with $h = 0.1$, $k = 0.01$ and $\lambda = (1)^2(0.01/(0.1)^2) = 1$ gives the results in the fifth column of Table 12.3. As can be seem from the sixth column, these results are worthless. ■

Custo Computacional versus Estabilidade:

Para cada ponto interno *m-2* realizam-se cerca de *3* multiplicações

Para cada nível de tempo *3*(m-2)* multiplicações, portanto para atingir

o instante *T*, são necessárias *3*(m-2)*T/k* multiplicações.

Quanto maior o *k*, menor o custo computacional.

Entretanto o aumento de *k* no Exemplo 1, em cerca de 20 vezes, produziu resultados inúteis.

Porque ?

Stability Considerations

Suppose that an error $\mathbf{e}^{(0)} = \left(e_1^{(0)}, e_2^{(0)}, \ldots, e_{m-1}^{(0)}\right)^t$ is made in representing the initial data

$$\mathbf{w}^{(0)} = \left(f(x_1), f(x_2), \ldots, f(x_{m-1})\right)^t$$

(or in any particular step, the choice of the initial step is simply for convenience). An error of $A\mathbf{e}^{(0)}$ propagates in $\mathbf{w}^{(1)}$, because

$$\mathbf{w}^{(1)} = A\left(\mathbf{w}^{(0)} + \mathbf{e}^{(0)}\right) = A\mathbf{w}^{(0)} + A\mathbf{e}^{(0)}.$$

This process continues. At the nth time step, the error in $\mathbf{w}^{(n)}$ due to $\mathbf{e}^{(0)}$ is $A^n\mathbf{e}^{(0)}$. The method is consequently stable precisely when these errors do not grow as n increases. But this is true if and only if for any initial error $\mathbf{e}^{(0)}$, we have $\left\|A^n\mathbf{e}^{(0)}\right\| \leq \left\|\mathbf{e}^{(0)}\right\|$ for all n. Hence, we must have $||A^n|| \leq 1$, a condition that, by Theorem 7.15 on page 446, requires that $\rho(A^n) = (\rho(A))^n \leq 1$. The Forward-Difference method is therefore stable only if $\rho(A) \leq 1$.

Raio Espectral

The eigenvalues of A can be shown (see Exercise 13) to be

$$\mu_i = 1 - 4\lambda\left(\sin\left(\frac{i\pi}{2m}\right)\right)^2, \quad \text{for each } i = 1, 2, \ldots, m-1.$$

So the condition for stability consequently reduces to determining whether

$$\rho(A) = \max_{1 \le i \le m-1} \left| 1 - 4\lambda\left(\sin\left(\frac{i\pi}{2m}\right)\right)^2 \right| \le 1,$$

and this simplifies to

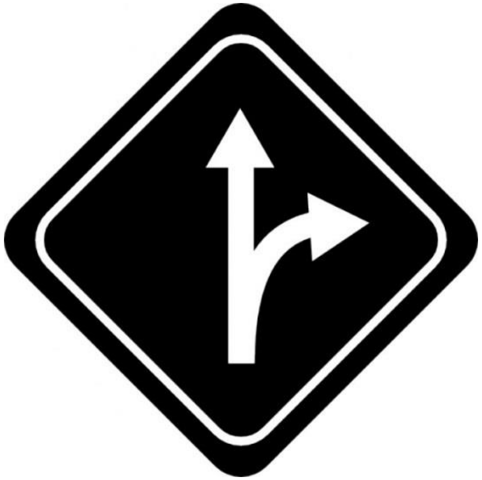
$$0 \le \lambda\left(\sin\left(\frac{i\pi}{2m}\right)\right)^2 \le \frac{1}{2}, \quad \text{for each } i = 1, 2, \ldots, m-1.$$

Stability requires that this inequality condition hold as $h \to 0$, or, equivalently, as $m \to \infty$. The fact that

$$\lim_{m \to \infty} \left[\sin\left(\frac{(m-1)\pi}{2m}\right)\right]^2 = 1$$

means that stability will occur only if $0 \le \lambda \le \frac{1}{2}$.

MAP2320



FTCS scheme

FTCS = forward in time centered in space

The FTCS scheme used for sim1.f uses the approximation

$$T_i^{n+1} = T_i^n + s\left(T_{i-1}^n + T_{i+1}^n - 2T_i^n\right)$$

Similar to $w_{i,j+1} = \left(1 - \frac{2\alpha^2 k}{h^2}\right) w_{ij} + \alpha^2 \frac{k}{h^2}(w_{i+1,j} + w_{i-1,j}),$ with $s = \frac{\alpha^2 k}{h^2} = \lambda$

Stability

For any numerical model small perturbations (such as truncation errors) should decay in time. However, for instance the FTCS scheme for the diffusion equation (sim1.f) will generate increasing errors for $s = 0.55$ as illustrated in Figure 5.3. The cause for the instability is an amplification of errors which occur and accumulate during a simulation.

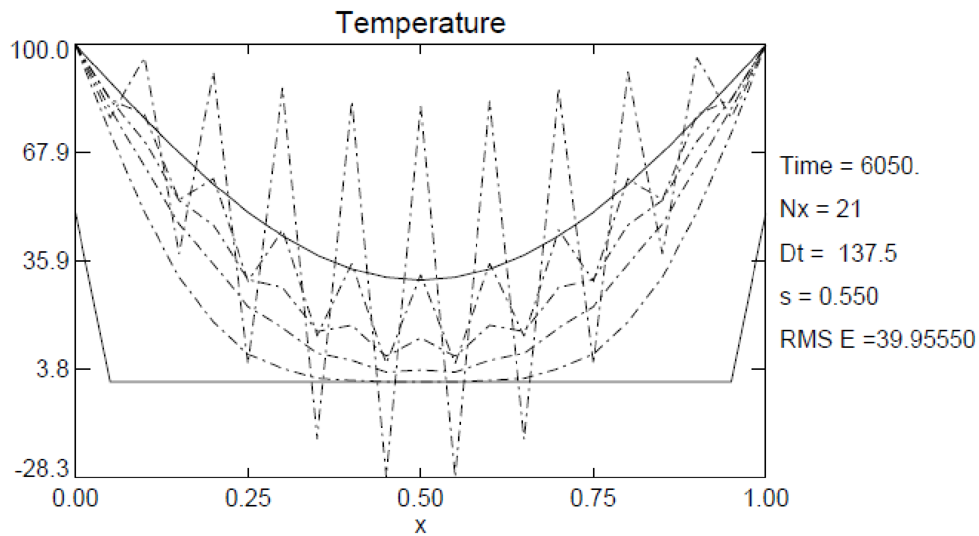


Figure 5.3: Grid oscillations increasing with time are typical for numerical instabilities.

Round-off errors occur at any stage in a computation such that the computational solution is not actually T_i^n but $\widetilde{T}_i^n$, i.e., a value which contains some newly generated round-off errors and the entire accumulation of prior errors. The error between the computed solution and the expected solution to the algebraic equation can be expressed as

$$\xi_i^n = T_i^n - \widetilde{T}_i^n \tag{5.10}$$

Similar to T_i^n, the solution $\widetilde{T}_i^n$ which includes the errors also satisfies the algebraic equations. For instance for the FTCS scheme applied to the diffusion equation

$$\widetilde{T}_i^{n+1} = (1-2s)\widetilde{T}_i^n + s\left(\widetilde{T}_{i-1}^n + \widetilde{T}_{i+1}^n\right)$$

If the algebraic equations are homogeneous and linear the error also satisfies the same equation which can be demonstrated by subtracting the equation for $\widetilde{T}_i^n$ from the one for T_i^n.

$$\xi_i^{n+1} = \xi_i^n + s\left(\xi_{i-1}^n + \xi_{i+1}^n - 2\xi_i^n\right) \tag{5.11}$$

For given initial conditions all initial errors ξ_i^0 are zero for all grid points i. Similarly Dirichlet boundary conditions imply that the error at the boundary points ξ_1^n and $\xi_{N_x}^n$ are zero for all times t_n.

The two most common methods to analyze stability are the matrix method and the von Neumann method.

Matrix method FTCS scheme

Consider the FTCS scheme for the diffusion equation with Dirichlet boundary conditions, i.e.,

$\xi_1^n = 0$ and $\xi_{N_x}^n = 0$.

The equations for the solution are

$$
\begin{aligned}
\xi_2^{n+1} &= (1-2s)\,\xi_2^n + s\xi_3^n \\
\xi_3^{n+1} &= s\xi_1^n + (1-2s)\,\xi_2^n + s\xi_3^n \\
\xi_i^{n+1} &= s\xi_{i-1}^n + (1-2s)\,\xi_i^n + s\xi_{i+1}^n \\
\xi_{N_x}^{n+1} &= s\xi_{N_x-1}^n + (1-2s)\,\xi_{N_x}^n
\end{aligned}
$$

Thus the solution equations can be written as $\underline{\xi}^{n+1} = \underline{\underline{\mathbf{A}}} \cdot \underline{\xi}^n$

with

$$
\underline{\xi} = \begin{pmatrix} \xi_2^n \\ . \\ \xi_i^n \\ . \\ . \\ \xi_{N_x-1}^n \end{pmatrix}, \quad \underline{\underline{\mathbf{A}}} = \begin{pmatrix} (1-2s) & s & 0 & & & \\ s & (1-2s) & s & & & \\ 0 & s & (1-2s) & s & & \\ & & . & . & . & \\ & & & . & . & . \\ & & & . & s & (1-2s) \end{pmatrix} \tag{5.13}
$$

It can be shown that all errors are bounded if all Eigenvalues are distinct and the absolute values are smaller than 1. If the Eigenvectors are given by

$$\underline{\xi}_k = \underline{\underline{\mathbf{T}}}^{-1} \cdot \underline{\eta}_k \tag{5.14}$$

where $\underline{\eta}_k$ is the unit vector in the kth direction the equation can be re-written as

$$\underline{\eta}^{n+1} = \left(\underline{\underline{\mathbf{T}}}\,\underline{\underline{\mathbf{A}}}\,\underline{\underline{\mathbf{T}}}^{-1}\right) \underline{\eta}^{n} \tag{5.15}$$

where the matrix $\underline{\underline{\widetilde{\mathbf{A}}}} = \underline{\underline{\mathbf{T}}}\,\underline{\underline{\mathbf{A}}}\,\underline{\underline{\mathbf{T}}}^{-1}$ is a diagonal matrix with the Eigenvalues on the diagonal. Thus for all Eigenvectors $\left|\underline{\eta}^{n+1}\right| < \left|\underline{\eta}^{n}\right|$ if all Eigenvalue are ≤ 1.

The Eigenvalues of a tridiagonal $(N_x - 2) \times (N_x - 2)$ matrix

$$\begin{pmatrix} b & c & 0 & 0 & & & \\ a & b & c & 0 & & & \\ 0 & a & b & c & . & & \\ & & & . & . & & \\ & & & . & . & & \\ & & & & a & b & c \\ & & & & 0 & a & b \end{pmatrix}$$

are given by

$$\lambda_k = b + 2\,(ac)^{1/2} \cos \frac{k\pi}{N_x - 1}$$

for $k = 2, ..., N_x - 2$. Thus the Eigenvalues of the FTCS algorithm are

$$\lambda_k = 1 - 2s + 2s \cos \frac{k\pi}{N_x - 1} = 1 - 4s \sin^2 \frac{k\pi}{2\,(N_x - 1)}$$

Thus the condition for stability is

$$-1 \leq 1 - 4s \sin^2 \frac{k\pi}{2\,(N_x - 1)} \leq 1 \tag{5.18}$$

or $s \leq 1/2$. Therefore convergence of the approximate solution requires $s \leq 1/2$ and in this case all perturbations decrease in amplitude over time.

$$1 - \cos(x) = 2 \sin^2(x/2)$$

Von Neumann method

The von Neumann method is usually easy to apply, straightforward, and dependable. However, similar to the matrix method the von Neumann method is applicable strictly only to linear equations with constant coefficients (which is already assumed in the definition for ξ). Nonlinear equations can be linearized to attempt to apply the method. However, in cases of larger sets of equations also the von Neumann methods can be challenging.

The basic idea is the following: Assume that the error can be expanded in a Fourier series

$$\xi_j^0 = \sum_{l=1}^{N_x} a_l \exp\left(i\Theta_l j\right)$$

with $\Theta_l = l\pi\Delta x$. Considering an initial error from one particular mode with amplitude 1 one can express

$$\xi_j^n = g^n \exp\left(i\Theta_l j\right) \tag{5.33}$$

where g is an amplification factor for this mode. **Stability requires** $|g| \leq 1$. Equivalent and maybe more intuitive we can assume that an error is given by a

$$\xi_j^n = \xi_0 \exp\left[i\left(kx_j + qt_n\right)\right], \tag{5.34}$$

i.e., a Fourier mode with amplitude ξ_0, wave number k, and frequency q.

$$\xi_j^{n+1} = \xi_0 \exp\left[i\left(kx_j + qt_n\right) + iq\Delta t\right]$$

(Again a general error is expanded by an appropriate superposition of all modes as in (5.32)). The amplification (or damping) of this mode is

$$\frac{\xi_j^{n+1}}{\xi_j^n} = \exp\left[iq\Delta t\right] = g \tag{5.35}$$

Similarly we can express

$$\xi_{j+1}^n = \xi_0 \exp\left[i\left(kx_j + qt_n\right) + ik\Delta x\right]$$

and

$$\frac{\xi_{j+1}^n}{\xi_j^n} = \exp\left[ik\Delta x\right] \tag{5.36}$$

The task now becomes to derive g for all modes from the discretized equations and to insure that $|g| \leq 1$ for all modes (all values of k).

FTCS scheme

The basic equation for the FTCS scheme is

$$\xi_i^{n+1} = (1-2s)\,\xi_i^n + s\left(\xi_{i-1}^n + \xi_{i+1}^n\right)$$

By substituting the single Fourier mode and division by ξ_j^n we obtain

$$\exp\left[iq\Delta t\right] = (1-2s) + s\left[\exp\left(ik\Delta x\right) + \exp\left(-ik\Delta x\right)\right]$$

Substituting the amplification factor g and applying the definition for the cosine function yields

$$g = 1 - 2s + 2s\cos\left(k\Delta x\right) = 1 - 4s\sin^2\left(k\Delta x/2\right) \tag{5.37}$$

where $1-\cos\left(x\right) = 2\sin^2\left(x/2\right)$ has been used. Thus the stability condition reads

$$-1 \leq 1 - 4s\sin^2\left(k\Delta x/2\right) \leq 1 \tag{5.38}$$

Note that **stability requires that the above condition is satisfied for all possible wave numbers k.** The condition on the right is always satisfied and the left inequality requires

$$s \leq \frac{1}{2} \tag{5.39}$$

MAP2320

By definition $\lambda = \alpha^2(k/h^2)$, so this inequality requires that h and k be chosen so that

$$\alpha^2 \frac{k}{h^2} \le \frac{1}{2}.$$

In Example 1 we have $\alpha^2 = 1$, so this condition is satisfied when $h = 0.1$ and $k = 0.0005$. But when k was increased to 0.01 with no corresponding increase in h, the ratio was

$$\frac{0.01}{(0.1)^2} = 1 > \frac{1}{2},$$

and stability problems became immediately apparent and dramatic.

Consistent with the terminology of Chapter 5, we call the Forward-Difference method **conditionally stable**. The method converges to the solution of Eq. (12.6) with rate of convergence $O(k + h^2)$, provided

$$\alpha^2 \frac{k}{h^2} \le \frac{1}{2}$$

and the required continuity conditions on the solution are met. (For a detailed proof of this fact, see [IK, pp. 502–505].)


EXTRA

Problema Homogêneo
C.C. de Dirichlet
homogêneas

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$
$$u(x,0) = f(x),\ 0 < x < L$$
$$u(0,t) = 0,\ u(L,t) = 0$$

Problema Homogêneo
C.C. de Dirichlet
não-homogêneas

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$
$$u(x,0) = f(x),\ 0 < x < L$$
$$u(0,t) = T_1,\ u(L,t) = T_2$$

Problema Homogêneo
C.C. de Neumann
homogêneas

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$
$$u(x,0) = f(x),\ 0 < x < L$$
$$\frac{\partial u}{\partial x}(0,t) = 0,\ \frac{\partial u}{\partial x}(L,t) = 0$$

Problema Homogêneo
C.C. de Mistas

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$
$$u(x,0) = f(x),\ 0 < x < L$$
$$u(0,t) = 0,\ \frac{\partial u}{\partial x}(L,t) = 0$$

Problema Não-Homogêneo

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} + g(x)$$
$$u(x,0) = f(x),\ 0 < x < L$$
$$u(0,t) = T_1,\ u(L,t) = T_2$$

Problema
Homogêneo
C.C. de Dirichlet
homogêneas

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

$$u(x,0) = f(x),\ 0 < x < L$$

$$u(0,t) = 0,\ u(L,t) = 0$$


```
h = L/m
lambda = alpha*alpha*k/(h*h)
jmax=T/k

do i = 0 , m
w(i,0)=f(x(i))
enddo

do  j = 0, jmax
        w(0,j)=0
        w(m,j)=0

        do i = 1 , m-1
                w(i,j+1) = (1-2*lambda)*w(i,j)+lambda*(w(i-1,j)+ w(i+1,j))
        enddo

enddo
```

Problema Homogêneo
C.C. de Dirichlet
homogêneas

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$
$$u(x,0) = f(x),\ 0 < x < L$$
$$u(0,t) = 0,\ u(L,t) = 0$$

Problema Homogêneo
C.C. de Dirichlet
não-homogêneas

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$
$$u(x,0) = f(x),\ 0 < x < L$$
$$u(0,t) = T_1,\ u(L,t) = T_2$$

Problema Homogêneo
C.C. de Neumann
homogêneas

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$
$$u(x,0) = f(x),\ 0 < x < L$$
$$\frac{\partial u}{\partial x}(0,t) = 0,\ \frac{\partial u}{\partial x}(L,t) = 0$$

Problema Homogêneo
C.C. de Mistas

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$
$$u(x,0) = f(x),\ 0 < x < L$$
$$u(0,t) = 0,\ \frac{\partial u}{\partial x}(L,t) = 0$$

Problema Não-Homogêneo

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} + g(x)$$
$$u(x,0) = f(x),\ 0 < x < L$$
$$u(0,t) = T_1,\ u(L,t) = T_2$$

Problema Homogêneo
C.C. de Dirichlet
não-homogêneas

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

$$u(x,0) = f(x),\ 0 < x < L$$

$$u(0,t) = T_1,\ u(L,t) = T_2$$

MAP2320

```
h = L/m
lambda = alpha*alpha*k/(h*h)
jmax=T/k

do i = 0 , m
w(i,0)=f(x(i))
enddo

do  j = 0, jmax
        w(0,j)=T1
        w(m,j)=T2

        do i = 1 , m-1
                w(i,j+1) = (1-2*lambda)*w(i,j)+lambda*(w(i-1,j)+ w(i+1,j))
        enddo

enddo
```

Problema Homogêneo
C.C. de Dirichlet
homogêneas

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$
$$u(x,0) = f(x),\ 0 < x < L$$
$$u(0,t) = 0,\ u(L,t) = 0$$

Problema Homogêneo
C.C. de Dirichlet
não-homogêneas

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$
$$u(x,0) = f(x),\ 0 < x < L$$
$$u(0,t) = T_1,\ u(L,t) = T_2$$

Problema Homogêneo
C.C. de Neumann
homogêneas

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$
$$u(x,0) = f(x),\ 0 < x < L$$
$$\frac{\partial u}{\partial x}(0,t) = 0,\ \frac{\partial u}{\partial x}(L,t) = 0$$

Problema Homogêneo
C.C. de Mistas

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$
$$u(x,0) = f(x),\ 0 < x < L$$
$$u(0,t) = 0,\ \frac{\partial u}{\partial x}(L,t) = 0$$

Problema Não-Homogêneo

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} + g(x)$$
$$u(x,0) = f(x),\ 0 < x < L$$
$$u(0,t) = T_1,\ u(L,t) = T_2$$

Problema Não-Homogêneo

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} + g(x)$$
$$u(x,0) = f(x),\ 0 < x < L$$
$$u(0,t) = T_1,\ u(L,t) = T_2$$

```
h = L/m
lambda = alpha*alpha*k/(h*h)
jmax=T/k

do i = 0 , m
w(i,0)=f(x(i))
enddo

do  j = 0, jmax
        w(0,j)=T1
        w(m,j)=T2

        do i = 1 , m-1
                w(i,j+1) = (1-2*lambda)*w(i,j)+lambda*(w(i-1,j)+ w(i+1,j))
                        + k*g(x(i))
        enddo

enddo
```

Problema Homogêneo
C.C. de Dirichlet
homogêneas

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$
$$u(x,0) = f(x),\ 0 < x < L$$
$$u(0,t) = 0,\ u(L,t) = 0$$

Problema Homogêneo
C.C. de Dirichlet
não-homogêneas

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$
$$u(x,0) = f(x),\ 0 < x < L$$
$$u(0,t) = T_1,\ u(L,t) = T_2$$

Problema Homogêneo
C.C. de Neumann
homogêneas

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$
$$u(x,0) = f(x),\ 0 < x < L$$
$$\frac{\partial u}{\partial x}(0,t) = 0,\ \frac{\partial u}{\partial x}(L,t) = 0$$

Problema Homogêneo
C.C. de Mistas

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$
$$u(x,0) = f(x),\ 0 < x < L$$
$$u(0,t) = 0,\ \frac{\partial u}{\partial x}(L,t) = 0$$

Problema Não-Homogêneo

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} + g(x)$$
$$u(x,0) = f(x),\ 0 < x < L$$
$$u(0,t) = T_1,\ u(L,t) = T_2$$

Para Condições de Contorno de Neumann (Não-Homogêneas)

$$\frac{\partial u}{\partial x}(0,t) = q_A \qquad \frac{\partial u}{\partial x}(L,t) = q_B$$

Usando operadores de diferenças-finitas (1ª ordem)

$$D_+u(\bar{x}) = \frac{u(\bar{x}+h) - u(\bar{x})}{h} \qquad D_-u(\bar{x}) \equiv \frac{u(\bar{x}) - u(\bar{x}-h)}{h}$$

temos

$$\frac{u(h,t) - u(0,t)}{h} = q_A, \qquad u(0,t) = u(h,t) - h \cdot q_A$$

$$\frac{u(L,t) - u(L-h,t)}{h} = q_B, \qquad u(L,t) = u(L-h,t) + h \cdot q_b$$

No caso particular de condições homogêneas:

$$q_A = q_B = 0, \qquad u(0,t) = u(h,t), \qquad u(L,t) = u(L-h,t)$$

Entretanto, como o esquema Forward Diference é de 2ª ordem no espaço, seria mais adequado, em termos de precisão, utilizar uma aproximação de 2ª ordem para as condições de contorno também. Logo:

$$\frac{\partial u}{\partial x}(0,t) = q_A \qquad\qquad \frac{\partial u}{\partial x}(L,t) = q_B$$

Usando operadores de diferenças-finitas (2ª ordem)

$$D_2 u(\bar{x}) = \frac{1}{2h}\left[-3u(\bar{x}) + 4u(\bar{x}+h) - u(\bar{x}+2h)\right]$$

$$D_2 u(\bar{x}) = \frac{1}{2h}\left[3u(\bar{x}) - 4u(\bar{x}-h) + u(\bar{x}-2h)\right]$$

temos

$$u(0,t) = \frac{1}{3}\left[-2h\cdot q_A + 4u(h,t) - u(2h,t)\right]$$

$$u(L,t) = \frac{1}{3}\left[2h\cdot q_B - 4u(L-h,t) + u(L-2h,t)\right]$$

No caso particular de condições homogêneas: $q_A = q_B = 0,$

$$u(0,t) = \frac{1}{3}\left[4u(h,t) - u(2h,t)\right]$$

$$u(L,t) = \frac{1}{3}\left[-4u(L,t) + u(L-2h,t)\right]$$

Verifiquem !

Problema Homogêneo
C.C. de Neumann
homogêneas

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

$$u(x,0) = f(x),\ 0 < x < L$$

$$\frac{\partial u}{\partial x}(0,t) = 0,\ \frac{\partial u}{\partial x}(L,t) = 0$$


```
h = L/m
lambda = alpha*alpha*k/(h*h)
jmax=T/k

do i = 0 , m
        w(i,0)=f(x(i))
enddo


do  j = 0, jmax

        do i = 1 , m-1
                w(i,j+1) = (1-2*lambda)*w(i,j)+lambda*(w(i-1,j)+ w(i+1,j))
        enddo

        w(0,j+1)=(4*w(1,j+1)-w(2,j+1))/3
        w(m,j+1)=(-4*w(m-1,j+1)+w(m-2,j+1))/3

enddo
```

Problema Homogêneo
C.C. de Dirichlet
homogêneas

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$
$$u(x,0) = f(x),\ 0 < x < L$$
$$u(0,t) = 0,\ u(L,t) = 0$$

Problema Homogêneo
C.C. de Dirichlet
não-homogêneas

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$
$$u(x,0) = f(x),\ 0 < x < L$$
$$u(0,t) = T_1,\ u(L,t) = T_2$$

Problema Homogêneo
C.C. de Neumann
homogêneas

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$
$$u(x,0) = f(x),\ 0 < x < L$$
$$\frac{\partial u}{\partial x}(0,t) = 0,\ \frac{\partial u}{\partial x}(L,t) = 0$$

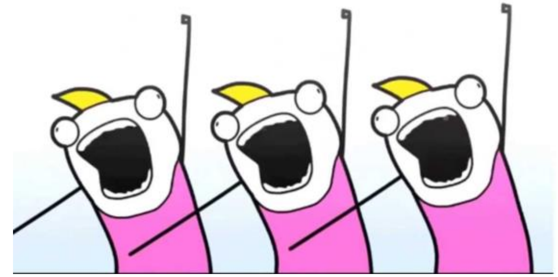
Problema Homogêneo
C.C. de Mistas

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$
$$u(x,0) = f(x),\ 0 < x < L$$
$$u(0,t) = 0,\ \frac{\partial u}{\partial x}(L,t) = 0$$

Problema Não-Homogêneo

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} + g(x)$$
$$u(x,0) = f(x),\ 0 < x < L$$
$$u(0,t) = T_1,\ u(L,t) = T_2$$

Numericamente podemos facilmente estender essa lista



O que queremos ?

- O menor custo computacional !

Porque não podemos ?

- Erro de discretização e Limite de Estabilidade

O que faremos?

Buscar métodos incondicionalmente estáveis !

MAP 2320 – MÉTODOS NUMÉRICOS EM EQUAÇÕES DIFERENCIAIS II

2º Semestre - 2019

Roteiro do curso

- Introdução
- Séries de Fourier
- **Método de Diferenças Finitas**
- **Equação do calor transiente (parabólica)**
- Equação de Poisson (elíptica)
- Equação da onda (hiperbólica)