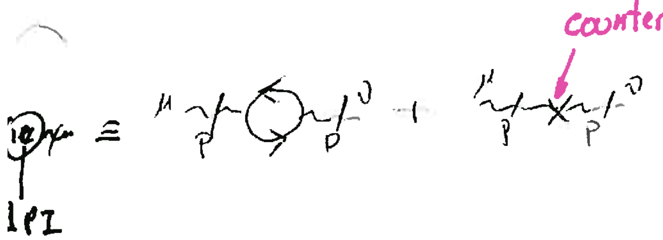
15. Renormalization Group

15.a Introduction

When renormalizing a model, we must choose a renormalization rescription R. However, this is arbitrary since there are infinite ther choices R'. Nevertheless, observables are independent of renormalization escriptions. A change in renormalization prescription must be compensated simultaneous changes in the renormalization parameters to have the ysical results renormalization-invariant. The renormalization group (RG) udies these changes!

The renormalization group eq's (RGE) are extremelly useful in summing series of logarithms ~~th~~ to all orders in perturbation theory! Let's ebbolate that. In QED, the vacuum polarization is given at one-loop by



$$\equiv i\Pi_{\mu\nu}(p) = i(p^2 g_{\mu\nu} - p_\mu p_\nu)\times$$

$$\underbrace{\times\left\{-\frac{\alpha}{\pi}\int_0^1 dx\, 2x(1-x)\left(\frac{2}{\epsilon}-\gamma_E+\ln\left(\frac{4\pi\mu^2}{m^2-q^2x(1-x)}\right)\right)\right.}_{\Pi^{ON}} \quad \delta Z_3 \qquad (1)$$

We learnt that for $-q^2 \gg m^2$ we ~~h~~have [(13.18)-(5.1)]

$$\Pi^{ON}(q^2) \longrightarrow \frac{\alpha}{\pi}\left[\frac{1}{3}\ln\frac{|q^2|}{m^2} - \frac{5}{9} + \mathcal{O}\left(\frac{m^2}{q^2}\right)\right] \qquad (2)$$

So, there is the appearance of a large logarithm for large $|q| \gg m$ that might spoil perturbation theory. The solution was to sum up the series

full

$$\overset{\cancel{X}}{\Longrightarrow} e_R^2 \longrightarrow \frac{e_R^2}{1-\Pi^{OW}(q^2)} \qquad (3)$$

he RGE performs the sum of potentially large logarithms using the one-loop result without worrying with summing a geometrical series or details!

Notice that the appearance of logarithms is a feature of loop contributions. For instance a divergent integral like

$$I(p) = \int_0^\infty \frac{k\,dk}{k+p} \Longrightarrow \frac{d^2I}{dp^2} = \int_0^\infty \frac{2k\,dk}{(k+p)^2} = \frac{1}{p}$$

Now, integrating twice $\frac{d^2I}{dp^2}$ up to Λ to regulate it, we get

$$I(p) = p\ln p - p(\ln\Lambda + 1) + c_1\Lambda \Longrightarrow$$ there are log's in the result!

In QED we have, at large $-p^2$,

$$\Pi(p^2) = -\frac{e_R^2}{12\pi^2}\left[2/\epsilon + \ln\frac{\mu^2}{-p^2} + \dots\right] - \delta Z_3 \quad (4)$$ in dimensional regularization.

Had we used a cut off as in ~~pass~~ Pauli-Villars we would get

$$\Pi(p^2) = -\frac{e_R^2}{12\pi^2}\left[\ln\frac{\Lambda^2}{-p^2} + \dots\right] - \delta Z_3 \quad (5)$$

otice that the $\ln(-p^2)$ in (4) and (5) comes accompanied by μ^2 or Λ^2 (unphysical parameters) due to dimensional analysis! When we focus on μ^2 we have the continuum RG while focusing on Λ^2 leads to the wilsonian RG! We will start with the former and comment on the latter after.

15.0
2'

Stueckelberg + Petermann 1953

Gell-man - Low 1959 → behaviour of QED at short distances

block-spin RG Kadanov 1966
⇓
Continuum ⟹ dif. eq. Wilson 1971
F.T. : lattice or path integral: messy

Tools : { Callan widespread applications.
{ Symanzik

15.b Renormalization Group Equation

Let's obtain the RGE for the $\lambda\varphi^4$ model

$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi_B\partial^\mu\varphi_B - \frac{1}{2}m_B^2\varphi_B^2 - \frac{\lambda_B}{4!}\varphi_B^4$$

$$= \frac{1}{2}\partial_\mu\varphi_R\partial^\mu\varphi_R - \frac{1}{2}m_R^2\varphi_R^2 - \frac{\lambda_R}{4!}\varphi_R^4 \qquad (6)$$

$$+\frac{1}{2}(Z_3-1)\partial_\mu\varphi_R\partial^\mu\varphi_R - (Z_0-1)\frac{1}{2}m_R^2\varphi_R^2 - (Z_1-1)\frac{\lambda_R}{4!}\varphi_R^4$$

where
$$\varphi_B = Z_3^{1/2}\varphi_R$$

$$\mu^{-\varepsilon}\lambda_B = Z_3^{-2}Z_1\lambda_R \qquad (7)$$

$$m_B^2 = Z_3^{-1}Z_0 m_R^2$$

We assume that the renormalization scheme is such that Z_i's are mass independent. For instance, we are using the MS scheme. Working in $d=4-\varepsilon$ dimension we know

explain why!

$$\Gamma_R^{(n)}(p_i,\lambda_R,m_R,\mu,\varepsilon) = Z_3^{n/2}\,\Gamma_B^{(n)}(p_i,\lambda_B,m_B,\varepsilon) \qquad (8)$$

where
$$Z_i = Z_i(\lambda_R,\varepsilon) \qquad (9)$$

Notice that $\Gamma_B^{(n)}$ does not depend on μ! So, $\mu\frac{d}{d\mu}$ (8) leads to

$$\mu\left(\frac{\partial}{\partial\mu} + \frac{d\lambda_R}{d\mu}\frac{\partial}{\partial\lambda_R} + \frac{dm_R}{d\mu}\frac{\partial}{\partial m_R}\right)\Gamma_R^{(n)}(p_i,\lambda_R,m_R,\mu,\varepsilon) = \qquad (9)$$

$$= \frac{n}{2}Z_3^{\frac{n}{2}-1}\mu\frac{dZ_3}{d\mu}\Gamma_B = \frac{n}{2}\frac{\mu}{Z_3}\frac{d}{d\mu}(Z_3)\,\Gamma_R^{(n)}$$

Now we define in order to use the method of characteristics $\boxed{15.4}$

$$\beta(\lambda_R,\varepsilon)\equiv \mu\frac{d\lambda_R}{d\mu} \overset{(7)}{=} \mu\frac{d}{d\mu}\left(\mu^{-\varepsilon} Z_3^{2} Z_1^{-1}\lambda_B\right) \tag{10}$$

$$= -\varepsilon\lambda_R + \lambda_R\,\mu\frac{d\ln}{d\mu}\underbrace{\left(Z_3^{2} Z_1^{-1}\right)}_{\frac{1}{Z_\lambda}} \xrightarrow{\varepsilon\to 0} \beta(\lambda_R)$$

$$\gamma(\lambda_R,\varepsilon)\equiv \frac{1}{2}\,\mu\frac{d}{d\mu}\ln(Z_3) \xrightarrow{\varepsilon\to 0} \gamma(\lambda_R) \tag{11}$$

$$\gamma_m(\lambda_R,\varepsilon)\equiv \frac{\mu}{m_R}\frac{dm_R}{d\mu} = \frac{1}{2}\,\mu\frac{d\ln}{d\mu}\left(Z_3 Z_0^{-1}\right) \xrightarrow[\varepsilon\to 0]{} \gamma_m(\lambda_R) \tag{12}$$

o we write (9) as

$$\mu\frac{\partial}{\partial\mu} + \beta(\lambda_R)\frac{d}{d\lambda_R} + \gamma_m(\lambda_R)\, m_R\frac{\partial}{\partial m_R} - n\gamma(\lambda_R)\Big]\,\Gamma_R^{(n)}(p_i,\lambda_R,m_R,\mu) = 0 \quad (13$$

this is the RGE!

To calculate β we write the perturbation theory result for

$$Z_\lambda = Z_3^{-2}Z_1 = 1 + \sum_\nu \frac{a_\nu(\lambda_R)}{\varepsilon^\nu} \tag{14}$$

and

$$a_\nu(\lambda) = \sum_j a_{\nu j}\lambda_R^j$$

[START BY THE DERIVATION OF eq'n (10) then do the series expansion!]

so $\lambda_B = \mu^{\varepsilon} Z_\lambda \lambda_R$ (15)

Now

$$\mu \frac{d}{d\mu} Z_\lambda = \mu \frac{d\lambda_R}{d\mu} \frac{dZ_\lambda}{d\lambda_R} = \beta(\lambda_R, \varepsilon) \frac{dZ_\lambda}{d\lambda_R}$$

that substituted in (10) yields

$$\beta(\lambda_R, \varepsilon) Z_\lambda + \varepsilon \lambda_R Z_\lambda + \beta(\lambda_R, \varepsilon) \frac{dZ_\lambda}{d\lambda_R} = 0 \qquad (16)$$

At this point, $\beta(\lambda, \varepsilon) = \sum_{k=1}^{M} \beta_k \varepsilon^k$ (17)

Notice that β is regular at $\varepsilon = 0$ since the theory is renormalizable!

Now, (14)+(16)+(17) $\Longrightarrow$

$$\sum_{k=1}^{M} \beta_k \epsilon^k \cdot \left(1 + \sum_\nu \frac{a_\nu}{\epsilon^\nu}\right) + \varepsilon \lambda_R \left(1 + \sum_\nu \frac{a_\nu}{\epsilon^\nu}\right) + \left(\sum_{k=1}^{M} \beta_k \epsilon^k\right) \sum_\nu \frac{\lambda_R}{\epsilon^\nu} \frac{da_\nu}{d\lambda_R} \qquad (18)$$

From (18) it follows:

- for $2 \le k \le M$ $\beta_k = 0$ from terms exhibiting ϵ^k
- the ϵ^1 component $\Longrightarrow$ $\beta_1 + \lambda_R = 0 \Longrightarrow \beta_1 = -\lambda_R$
- $\epsilon^0 \Longrightarrow \beta_0 + \beta_1 \cancel{a_1} + \lambda_R \cancel{a_1} + \beta_1 \lambda_R \frac{da_1}{d\lambda_R} = 0 \Longrightarrow \beta_0 = \beta(\lambda) = \lambda_R^2 \frac{da_1}{d\lambda_R}$ (1
- Notice that $\beta(\lambda_R, \varepsilon) = -\varepsilon \lambda_R + \beta(\lambda_R)$ (20)
- From $\epsilon^{-\nu}$:

$$\beta_0 a_\nu + \beta_1 \cancel{a_{\nu+1}} + \cancel{\lambda_R} a_{\nu+1} + \lambda_R \beta_0 \frac{da_\nu}{d\lambda_R} + \lambda_R \overset{-\lambda_R}{\beta_1} \frac{da_{\nu+1}}{d\lambda_R} = 0$$

$$\Rightarrow \lambda_R^2 \frac{d a_{\nu+1}}{d\lambda_R} = +\beta(\lambda_R)\frac{d}{d\lambda_R}(\lambda_R a_\nu) \qquad (21)$$

Comments:

(i) $\beta(\lambda_R)$ is determined by the residua of the simple poles in ϵ of Z_λ!

(ii) **Bonus:** higher order poles of Z_λ can be obtained from a_1 using (21)! At each order in perturbation theory the new divergence arises in the $\frac{1}{\epsilon}$ form! Higher order terms $\frac{1}{\epsilon^N}$ are determined by lower order perturbation expansion!

Expressions similar to (19) can be obtained for $\gamma(\lambda_R)$ and $\gamma_m(\lambda_R)$:

$$Z_3 = 1 + \sum_\nu \frac{z_3^{(\nu)}}{a^\nu} \Rightarrow \gamma(\lambda_R) = -\frac{\lambda_R}{2}\frac{d}{d\lambda_R} z_3^{(1)}(\lambda_R) \qquad (22)$$

$$m_B^2 = Z_m m_R^2 \quad \text{with} \quad Z_m = 1 + \sum_\nu \frac{z_m^{(\nu)}}{\epsilon^\nu} \Rightarrow \gamma_m(\lambda_R) = \frac{\lambda_R}{2}\frac{d z_m^{(1)}}{d\lambda_R} \qquad (23$$

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15.c Solution of the RGE

Before solving the RGE (13), let's ~~analyse~~ write $\mu = \mu_0 e^{t}$

$$\Rightarrow \mu\frac{\partial}{\partial\mu} \to -\frac{\partial}{\partial t}$$ so (13) reads

$$\left[-\frac{\partial}{\partial t}+\beta(\lambda_R)\frac{\partial}{\partial \lambda_R}+\gamma_m(\lambda_R)\,m_R\frac{\partial}{\partial m_R}-n\gamma_\phi(\lambda_R)\right]\Gamma_R^{(n)}(p_i,\lambda_R,m_R,\mu_0 e^{-t})=0 \quad (24$$

As an auxiliary step let's solve

$$\frac{\partial}{\partial t}\bar\lambda(t,\lambda_R)=\beta(\bar\lambda) \qquad (25)$$

,

with $\bar\lambda(0,\lambda_R)=\lambda_R$

nd

$$\frac{\partial \bar m}{\partial t}(t,\lambda_R, m_R)=\gamma_m(\bar\lambda)\,\bar m \qquad (26)$$

,

with $\bar m(0,\lambda_R,m_R)=m_R$

From $(25)\overset{\int_0^t dt'}{\Longrightarrow}$ $$t=\int_{\lambda_R}^{\bar\lambda(t,\lambda_R)}\frac{dx}{\beta(x)} \qquad (27)$$

lso, integrating $(26)\Rightarrow \bar m(t,\lambda_R,m_R)=m_R\exp\left[\int_0^t dt'\,\gamma_m(\bar\lambda(t'))\,dt'\right]$ (28

$$\overset{\text{"(25)"}}{=} m_R\exp\left[\int_{\lambda_R}^{\bar\lambda(t,\lambda_R)}dx\,\frac{\gamma_m(x)}{\beta(x)}\right] \qquad (29)$$

Notice that

$$\frac{\partial}{\partial\lambda_R}(27)\Rightarrow 0=-\frac{1}{\beta(\lambda_R)}+\frac{1}{\beta(\bar\lambda(\lambda_R,t)}\frac{\partial\bar\lambda}{\partial\lambda_R}\overset{(25)}{\Longrightarrow}\left(-\frac{\partial}{\partial t}+\beta(\lambda_R)\frac{\partial}{\partial\lambda_R}\right)\bar\lambda(\lambda_R,t)=$$

(30)

[15.7.]

$$\frac{\partial \bar{m}}{\partial \lambda_R} = \bar{m} \left[-\frac{\gamma_m(\lambda_R)}{\beta(\lambda_R)} + \frac{\gamma_m(\bar{\lambda})}{\beta(\bar{\lambda})} \underbrace{\frac{\partial \bar{\lambda}}{\partial \lambda_R}}_{\big\downarrow \, (25)+(30) \;\; \frac{\beta(\bar{\lambda})}{\beta(\lambda_R)}} \right]$$

$$= \bar{m} \left[-\frac{\gamma_m^{1}(\lambda_R)}{\beta(\lambda_R)} + \frac{\gamma_m(\bar{\lambda})}{\beta(\lambda_R)} \right] \qquad (30a)$$

Initially let's solve (24) for $\gamma \equiv 0$. The sol'n is

$$\Gamma_R^{(n)}(p_i, \lambda_R, m_R, \mu_0 e^{-t}) = \Gamma_R^{(n)}(p_i, \bar{\lambda}(t), \bar{m}(t), \mu_0) \qquad (31)$$

in fact, ~~substituting [illegible] (24) with $\gamma=0$ we have~~

$$\left(\frac{\partial}{\partial t} + \beta(\lambda_R)\frac{\partial}{\partial \lambda_R} + \gamma_m(\lambda_R)\, m_R \frac{\partial}{\partial m_R}\right)\Gamma_R^{(n)}(p_i, \bar{\lambda}, \bar{m}, \mu_0) =$$

$$= \left[\frac{\partial \Gamma_R^{(n)}}{\partial \bar{\lambda}}\left(-\frac{\partial \bar{\lambda}}{\partial t} + \beta(\lambda_R)\frac{\partial \bar{\lambda}}{\partial \lambda_R}\right) + \frac{\partial \Gamma_R^{(n)}}{\partial \bar{m}}\left(-\frac{\partial \bar{m}}{\partial t} + \gamma_m(\lambda_R)\, \overbrace{m_R}^{(29)\ \bar{m}} \frac{\partial \bar{m}}{\partial m_R} + \beta(\lambda_R)\frac{\partial \bar{m}}{\partial \lambda_R}\right)\right.$$

$\downarrow$ (3[illegible]) $\qquad\qquad$ (26) $\qquad\qquad$ $\downarrow$ (30a)

$$= \frac{\partial \Gamma_R^{(n)}}{\partial \bar{\lambda}}\left(-\beta(\lambda_R)\frac{d\bar{\lambda}}{dt} + \beta(\lambda_R)\frac{d\bar{\lambda}}{dt}\right) + \frac{\partial \Gamma_R^{(n)}}{\partial \bar{m}}\, \bar{m}\left(-\gamma_m(\bar{\lambda}) + \gamma_m(\lambda_R) - \gamma_m(\lambda_R) + \gamma_m(\bar{\lambda}\right.$$

$$\equiv 0 \;!!! \implies \text{Eq'n (31) is correct!}$$

The sol'n to the full (24) is

$$\Gamma_R^{(n)}(p_i, \lambda_R, m_R, \underbrace{\mu_0 e^{-t}}_{\mu}) = \Gamma_R^{(n)}(p_i, \bar{\lambda}(t), \bar{m}(t), \underbrace{\mu_0}_{\mu e^{t}}) \exp\left[-n\int_0^t dt'\, \gamma(\bar{\lambda}(t'))\right] \qquad (32)$$

which follows direct from the meaning of $\Gamma_R^{(n)}(p_i, \bar{\lambda}, \bar{m}, \mu_0)$ and

$$\left(\frac{\partial}{\partial t} + \beta(\lambda_R)\frac{\partial}{\partial \lambda_R} - n\gamma(\lambda_R)\right) e^{-n\int_0^t dt'\, \gamma(\bar{\lambda}(t'))} = e^{-n[\ldots]} \left[+n\gamma(\bar{\lambda}) - n\beta(\lambda_R)\left(-\frac{\gamma(\lambda_R)}{\beta(\lambda_R)} + \frac{\gamma(\bar{\lambda})}{\beta(\lambda_R)}\right.\right.$$

$$\left. - n\gamma(\lambda_R)\right] = 0 \,. \qquad (33)$$

15.d Equation for physical observables

Physical quantities can be evaluated as a function $P(\lambda_R, m_K, \mu)$ however, they do not depend on the μ parameter. Therefore,

$$\mu \frac{dP}{d\mu} = \left[\mu \frac{\partial}{\partial\mu} + \beta(\lambda_R) \frac{\partial}{\partial\lambda_R} + \gamma_m(\lambda_R) m_K \frac{\partial}{\partial m_K} \right] P(\lambda_R, m_K, \mu) = 0 \quad (3$$

So, we know that P satisfies (31), i.e,

$$P(p_i, \lambda_R, m_K, \mu_0 e^{-t}) = P(p, \bar{\lambda}(t), \bar{m}(t), \mu_0) \qquad (35)$$

~~15.e Green's functions for rescaled momenta~~

~~We want to study~~

For example, let's consider the Laurent series of scalar propagator $\Delta(p^2)$:

$$\Delta(p^2) = \frac{R^2}{p^2 - m_p^2} + \bar{\Delta}(p^2, \lambda_R, m_K, \mu) \qquad (36)$$

that satisfies

$$\Big[\overbrace{\mu \frac{\partial}{\partial\mu} + \beta(\lambda_K) \frac{\partial}{\partial\lambda_R} + \gamma_m(\lambda_R) m_K \frac{\partial}{\partial m_K}}^{\equiv D} + 2\gamma(\lambda_K) \Big] \Delta = 0 \qquad (37)$$

Notice that we have $2\gamma(\lambda_K)$ since Δ is the connected Green's function and not $\Gamma_R^{(2)}$! This is due to

$\Delta_R = (Z_3^{1/2})^{-2} \Delta_B$ in contrast with (8)!

(37) into (36) ⇒

$$(D+2\gamma)\bar{\Delta} + \frac{1}{p^2-m_F^2}(D+2\gamma)R^2 + \frac{2m_F(Dm_F)}{(p^2-m_F^2)^2} = 0$$

ment ⇒ ms

$$(D+2\gamma)\bar{\Delta} = 0$$

$$(D+2\gamma)R^2 = 0$$

$$Dm_F = 0 \qquad (38)$$

As expected, the position of the pole (physical mass) satisfies (3

15.e Green's functions for rescaled momenta

e want to study the effect of $p_i \to \rho p_i$ in $\Gamma_R^{(n)}(\rho p_i, \lambda_R, m_R, \mu)$. We know the canonical dimension of $\Gamma_R^{(n)}$: $[\Gamma_R^{(n)}] = [M]^D$. So, using dimensional analysis we now that

$$\Gamma_R^{(n)}(\rho p_i, \lambda_R, m_R, \mu) = \mu^D f\left(\rho^2 \frac{p_i \cdot p_j}{\mu^2}, \frac{m_R}{\mu}, \lambda_R\right) \qquad (39)$$

Since $\Gamma_R^{(n)}$ is an homogeneous function of order D in the variables ρ, m_R and μ we know that

$$\left(\rho\frac{\partial}{\partial \rho} + m_R\frac{\partial}{\partial m_R} + \mu\frac{\partial}{\partial \mu} - D\right)\Gamma_R^{(n)}(\rho p_i, \lambda_R, m_R, \mu) = 0 \qquad (40)$$

:xtracting $\mu\frac{\partial\Gamma_R^{(n)}}{\partial\mu}$ from $(\overset{40}{\blacksquare})$ and inserting into $(\overset{13}{\blacksquare})$ $\Rightarrow$

$\boxed{15.11}$

$$\left[-\rho\frac{\partial}{\partial\rho}+\beta(\lambda_R)\frac{\partial}{\partial\lambda_R}+(\gamma_m(\lambda_R)-1)\frac{\partial}{\partial m_K}-n\gamma(\lambda_R)+D\right]\Gamma_R^{(n)}(\rho p_i,\lambda_R,m_K,\mu)=$$ (41

ow $t=\ln\rho \Rightarrow \frac{\partial}{\partial t}=\rho\frac{\partial}{\partial\rho}$. Analogously to the previous notes

$$\Gamma_R^{(n)}(e^t p_i,\lambda_R,m_K,\mu)=\Gamma_R^{(n)}(p_i;\bar\lambda(t),\bar{\bar m}(t),\mu)\exp\left[\underset{\uparrow}{Dt}-n\int_0^t\gamma(\bar\lambda(t'))dt'\right] \quad (42)$$

Scaling
for $m_K=\beta=\gamma=0$ (massless non-interac

where $\bar\lambda$ is given by (25) and as in (26).

$$\bar{\bar m}=\frac{\bar m(t)}{e^t}$$

le could also do

$$\Gamma_R^{(n)}(e^t p_i,\lambda_R,m_K,\mu)\overset{(32)}{=}\Gamma_R^{(n)}(e^t p_i,\bar\lambda(t),\bar m(t),e^t\mu)\exp\left[-n\int_0^t\gamma(t')dt'\right]$$

$$=\Gamma_R^{(n)}(e^t p_i,\bar\lambda(t),e^t e^{-t}\bar m(t),e^t\mu)\exp\left[-n\int_0^t\gamma(t')dt'\right]$$

s homogeneous

$$=(e^t)^D\,\Gamma_R^{(n)}(p_i,\bar\lambda(t),\bar{\bar m}(t),\mu)\exp\left[-n\int_0^t\gamma(t')dt'\right]$$

QCD!

Anomalous dimensions:

$\boxed{15.11.}$

Let's focus on the last term of (42). In d = 4

$[\varphi] = M^1$ $[m] = M^1$ and the action

$$S = \int d^4x \left[-\frac{1}{2}\varphi(\Box + m^2)\varphi - \frac{\lambda}{4!}\varphi^4 \right.$$

S is invariant under $x^\mu \to \frac{1}{\lambda}x^\mu$, $m \to \lambda m$, $\varphi \to \lambda\varphi$ ⊛

$\partial_\mu \to \lambda\partial_\mu$, $\lambda \to \lambda$

This transformation is called Dilation (D)

$D: \varphi \to \lambda^{d_0}\varphi$ where d_0 is the canonical scaling dimension of the field.

~~we~~ Under D: $G_c^{(n)}(x \dots x_n) \to \lambda^{n d_0} G_c(\dots)$

e expect

$\Rightarrow \Gamma^{(n)}(x \dots x_n) \Rightarrow \lambda^{-n d_0} \Gamma^{(n)}(\dots)$

ince we have to amputate n external legs with $G^{(2)^{-1}}$! We expect classically that

$\Gamma^{(n)} \cong m^a \lambda^b x_1^{c_1} \dots x_n^{c_n}$ with $a \cancel{+} - c_1 - \dots - c_n = -n$. However, in the entum theory we ~~[illegible]~~ also have the renormalization scale μ:

$$\Gamma^{(n)} \cong m^a \lambda^b x_1^{c_1} \dots x_n^{c_n} \mu^\gamma$$

hus, we have $D: \Gamma_n \to \lambda^{-n-\gamma}\Gamma_n$ since just m and x are scaled in ⊛. This is the origin of the last term in (42)!

12.7 Example: $\lambda\phi^4$ at 1 loop

15.12

From TQC 1, we know that

$$Z_1 = 1 + \frac{3\lambda_R}{16\pi^2\epsilon}$$

$$Z_0 = 1 + \frac{\lambda_R}{16\pi^2\epsilon} \qquad (43)$$

$$Z_3 = 1$$

For the β function: $a_1 = \frac{3\lambda_R}{16\pi^2}$

$Z_\lambda = Z_3^{-2} Z_1$

$$\overset{(19)}{\Longrightarrow} \beta(\lambda_R) = \lambda_R^2 \frac{da_1}{d\lambda_R} = \frac{3\lambda_R^2}{16\pi^2} \qquad (44)$$

For the γ function: $Z_3^{(1)} \equiv 0 \overset{(22)}{\Longrightarrow} \gamma = 0 \quad (45)$

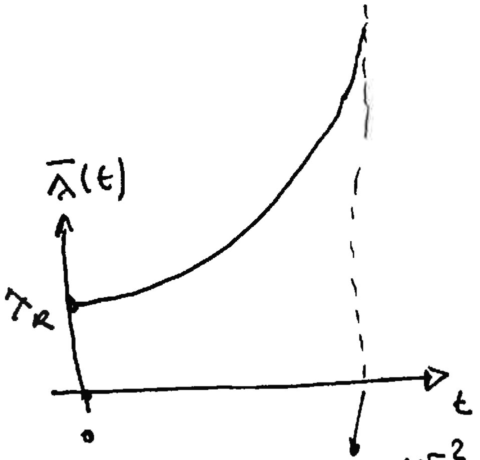
or the γ_m function: $Z_m^{(1)} = \frac{\lambda_R}{16\pi^2} \overset{(23)}{\Longrightarrow} \gamma_m(\lambda_R) = \frac{1}{2}\frac{\lambda_R}{16\pi^2} \quad (46)$

low turning to the evolution eq'ns:

$$\frac{\partial\bar\lambda}{\partial t} = \frac{3\cdot\bar\lambda^2}{16\pi^2} \Longrightarrow \int_{\lambda_R}^{\bar\lambda(t)} \frac{d\lambda}{\lambda^2} = \frac{3}{16\pi^2}\int_0^t dt'$$

$$\Longrightarrow -\frac{1}{\bar\lambda} + \frac{1}{\lambda_R} = \frac{3t}{16\pi^2}$$

$$\Longrightarrow \bar\lambda(t) = \frac{\lambda_R}{1 - \frac{3t\lambda_R}{16\pi^2}} \qquad (47)$$



$\bar t = \frac{16\pi^2}{3\lambda_R}$

Landau pole!

In the other hand

$$\frac{d\bar{m}}{dt} = \gamma_m(\bar{\lambda})\,\bar{m} = \frac{1}{32\pi^2}\;\frac{\lambda_R}{1-\frac{3\lambda_R}{16\pi^2}t}\;\bar{m}$$

$$\int_{m_R}^{\bar{m}} \frac{dy}{y} = \int_0^t dt' \,\frac{\cancel{\lambda_R}}{32\pi^2}\;\frac{1}{1-\frac{3\lambda_R}{16\pi^2}t}$$

$$\ln\left(\frac{\bar{m}}{m_R}\right) = \frac{\cancel{\lambda_R}}{32\pi^2}\left(-\frac{16\pi^2}{3\cancel{\lambda_R}}\right) \ln\left(1-\frac{3\lambda_R t}{16\pi^2}\right)$$

$$\bar{m}(t) = m_R\left(1-\frac{3\lambda_R t}{16\pi^2}\right)^{-1/6} \quad (48)$$

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15.6 Renormalization Group Flow

Starting at small couplings we can have:

i) $\beta>0$ (like $\lambda\phi^4$, QED), as we saw the coupling grows as indicated in the previous page.

ii) $\beta=0 \Rightarrow$ the coupling constant is scale independent

iii) $\beta<0 \Rightarrow$ the coupling decreases at high energies.

Let's make our analyses more general. ~~Ka~~ and concentrate on the IR ($t\to-\infty$) and UV ($t\to+\infty$) limits. Here, we assume that (25) is valid for $-\infty<t<+\infty$. From ~~them~~ the eq'n (27)

$$t = \int_{\lambda_R}^{\bar{\lambda}(t,\lambda_R)} \frac{dx}{\beta(x)} \qquad (27)$$

For $t \to \pm\infty$, (27) implies that either

(i) $\bar\lambda$ ~~[illegible]~~ goes to infinity

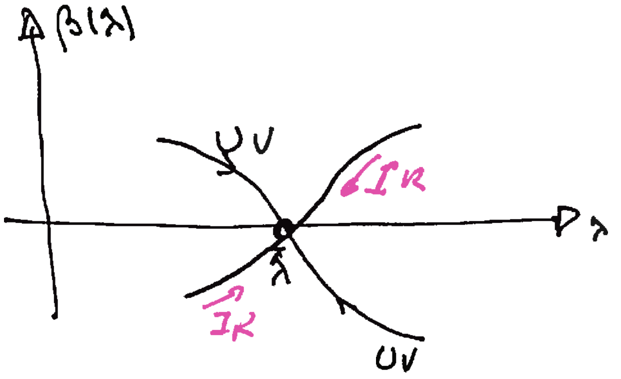
or (ii) $\bar\lambda$ approaches a zero of β

A zero of $\beta(x)$ is called a fixed point, that is classified as:

(i) UV stable fixed point if $\lim_{t\to+\infty} \bar\lambda(t,\lambda_\kappa) = \bar\lambda \equiv \lambda_+$

(ii) IR stable fixed point for $\lim_{t\to-\infty} \bar\lambda(t,\lambda_\kappa) = \bar\lambda \equiv \lambda_-$

Graphically



$\Longrightarrow$

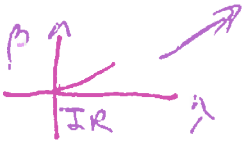
$\beta'(\lambda)\big|_{\bar\lambda} < 0 \Longrightarrow$ UV fixed poi

$\beta'(\lambda)\big|_{\bar\lambda} > 0 \Longrightarrow$ IR fixed poi

le use P.T around $\lambda=0$ to obtain β. We classify the theory as

i) asymptotically free theory if $\lambda=0$ is an UV stable fix
point (eg QCD)

ii) IR stable if $\lambda=0$ is an IR stable fixed point.
(eg $\lambda\varphi^4$, QED)



Consider QED in the gauge $-\frac{1}{a}(\partial_\mu A^\mu)^2$ with a mass independent renormalization scheme. We have

$$\Gamma_R^{(n_A, n_F)}(p_i, \alpha_R, m_R, \mu, a_R) = Z_3^{\frac{n_A}{2}} Z_2^{\frac{n_F}{2}} \quad \Gamma_B^{(n_A, n_F)}(p_i, \alpha_B, m_B, a_B) \qquad (49)$$

with $\alpha_R = \frac{e_R^2}{4\pi}$

$$\alpha_B = \mu^\varepsilon Z_3^{-1} \alpha_R$$

$$m_B = Z_m^{-1} m_R$$

$$A_{\mu B} = Z_3^{1/2} A_{\mu R} \qquad (50)$$

$$\psi_B = Z_2^{1/2} \psi_R$$

$$a = Z_3^{-1} a_B$$

Analogously to the procedure in section (15.6), we have that $\mu \frac{d}{d\mu}$ (49) leads to

$$\frac{\partial}{\partial \mu} + \beta(\alpha_R, a_R) \frac{\partial}{\partial \alpha_R} + \gamma_m(\alpha_R, a_R) m_R \frac{\partial}{\partial m_R} + \delta(\alpha_R, a_R) \frac{\partial}{\partial a_R} - n_A \gamma_A(\alpha_R, a$$

$$- n_F \gamma_F(\alpha_R, a_R) \Big] \Gamma_R^{(n_A, n_F)}(p_i, \alpha_R, m_R, \mu, a_R) = 0 \qquad (51)$$

From now on we drop the subscript R! We have that

$$\beta(\alpha, a) = \lim_{\epsilon \to 0} \beta(\alpha, a, \epsilon) = \lim_{\epsilon \to 0} \mu \frac{d}{d\mu} \alpha$$

$$\gamma_m(\alpha, a) = \lim_{\epsilon \to 0} \mu \frac{d}{d\mu} \ln Z_m$$

$$\gamma_A(\alpha, a) = \lim_{\epsilon \to 0} \frac{1}{2} \mu \frac{d}{d\mu} \ln Z_3 \qquad (52)$$

$$\gamma_F(\alpha, a) = \lim_{\epsilon \to 0} \frac{1}{2} \mu \frac{d}{d\mu} \ln Z_2$$

$$\delta(\alpha, a) = \lim_{\epsilon \to 0} \mu \frac{da}{d\mu} = \lim_{\epsilon \to 0} \left(-a \mu \frac{d}{d\mu} \ln Z_3 \right)$$

The sol'n of (51) ~~[illegible]~~ requires solving

$$\frac{\partial}{\partial t} \bar{\alpha}(t, \alpha) = \beta(\bar{\alpha}) \quad \text{with} \quad \bar{\alpha}(0, \alpha) = \alpha$$

$$\frac{\partial}{\partial t} \bar{m}(t, \alpha, m) = \bar{m}\, \gamma_m(\bar{\alpha}) \quad \text{with} \quad \bar{m}(0, \overset{\alpha}{,} m) = m \qquad (53)$$

$$\frac{\partial}{\partial t} \bar{a}(t, \alpha, a) = \delta(\bar{\alpha}, \bar{a}) \quad \text{with} \quad \bar{\alpha}(0, \alpha) = a$$

to obtain that (for ~~[illegible]~~ $\mu = \mu_0 e^{-t}$) $\left(t = -\ln \frac{\mu}{\mu_0} \right)$

$$_R^{(n_A, n_F)}\left(P_i, \overset{\alpha}{\cancel{\alpha}}_R, m_R, a_R, \mu_0 e^{-t}\right) = \Gamma_R^{(n_A, n_F)}\left(P_1, \bar{\alpha}(t), \bar{m}(t), \bar{a}(t)\, \mu_0\right) \otimes$$

$$\otimes \exp\left[-n_A \int_0^t dt'\, \gamma_A(\bar{\alpha}(t'), \bar{a}(t'))\, dt' - n_F \int_0^t dt'\, \gamma_F(\bar{\alpha}(t'), \bar{a}(t'))\, dt' \right] \quad (5$$

15.16-1

$$\beta = \mu \frac{d}{d\mu} \alpha = \mu \frac{d}{d\mu}\left(\mu^{-\varepsilon} Z_3 \, \alpha_B\right)$$

$$= -\varepsilon \alpha_R + \alpha_R \mu \frac{d \cancel{Z_3}}{d\mu} \ln Z_3 \qquad (10.\text{QED})$$

$$\gamma_A = \frac{1}{2} \mu \frac{d}{d\mu} \ln Z_3 \quad \checkmark$$

Since $\alpha_B = \mu^{\varepsilon} Z_3^{-1} \alpha$

and comparing (10) with (14) $\Bigg\}$ $\Rightarrow$ $Z_\lambda^{QED} = Z_3^{-1}$ (55)

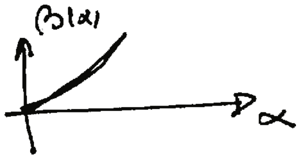
$$\Rightarrow \beta_{QED} = \alpha_R^2 \frac{d Z_\lambda^{QED(1)}}{d\alpha_R} \quad (56)$$

We know that, at the one-loop level, $Z_3 = 1 - \frac{\alpha 2}{3\pi\varepsilon}$ in MS.

see page 13.17, eq (47)) $\Rightarrow$ $Z_\lambda^{QED} = 1 + \frac{2\alpha}{3\pi\varepsilon}$

$$\Rightarrow \beta_{QED}^{(\alpha)} = \frac{2\alpha^2}{3\pi} \quad (57)$$

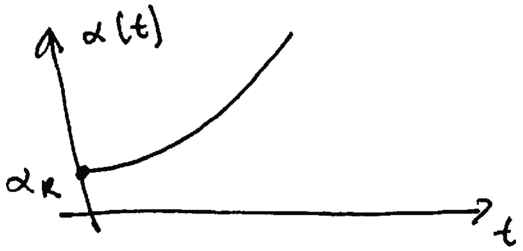
Notice that $\beta_{QED}^{(\alpha)} > 0$ $\Rightarrow$ QED is IR stable.



In this case, the RGE for α reads $\overset{(63)}{\Longrightarrow}$

$$\frac{d\bar{\alpha}}{dt} = \frac{2\bar{\alpha}^2}{3\pi} \Rightarrow -\frac{1}{\bar{\alpha}(t)} + \frac{1}{\alpha_R} = \frac{2t}{3\pi}$$

$$\Rightarrow \bar{\alpha} = \frac{\alpha_R}{1 - \frac{2\alpha_R t}{3\pi}} \quad (58) \Rightarrow$$



ompare with eq'n (3) on page(15.1).

<u>**Notice:**</u> Z_3 is gauge-independant since it renormalizes the gauge invariant operator $F_{\mu\nu}F^{\mu\nu}$. Therefore, β_{QED} is also gauge independent; see (55-56)!

Instead of α we could have considered e. 15.18

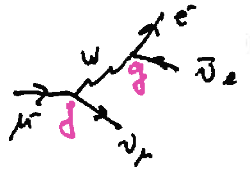
In this case

$$\beta(e) = \mu \frac{de}{d\mu} \Rightarrow \beta(\alpha) = \mu \frac{d\alpha}{d\mu} = \frac{2e}{4\pi} \mu \frac{de}{d\mu} \Rightarrow \beta(e) = \frac{2\pi}{e} \beta(\alpha$$

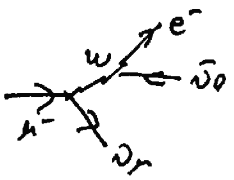
$$\Rightarrow \beta(e) \overset{\overset{\text{1-loop}}{\downarrow}}{=} \frac{4}{3} \frac{e^3}{16\pi^2} \qquad (59)$$

5.I RGE for higher-order operators

otivation : In the SM, μ^- decay rate is given by

$(\mu^- \to \bar\nu_e e^- \nu_\mu) = \frac{1}{2m_\mu} \int d\phi_3 \Big|$  $\Big|^2 = \left(\frac{\sqrt{2}\, g^2}{8 m_W^2}\right)^2 \frac{m_\mu^5}{192\,\pi^3}$ (59a)

Large log's can appear since $m_\mu \gg m_e$. Consider $m_W \gg m_\mu$

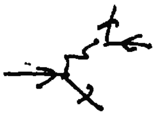
$(\mu^- \to \bar\nu_e e^- \nu_\mu) = \frac{1}{2m_\mu} \int d\phi_3 \Big|$  $+$  $\Big|^2$

$$= \left(\frac{\sqrt{2}\, g^2}{8 m_W^2}\right)^2 \frac{m_\mu^5}{192\pi^3} \left[1 + A \frac{\alpha}{4\pi} \ln \frac{m_W}{m_\mu} + \dots \right] \qquad (60)$$

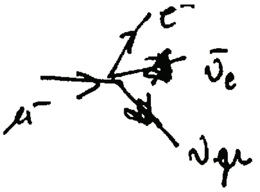
et's use the RGE to not only estimate A but also to some terms

$\left(A \frac{\alpha}{4\pi} \ln \frac{m_W}{m_\mu}\right)^n$!

Effective operator (Fermi theory)

Since $M_W \gg m_\mu$, the propagator $\frac{i}{p^2 - M_W^2}$ in  is effectively $-\frac{i}{m_W^2}$

$\boxed{10.19}$

ading to  which is associated to the

operator

$$\mathcal{L}_{4F} = \frac{G_F}{\sqrt{2}} \bar\Psi_\mu \gamma^\alpha P_L \Psi_\nu \; \bar\Psi_e \gamma_\alpha P_L \Psi_{\nu_e} + h.c. \qquad (61)$$

ith $P_L = \frac{1-\gamma_5}{2}$, $G_F = \frac{\sqrt{2} g^2}{8 m_W^2} = 1.166 \times 10^{-5}\ \text{GeV}^{-2}$. From (61) we can obtain

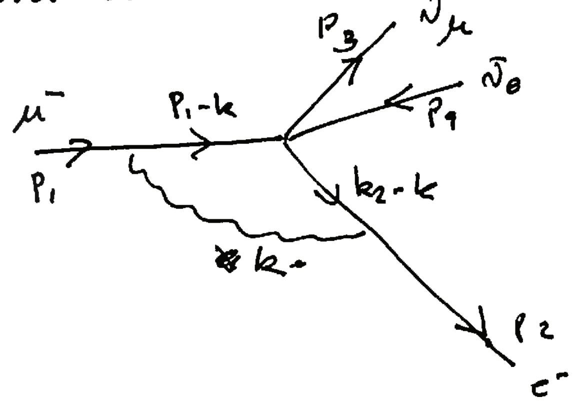
asily (59a). Do it! (61) is the Fermi 4-fermion model!

o simplify our life let's consider

01|10|19

$$\mathcal{L}_4 = \frac{G}{\sqrt{2}} (\bar\Psi_\mu \Psi_e)(\bar\Psi_{\nu_e} \Psi_{\nu_\mu}) + h.c. \qquad (62)$$

Let's evaluate

$iM =$ 

$$= \frac{G}{\sqrt{2}} e_R^2 \mu^\varepsilon \int \frac{d^dk}{(2\pi)^d} \frac{\bar u(p_2) \gamma^\mu (\not{p}_2 - \not{k} + m_e)(\not{p}_1 - \not{k} + m_\mu) \gamma_\mu u(p_1)\, \bar u(p_3) v(p_4)}{[(p_1 - k)^2 - m_\mu^2]\,[(p_2 - k)^2 - m_e^2]\, k^2} \qquad (63)$$

Since we just need the pole to get the β function we set all masses and external momenta to zero!

$$M = M_0 \left(-i e_R^2 \mu^\varepsilon \int \frac{d^dk}{(2\pi)^d} \frac{d}{k^4} \right) + \text{finite}$$

15.20

ith $M_0 = \frac{G}{\sqrt{2}} \bar{u}(p_2) u(p_1) \bar{u}(p_3) v(p_4)$

So we have $M = M_0 \left(\frac{e_R^2}{2\pi^2} \mu^{\epsilon} \frac{1}{\epsilon} \right) + \text{finite}$ (64)

o remove the divergence we write

$$\mathcal{L} = \frac{G_R}{\sqrt{2}} Z_G (\bar{\psi}_\mu^R \psi_e^R)(\bar{\psi}_{\nu e}^R \psi_e^R| \quad + h$$

So to render (64) finite we have i

$$Z_G = 1 - \frac{e_R^2}{16\pi^2} \frac{8}{\epsilon} \qquad (66)$$

adding $\mu^{-\epsilon}$ to $\mu \frac{d}{d\mu} G_{Bare}$

=? ϵ term in (67) that disappears in the limit $\epsilon \to 0$!

Notice that (65) is

$$\mathcal{L} = \frac{G_R}{\sqrt{2}} \frac{Z_G}{(Z_{2\mu} Z_{2e} Z_{2\nu e} Z_{2\nu\mu})^{1/2}} (\bar{\psi}_\mu^B \psi_e^B)(\bar{\psi}_{\nu e}^B \psi_{\nu\mu}^B) \qquad (66)$$

Since the $\nu_{e,\mu}$ are neutral $Z_{2\nu e} = Z_{2\nu\mu} = 1$. Moreover,

$$Z_{2e} = Z_{2\mu} = Z_2$$

The RGE group reads

$$0 = \mu \frac{d}{d\mu} \Big(\underbrace{\frac{G_R Z_G}{Z_2}}_{G_{BARE}!} \Big)$$

$\Downarrow$

$$0 = \frac{G_R Z_G}{Z_2} \left[\frac{\mu}{G_R} \frac{dG_R}{d\mu} + \frac{1}{Z_G} \frac{dZ_G}{de_R} \mu \frac{de_R}{d\mu} - \frac{1}{Z_2} \frac{\partial Z_2}{\partial e_R} \mu \frac{de_R}{d\mu} \right]$$

$$\Rightarrow \frac{\mu}{G_R}\frac{dG_R}{d\mu} = \left(-\frac{1}{Z_G}\frac{\partial Z_G}{\partial e_R} + \frac{1}{Z_2}\frac{\partial Z_2}{\partial e_R}\right)\beta(e_R) \qquad (67)$$

Now let's analyze (67) to the lowest order:

$$Z_2 = 1 - \frac{e_R^2}{16\pi^2}\frac{2}{\epsilon}$$

$$Z_G = 1 - \frac{e_R^2}{16\pi^2}\frac{8}{\epsilon} \qquad \Rightarrow$$

$$\beta(e_R) = -\frac{\epsilon}{2}e_R + \frac{e_R^3}{12\pi^2}$$

lowest order β

$$\frac{\mu}{G_R}\frac{dG_R}{d\mu} = \left(-\frac{\epsilon}{2}e_R\right)\times\left(\overset{-2}{\cancel{\frac{-1}{e_R}}}\left(-\frac{e_R}{16\pi^2}\frac{8}{\epsilon}\right) - 2\frac{e_R}{16\pi^2}\frac{2}{\epsilon}\right)$$

$$\Rightarrow \frac{\mu}{G_R}\frac{dG_R}{d\mu} = -\frac{3e_R^2}{8\pi^2} = -\frac{3\alpha_R}{2\pi} \equiv \gamma_G$$

$$\Rightarrow \frac{dG_R}{G_R} = \gamma_G(\alpha(\mu))\frac{d\mu}{\mu} \underset{\uparrow}{=} \frac{\gamma(\bar{\alpha})}{\beta(\bar{\alpha})}d\alpha$$

$$d\alpha = \beta(\alpha)\frac{d\mu}{\mu}$$

μ
$d\mu \Rightarrow$
μ_0

$$\Rightarrow G_R(\mu) = G_R(\mu_0)\exp\left[\int_{\alpha(\mu_0)}^{\bar{\alpha}(\mu)} d\alpha\,\frac{\gamma_G(\alpha)}{\beta(\alpha)}\right]$$

As seen, $\beta(\alpha) = \frac{2\alpha^2}{3\pi}$

$$\Rightarrow \frac{G(\mu)}{G(\mu_0)} = \exp\int_{\alpha(\mu_0)}^{\alpha(\mu)} d\alpha\,\frac{-\frac{3\alpha}{2\pi}}{\frac{2\alpha^2}{3\pi}}$$

$$\Rightarrow \quad G(\mu) = G(\mu_0)\left(\frac{\alpha(\mu)}{\alpha(\mu_0)}\right)^{-\frac{7}{4}} \qquad (68)$$

15.J Wilsonian RGE

Basic idea: remove "high energy" modes while preserving
heir effect in the "low energy" dynamics. To make the ~~[illegible]~~ distinctio
etween low and high energy we work in the euclidean space. For the
φ^4 model

$$S[\varphi] = \int d^4x \left\{ \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}m^2\varphi^2 - \frac{\lambda}{4!}\varphi^4 \right\} \qquad (69)$$

e perform the wick rotation $x_4 = i\,x_0 \Rightarrow dx_0 = -i\,dx_4$ and (69) reads

$$S_E[\varphi] = i\int d^4x \left\{ \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi + \frac{m^2}{2}\varphi^2 + \frac{\lambda}{4!}\varphi^4 \right\} \qquad (70)$$

$_E$ is the hamiltonian of the system, apart from the dx_4 integration!
ow, we write the generating functional as

$$Z = \int D\varphi \; e^{-S_E[\varphi]} \qquad (71)$$

o, Z is the "partition function" of the system! To continue, we consider
n overall cutoff Λ

$$Z = \int [D\varphi]_\Lambda \; e^{-S_E[\varphi]} \qquad (72)$$

with

$$[D\varphi]_\Lambda = \prod_{|k|<\Lambda} d\phi(k) \qquad (73)$$

⟶ Fourier modes of the system

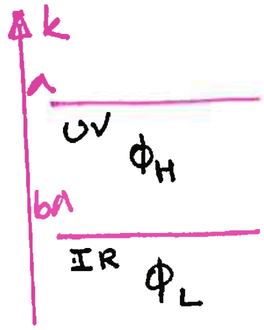
assume that $\Lambda \gg m$. We can separate the low and high $\boxed{15.20}$
nergy modes as

$$\phi(k) = \phi_L(k) + \phi_H(k) \qquad (73)$$

here $$\varphi(x) = \int \frac{d^dk}{(2\pi)^d}\, e^{-ikx}\, \phi(k) \qquad (74)$$

nd

$$\phi_H(k) \equiv \begin{cases} \phi(k) & \text{for } |k| > b\Lambda \\ 0 & \text{for } |k| < b\Lambda \end{cases} \qquad (75)$$



$$\phi_L(k) \equiv \begin{cases} \phi(k) & \text{for } |k| < k\Lambda \\ 0 & \text{for } |k| > b\Lambda \end{cases} \qquad (76)$$

ith $\alpha b<1$.

Idea: we split the integration

$$[D\varphi]_\Lambda = D\phi_L\, D\phi_H \qquad (77)$$

nd we integrate out the high-energy modes! In practical terms,

$$Z = \int D\phi_L \int D\phi_H\, e^{-\int d^dx \left\{ \frac{1}{2}(\partial_\mu\phi_L + \partial_\mu\phi_H)^2 + \frac{1}{2}m^2(\phi_L+\phi_H)^2 + \frac{\lambda}{4!}(\phi_L+\phi_H)^4 \right\}}$$

ue to the ortogonality conditions in momentum space, terms like $\partial_\mu\phi_L\,\partial_\mu\phi_H$ and $\phi_L\phi_H$ do not contribute and we are left with:

$$Z = \int D\phi_L\, e^{-S_E[\phi_L]} \int D\phi_H\, e^{-\int d^dx \left\{ \frac{1}{2}(\partial_\mu\phi_H)^2 + \frac{m^2}{2}\phi_H^2 + \lambda\left(\frac{1}{6}\phi_H\phi_L^3 + \frac{1}{4}\phi_L^2\phi_H^2 + \frac{1}{6}\phi_L\phi_H^3 + \frac{1}{4!}\phi_H^4\right)\right\}} \qquad (7$$

n principle, we can perform the $D\phi_H$ integral obtaining that

$$Z = \int D\phi_L\, e^{-S_E^{eff}[\phi_L]} \qquad (79)$$

here

$$S_E^{\text{eff}}[\phi_L] = \int d^dx \left\{ \frac{1}{2}\partial_\mu\phi_L\partial_\mu\phi_L + \frac{m^2}{2}\phi_L^2 + \frac{\lambda}{4!}\phi_L^4 + \mathcal{O}(\lambda)\text{ corrections} \right\} \quad (80)$$

from $D\phi_H$!

How do we evaluate that?

ince $\Lambda \gg m^2$ we can neglect the $m^2\phi_H^2$ term in (78), thus the quadratic term reads

$$d^dx\, \frac{1}{2}\partial_\mu\phi_H\partial_\mu\phi_H = \frac{1}{2}\int d^dx \int\frac{d^dk}{(2\pi)^d}\int\frac{d^dk'}{(2\pi)^d}\, \phi_H(k)(-ik)\cdot(-ik')\,\phi_H(k')\, e^{i(k+k')\cdot x}$$

$$= \frac{1}{2}\int\frac{d^dk}{(2\pi)^d}\, \phi_H(k)\, k^2\phi_H(-k) \quad (81)$$

or a free theory

$$Z_{\text{free}} = \int D\phi_H\; e^{-\int\frac{d^dk}{(2\pi)^d}\left\{\frac{1}{2}\phi_H(k)\,k^2\phi_H(-k) + J(k)\,\phi_H(k)\right\}} \quad (82)$$

herefore, the propagator in momentum space is

$$D_F(p+q) = \frac{(2\pi)^d}{k^2}\,\delta^{(d)}(k+p) \quad (83)$$

xpanding the exponential in (78) we get terms like:

i) $-\frac{\lambda}{4}\int d^dx\, \phi_L^2(x)\,\phi_H(x)\,\phi_H(x) \Longrightarrow \int D\phi_H\, e^{-\frac{1}{2}\int\frac{d^dk}{(2\pi)^d}\phi_H(k)\,k^2\phi_H(-k)}\,\phi_H(q)\,\phi_H(p) \longrightarrow D_F(q+p$

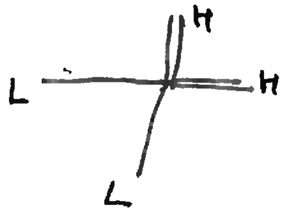
$$\Longrightarrow -\frac{\lambda}{4}\int^{b\Lambda}\frac{d^dk_1}{(2\pi)^d}\,\phi_L(k_1)\phi_L(-k_1)\int_{b\Lambda}^{\Lambda}\frac{d^dk_2}{(2\pi)^d}\,\frac{1}{k_2^2} = -\frac{\mu}{2}\int d^dx\,\phi_L^2(x)$$

$$\text{with}\quad \mu = \frac{\lambda}{2}\int_{b\Lambda}^{\Lambda}\frac{d^dk_2}{(2\pi)^d}\,\frac{1}{k_2^2} = \lambda\,\frac{1}{(4\pi)^d\,\Gamma(d/2)}\,\frac{(1-b^{d-2})\,\Lambda^{d-2}}{d-2} \quad (84)$$

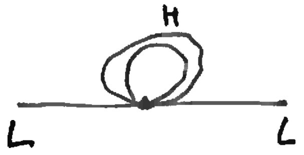
) To ~~see that write (82) as~~

$$\cancel{\int\frac{d^dk}{(2\pi)^d}\int\frac{d^dp}{(2\pi)^d}\,\frac{1}{2}\,\phi_H(k)\,\mathcal{O}\,\phi_H(p)\ \text{with}\ \mathcal{O} = k^2\,\delta^{(d)}(p+k)}$$

This corresponds to ~~[illegible]~~ start from



end



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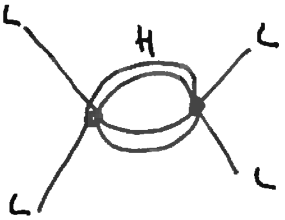
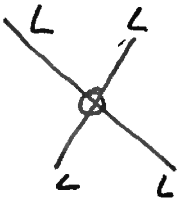
.

urthermore, this one contribution to the expansion $e^{-\frac{\mu}{2}\int d^dx\,\phi_L^2(x)}$

ii) Analogously, in order λ^2 we have contributions of the form



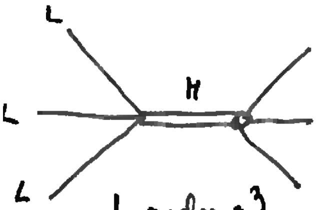
$\longrightarrow$



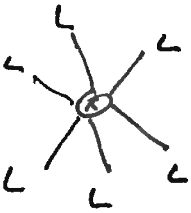
$\longleftrightarrow \; -\frac{1}{4!}\int d^dx\; \zeta\, \phi_L^4(x)$

with $\zeta = -\frac{3\lambda^2}{(4\pi)^{d/2}}\,\frac{1}{\Gamma(\frac{d}{2})}\,\frac{(1-b^{d-4})}{d-4}\,\Lambda^{d-4}$

iii) Another possibility ~~to~~ at order λ^2 is

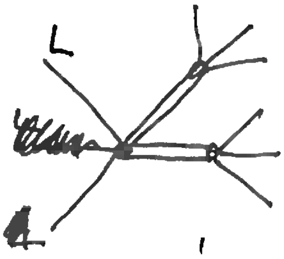


$\longrightarrow$



$\sim \phi_L^6(x)$

at order λ^3



$\Longrightarrow \phi_L^8(x)$

⋮

At the end we can write

$$S_E^{eff}[\phi_L] = \int d^dx \left\{ \frac{1}{2}(1+\Delta Z)(\partial_\mu\phi_L)^2 + \frac{1}{2}(m^2+\Delta m^2)\phi_L^2 + \frac{1}{4!}(\lambda+\Delta\lambda)\phi_L^4 + \Delta C(\partial_\mu\phi_L\partial_\mu\phi_L)^2 + \Delta D\,\phi_L^6 + \dots \right. \quad (B.$$

here the integration over ϕ_H gives rise to $\Delta z, \Delta m^2, \Delta\lambda, \Delta c, \Delta D, \ldots$

Notice, that we started with a renormalizable theory but S_E^{eff} is not renormalizable in the usual sense!

Renormalization flow: let's analyze the effects of changing b; for that

$$k' = \frac{k}{b} \qquad \text{and} \qquad x' = bx \qquad \text{(with this } |k'| < \Lambda)$$

$$\Downarrow$$

$$d^d x = b^{-d} d^d x' \qquad \frac{\partial}{\partial x^\mu} = b \frac{\partial}{\partial x'^\mu} \equiv b\,\partial'_\mu \qquad (86)$$

$$\Downarrow$$

$$S_E^{eff}[\phi_L] = \int d^d x'\, b^{-d} \Big\{ \frac{1}{2}(1+\Delta z) b^2 (\partial'_\mu \phi_L)^2 + \frac{1}{2}(m^2+\Delta m^2)\phi_L^2 + \frac{1}{4!}(\lambda+\Delta\lambda)\phi_L^4$$

$$+ \Delta c\, b^2 (\partial'_\mu \phi_L)^2 + \Delta D\, \phi_L^6 + \ldots \Big\} \qquad (87)$$

To impose that the field ϕ_L is canonically normalized we also define

$$\phi'_L = \left[b^{2-d}(1+\Delta z)\right]^{1/2} \phi_L \qquad (88)$$

so that $(88)+(87) \Longrightarrow$

$$S_E^{eff}[\phi'_L] = \int d^d x' \Big\{ \frac{1}{2}(\partial'_\mu \phi'_L)^2 + \frac{1}{2} m'^2 \phi_L'^2 + \frac{\lambda'}{4!}\phi_L'^4 + c'(\partial'_\mu \phi'_L)^2 + D' \phi_L'^6 + \ldots \Big\} ($$

with

$$m'^2 = (m^2+\Delta m^2)(1+\Delta z)^{-1} b^{-2}$$

$$\lambda' = (\lambda+\Delta\lambda)(1+\Delta z)^{-2} b^{d-4} \qquad (90)$$

$$c' = \Delta c\,(1+\Delta z)^{-2} b^{d}$$

$$D' = \Delta D\,(1+\Delta z)^{-3} b^{2d-6}$$

The combination of the integration ~~and the~~ of higher $\boxed{15.27}$
energy modes and the re-scaling leads to a continuous transformatio
~~that is the RG~~. of the Lagrangian that is the RG. Considering
b infinitesimaly close to 1 we get the RGE.

For further details see Peskin section 12.1.