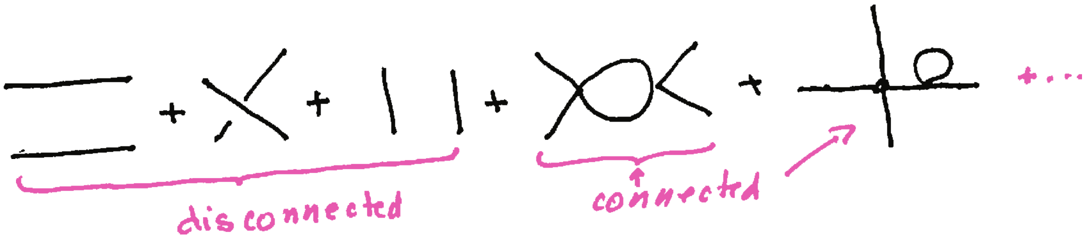
10. Green's functions and their generating functionals.

10.a Generating functionals

We have already encoutered the Green's functions

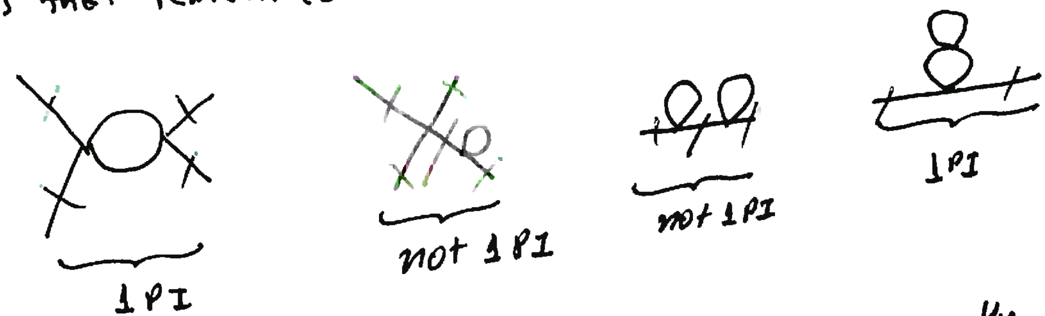
$$G^{(n)}(x_1 \dots x_n) = \langle 0 | T\, \varphi(x_1) \dots \varphi(x_n) | 0 \rangle \times \frac{1}{\langle 0 | 0 \rangle} \qquad (1)$$

(for definiteness we will consider a real scalar field). We have seen that $G^{(n)}$ contains disconnected diagrams (with more than 1 independent piece) and connected ones. For instance, for $G^{(4)}$:



The connected Green's functions $G^{(n)}_{conn}(x_1 \dots x_n)$ are defined as the connected part of $G^{(n)}(x_1 \dots x_n)$.

We can also define the connected proper vertex functions $\Gamma^{(n)}(x_1 \dots x_n)$, also called the one-particle-irreducible (1PI) amputed Green's functions. The 1PI functions are the ones that remain connected even when we cut one arbitrary internal line:



/ ≡ removed ≡ amputed

We say that they are amputed because we remove the external lines

To each type of Green's functions we associate generating functional:

$$Z[J] = \sum_{j=0}^{\infty} \frac{i^n}{n!} \int d^4x_1 .. d^4x_j \; G^{(j)}(x_1 .. x_j) \, J(x_1) .. J(x_j) \qquad (2)$$

with

$$G^{(n)}(x_1 .. x_n) = \left(\frac{1}{i}\right)^n \frac{\delta^n Z}{\delta J(x_1) .. \delta J(x_n)} \bigg|_{J=0} \qquad (3)$$

In this case we've seen that

$$Z[J] = \underbrace{Z[0]}_{\text{to eliminate vacuum bubbles} \to Z[0]=1} \int \mathcal{D}\varphi \; \exp\left\{ i S[\varphi] + i \int d^4z \, J(z) \varphi(z) \right\} \qquad (4)$$

On the other hand, for the connected Green's functions the generating function is

$$W[J] = \sum_{j=1}^{\infty} \frac{i^{j-1}}{j!} \int d^4x_1 .. d^4x_j \; G^{(j)}_{conn}(x_1 .. x_j) \, J(x_1) .. J(x_j) \qquad (5)$$

and

$$G^{(n)}_{conn} = \left(\frac{1}{i}\right)^{n-1} \frac{\delta^n W[J]}{\delta J(x_1) \dots \delta J(x_n)} \bigg|_{J=0} \qquad (6)$$

30/8/19

Finally, for the 1PI Green's functions, the generating functional is

$$i\Gamma[\Phi] = \sum_{j=1}^{\infty} \frac{1}{j!} \int d^4x_1 \dots d^4x_j \; \Gamma^{(j)}(x_1 .. x_j) \, \Phi(x_1) .. \Phi(x_j) \qquad (7)$$

and

$$\Gamma^n(x_1 .. x_n) = \frac{\delta^n i\Gamma[\Phi]}{\delta \Phi(x_1) \dots \delta \Phi(x_n)} \bigg|_{\Phi=0} \qquad (8)$$

We shall prove that

$$Z[J] = \exp\left[i\, W[J]\right] \qquad (9) \qquad \text{or} \qquad W[J] = -i \ln Z[J] \qquad (9')$$

and

$$\Gamma[\Phi_{CL}] = W[J] - \int d^4x\, J(x)\, \Phi_{CL}(x) \qquad (10)$$

i Legendre transformation

with

$$\left.\frac{\delta \Gamma[\Phi]}{\delta \Phi(x)}\right|_{\Phi=\Phi_{cl}} = -J \qquad (11)$$

$$\left.\frac{\delta W[J]}{\delta J}\right|_{J} = \varphi_{cl}$$

Let's prove (9) is the generating functional of connected Green's functions:

$$(9') \Longrightarrow \frac{\delta^2 W}{\delta J(x_1)\,\delta J(x_2)} = \frac{-i}{Z} \frac{\delta^2 Z}{\delta J(x_1)\,\delta J(x_2)} + \frac{i}{Z^2} \frac{\delta Z}{\delta J(x_1)} \frac{\delta Z}{\delta J(x_2)} \qquad (12)$$

Assuming that $\langle \varphi(x) \rangle = 0$, the evaluation of (12) for $J=0$ leads to

$$\left.\frac{\delta^2 W}{\delta J(x_1)\,\delta J(x_2)}\right|_{J=0} = -i \left.\frac{\delta^2 Z}{\delta J(x_1)\,\delta J(x_2)}\right|_{J=0} = i \langle 0 | T \varphi(x_1)\, \varphi(x_2) | 0 \rangle$$

so $\left. -i \dfrac{\delta^2 W}{\delta J(x_1)\,\delta J(x_2)}\right|_{J=0}$ is the propagator that has NO disconnected piece ☺

(Consider the 3-point Green's function)

Now $\dfrac{\delta}{\delta J(x_3)} \dfrac{\delta}{\delta J(x_4)}$ (12) $\overset{J=0}{\Longrightarrow}$

$$\left.\frac{\delta^4 W}{\delta J(x_1)\,\delta J(x_2)\,\delta J(x_3)\,\delta J(x_4)}\right|_{J=0} = \frac{-i}{Z} \left.\frac{\delta^4 Z}{\delta J(x_1) \ldots \delta J(x_4)}\right|_{J=0}$$

$$+ \frac{i}{Z^2} \left\{ \left.\frac{\delta^2 Z}{\delta J(x_1)\,\delta J(x_2)}\right|_{J=0} \left.\frac{\delta^2 Z}{\delta J(x_3)\,\delta J(x_4)}\right|_{J=0} + \left.\frac{\delta^2 Z}{\delta J(x_1)\,\delta J(x_3)}\right|_{J=0} \left.\frac{\delta^2 Z}{\delta J(x_2)\,\delta J(x_4)}\right|_{J=0} + \left.\frac{\delta^2 Z}{\delta J(x_1)\,\delta J(x_4)} \frac{\delta^2 Z}{\delta J(x_2)\,\delta J(x_3)}\right|_{J=0} \right\}$$

So

$$\left.\frac{\delta^4 W}{\delta J(x_1)\delta J(x_2)\delta J(x_3)\delta J(x_4)}\right|_{J=0} = -i\Big\{\langle 0|T\varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4)|0\rangle -$$

$$-\langle 0|T\varphi(x_1)\varphi(x_2)|0\rangle\langle 0|T\varphi(x_3)\varphi(x_4)|0\rangle - \langle 0|T\varphi(x_1)\varphi(x_3)|0\rangle\langle 0|T\varphi(x_2)\varphi(x_4)|0\rangle$$

$$-\langle 0|T\varphi(x_1)\varphi(x_4)|0\rangle\langle 0|T\varphi(x_2)\varphi(x_3)|0\rangle\Big\} \qquad (13)$$

The last two lines of the RHS of (13) cancel out the disconnected pieces of $\langle 0|T\varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4)|0\rangle$ leaving only the connected part!!

It can be shown by induction that

$$G^{(n)}_{conn}(x_1\ldots x_n) = \left(\frac{1}{i}\right)^{n-1}\left.\frac{\delta^N W[J]}{\delta J(x_1)\ldots\delta J(x_n)}\right|_{J=0}$$, ie, (6) is valid!

In the case of free field: $W[J] = -\frac{1}{2}\int d^4x\, d^4y\; J(x)\, D_F(x-y)\, J(y)$

$\Rightarrow$ the only connected Green function is $G^{(2)}(x_1, x_2)$ that comes at no surprise!

Now let's "prove" (7-8): From (10);

explain

$$\frac{\delta W[J]}{\delta J(x)} = \Phi(x) \quad \text{and} \quad \frac{\delta \Gamma[\Phi]}{\delta \Phi(x)} = -J(x) \qquad (14)$$

$$\Rightarrow \frac{\delta^2 W[J]}{\delta J(x_1)\delta J(x_2)} = \frac{\delta\Phi(x_1)}{\delta J(x_2)} \quad \text{and} \quad \frac{\delta^2\Gamma[\Phi]}{\delta\Phi(x_1)\delta\Phi(x_2)} = -\frac{\delta J(x_1)}{\delta\Phi(x_2)} \qquad (15)$$

$$\Rightarrow \int d^4z\, \frac{\delta^2 W[J]}{\delta J(x_1)\delta J(z)}\,\frac{\delta^2\Gamma[\Phi]}{\delta\Phi(z)\delta\Phi(x_2)} = \int d^4z\, \frac{\delta\Phi(x_1)}{\delta J(z)}\,\frac{\delta J(z)}{\delta\Phi(x_2)} = -\frac{\delta\Phi(x_1)}{\delta\Phi(x_2)} = -\delta^{(4)}(x_1-x_2) \qquad (16)$$

Hence,

$$-\frac{\delta^2\Gamma[\phi]}{\delta\phi(y)\,\delta\phi(x_2)} \quad \text{is the inverse of} \quad \frac{\delta^2 W[J]}{\delta J(x_1)\,\delta J(x_2)}$$

taking $J=0$ ($\Rightarrow \phi = 0$) $\Rightarrow$ use 9 and 14

$$-i\left.\frac{\delta^2\Gamma[\phi]}{\delta\phi(y)\,\delta\phi(x_2)}\right|_{\phi=0} \quad \text{is the inverse of the propagator (17)}$$

Now, $\frac{\delta}{\delta J(x_3)}$ (16) $\Rightarrow$

$$\int d^4z\, \frac{\delta^3 W[J]}{\delta J(x_1)\,\delta J(z)\,\delta J(x_3)}\,\frac{\delta^2\Gamma}{\delta\phi(z)\,\delta\phi(x_2)} = -\int d^4z\, d^4z_1\, \frac{\delta^2 W[J]}{\delta J(x_1)\,\delta J(z)}\,\frac{\delta^3\Gamma[\phi]}{\delta\phi(z)\,\delta\phi(x_2)\,\delta\phi(z_1)}\,\frac{\delta\phi(z_1)}{\delta J(x_3)} \quad (18)$$

$$\frac{\delta^2 W}{\delta J(z_1)\,\delta J}$$

Now, $\int d^4x_1\, d^4x_3\, \frac{\delta^2\Gamma}{\delta\phi(x_1)\,\delta J(y_1)}\,\frac{\delta^2\Gamma}{\delta\phi(x_3)\,\delta J(y_3)} \Rightarrow$

$$\int d^4z\, d^4x_1\, d^4x_3\, \frac{\delta^2\Gamma}{\delta\phi(x_2)\,\delta\phi(t)}\,\frac{\delta^2\Gamma}{\delta\phi(y_1)\,\delta\phi(x_1)}\,\frac{\delta^2\Gamma}{\delta\phi(y_3)\,\delta\phi(x_3)}\,\frac{\delta^3 W}{\delta J(x_1)\,\delta J(z)\,\delta J(x_3)}$$

$$= -\int d^4z\, d^4z_1\, d^4x_1\, d^4x_3\, \underbrace{\frac{\delta^2 W}{\delta J(x_1)\,\delta J(z)}\,\frac{\delta^2\Gamma}{\delta\phi(x_1)\,\delta\phi(y_1)}}_{\int d^4x_1 \Rightarrow \delta(z-y_1)}\;\underbrace{\frac{\delta^2 W}{\delta J(z_1)\,\delta J(x_3)}\,\frac{\delta^2\Gamma}{\delta\phi(x_3)\,\delta\phi(y_3)}}_{\int d^4x_3 \Rightarrow -\delta(z_1-y_3)}\;\frac{\delta^3\Gamma}{\delta\phi(t)\,\delta\phi(x_2)\,\delta\phi}$$

Changing $x_2 \to y_2$
$z \to x_2$

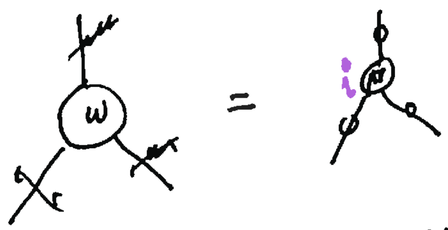
$i \times$ inverse of the propagator

$$\int d^4x_1\, d^4x_2\, d^4x_3 \; \frac{\delta^2\Gamma}{\delta\phi(y_1)\,\delta\phi(x_1)} \; \frac{\delta^2\Gamma}{\delta\phi(y_2)\,\delta\phi(x_2)} \; \frac{\delta^2\Gamma}{\delta\phi(y_3)\,\delta\phi(x_3)} \; \underbrace{\frac{\delta^3 W}{\delta J(x_1)\,\delta J(x_2)\,\delta J(x_3)}}_{-\cancel{i}\, G^{(3)}_{conn}}$$

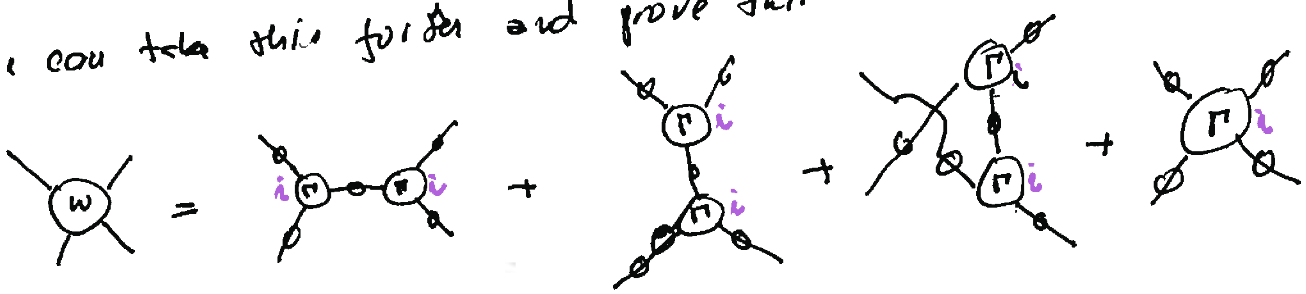
$$= -\underbrace{\frac{\delta^3\Gamma}{\delta\phi(y_1)\,\delta\phi(y_2)\,\delta\phi(y_3)}}_{1PI^{(3)}} \qquad (19)$$

This means, using (17), that the amputed 3 ordinal 1PI function is the RHS of (19) as it should be. Graphically

$-\!\circ\!-\; \equiv$ propagator



We can take this further and prove that



(20)

Again, the general case can be proven by induction.

In the case of QED we define

$$W[\eta, \bar\eta, J_\mu] = -i \ln Z[\eta, \bar\eta, J_\mu] \qquad (21)$$

and

$$\Gamma[\bar\Psi, \Psi, A_\mu] = W[\eta, \bar\eta, J_\mu] - \int d^4x\, \bar\Psi \eta - \int d^4x\, \bar\eta \Psi - \int d^4x\, J_\mu A^\mu \qquad (22)$$

with

$$\frac{\delta W}{\delta J_\mu(x)} = A_\mu(x)$$

$$\frac{1}{i}\frac{\delta W}{\delta \bar\eta(x)} = \psi(x) \qquad (23)$$

$$i\,\frac{\delta W}{\delta \eta(x)} = \bar\psi(x)$$

References: Nash, "Relativistic Quantum Fields", pages 50 – 55

Pokorski, sections 2.6

10.b Organizing Calculations

~~Relation (20)~~ Relations like (20) allow us to simplify calculations. For instance suppose that we have focusing on 1PI functions. For instance suppose that we have calculated the 1PI correction to ~~a~~ 1PI 2-point function

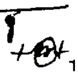
We can write the complete two-point function as:

 $= \frac{i}{p^2-m^2+i\epsilon}$  $\frac{i}{p^2-m^2+i\epsilon}$ ($i/p^2-m^2+i\epsilon + \ldots$)

$+\ \frac{i}{p^2-m^2+i\epsilon}$  $\frac{i}{p^2-m^2+i\epsilon}$  $\frac{i}{p^2-m^2+i\epsilon}$

$+ \ldots$

geometric series!

$= \frac{i}{p^2-m^2+i\epsilon}\sum_{n=0}\Big($  $\frac{i}{p^2-m^2+i\epsilon}\Big)^n = \frac{i}{p^2-m^2+i\epsilon}\,\frac{1}{1-}$  $\frac{i}{p^2-m^2+i\epsilon}$ $= \frac{i}{p^2-m^2-}$ 

10.C Ward Identities

In QED we have

$$Z[J_\mu,\eta,\bar\eta] = \int \mathcal{D}A_\mu \mathcal{D}\bar\psi \mathcal{D}\psi \, \exp\left\{ i \underbrace{\int d^4x \left[\mathcal{L}_{QED} + \mathcal{L}_{GF} + J_\mu A^\mu + \bar\eta\psi + \bar\psi\eta \right]}_{S_{eff}} \right\} \quad (24)$$

with

$$\mathcal{L}_{QED} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \bar\psi(i\gamma - qe\not{A})\psi - m\bar\psi\psi \quad (25a)$$

$$\mathcal{L}_{GF} = -\frac{1}{2a}(\partial_\mu A^\mu) \quad (25.b)$$

gauge fixing

Now we perform the change of variables

$$\psi \longrightarrow \psi' = e^{-iq\theta}\psi \quad (26)$$

$$A_\mu \to A'_\mu = A_\mu + \frac{1}{e}\partial_\mu\theta$$

that does not change $Z[J_\mu,\eta,\bar\eta]$! Notice that (26) is a gauge transf... The ...ation, therefore, $\mathcal{L}_{QED}$ is invariant, as well as the integration measure. The only terms that change are the ~~...~~ gauge fixing one and the source terms. So,

$$\frac{\delta Z}{\delta\theta(y)} \int \mathcal{D}A'_\mu \mathcal{D}\bar\psi'\mathcal{D}\psi' \exp\left\{ i\int d^4x \left[\mathcal{L}_{QED} - \frac{1}{2a}\left(\partial_\mu A'^\mu - \frac{1}{e}\partial_\mu\partial^\mu\theta\right)^2 + \bar\eta\psi' e^{+iq\theta} + \bar\psi'\eta\, e^{-iq\theta} + J_\mu A'^\mu - J_\mu \frac{1}{e}\partial^\mu\theta \right]\right\}\Bigg|_{\theta=0} = 0 \quad (27)$$

This leads to

$$0 = \int \mathcal{D}A'_\mu \mathcal{D}\bar\psi'\mathcal{D}\psi \, e^{iS_{eff}} \, i\int d^4x \left\{ \frac{1}{a}\partial_\mu A'^\mu \left(\frac{1}{e}\right) \partial_\nu\partial^\nu \delta(x-y) + iq\left[\bar\eta\psi' - \bar\psi'\eta\right]\delta(x-y) - \frac{1}{e} J_\mu \partial^\mu \delta(x-y) \right.$$

Now integrating by parts and dropping the surface terms we obtain

$$0=\int \mathcal{D}A'_\mu \mathcal{D}\bar{\Psi}'\mathcal{D}\Psi \left\{+\frac{1}{ea}\partial_{\nu}^{(y)}\partial^{\nu}_{(y)}(\partial_\mu A'^\mu)+iq\,[\bar{\eta}\,\Psi'-\bar{\Psi}'\eta]+\frac{1}{e}\partial_\mu J^\mu(y)\right\}e^{iS_{eff}}$$

Now making the changes: $\frac{1}{i}\frac{\delta}{\delta J_\mu}$ $\frac{1}{i}\frac{\delta}{\delta\bar{\eta}}$ $i\frac{\delta}{\delta\eta}$

we can perform the functional integral obtaining that

$$\partial_\mu J^\mu Z+\frac{1}{a}\frac{\partial_\nu\partial^\nu}{i}\left(\partial_\mu\frac{\delta Z}{\delta J_\mu}\right)+qe\left(\bar{\eta}\frac{\delta Z}{\delta\bar{\eta}}+\frac{\delta Z}{\delta\eta}\eta\right)=0 \quad (28)$$

10/9/19

Now using that $Z=e^{iW[J]}$ we get

$$\partial_\mu J^\mu+\frac{1}{a}\partial_\nu\partial^\nu\left(\partial_\mu\frac{\delta W}{\delta J_\mu}\right)+iqe\left(\bar{\eta}\frac{\delta W}{\delta\bar{\eta}}+\frac{\delta W}{\delta\eta}\eta\right)=0 \quad (29)$$

(29) is the general Ward-Takahashi identity for the covariant gauges in (25.b)

We can also express these identities using $\Gamma[A_\mu,\Psi,\bar{\Psi}]$. From (22) we have that

$$J_\mu=-\frac{\delta\Gamma}{\delta A^\mu},\qquad \bar{\eta}=\frac{\delta\Gamma}{\delta\Psi}\quad\text{and}\quad \eta=-\frac{\delta\Gamma}{\delta\bar{\Psi}} \quad (30)$$

$A_\mu=\frac{\delta W}{\delta J_\mu}$, $\Psi=\frac{\delta W}{\delta\bar{\eta}}$ $\bar{\Psi}=-\delta W/\delta\eta$

(notice that I'm dropping the subscripts c on the "classical" fields) leading to

$$\frac{1}{a}\partial^2\partial_\mu A^\mu-\partial_\mu\frac{\delta\Gamma}{\delta A_\mu}+ieq\left(\frac{\delta\Gamma}{\delta\Psi}\Psi+\bar{\Psi}\frac{\delta\Gamma}{\delta\bar{\Psi}}\right)=0 \quad (31)$$

To see the meaning of (28) or (29) or (31) let's analyze a few examples.

Example 1:

In the gauge choice (25.5) we know that the tree level propagator is given by

$$\tilde{D}_{\mu\nu} = -\frac{i}{k^2}\left[g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right] - ia\frac{k_\mu k_\nu}{k^4} \qquad (32)$$

Let's see how higher order corrections modify the last term in this propagator. For that,

as a function of

$$\frac{\delta}{\delta J^\nu(y)}(29)(x)\Bigg|_{J_\mu=\eta=\bar\eta=0} \Rightarrow \partial_\nu^{(x)}\delta(x-y) + \frac{1}{a}\partial^2_{(x)}\partial^{(x)\mu}\underbrace{\frac{\delta^2 W}{\delta J^\mu(x)\,\delta J^\nu(y)}\Bigg|_{J=\bar\eta=\eta=0}}_{i\,G^c_{\mu\nu}(x-y)} = 0$$

$\partial_m \partial^\beta$

$$\Rightarrow \partial_\nu^{(x)}\delta(x-y) = -\frac{i}{a}\partial^2_{(x)}\partial^\mu_{(x)}G^c_{\mu\nu}(x-y) \qquad (33)$$

Now using that
$$G^c_{\mu\nu}(x-y) = \int\frac{d^4p}{(2\pi)^4}\tilde{D}_{\mu\nu}(p)\,e^{-ip(x-y)}$$

we obtain
$$\frac{p^2p^\mu\tilde{D}_{\mu\nu}}{a} = -i\,P_\nu \qquad (34)$$

The solution of (34) is
$$\tilde{D}_{\mu\nu} = -ia\frac{P_\mu P_\nu}{p^4} + \tilde{D}^T_{\mu\nu}(p) \qquad (35)$$

where $\tilde{D}^T_{\mu\nu}(p) = \left(g_{\mu\nu} - \frac{P_\mu P_\nu}{k^2}\right)f(p^2)$ $\qquad (36)$

From (35) we can learn that the "gauge fixing" term

$$-ia\frac{p_\mu p_\nu}{p^4}$$ in (32) is not renormalized by higher order corrections!! See eq'n (12) on page (13.4)!

Example 2: Let's reobtain the relation that leads to $Z_1 = Z_2$.

$$\frac{\delta^2}{\delta\bar{\eta}(y)\,\delta\eta(z)}(29)(x) \Longrightarrow \frac{1}{a}\partial^2_{(x)}\partial^{(x)}_\mu \left.\frac{\delta^3 W}{\delta\bar{\eta}(y)\delta\eta(z)\,\delta J_\mu(x)}\right|_{J_\nu=\eta=\bar{\eta}=0} = -iqe\left[-\frac{\delta^2 W}{\delta\eta(z)\delta\bar{\eta}(x)}\delta(x-y) - \frac{\delta^2 W}{\delta\bar{\eta}(y)\delta\eta(x)}\delta(x-z)\right]_{\bar{\eta}=J=\eta=0}$$

$$\Rightarrow +\frac{1}{a}\partial^2_{(x)}\partial^{(x)}_\mu\langle 0|T A^\mu(x)\Psi(y)\bar{\Psi}(z)|0\rangle = qe\Big[\delta(y-x)\big(-\langle 0|T\,\bar{\Psi}(z)\Psi(x)|0\rangle\big) - \langle 0|T\Psi(y)\bar{\Psi}(x)|0\rangle\delta(x-z)\Big] \quad (37)$$

Now, using the translation invariance we write,

$$\langle 0|TA^\mu(x)\Psi(y)\bar{\Psi}(z)|0\rangle = \langle 0|TA^\mu(x-z)\,\Psi(y-z)\,\bar{\Psi}(0)|0\rangle \quad (38)$$

$$= \int\frac{d^4p}{(2\pi)^4}\int\frac{d^4q}{(2\pi)^4}\, e^{-ip(y-z)}\, e^{-iq(x-z)}\, V^\mu(p,q)$$

and,

$$\langle 0|T\,\Psi(x)\bar{\Psi}(z)|0\rangle = \langle 0|T\Psi(x-z)\bar{\Psi}(0)|0\rangle = \int\frac{d^4k}{(2\pi)^4}\, e^{-ik(x-y)}\, iS_F(k) \quad (39)$$

and

$$\langle 0|T\psi(y)\bar{\psi}(x)|0\rangle = \langle 0|T\psi(y-x)\bar{\psi}(0)|0\rangle = \int \frac{d^4k}{(2\pi)^4} e^{-ik(y-x)} iS_F(k) \quad (40)$$

Substituting (38–40) into (37) we get

$$\frac{1}{a}\int\frac{d^4p}{(2\pi)^4}\int\frac{d^4q}{(2\pi)^4}\; i q^2 q_\mu\, e^{-ip(y-z)} e^{-iq(x-z)} V^\mu(p,q) =$$

$$= qe\left[\delta(y-x)\int\frac{d^4k}{(2\pi)^4} e^{-ik(x-z)} iS_F(k) - \delta(x-z)\int\frac{d^4k}{(2\pi)^4} e^{-ik(y-x)} iS_F(k)\right] \quad (4$$

Now

$$\int dx \int dy \int dz\; e^{ip'y} e^{iq'x} e^{ik'z} \quad (41) \Rightarrow$$

$$\Rightarrow -\frac{i q'^2 q'_\mu}{a} V^\mu(p',q')\,(2\pi)^4\delta(k'+p'+q') = qe\Big[iS_F(\bar{k}')(2\pi)^4\delta(k'+p'+q')$$

$$- iS_F(p')(2\pi)^4\delta(k'+p'+q')\Big]$$

Factoring out $(2\pi)^4\,\delta(k'+p'+q')$

$$+\frac{1}{a} q'^2 q'_\mu V^\mu(p',q') = qe\left[S_F(p'+q') - S_F(p')\right] \quad (41)$$

10.d Schwinger-Dyson eq'ns (SDE)

SDE are integral eq'ns ~~that~~ obeyed by the Green's function of a QF. They are obtained without approximations, ie, they are exact. Therefore they can be used to study non-perturbative effects.

For simplicity, we shall consider a single real scalar field φ whose Lagrangian density is

$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi - \frac{m^2}{2}\varphi^2 - \frac{\lambda}{4!}\varphi^4 \qquad (42)$$

Due to the boundary conditions

$$\int D\varphi\,\frac{\delta}{\delta\varphi}F[\varphi] = 0$$

so,

$$0 = \int D\varphi\,\frac{\delta}{\delta\varphi(y)}\exp\left\{i\int d^4x\left[\mathcal{L} + J(x)\varphi(x)\right]\right\} \qquad (43)$$

$$\Rightarrow 0 = \int D\varphi\; i\int d^4x\left[\partial^x_\mu\varphi\,\partial^\mu_x\delta(x-y) - m^2\varphi(x)\delta(x-y) - \frac{\lambda}{3!}\varphi^3(x)\delta(x-y) + J(x)\delta(x-y)\right]\times$$

$$\times\exp\left\{i\int d^4x\left[\mathcal{L} + J(x)\varphi(x)\right]\right\}$$

Integrating by parts and substituting $\varphi(y)$ by $\frac{1}{i}\frac{\delta}{\delta J(y)}$ leads to

$$-\partial^{(y)}_\mu\partial^\mu_{(y)}\frac{1}{i}\frac{\delta Z}{\delta J(y)} - m^2\frac{1}{i}\frac{\delta Z}{\delta J(y)} - \frac{\lambda}{3!}\left(\frac{1}{i}\frac{\delta}{\delta J(y)}\right)^3 Z + J(y)Z = 0$$

thus, finally we get the master eq'n

$$\left(\partial^{(y)}_\mu\partial^\mu_{(y)} + m^2\right)\frac{\delta Z}{\delta J(y)} + \frac{\lambda}{3!}\frac{\delta^3 Z}{\delta J(y)^3} - iJ(y)Z = 0 \qquad (44)$$

Now $Z = e^{iW[J]}$ into (44) yields

$$e^{iW}\Big[i\left(\partial^{(y)}_\mu \partial^\mu_{(y)} + m^2\right)\frac{\delta W}{\delta J(y)} - iJ(y) - \frac{\lambda}{3!}\Big(i\frac{\delta^3 W}{\delta J(y)\delta J(y)\delta J(y)} + 3i^2 \frac{\delta^2 W}{\delta J(y)\delta J(y)}\frac{\delta W}{\delta J(y)} + i^3\Big(\frac{\delta W}{\delta J(y)}\Big)^3\Big)\Big] = 0$$

$$\Rightarrow \left(\partial_\mu\partial^\mu + m^2\right)\frac{\delta W}{\delta J} - J - \frac{\lambda}{3!}\frac{\delta^3 W}{\delta J\delta J\delta J} - i\frac{\lambda}{2!}\frac{\delta^2 W}{\delta J\delta J}\frac{\delta W}{\delta J} + \frac{\lambda}{3!}\Big(\frac{\delta W}{\delta J}\Big)^3 = 0 \qquad (45)$$

where everything is evaluated at the point y.

In order to obtain an eq'n for the 2 point function we do

$$\frac{\delta}{\delta J(x)}(45)\Big|_{J=0} \overset{\langle\varphi\rangle = 0}{\Longrightarrow} \left(\partial^y_\mu\partial^\mu_y + m^2\right) i G^{(2)}_c(x,y) - \delta^{(4)}(x-y) - \frac{\lambda}{3!} i^3 G^{(4)}_c(x,y,y,y)$$

$$- i\frac{\lambda}{2!}\, i G^{(2)}_c(y,y)\, i G^{(2)}_c(x,y) = 0 \qquad (46)$$

Now using that $(\Box_y + m^2) D_F(x-y) = -i\delta^{(4)}(x-y)$ — free

with $D_F(x-y) = \langle 0|T\varphi(x)\varphi(y)|0\rangle$

and $\int d^4y\, D_F(x_1-y)\ (46) \Rightarrow$

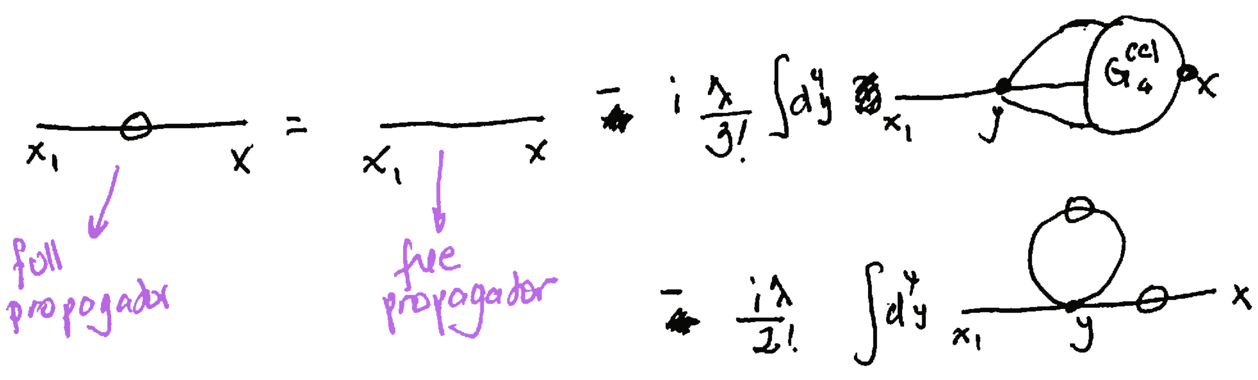
$$(-i)\, i G^{(2)}_c(x_1,x) - D_F(x_1-x) - \frac{\lambda}{3!} i^3 \int d^4y\, D_F(x_1-y)\, G^{(4)}_c(x,y,y,y)$$

$$- i^3 \frac{\lambda}{2!} \int d^4y\, G^{(2)}_c(y,y)\, G^{(2)}_c(x,y)\, D_F(x_1-y) = 0$$

imaly,

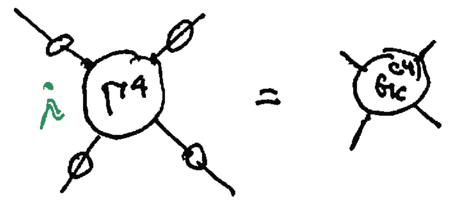
$$G_c^{(2)}(x_1,y) = D_F(x_1-y) - \frac{i\lambda}{3!}\int d^4y\, D_F(x_1-y_1)\, G_c^{(4)}(y,y,y,x) - \frac{i\lambda}{2!}\int d^4y\, G_c^{(2)}(y,y) G_c^{(2)}(y,x) \times D_F(x_1-y) \quad (47)$$

e represent (47) grafically as

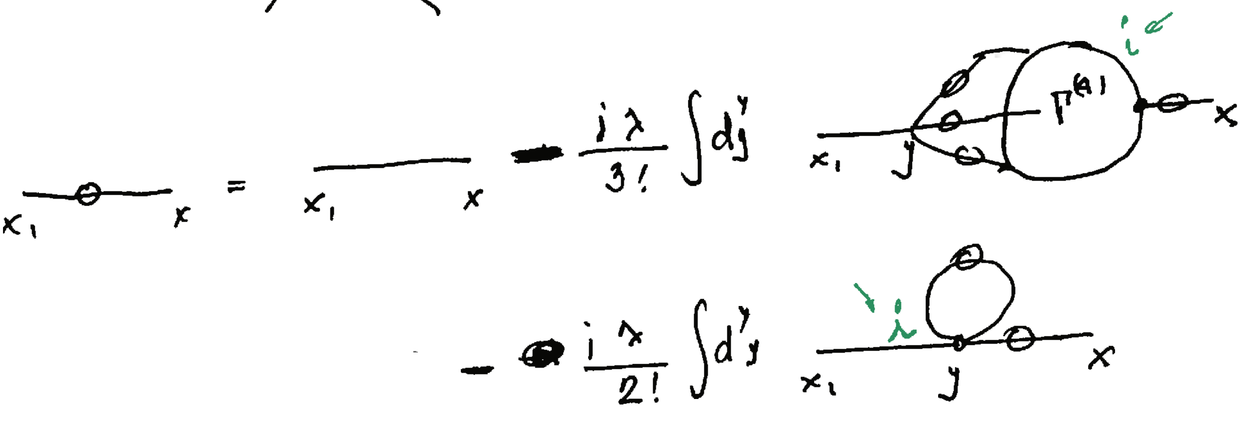


Due to the symmetry $\varphi \leftrightarrow -\varphi$ of (42) $G^{(2n+1)} \equiv 0 \equiv \Gamma^{(2n+1)}$

and



leading to



Exercise: Obtain the DSE for $\Gamma^{(4)}$

10.e Effective action/potential

13/09/19

From (14) we have that

$$\Phi_c(x) = \frac{\delta W}{\delta J(x)} = \frac{1}{Z}\frac{1}{i}\frac{\delta Z}{\delta J(x)}$$

Now, using (4) and the interpretation/construction of the path integral we have that

$$\Phi_c = \frac{\langle 0|\varphi|0\rangle_J}{\langle 0|0\rangle_J} \qquad (48)$$

where the subscript J indicates ~~the~~ the presence of the source J. So, for $J \to 0$ Φ_c goes to the vacuum expectation value (VEV) of the field φ.

Instead of expanding $\Gamma[\Phi_c]$ in powers of Φ_c, as in (7), we can ~~[illegible]~~ in powers of momentum (derivatives):

$$\Gamma[\Phi_c] = \int d^4x \left[-V(\Phi_c) + \frac{1}{2} Z[\Phi_c]\, \partial_\mu \Phi_c\, \partial^\mu \Phi_c + \dots \right] \qquad (49)$$

$V(\Phi_c)$, that is an ordinary function, is called effective potential.

Notice that,

$$\Gamma[\Phi_c] = -\Omega\, V(\Phi_c) \qquad (50)$$

for constant Φ_c, where Ω is the volume of the space-time.

Writing

$$\Phi_c(x_j) = \int \frac{d^4 p_j}{(2\pi)^4}\, e^{+i p_j x_j}\, \tilde{\Phi}_c(p_j) \qquad (51)$$

and substituting into (7) leads to

$|\langle 0|\,|\rangle|$

$$\Gamma[\Phi_c] = i\sum_{j=1}^{\infty}\frac{1}{j!}\int dx_1\ldots dx_j \int\frac{d^4p_1}{(2\pi)^4}\ldots\frac{d^4p_j}{(2\pi)^4}\ \exp\{i(p_1x_1+\ldots+p_jx_j)\}\ \Gamma^n(x_1\ldots x_j)\,\tilde{\Phi}_c(p_1)\ldots\tilde{\Phi}_c(p_j)$$

But $\int d^4x_1\ldots d^4x_j \exp\{i(p_1x_1+\ldots+p_jx_j)\}\,\Gamma^n(x_1\ldots x_j) = (2\pi)^4\,\delta(p_1+\ldots p_j)\,\tilde{\Gamma}(p_1,p_2\ldots,p_j)$

$$\Gamma(\Phi_c) = -i\sum_{j=1}^{\infty}\frac{1}{j!}\int\frac{d^4p_1}{(2\pi)^4}\ldots\frac{d^4p_j}{(2\pi)^4}\,(2\pi)^4\,\delta(p_1+\ldots+p_j)\,\tilde{\Gamma}(p_1\ldots p_j)\,\tilde{\Phi}_c(p_1)\ldots\tilde{\phi}(p_j) \quad (52)$$

For Φ_c constant, (51) implies that

$$\tilde{\Phi}_c(p_j) = (2\pi)^4\,\Phi_c\,\delta(p_j) \quad (53)$$

So, $\begin{pmatrix}50+\\52+53\end{pmatrix} \Rightarrow$

$$\Rightarrow -\Omega V(\Phi_c) = -i\sum_{j=1}^{\infty}\frac{1}{j!}\,\phi_c^{\,j}\,\underbrace{(2\pi)^4\delta(0)}_{\Omega}\,\tilde{\Gamma}^n(0,\ldots,0)$$

$$\Rightarrow V(\Phi_c) = +i\sum_j\frac{1}{j!}\,\phi_c^{\,j}\,\tilde{\Gamma}^{(n)}(0,\ldots0) \quad (54)$$

Now, ~~[illegible]~~ we can express the renormalization conditions as

$$m_R^2 = \left.\frac{d^2V(\Phi_c)}{d\Phi_c^2}\right|_{\Phi_c=0} \quad \text{(renormalized mass)} \quad (55)$$

$$\lambda_R = +\left.\frac{d^4V(\Phi_c)}{d\Phi_c^2}\right|_{\Phi_c} = 0 \quad \text{(renormalized coupling constant)} \quad (56)$$

10.f The physical meaning of the effective potential

We shall prove that $V(\Phi_c)$ is an energy density, therefore, it can be useful to ~~evaluate~~ obtain the ground state of the system. In analogy to (4…

we can also write

$$W[J] = \int d^4x \left\{ -\mathcal{E}(J) + \chi(J)\, \partial_\mu J \partial^\mu J + \dots \right\} \qquad (57)$$

Now we consider J constant within a box of size L^3 during the time T, and that it goes smoothly to zero outside this region. Thus,

$$e^{iW} \cong e^{-iL^3 T \mathcal{E}(J)} \qquad (58)$$

However,

$$e^{iW} = \langle 0 | e^{i\int d^4x J\varphi} | 0 \rangle \qquad (59)$$

which can be interpreted as the vacuum to vacuum transition when we replace the original hamiltonian (density) $\mathcal{H}$ by $\mathcal{H} - J\varphi$. Therefore, $\mathcal{E}(J)$ is the ~~all~~ $\mathcal{E}(J)$ is the energy per unit volume of the ground state of the perturbed hamiltonian. Think in terms of adiabatic approximation.

Let's save this result for the time being! We'll develop an argument for QM to simplify the notation! In order to obtain the state $|a\rangle$ that leaves stationary

$$\langle a | H | a \rangle$$

subject to the constraints $\langle a|a\rangle = 1$

and $\langle a | A | a \rangle = A_c$

~~We~~ introduce Lagrange multipliers E and J, and vary without constraint

$$\langle a | H - E - JA | a \rangle \quad . \quad (60)$$

So, we get

$$(H - E - JA) | a \rangle = 0 \quad (61)$$

$\Rightarrow |a\rangle$ is the eigenvalue of the perturbed hamiltonian $H - JA$.

This procedure gives $E(J)$ but we want this as a function of A_c! Now

$$\frac{d}{dJ}(60) \Rightarrow 0 = \frac{d\langle a|}{dJ} \underbrace{(H - E - JA | a \rangle}_{0} - \langle a | \frac{dE}{dJ} + A | a \rangle + \underbrace{\langle a | H - E - JA)}_{0} \frac{d}{dJ} | a \rangle$$

$$\Rightarrow \quad \langle a | A | a \rangle = A_c = - \frac{dE}{dJ} \qquad (62)$$

So, $\overset{61}{()} \Rightarrow \quad \langle a | H | a \rangle = E + J \langle a | A | a \rangle$

$$= E + J A_c$$

$$\langle a | H | a \rangle = E - J \frac{dE}{dJ} \qquad (63)$$

This is exactly the procedure to obtain the effective potential with the changes

$$V(\phi_c) = \langle a | \mathcal{H} | a \rangle$$

with $\langle a | \mathcal{H} | a \rangle$ stationary under the constraints

$$\langle a | a \rangle = 1 \qquad\qquad \langle a | \phi | a \rangle = \phi_c \text{ !!}$$

10.g The loop expansion

The loop expansion is an alternative approximation method. Let's assu[me] that the Lagrangian density is such that

$$\mathcal{L} = \frac{1}{a} \mathcal{L}(\varphi, \partial_\mu \varphi) \qquad (64)$$

[w]here a is a parameter. The loop expansion is a power-series ~~ex~~ expansion [i]n a. In fact, reintroducing $\hbar$ ~~into~~ back into our eq^n:

$$Z[J] = \int D\varphi \, \exp\left[\frac{i}{\hbar} \int d^4x \left(\mathcal{L} + \hbar J(x) \varphi(x) \right)\right] \qquad (65)$$

[s]o $\hbar$ is the a ~~parameter~~ parameter! To show that the loop expansion is a power series [i]n a notice that:

- Propagators carry a factor a since they are the inverse of the differential operator in the quadratic form.
- Vertices have a factor a^{-1}.

So the power P of the expansion is

$$P = I - V$$

where I = number of internal lines
V = number of vertices

But we know that the number of loops is

$$L = I - V + 1$$

so $P = L - 1$.

Comments:

- the loop expension requires summing up an infinity number of diagrams in each order.

- Since the loop expansion is on a parameter multiplying $\mathcal{L}$, it is ~~not~~ affected by shift in the fields or choice of the free Lagrangian.

Example: $\lambda\varphi^4$ model: $\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{m^2}{2}\varphi^2 - \frac{\lambda}{4!}\varphi^4$ (66)

or $L=0$ we have $\frac{m^2}{2}\varphi_c^2 + \frac{\lambda}{4!}\varphi_c^4 = V_0(\varphi_c)$

at one-loop level: $V_1(\varphi_c) \overset{(54)}{=} i\sum_{n=1}^{\infty}\frac{1}{n!}\varphi_c^n\,\tilde{\Gamma}^{(n)}(0,..,0)$ (67)

φ_c^2 +

φ_c^4 +

φ_c^6 + ...

~~Notice that a diagram with (2k) external legs~~

Facts:
- a diagram with $(2k)$ external legs has a combinatorial factor of $\frac{1}{2}$ (prove it)

- there are $(2k-1)(2k-3)\ldots 1\,(k-1)!$ different diagrams that are equivalent ~~to be~~ for zero momenta. So

$$\frac{1}{\underbrace{(2k}_{n}!)}(2k-1)(2k-3)\ldots 1\times(k-1)! = \frac{1}{2k}\left(\frac{1}{2}\right)^{k}$$

therefore,

$$V_1(\varphi_c) = i\sum_{k=1}^{\infty}\frac{1}{2k}\int\frac{d^4q}{(2\pi)^4}\left(-i\lambda\frac{i}{q^2-m^2+i\epsilon}\frac{\varphi_c^2}{2}\right)^k \quad (68)$$

$$V_1(\varphi_c) = -\frac{i}{2}\int \frac{d^4q}{(2\pi)^4}\, \ln\left(1 - \frac{\frac{\lambda\varphi_c^2}{2}}{q^2-m^2+i\epsilon}\right) \qquad (69)$$

here we used that. $\ln(1-x) = -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \dots\right)$

Notice that $V_1(\varphi_c)$ exhibits UV divergences that are canceled by counterterms $\frac{\delta m^2}{2}\varphi^2 + \frac{\delta\lambda}{4!}\varphi^4$ that must be added to V_0.

To fix the counterterms we can use the renormalization conditions (55) and (56). This procedure leads to, prove it,

$$V_{eff}(\varphi_c) = \frac{m^2}{2}\varphi^2 + \frac{\lambda}{4!}\varphi^4 + \frac{1}{64\pi^2}\left\{\left(\frac{\lambda\varphi_c^2}{2}+m^2\right)^2 \ln\left(\frac{\frac{\lambda\varphi_c^2}{2}+m^2}{m^2}\right)\right.$$

$$\left. -\lambda\frac{m^2}{2}\varphi_c^2 - \frac{3}{8}\lambda^2\varphi_c^4\right\} \qquad (70)$$

Exercise: obtain the above result.

10.h Saddle-point approximation

We can also obtain (69) using the saddle-point approximation for (4). In the limit that the parameter a ($\hbar$) in (64) goes to zero the functional integral is dominated by its saddle-point Φ_0

$$\left.\frac{\delta S[\varphi]}{\delta\varphi(x)}\right|_{\Phi_0} = -J(x) \qquad (71).$$

writing $\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - V(\varphi)$ (72)

(71) $\Rightarrow$

$$\partial_\mu\partial^\mu\Phi_0 + V'(\Phi_0) = J(x) \qquad (73)$$

So, we expand $S[\varphi]$ around ϕ_0:

$$\varphi = \phi_0 + \eta\,(\hbar) \qquad (74)$$

higher order corrections

and

$$S[\varphi] + \int d^4x\, J\varphi = S[\phi_0] + \int d^4x\, J(x)\phi_0 + \frac{1}{2}\int d^4x\, d^4y\, \frac{\delta^2 S[\varphi]}{\delta\varphi(x)\,\delta\varphi(y)}\,\eta(x)\eta(y) + R_2 \qquad (75)$$

Neglecting the higher order corrections R_2 we have from (4) after

$$D\varphi \to D\eta$$

$$Z[J] = \exp\left\{\frac{i}{\hbar}S[\phi_0] + \frac{i}{\hbar}\int d^4x\, J(x)\phi_0(x)\right\}\int D\eta\; e^{\frac{i}{\hbar}\int d^4x\left[\frac{1}{2}\partial_\mu\eta\,\partial^\mu\eta - \frac{1}{2}\eta^2 V''(\phi_0)\right]} \qquad (76)$$

(76) contains a Gaussian integral that we know how to do (!!)

$$Z[J] = \exp\left\{iS[\phi_0] + i\int d^4x\, J\phi_0\right\}\left[\det\left(\partial^\mu\partial_\mu + V''(\phi_0)\right)\right]^{-1/2} \qquad (77)$$

Using that

$$\det \overset{\text{operator}}{A} = \exp\left[\mathrm{Tr}(\ln A)\right] \qquad (78)$$

we have

$$Z[J] = e^{iW[J]} = \exp\left\{iS[\phi_0] + i\int d^4x\, J\phi_0 - \frac{1}{2}\mathrm{Tr}\left[\ln\left(\partial_\mu\partial^\mu + V''(\phi_0)\right)\right]\right\}. \qquad (79)$$

$$\Rightarrow W[J] = S[\phi_0] + \int d^4x\, J\phi_0 + \frac{i\hbar}{2}\mathrm{Tr}\left[\ln\left(\partial_\mu\partial^\mu + V''(\phi_0)\right)\right] \qquad (80)$$

To obtain the effective action

$$\phi_c = \frac{\delta W[J]}{\delta J} \Rightarrow \phi_c = \phi_0 + \underbrace{\frac{\delta\left[S[\phi_0] + \int d^4x\, J\phi_0\right]}{\delta\phi_0}}_{O(\hbar)}\frac{\delta\phi_0}{\delta J} + \left(\begin{array}{l}\text{higher order}\\ \text{in } \hbar\end{array}\right. \equiv \phi_1$$

Now

$$S[\phi_0] = S[\phi_c - \phi_1 \hbar] = S[\phi_c] - \int d^4x \, \underbrace{\frac{\delta S}{\delta \varphi}\bigg|_{\phi_0}}_{-J} \phi_1 + \mathcal{O}(\hbar^2) \qquad (82)$$

$\phi_0 \to \phi_c$ to order $\hbar$

$$= S[\phi_c] + \int d^4x \, \phi_1 J \qquad (83)$$

Using (80), (81) and (83)

$$\Gamma[\phi_c] = S[\phi_c] + \frac{i}{2}\hbar \, \mathrm{Tr}\left[\ln\left(\partial_\mu \partial^\mu + V''(\phi_c)\right)\right] + \mathcal{O}(\hbar^2) \qquad (84)$$

For $\phi_c =$ constant, we obtain the effective potential. For that we need

$$\mathrm{Tr}\left(\ln\left(\partial_\mu\partial^\mu + V''(\phi_c)\right)\right) = \int d^4x \, \langle x | \ln\left(\partial_\mu \partial^\mu + V''(\phi_c)\right) | x \rangle$$

$$= \int d^4x \int d^4p \int d^4k \, \langle x | p \rangle \langle p | \ln\left(\overbrace{\partial_\mu \partial^\mu}^{-k^2} + V''(\phi_c)\right) | k \rangle \langle k | x \rangle$$

and $\langle k | p \rangle = \delta^4(p-k) \Longrightarrow$

$$\Longrightarrow \mathrm{Tr}\left[\ln\left(\partial_\mu\partial^\mu + V''(\phi_c)\right)\right] = \int d^4x \int \frac{d^4k}{(2\pi)^4} \ln\left(-k^2 + V''(\phi_c)\right) \qquad (85)$$

$$\Rightarrow V_{eff}^{(\phi)} = V(\phi_c) - \frac{i}{2} \int \frac{d^4k}{(2\pi)^4} \ln\left(-k^2 + V''(\phi_c)\right) \qquad (86)$$

In order for the argument of the ln in (86) to be dimensionless we subtract the constant

$$+\frac{i}{2}\int\frac{d^4k}{(2\pi)^4}\ln(-k^2)$$

leading to

$$V_{eff}(\phi_c) = V(\phi_c) - \frac{i}{2}\int\frac{d^4k}{(2\pi)^4}\ln\left(\frac{-k^2+V''(\phi_c)}{-k^2}\right) \qquad (87)$$

Notice that (87) reproduces (69)!

10.i Bonus: Alternative proof of (7-8) but General!

Consider a fictitious model whose action is $\Gamma[\phi]$:

$$e^{iU[J,a]} = \int D\phi\; e^{\frac{i}{a}\{\Gamma(\phi)+\int d^4x\, J(x)\phi(x)\}}. \qquad (88)$$

In the limit $a \to 0$ the integral is dominated by its saddle point ϕ_c:

$$\left.\frac{\delta\Gamma}{\delta\phi}\right|_{\phi_c} + J = 0 \qquad (99)$$

Therefore, in this limit (ie $a\to 0$)

$$a\,U[J,a] = \Gamma(\phi_c) + \int d^4x\, J(x)\phi_c(x) \qquad (100)$$

Comparing (10) and (100) we have

$$\lim_{a\to 0} a\,U[J,a] = W[J] \qquad (101)$$

The right side of (101) is the sum of connected Feynman diagrams of the original theory. The left side is the sum of tree diagrams whose vertices are Γ^{n} $(n>2)$ ~~and~~ with the propagator that is $(i\,\Gamma^{(2)})^{-1}$!
So it's exactly what we want to prove!!

References:

- Section 10.a: Pokorski (Gauge Field Theories) section 2.6
  C. Nash (Relativistic Quantum Fields) chapter 1 p. 5
- Section 10.c: Pokorski section 5.1
- Section 10.d: Nash chapter 1 page 55
  Kaku (Quantum Field Theory) section 8.7
- Sections 10.e,f,g: Pokorski section ~~[illegible]~~ 2.6
  Coleman (Aspects of Symmetry) chapter 5 sections 3.3, 3.4, 3.5, 3.7
- Section 10.h. K. Huang (Quantum field theory) section 15.8
- Section 10.1 Huang section 15.7