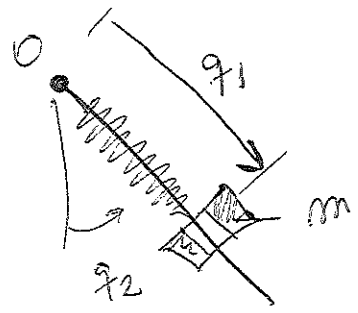


$$\vec{v}_m = \vec{v}_0 + \vec{\omega} \times \vec{r}_{M/O} + \vec{v}_{M/O}$$

$$\vec{v}_m = 0 + \dot{q}_2 \hat{k} \times \left(q_1 \sin q_2 \hat{i} - q_1 \cos q_2 \hat{j} \right) + \dot{q}_1 \sin q_2 \hat{i} - \dot{q}_1 \cos q_2 \hat{j}$$



$$\vec{v}_m = \dot{q}_2 q_1 \sin q_2 \hat{j} + \dot{q}_2 q_1 \cos q_2 \hat{i} \ldots$$



$$\vec{v}_m = \left(\dot{q}_2 q_1 \sin q_2 - \dot{q}_1 \cos q_2 \right) \hat{j} + \left(\dot{q}_2 q_1 \cos q_2 + \dot{q}_1 \sin q_2 \right) \hat{i}$$

$$v_m^2 = \dot{q}_2^2 q_1^2 \sin^2 q_2 - 2 q_1 \dot{q}_2 \dot{q}_1 \sin q_2 \cos q_2 + \dot{q}_1^2 \cos^2 q_2 + \dot{q}_2^2 q_1^2 \cos^2 q_2 + 2 q_1 \dot{q}_2 \dot{q}_1 \sin q_2 \cos q_2 + \dot{q}_1^2 \sin^2 q_2 =$$

$$= \dot{q}_2^2 q_1 + \dot{q}_1^2$$

$$T = \left(\dot{q}_2^2 q_1^2 + \dot{q}_1^2 \right) \frac{m}{2} \qquad V = \frac{1}{2} k q_1^2 - m g \cos q_2 q_1$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_1} \right) - \frac{\partial T}{\partial q_1} + \frac{\partial V}{\partial q_1} = 0$$

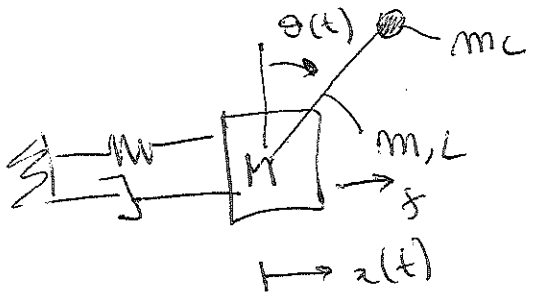
$$\frac{d}{dt}\left(m \dot{q}_1 \right) - \left(\frac{m}{2} 2 \dot{q}_2^2 q_1 \right) + k q_1 - m g \cos q_2 = 0$$

$$\boxed{m \ddot{q}_1 - m q_1 \dot{q}_2^2 - m g \cos q_2 + k q_1 = 0}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_2} \right) - \frac{\partial T}{\partial q_2} + \frac{\partial V}{\partial q_2} = 0$$

$$\frac{d}{dt}\left(\frac{m}{2} 2 \dot{q}_2 q_1^2 \right) - \cancel{\left(\frac{m}{2} 2 \dot{q}_2 \dot{q}_1 \right)} + m g \sin q_2 q_1 = 0$$

$$\boxed{m \ddot{q}_2 q_1^2 + m \dot{q}_2 2 q_1 \dot{q}_1 - m \cancel{\dot{q}_2} \dot{q}_1 + m g q_1 \sin q_2 = 0}$$



2 GDL

2 coord gen: x e θ

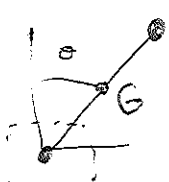
Energia cinética

M — Translação pto material $T = \frac{1}{2} M \dot{x}^2$

Translação + rotação corpo $T = \frac{1}{2} I_G \dot{\theta}^2 + \frac{1}{2} m v_G^2$

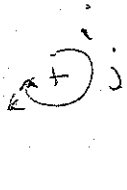
Translação pto mat $T = \frac{1}{2} m v_c^2$

v_G^2 e v_c^2 ?



mesmo corpo

$\vec{v}_c = \vec{v}_P + \vec{\omega} \times \vec{r}$



$= \dot{x}\,\hat{i} + (-\dot{\theta}\,\hat{k}) \times (L \sin\theta\,\hat{i} + L\cos\theta\,\hat{j})$

$= (\dot{x} + L\dot{\theta}\cos\theta)\,\hat{i} - L\dot{\theta}\sin\theta\,\hat{j}$

$v_c^2 = \dot{x}^2 + 2\dot{x}\dot{\theta}L\cos\theta + \dot{\theta}^2 L^2 \cos^2\theta + L^2\dot{\theta}^2\sin^2\theta$

$\dot{x}^2 + 2\dot{x}\dot{\theta}L\cos\theta + \dot{\theta}^2 L^2$

similar $\left(\frac{L}{2}\right)$ $v_\theta^2 = \dot{x}^2 + \dot{x}\dot{\theta}L\cos\theta + \left(\frac{L}{2}\right)^2\dot{\theta}^2$

$T = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}mc\left(\dot{x}^2 + 2L\dot{\theta}\dot{x}\cos\theta + \dot{\theta}^2L^2\right) + \frac{1}{2}m\left(\dot{x}^2 \cdots\right) + \frac{1}{2}I_G\dot{\theta}^2$

Energia potencial

$V = V_e + V_g = \frac{1}{2}kx^2 + mg\frac{L}{2}\cos\theta + mcgL\cos\theta$

Forças não conservativas $Q_{NC_x} = f(t) - b\dot{x}$ $\quad \delta W_{NC} = \underbrace{(f(t) - b\dot{x})}_{Q_{NC_x}}\delta x$

(x) $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) - \frac{\partial T}{\partial x} + \frac{\partial V}{\partial x} = Q_{NCx}$

$\frac{d}{dt}\left(\frac{1}{2}M2\dot{x} + \frac{1}{2}m_c 2\dot{x} + \frac{1}{2}m_c 2L\dot{\theta}\cos\theta + \frac{1}{2}m2\dot{x} + \frac{1}{2}mL\dot{\theta}\cos\theta\right) - \dots$

$\dots 0 + kx = f - b\dot{x}$

$\underbrace{(M + m_c + m)}_{m_t}\ddot{x} + \underbrace{\left(\frac{m}{2} + m_c\right)}_{m_o} L\cos\theta\,\ddot{\theta} - \left(\frac{m}{2} + m_c\right)L\dot{\theta}^2\sin\theta + b\dot{x} + kx = f$

$\cos\theta \approx 1$
$\sin\theta = \theta$
$\dot{\theta}^2 \Rightarrow 0$

$$\boxed{m_t\ddot{x} + m_o L\ddot{\theta} + b\dot{x} + kx = f}$$

(θ) $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} + \frac{\partial V}{\partial \theta} = Q_{NC\theta}$

$\frac{d}{dt}\left(\frac{1}{2}m_c 2L\dot{x}\cos\theta + \frac{1}{2}m_c 2\dot{\theta}L^2 + \frac{1}{2}m\dot{x}L\cos\theta + \frac{1}{2}m\left(\frac{L}{2}\right)^2 2\dot{\theta} + \frac{1}{2}I_G 2\dot{\theta}\right)$

$-\left(-\frac{1}{2}m_c 2L\dot{\theta}\dot{x}\sin\theta - \frac{1}{2}m\dot{x}\dot{\theta}L\sin\theta\right) + \left(-mg\frac{L}{2}\sin\theta - m_c gL\sin\theta\right) = 0$

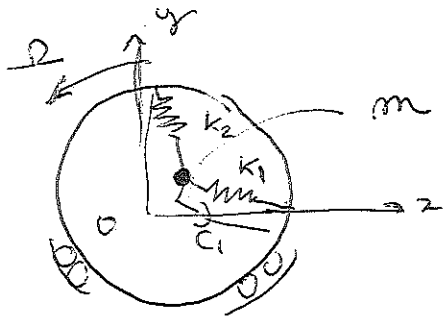
$\frac{d}{dt}\left(\left(m_c + \frac{m}{2}\right)L\dot{x}\cos\theta + \left(m_c L^2 + m\left(\frac{L}{2}\right)^2\right)\dot{\theta} + \frac{mL^2}{12}\dot{\theta}\right)$ (I_G)

$+\left(m_c + \frac{m}{2}\right)\dot{x}\dot{\theta}L\sin\theta - \left(\frac{m}{2}g + m_c\right)gL\sin\theta = 0$

$m_o L\ddot{x}\cos\theta - m_o L\dot{x}\sin\theta\dot{\theta} + I_p\ddot{\theta} + m_o L\dot{x}\sin\theta\dot{\theta} - m_o L\sin\theta = 0$

$I_p\ddot{\theta} + m_o L\ddot{x}\cos\theta - m_o gL\sin\theta = 0$

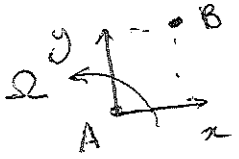
$$\boxed{I_o\ddot{\theta} + m_o L\ddot{x} - m_o gL\theta = 0}$$



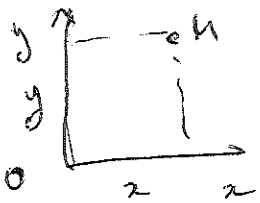
a massa m pode se mover em x e y : 2 GDL.

p/ usar o Lagrange o gente precisa escrever T que depende da velocidade de m.

ref em rotação : pontos que não estão conectados!



$$\vec{v}_B = \vec{v}_A + \vec{\omega} \times \vec{r}_{B/A} + \vec{v}_{B/A}$$

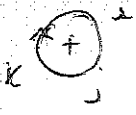


$$\vec{v}_{M/O} = \dot{x}\hat{i} + \dot{y}\hat{j}$$

$$\vec{v}_O = 0$$

$$\vec{\omega} \times \vec{r}_{M/O} = \Omega \hat{k} \times (x\hat{i} + y\hat{j})$$

$$\vec{v}_M = \vec{v}_O + \vec{\omega} \times \vec{r}_{M/O} + \vec{v}_{M/O}$$



$$= 0 + \Omega\hat{k} \times (x\hat{i} + y\hat{j}) + \dot{x}\hat{i} + \dot{y}\hat{j}$$

$$= \Omega x\hat{j} - \Omega y\hat{i} + \dot{x}\hat{i} + \dot{y}\hat{j}$$

$$= \underbrace{(\dot{x} - \Omega y)}_{v_{Mx}}\hat{i} + \underbrace{(\dot{y} + \Omega x)}_{v_{My}}\hat{j}$$

$$v_M^2 = v_{Mx}^2 + v_{My}^2 = \dot{x}^2 - 2\dot{x}\Omega y + \Omega^2 y^2 + \dot{y}^2 + 2\dot{y}\Omega x + \Omega^2 x^2$$

$$T = \frac{1}{2} m v_M^2 = \frac{1}{2} m (\dots)$$

$V = \frac{1}{2} k_1 x^2 + \frac{1}{2} k_2 y^2$ $\qquad Q_{nc_x} = -c_1 \dot{x}$

$Q_{nc_y} = 0$

2 coord gen : x, y $\qquad L = T - V$

$$L = \frac{1}{2} m \left(\dot{x}^2 - 2\dot{x}\Omega y + \Omega^2 y^2 + \dot{y}^2 + 2\dot{y}\Omega x + \Omega^2 x^2 \right) - \frac{1}{2} k_1 x^2 - \frac{1}{2} k_2 y^2$$

(x) $\quad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) - \frac{\partial T}{\partial x} + \frac{\partial V}{\partial x} = Q_{nc_x}$

$$\frac{d}{dt}\left(\frac{1}{2} m 2\dot{x} - \frac{1}{2} m 2 \Omega y \right) - \left(\frac{1}{2} m 2\dot{y}\Omega + \frac{1}{2} m 2 \Omega^2 x \right) + \frac{1}{2} k_1 2x = -c_1 \dot{x}$$

$$m\ddot{x} - m\Omega\dot{y} - m\Omega\dot{y} - m\Omega^2 x + k_1 x + c_1 \dot{x} = 0$$

$$\boxed{m\ddot{x} - 2m\Omega\dot{y} + c_1\dot{x} + k_1 x - m\Omega^2 x = 0}$$

(y) $\quad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{y}}\right) - \frac{\partial T}{\partial y} + \frac{\partial V}{\partial y} = Q_{nc_y}$

$$\frac{d}{dt}\left(\frac{1}{2} m 2\dot{y} + \frac{1}{2} m 2 \Omega x \right) - \left(\frac{1}{2} m 2\dot{x}\Omega + \frac{1}{2} m\Omega^2 2y \right) + \frac{1}{2} k_2 2y = 0$$

$$m\ddot{y} + m\Omega\dot{x} + m\Omega\dot{x} - m\Omega^2 y + k_2 y = 0$$

$$\boxed{m\ddot{y} + 2m\Omega\dot{x} + k_2 y - m\Omega^2 y = 0}$$

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{x} \\ \ddot{y} \end{Bmatrix} + \begin{bmatrix} c_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} \dot{x} \\ \dot{y} \end{Bmatrix} + \begin{bmatrix} 0 & -2m\Omega \\ 2m\Omega & 0 \end{bmatrix} \begin{Bmatrix} \dot{x} \\ \dot{y} \end{Bmatrix} + \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix} + \begin{bmatrix} -\Omega^2 m & 0 \\ 0 & -\Omega^2 m \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix} = 0$$

$$M\ddot{q} + (G+C)\dot{q} + (K - \Omega^2 H)q = 0$$

G: forças giroscópicas