14. Renormalization III: further details

13/08/19

14.1 Going beyond one loop

Let's use QED as an example ($q=-1$)

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\Psi}(i\not{\partial} - m_R)\Psi + e_R\bar{\Psi}\not{A}\Psi - \frac{1}{2a}(\partial_\mu A^\mu)^2 \qquad (14.1)$$

$$-\frac{1}{4}(Z_3-1)F_{\mu\nu}F^{\mu\nu} + (Z_2-1)\bar{\Psi}i\not{\partial}\Psi - m_R(Z_0-1)\bar{\Psi}\Psi + e_R(Z_1-1)\bar{\Psi}\not{A}\Psi$$

In the on-mass-shell renormalization scheme, the Ward identity leads to $Z_1 = Z_2$. The counterterms $\delta Z_2 = (Z_2-1)$, $\delta Z_0 = Z_0 - 1$ and $\delta Z_3 = Z_3 - 1$ are written as a power series on e_R:

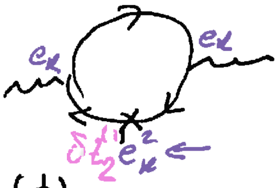
$$\delta Z = \sum_{j=2}^{\infty} \delta Z_{(j)}\, e_R^{j} \qquad (14.2)$$

In QFT1 evaluated δZ to order e_R^2. Let's consider the photon self-energy to 2-loops (e_R^4) that receives contributions from

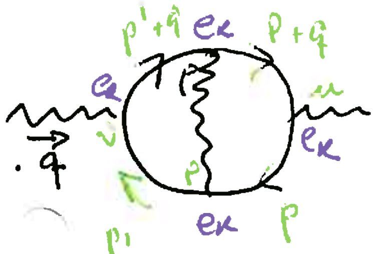
(a)

(b)

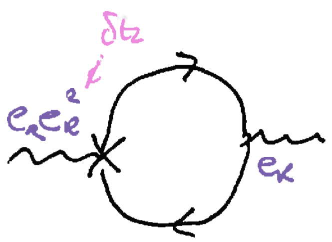
(c)



(d)

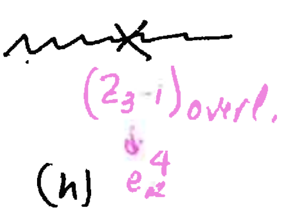


(e)



(f)

(g)



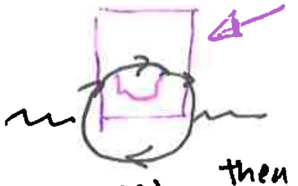
(h)

Notice that to this order the counter-term vertices appear in the loops!

In QED ~~all~~ these diagrams have a ~~logarithmic~~ superficial degree of divergence

$$D = 4 - \sum_p E_p(\ell_p + 1) = 2 \qquad \overset{2\gamma's = 0}{}$$

Moreover, there are ~~degen~~ divergent sub-diagrams like ~~as~~ (a)



(a)

. This is rendered finite by

(b)

then the renormalization of the whole diagrams follow the usual path.

~~The counter-term at 2 loop leaves this subdiagram~~ The control of divergences in sub-diagrams is not always clear.

Notice that there ~~are~~ is also an overlaping divergence in diagram (e); that we sum with the contribution from f→h.

(e)

$$\Pi^{\text{overlap.}}_{\mu\nu}(q) = -\frac{e_R^4}{(2\pi)^8}\int d^4_p \int d^4_{p'} \frac{1}{(p-p')^2 + i\epsilon} \mathrm{Tr}\Big[S_F(p')\gamma_\nu S_F(p'+q)\gamma^\rho S_F(p+q)\gamma_\mu S_F(p)\gamma$$

(f+g)

$$- 2\,\delta Z_2^{(1)} \frac{i e_R^2}{(2\pi)^4}\int d^4_p\, \mathrm{Tr}\left[\gamma_\nu S_F(p+q)\gamma_\mu S_F(p)\right] \tag{14.3}$$

$$- i\,\delta Z_3^{\text{ovul}}\,(q^2 g_{\mu\nu} - q_\mu q_\nu)$$

(h)

We can have two interpretations of diagram (e) :

(i) it's an insertion of a vertex correction given by $\int dp'$ in a photon self-energy diagram $\int dp$

(ii) vice-versa $(\int dp' \Leftrightarrow \int dp)$

However, it is (i) <u>OR</u> (ii), not both

et's analyze $\Pi_{\mu\nu}^{\text{overlap}}$ in more detail. From the vertex renormali- $\boxed{14.3}$
zation at one loop we know that

$$\left[\delta Z_2^{(1)} + R_2\right]\gamma_\mu = + \frac{i e_R^4}{(2\pi)^4} \int d^4p' \, \frac{1}{p'^2 + i\epsilon} \, \gamma_\rho S_F(p') \gamma_\mu S_F(p') \gamma^\rho \quad (14.4$$

$\hookrightarrow$ finite part

coming from  +  .

Substituting (14.4) into (14.3) we obtain

$$\Pi_{\mu\nu}^{\text{overlap}}(q) = - \frac{e_R^4}{(2\pi)^8} \int d^4p \int d^4p' \Big\{ \frac{1}{(p-p')^2} \mathrm{Tr}\left[S_F(p') \gamma_\nu S_F(p'+q) \gamma^\rho S_F(p+q) \gamma_\mu S_F(p)\right.$$

$T_1 \nearrow$

$$T_2 \longrightarrow \quad - \frac{1}{p'^2} \mathrm{Tr}\left[S_F(p') \gamma_\nu S_F(p') \gamma^\rho S_F(p+q) \gamma_\mu S_F(p)\gamma\right.$$

$$T_3 \longrightarrow \quad - \frac{1}{p^2} \mathrm{Tr}\left[S_F(p') \gamma_\nu S_F(p'+q) \gamma^\rho S_F(p) \gamma_\mu S_F(p)\gamma\right.$$

$$T_4 \longrightarrow \quad - R \, \mathrm{Re} \, \frac{i e_R^2}{(2\pi)^4} \int d^4p \, \mathrm{Tr}\left[\gamma_\nu S(p+q) \gamma_\mu S(p)\right]$$

$$(14.5$$

$$T_5 \longrightarrow \quad - i \, (\cancel{Z}_2 \, \delta Z_3^{\text{overl}} (q^2 g_{\mu\nu} - q_\mu q_\nu))$$

Notice that : $\rightarrow \int d^4p' (T_1 + T_2)$ is finite since $S_F(p'+q) - S_F(p') \propto \frac{1}{p'^2 - m}$

$\rightarrow \int d^4p \, T_3$ is logarithmic divergent being a polynomial on q

$\rightarrow T_5$ cancels the divergence in T_3

In general theories with

$$\Delta_i = 4 - d_i - \sum_p n_{ip}(s_p + 1) \geq 0 \text{ are renormaligable}$$
$$D = 4 - \sum_p E_p(s_p + 1) - \sum_i N_i \Delta_i$$

and the divergences can be absorved in terms of renormalized masses, fields and couplings. A specific general prescription to eliminate ultraviolet divergences was proposed by Bogoliubov, and Parasiuk and also by Hepp. Zimmerman showed that this procedure eliminate all superficial divergences as well as the ones in subintegrations. The method is called BPHZ.

By symmetry the same holds for the integration $\int d^4 l$!

So, all subdiagrams are convergent!

On the other hand, the integration over p and p' simultanea
can be made finite by the counter-term $\delta Z_3^{ovl.}$ due to ~~the~~ its
superficial divergence degree!

Therefore, we have a finite contribution to the photon self-energy.

$\Longrightarrow$ see (14.4A)

Suggested reading: S. Weinberg, volume I, sections 12.1 and 12.

14.2 Other renormalization prescriptions

We have defined the on-~~shell~~ mass-shell renormalization scheme where we do perturbation theory using the physical parameters m_R, and e_R; see (14.1). However, this is not the only possibility since the main goal in renormalization is to deal with the divergences. Let's analyze the 1-loop electron self-energy:

$$\Rightarrow \Sigma = \frac{e^2 m}{16\pi^2}\left\{\frac{8}{\varepsilon} + 4\int_0^1 dx\, \ln\left(\frac{4\pi\mu^2}{x(x-1)p^2 + xm^2}\right) - 2 - 4\gamma_E\right\} \qquad (14.6)$$

$$+ \quad + \not{p}\,\frac{e^2}{16\pi^2}\left\{-\frac{2}{\varepsilon} - 2\int_0^1 dx\,(1-x)\left[\ln\left(\frac{4\pi\mu^2}{x(x-1)p^2 - xm^2}\right) - 1 + \gamma_E\right]\right\} - \delta Z_2 \not{p} + m\delta$$

In the minimum subtraction (MS) scheme the counterterms just cancel the divergences and have NO finite part. For instance, applying the MS prescription to (14.6) leads to

$$\delta Z_0^{MS} = -\frac{e^2}{2\pi^2}\frac{1}{\epsilon}$$

(14.7)

$$\delta Z_2^{MS} = -\frac{e^2}{8\pi^2}\frac{1}{\epsilon}$$

Notice that with this choice the renormalized mass m is not the physical mass. So, we still have to express the physical mass (pole in the propagator) with the renormalized mass parameter m.

A variant of the $\overline{MS}$ scheme is that counterterms hav a finite piece to cancel the factor $\ln 4\pi$ and γ_E! For instance, for (14.6)

$$\delta Z_0^{\overline{MS}} = -\frac{e^2}{2\pi^2}\frac{1}{\epsilon} + \frac{e^2}{4\pi^2}\left(\gamma_E - \ln 4\pi\right)$$

(14.8)

$$\delta Z_2^{\overline{MS}} = +\frac{e^2}{16\pi^2}\left[\frac{-2}{\epsilon} - \ln 4\pi + \gamma_E\right]$$

Why do we introduce these prescriptions?

- It's simple
- We might not be able to have a particle on the mass shell, e.g., $m=0$ (infrared divergence) or confinement.
- the scheme is mass independent, ie, these renormaliza scheme does not introduce a mass scale.

14.3 Infrared "Divergences"

16/08/19

Renormalization deals with the ultraviolet (UV) (short distance) behavior of the model. However, we already encountered infrared (IR) divergences in QED that are related to the long distance behavior of the model! At one loop we obtained (before introducing a photon mass!)

$$\delta Z_2 = \frac{\alpha}{4\pi}\Bigg\{-8\int_0^1 dx\,\frac{x-1}{x} + \left(-\frac{2}{\varepsilon} - 2\int_0^1 dx\,(1-x)\Big[\ln\left(\frac{4\pi\mu^2}{x^2m^2}\right)-1+\gamma_E\right)$$

IR divergences

$$-4\int_0^1 dx\,\frac{(1-x)^2}{x}\Bigg\} \qquad (14.9)$$

Introducing a photon mass m_γ we obtained

$$\delta Z_2 = \frac{\alpha}{4\pi}\left[-\frac{2}{\varepsilon}+\gamma_E-4+\ln\left(\frac{m^2}{4\pi\mu^2}\right)-2\ln\left(\frac{m_\gamma^2}{m^2}\right)\right] \qquad (14.10)$$

t this point we must understand how to cope with IR divergences! As general principle IR divergences (as the UV ones cancel only for physically observable quantities! In the case of IR ivergences, they cancel after cross sections involving different initial nd final states are combined!

1.3.9 Example of IR finite observable

Let's consider the productions

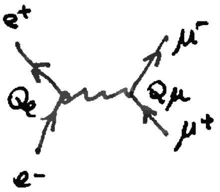
$$e^+e^- \to \mu^+\mu^- \quad \text{and} \quad e^+e^- \to \mu^+\mu^-\gamma$$

he total cross sections of the processes to order $\alpha^3 (e^6)$ is IR divergent, 14.7

ut their sum is finite! Let's describe the elements of the calculation. We assume that the charge of the electron (muon) is Q_e (Q_μ) and that those are free parameters.

i) Process $e^+e^- \to \mu^+\mu^-$

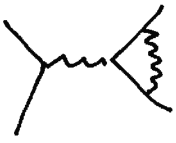
At tree level (e_R^4)

$M_0 =$ 

$\longrightarrow$ finite and $\mathcal{O}(Q_e Q_\mu e_R^2)$

The one-loop contribution is

$M_1 =$ (



IR divergent $\mathcal{O}(Q_e Q_\mu^3 e_R^4)$

+

) $\to M_\Gamma$

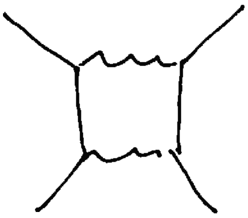
+

(

+

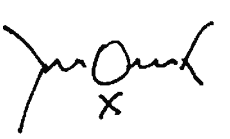
) IR divergent $\mathcal{O}(Q_e^3 Q_\mu e_R^4)$

+



+ ... $\mathcal{O}(Q_e^2 Q_\mu^2 e_R^4)$

+



$(Q_e Q_\mu Q_X^2 e_R^4)$

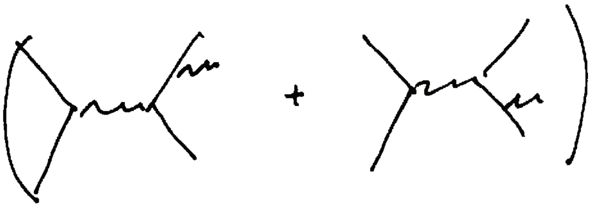
To order α^3 the amplitude for this process is

neglect

$$|M|^2 = \underset{\mathcal{O}(\alpha^3)}{\underset{\uparrow}{|M_0+M_1|^2}} = \underset{\mathcal{O}(\alpha^2)}{\underset{\uparrow}{|M_0|^2}} + \underset{\mathcal{O}(\alpha^3)}{\underset{\uparrow}{2\,\mathrm{Re}\,M_0^* M_1}} + \underset{\mathcal{O}(\alpha^4)}{\underset{\uparrow}{\cancel{|M_1|^2}}} \qquad (14.11)$$

ii) Process $e^+e^- \to \mu^+\mu^-\gamma$

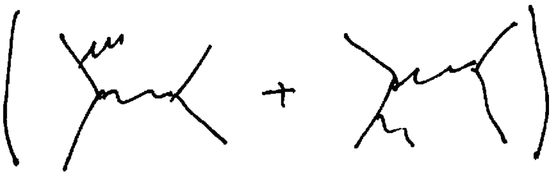
$M =$



$\mathcal{O}(Q_e Q_\mu^2 e_R^3)$

to (14.8A)

$+$



$\mathcal{O}(Q_e^2 Q_\mu e_R^3)$

so $|M|^2$ is of order α^3 and it is IR divergent!

We will focus on $\mathcal{O}(Q_e^2 Q_\mu^4 \alpha^3)$ and show that ~~they are~~ their total cross section is IR finite!

The same happens for other combinations of Q_e and Q_μ!

We will use dimensional regularization to regulate the IR and UV divergences!

(14.8A

$P_e^2 = m_e^2 \quad P_\gamma^2 = 0$

$$\frac{i\,(\cancel{P}_e + \cancel{P}_\gamma + m_e)}{(P_e + P_\gamma)^2 - m_e^2} \downarrow = \frac{i\,(\cancel{P}_e + \cancel{P}_\gamma + m_e)}{2\,P_e \cdot P_\gamma}$$

if $\quad P_\gamma = E_\gamma\,(1, 0, 0, 1)$

$P_e = E_e\,(1, \beta \sin\theta, 0, \beta\cos\theta) \qquad \beta =$ electron velocity

$$P_e \cdot P_\gamma = E_e E_\gamma\,(1 - \beta\cos\theta)$$

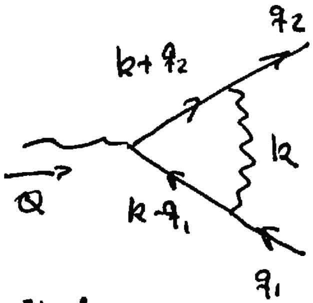
Divergences:

i) $E_\gamma \to 0 \qquad$ soft divergence

ii) $\left.\begin{array}{l} \theta \to 0 \\ \text{for} \\ \beta = 1 \\ \downarrow \\ (m_e = 0) \end{array}\right\} \to$ colinear divergence

14.3.b Vertex Correction

We have to evaluate



$$+ie_K\mu^{\frac{4-d}{2}}\,\bar u(q_2)\,\Gamma_2^\mu\,v(q_1) = +\left(e_K\mu^{\frac{4-d}{2}}\right)^3\int\frac{d^dk}{(2\pi)^d}\,\frac{\bar u(q_2)\gamma^\nu(\not k+\not q_2)\gamma^\mu(\not k-\not q_1)\gamma_\nu v(q_1)}{[(k+q_2)^2+i\epsilon]\,[(k-q_1)^2+i\epsilon]\,(k^2+i\epsilon)} \qquad (14.12)$$

where we set $m=0$. Now $\not q_1 v(q_1) = 0 = \bar u_{q_2}(\not q_2)$ and $Q^2 = 2q_1\cdot q_2$

Now analogously to what we have done in QFT1 we obtain after $k^\mu \to k^\mu - xq_2^\mu + yq_1^\mu$

$$\Gamma_2^\mu = -2i\gamma^\mu e_K^2\mu^{4-d}\int_0^1 dx\int_0^{1-x}dy\,\frac{d^4k}{(2\pi)^d}\,\frac{\frac{(d-2)^2}{d}k^2 + Q^2\big((2-d)xy+2x+2y-2\big)}{(k^2+Q^2xy+i\epsilon)^3} \qquad (14.13)$$

Notice that:
- k^2 term is UV divergent
- Q^2 term is IR divergent

explain!

Now using that $\displaystyle\int\frac{d^dk}{(2\pi)^d}\frac{(k^2)^\alpha}{(k^2-\Delta+i\epsilon)^n} = i(-1)^{\alpha-n}\Delta^{\alpha-n+\frac{d}{2}}\frac{\Gamma(n-\alpha-\frac{d}{2})\Gamma(\frac{d}{2}+\alpha)}{\Gamma(n)\Gamma(\frac{d}{2})}$

(14.14)

We obtain that (for $d=4-\epsilon_{UV}$)

explain

$$\int_0^1 dx\int_0^{1-x}dy\int\frac{d^dk}{(2\pi)^d}\,\frac{\frac{(d-2)^2}{d}k^2}{(k^2-(-Q^2xy)+i\epsilon)^3} = \frac{i}{16\pi^2}\left(\frac{4\pi}{-Q^2}\right)^{\frac{4-d}{2}}\frac{\Gamma(\frac{4-d}{2})\,\Gamma(\frac{d}{2})^2}{\Gamma(d-1)}$$

$$= \frac{i}{16\pi^2}\left(\frac{4\pi}{-Q^2}\right)^{\frac{4-d}{2}}\left[\frac{4}{\epsilon_{UV}}-\frac{\gamma_E}{2}+\frac{1}{2}+\mathcal{O}(\epsilon_{UV})\right] \checkmark \qquad (14.15)$$

Notice that for the UV integral to converge we need $\epsilon_{UV} > 0$.

formal name

For the integral $\int \frac{d^dk}{k^6}$ to converge in the IR we need $d > 4$. We perform the integral over the Q^2 term writing $d = 4 - \epsilon_{IR}$ where we assume that $\epsilon_{IR} < 0$ for the integral to be finite. So, we have

$$\int_0^1 dx \int_0^{1-x} dy \int \frac{d^dk}{(2\pi)^d} \frac{Q^2((2-d)xy + 2x + 2y - 2)}{(k^2 + Q^2xy + i\epsilon)^3} =$$

$$= \frac{i}{16\pi^2}\left(\frac{4\pi}{-Q^2}\right)^{\frac{4-d}{2}} \frac{\Gamma(\frac{4-d}{2})\Gamma(\frac{d-4}{2})\Gamma(\frac{d}{2})}{\Gamma(d-2)}\left(\frac{d^2+8d+24}{4(d-2)}\right)$$

$$= \frac{i}{16\pi^2}\left(\frac{4\pi}{-Q^2}\right)^{\frac{4-d}{2}}\left(-\frac{4}{\epsilon_{IR}^2} + \frac{-4+2\gamma_E}{\epsilon_{IR}} + \frac{-54+24\gamma_E-6\gamma_E^2+\pi^2}{12} + \mathcal{O}(\epsilon_{IR})\right) \quad (14.16)$$

The $\frac{1}{\epsilon_{IR}^2}$ pole is associated to soft-colinear divergences!

To remove the UV divergences we must add

$$= +i\delta Z_1 \left(e_R \mu^{\frac{4-d}{2}}\right) \bar{u}(q_2)\gamma^\mu v(q_1) \quad (14.17)$$

To fix δZ_1 we impose that  +  must vanish for $Q \to 0$

$$\Rightarrow \delta Z_1 = i e_R^2 \mu^{4-d} \frac{(d-2)^2}{d} \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^4} \qquad (14.18)$$

In dimensional regularization $\int \frac{d^d k}{k^4} \equiv 0$! Let's analyze this more carefully. In the euclidean space

$$\int \frac{d^d k_E}{k_E^4} = \Omega_d \int_0^\infty dk_E \, k_E^{d-5} \qquad (14.19)$$

Where the integral over the angular variables ~~yield~~ yield $\Omega_d = \frac{2\pi^{d/2}}{\Gamma(\frac{d}{2})}$

Let's separate (14.19) in the UV and IR by introducing an arbitrary scale Λ

$$\int \frac{d^d k_E}{k_E^4} = \Omega_d \left[\int_0^\Lambda dk_E \, k_E^{\overset{\overset{4-\varepsilon_{\cancel{UV}}\ IR}{\uparrow}}{d-5}} + \int_\Lambda^\infty dk_E \, k_E^{\overset{4-\varepsilon_{UV}}{d-5}} \right]$$

$$= \Omega_d \left(\ln \Lambda - \frac{1}{\varepsilon_{IR}} \right) + \Omega_d \left(\frac{1}{\varepsilon_{UV}} - \ln \Lambda \right) \qquad (14.20)$$

If we set $\varepsilon_{IR} = \varepsilon_{UV} = \varepsilon$ we have that $\int \frac{d^d k_E}{k_E^4} \equiv 0$.

Back to (14.18)

$$\delta Z_1 = i e_R^2 \mu^{4-d} \frac{(d-2)^2}{d} \frac{1}{(2\pi)^d} \frac{2\pi^{d/2}}{\Gamma(\frac{d}{2})} \left(\frac{1}{\varepsilon_{UV}} - \frac{1}{\varepsilon_{IR}} \right) \quad (14.21)$$

Putting together (14.15), (14.16) and (14.17) we obtain that (we use $\varepsilon_{UV} = \varepsilon_{IR} = \varepsilon$)

$$\Gamma_2^{\mu} = \gamma^{\mu} F(Q)$$

$$= \gamma^{\mu} \left\{ -\frac{e_R^2}{2\pi^2} \left(\frac{4\pi e^{-\gamma_E} \mu^2}{-Q^2} \right)^{\frac{4-d}{2}} \left(\frac{1}{\varepsilon^2} + \frac{3}{4\varepsilon} + 1 - \frac{\pi^2}{48} + \mathcal{O}(\varepsilon) \right) \right\}$$

$$(14.22)$$

$$= \gamma^{\mu} \left\{ -\frac{e_R^2}{2\pi^2} \left(\frac{4\pi e^{-\gamma_E} \mu^2}{Q^2} \right)^{\frac{4-d}{2}} \left(\frac{1}{\varepsilon^2} + \frac{\frac{3}{4} + i\pi/2}{\varepsilon} - \frac{7\pi^2}{48} + 1 + \frac{3\pi i}{8} + \mathcal{O}(\varepsilon) \right) \right.$$

<u>omment</u>: after renormalization we are left with the IR divergences only.

Notice that $M_1 = F(Q) M_0$ in (14.11), $\Rightarrow |M|^2 = |M_0|^2 (1 + 2 \operatorname{Re} F(Q))$

$\uparrow$ $\mathcal{O}(\alpha^3 Q_e^2 Q_\mu^4)$

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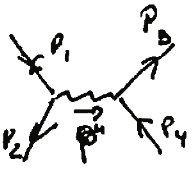
14.3.c Tree level processes in dimensional regularization (DR)

We know

$i M_0 =$ 

$$= \frac{i^{\mu^{2(d-4)}} e_R^2}{Q^2} \bar{u}(p_2) \gamma^{\mu} u(p_1) \bar{u}(p_3) \gamma_{\mu} v(p_4) \; (Q_e Q_\mu)$$

ith $Q^2 = (p_1+p_2)^2 = s$. We shall consider $Q^2 \gg m_e, m_\mu$. We have to evaluate this process n DR to keep the powers of ϵ since it's multiplied by M_1 at the order we e working!

We write

$$\sigma_0 = \frac{1}{2Q^2} \int d\phi_2 \, |M_0|^2 = \frac{e_R^4 \mu^{2(d-4)}}{2Q^6} L^{\mu\nu} X_{\mu\nu} \qquad (14.23)$$

$\uparrow$ phase space

where

14.13

$$-^{\mu\nu} = \frac{1}{4}\sum_{S1} \bar{v}(p_2)\gamma^\mu u(p_1)\,\bar{u}(p_1)\gamma^\nu v(p_2) = \frac{1}{4}\mathrm{Tr}[\not{p}_2\gamma^\mu\not{p}_1\gamma^\nu] = p_1^\mu p_2^\nu + p_1^\nu p_2^\mu - \frac{1}{2}Q^2 g^{\mu\nu}$$

ud

$$X^{\mu\nu} = \int d\phi_2 \sum_{\text{spi}} \bar{u}(p_3)\gamma^\mu v(p_4)\,\bar{v}(p_4)\gamma^\nu u(p_3) \qquad (14.24)$$

Since $X^{\mu\nu}$ originates from  $\xRightarrow{\text{ward}}$ $p_\mu X^{\mu\nu} = 0 \Rightarrow X^{\mu\nu} = (p^\mu p^\nu - p^2 g^{\mu\nu})X(p^2)$

(14.25)

~~So, we have~~

~~$g_{\mu\nu}X^{\mu\nu}$ $(p^2 - p^2 g_{\mu\nu}g^{\mu\nu}) = Q^2(1-d)$~~

~~$L^{\mu\nu}X_{\mu\nu} = Q^4 X(Q^2) = -\frac{Q^2}{(d-1)}g^{\mu\nu}X_{\mu\nu}$ (14.26)~~

~~'ow, (14.23) + (14.25) $\Rightarrow$ $\sigma_0 = \frac{e_R^4}{2Q^4(d-1)}g_{\mu\nu}X^{\mu\nu}$ (14.27)~~

Now $\quad X^{\mu\nu}L_{\mu\nu} = \left(p_1^\mu p_2^\nu + p_1^\nu p_2^\mu - \frac{Q^2}{2}g^{\mu\nu}\right)(p_\mu p_\nu - Q^2 g_{\mu\nu})$

$= p_1 + p_2 \oplus m_e = m_\mu = 0$

$$= \Big[2\underbrace{(p_1\cdot p_2)}_{Q^2/2}{}^2 - Q^2 2p_1\cdot p_2 - \frac{Q^4}{2} + \frac{Q^4 d}{2}\Big]X = \frac{(d-2)}{2}Q^2 X(Q^2) \quad (14.2$$

$$X(Q^2) = -\frac{1}{(d-1)Q^2}g_{\mu\nu}X^{\mu\nu}$$

$$= -\frac{1}{2}\frac{(d-2)}{(d-1)}Q^2 g_{\mu\nu}X^{\mu\nu}$$

:herefore,

$$\sigma_0 = -\frac{e_R^4}{4Q^4}\mu^{2(4-d)}\left(\frac{d-2}{d-1}\right)g_{\mu\nu}X^{\mu\nu} \qquad (14.27)$$

so all we need to know is $X^{\mu\nu}$. This is quite general!

in the case of Mo we know that

no average over spins

$$X^{\mu\nu} = \int \left(p_3^{\mu} p_4^{\nu} + p_3^{\nu} p_4^{\mu} - g^{\mu\nu} \underbrace{p_3 \cdot p_4}_{\frac{Q^2}{2}} \right) 4 \, d\Phi_2 \qquad (14.28)$$

$$\Rightarrow -g_{\mu\nu} X^{\mu\nu} = 2(d-2) Q^2 \int d\Phi_2 \qquad (14.29)$$

et's focus on $\int d\Phi_2$:

$$\int d\Phi_2 = \int \frac{d^{d-1} p_3}{(2\pi)^{d-1}} \frac{d^{d-1} p_4}{(2\pi)^{d-1}} \frac{1}{2E_3 \, 2E_4} \delta^{(d)}(p_3 + p_4 - P)(2\pi)^d \qquad (14.30)$$

ow $p_i = \frac{Q}{2} \hat{p}_i$ (dimensionless) and $x_i = \frac{2}{Q} E_i$. We evaluate $d^{d-1} p_4$ using the spatial part of the δ^d, obtaining

$$\int d\Phi_2 = (2\pi)^{2-d} \left(\frac{Q}{2}\right)^{d-2} \frac{1}{Q^2} \int \frac{d^{d-1}\hat{p}_3}{x_3 x_4} \delta(x_3 + x_4 - 2)$$

otice that x_4 depends on $\hat{p}_3$ since it is determined by momentum nservation! However, in the COM

$x_4 = |\vec{\hat{p}}_3| = x_3$ leading to

$$\int d\Phi_2 = \left(\frac{Q}{4\pi}\right)^{d-2} \frac{1}{Q^2} (2\pi)^{2-d} \int \frac{d\, x_3 \, x_3^{\cancel{x_3}\, d-2}}{x_3} \delta(2x_3 - 2) \int d\Omega_{d-1}$$

$$\frac{2\pi^{(d-1)/2}}{\Gamma\left(\frac{d-1}{2}\right)}$$

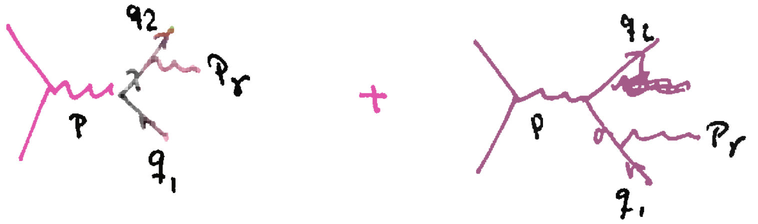
$$= \left(\frac{Q}{4\pi}\right)^{d-2} \frac{1}{2Q^2} \Omega_{d-1} = \left(\frac{4\pi}{Q^2}\right)^{\frac{\cancel{d-3}\; 4-d}{2}} \frac{2^{-d}}{\sqrt{\pi}\,\Gamma\left(\frac{d-1}{2}\right)} \qquad (14.31)$$

ually, (14.31) ⊕ (14.27) ⊕ (14.29)

$$\sigma_0^d(e^+e^-\to\mu^+\mu^-) = \sigma_0\,\mu^{2(4-d)}\left(\frac{4\pi}{Q^2}\right)^{\frac{4-d}{2}}\frac{3\sqrt{\pi}(d-2)^2}{2^d\,\Gamma\left(\frac{d+1}{2}\right)} \qquad (14.32)$$

with $$\sigma_0 = \sigma_0^{d=4} = \frac{e_R^4}{12\pi Q^2} \qquad (14.33)$$

Our job is not over, we still have to evaluate $e^+e^-\to\mu^+\mu^-\gamma$ to order $Q_0^2Q_\mu$



Notice that (14.23) is valid replacing $d\Phi_2$ with $d\Phi_3$ and

explain!!

$$X^{\mu\nu} \longrightarrow X^{\mu\nu} = -\mu^{4-d}\int d\Phi_3\ \mathrm{Tr}\left[\not q_1\, S^{\mu\alpha}\not q_2\, S^{\alpha\nu}\right] \qquad (14.34)$$

with $$S^{\mu\alpha} = +ie_R\left[\gamma^\alpha\frac{i}{\not q_2+\not p_\gamma}\gamma^\mu - \gamma^\mu\frac{i}{\not q_1+\not p_\gamma}\gamma^\alpha\right] \qquad (14.35)$$

Again, by the ward identity

$$X^{\mu\nu} = (p^\mu p^\nu - Q^2 g^{\mu\nu})\,X(Q^2)$$

Performing the trace and defining

$$x_i = \frac{2q_i\cdot p}{Q^2} \underset{\text{COM}}{\overset{\uparrow}{=}} \frac{2E_i}{Q} \qquad (14.36) \qquad i = 1,2,\gamma$$

e find that

$$-g_{\mu\nu}X^{\mu\nu} = \mu^{4-d}\,4e_R^2(d-2)\int d\Phi_3\,\frac{x_1^2+x_2^2+\frac{d-4}{2}x_\gamma^2}{(1-x_1)(1-x_2)} \qquad (14.37)$$

otice that ~~these~~ x_γ depends on one angle

$$x_\gamma = \frac{2E_\gamma}{Q} = \frac{2}{Q}\sqrt{(\vec{q}_1+\vec{q}_2)^2} = \frac{2}{Q}\sqrt{E_1^2+E_2^2-2E_1E_2\cos\theta} = \sqrt{x_1^2+x_2^2-2x_1x_2\cos\theta} \qquad (14.38)$$

o we have to write carefully the phase space $d\Phi_3$.

$$\int d\Phi_3 = (2\pi)^{3-2d}\int\frac{d^{d-1}q_1}{2E_1}\int\frac{d^{d-1}q_2}{2E_2}\int\frac{d^{d-1}p_\gamma}{2E_\gamma}\,\delta^{(d)}(q_1+q_2+p_\gamma-P) \qquad (14.39)$$

Like before (COM) we perform $\int d^{d-1}p_\gamma$ using the spatial part of the $\delta^{(d)}$

$q_i = \frac{Q}{2}\hat{q}_i$ $\qquad x_i = 2\frac{E_i}{Q} = |\hat{q}_i|$ $\qquad x_\gamma = \frac{2E_\gamma}{Q}$, leading to

$$\int d\Phi_3 = \left(\frac{Q}{4\pi}\right)^{2d-3}\frac{1}{Q^3}\int x_1^{d-2}dx_1\,d\Omega_{d-1}\int x_2^{d-2}dx_2\,d\Omega_{d-1}\int\frac{1}{x_1x_2x_3}\,\delta(x_1+x_2+x_3-2) \qquad (14.40)$$

Since the angle between $\vec{q}_1$ and $\vec{q}_2$ appears in x_γ we write for $\vec{q}_1$:

$$d\Omega_{d-1} = d\Omega_{d-2}\sin^{d-3}\theta\,d\theta \underset{\substack{\uparrow\\ z=\cos\theta}}{=} d\Omega_{d-2}\,(1-z^2)^{\frac{d-4}{2}}dz$$

he integrand in (14.37) depends only on x_i's and z, so.

$$\int d\Phi_3 = \left(\frac{Q}{4\pi}\right)^{2d-3} \frac{\Omega_{d-2}\,\Omega_{d-1}}{Q^3} \int dx_1\, x_1^{d-3} \int dx_2\, x_2^{d-3} \int_{-1}^{1} dz\, (1-z^2)^{\frac{d-4}{2}} \frac{1}{x_\gamma}\delta(x_1+x_2+x_\gamma$$

$$(14.41)$$

lotice that $z = \dfrac{x_1^2 + x_2^2 - x_\gamma^2}{2x_1x_2} \underset{\substack{\uparrow \\ \text{using} \\ x_1+x_2+x_\gamma = 2}}{\Longrightarrow} (1-z^2) = 4\,\dfrac{(1-x_1)(1-x_2)(1-x_\gamma)}{x_1^2x_2^2}$

allowing us to trade $dz \leftrightarrow dx_\gamma \Rightarrow$

$$d\Phi_3 = \frac{Q^2\left(\frac{Q^2}{4\pi}\right)^{d-4}}{128\pi^3\,\Gamma(d-2)} \int dx_1\,dx_2\,dx_\gamma\; \delta(x_1+x_2+x_\gamma-2)\left(\frac{1}{(1-x_1)(1-x_2)(1-x_\gamma)}\right)^{\frac{4-d}{2}}$$

$$= \left(\frac{Q^2}{4\pi}\right)^{d-4} \frac{Q^2}{128\pi^3\,\Gamma(d-2)} \int_0^1 dx_1 \int_{1-x_1}^{1} dx_2 \left[\frac{1}{(1-x_1)(1-x_2)(1-x_\gamma)}\right]^{\frac{4-d}{2}} \qquad (14.42)$$

with $x_\gamma = 2 - x_1 - x_2$

Now we can evaluate (14.37)

$$\int_0^1 dx_1 \int_{1-x_1}^{1} dx_2 \; \frac{4(d-2)\left(x_1^2 + x_2^2 + \frac{d-4}{2}x_\gamma^2\right)}{(1-x_1)^{\frac{3-d}{2}}\,(1-x_2)^{3-\frac{d}{2}}\,(1-x_\gamma)^{2-d/2}} = 4(d-3)(d^2-4d+8)\,\frac{\Gamma\left(\frac{d-4}{2}\right)^2 \Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{3d-6}{2}\right.}$$

$$= \frac{64}{\varepsilon^2} + \frac{16}{\epsilon} - 8\pi^2 + 52 + \mathcal{O}(\epsilon) \qquad (14.43)$$

substituting (14.43) + (14.37) into (14.27) we get

$${}^{d}_{RAD} = \sigma_0 e_R^2 \left(\frac{Q^2}{4\pi\mu^2}\right)^{d-4} \frac{3}{32\pi^2} \frac{(d-3)(d-2)(d^2-9d+8)}{d-1} \frac{\Gamma\left(\frac{d-4}{2}\right)^2 \Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{3d-6}{2}\right)\Gamma(d-2)}$$

(14.44)

$$= \sigma_0 \frac{e_R^2}{\pi^2} \left(\frac{4\pi e^{-\gamma_E}\mu^2}{Q^2}\right)^{4-d} \left(\frac{1}{\epsilon^2} + \frac{13}{12\epsilon} - \frac{5\pi^2}{24} + \frac{251}{144} + \mathcal{O}(\epsilon) \right)$$

14.3.C Adding $2\to3\big|_{tree} + 2\to2\big|_{tree+loop}$ to order e_R^6

The contribution from 1-loop diagrams at order e_R^6 takes place through the term

$$2\,\mathrm{Re}\, M_0^* M_1$$

However, the result (14.22) allows us to write

$$M_1 = F(Q)\, M_0$$

leading to

$$\sigma^d_{VIRTUAL} = \sigma_0^d\, 2\,\mathrm{Re}\, F(Q)$$

(14.45)

$$= -\sigma_0 \frac{e_R^2}{\pi^2} \left(\frac{4\pi e^{-\gamma_E}\mu^2}{Q^2}\right)^{4-d} \left(\frac{1}{\epsilon^2} + \frac{13}{12\epsilon} - \frac{5\pi^2}{24} + \frac{29}{8} + \mathcal{O}(\epsilon$$

Therefore, $\sigma^d_{Virtual} + \sigma^d_{RAD} = \sigma_0 \dfrac{3e_R^2}{16\pi^2}$ (14.46)

14.20.

14.47) is called an inclusive cross section since we sum over all possibility for the emitted photon. We can have a more exclusive cross section limiting the phase space for the photon.

In a real wolrd situation, a detector can not observe a photon if it is lose to the $\mu^\pm$ $(\theta < \theta_{res})$ and/or it is too soft $(E_\gamma < E_{res})$

We can separate ~~asituation~~ two cases:

i) the final state photon is observed $(E > E_{res}$ and/or $\theta_\gamma > \theta_{res})$ $[\sigma_{2\to3}$

ii) just a pair $\mu^+\mu^-$ is observed. that happens for the processes $[\sigma_{2\to2}]$

$$e^+e^- \to \mu^+\mu^-$$

$$e^+e^- \to \mu^+\mu^-\gamma \quad \text{with} \quad E_\gamma < E_{res} \text{ and/or } \theta < \theta_{res}$$

Notice that $\sigma_{2\to3}$ is IR finite by the requirements $E > E_{res}$ and $\theta_\gamma > \theta$ (hard photon and away from $\mu^\pm$). Therefore,

$$\sigma_{22} = \sigma_{tot} - \sigma_{23} \quad \text{is IR finite also!}$$

σ_{22} contain IR divergent contributions from loops and soft/colinear radiation!

It is possible to compute

$$\sigma_{2\to3} = \sigma_0 \frac{e^2}{8\pi^2}\left\{ \ln\frac{1}{\theta_{res}}\left[\ln\left(\frac{Q}{2E_{res}} - 1\right) - \frac{3}{4} + 3\frac{E_{res}}{Q}\right] + \frac{\pi^2}{12} - \frac{17}{16} - \frac{E_{res}}{Q} + \frac{3}{2}\left(\frac{E_{res}}{Q}\right)^2 + \dots \right\}$$

Therefore, the $Q_e^2 Q_\mu^4$ correction to the tree level $\sigma(e^+e^- \to \mu^+\mu^-)$ is finite! Finally, to order $e_R^6 Q_e^2 Q_\mu^4$

$$\sigma_{TOT} = \sigma_0 \left(1 + \underbrace{\frac{3 e_R^2}{16\pi^2}}_{\approx 1.7 \times 10^{-3}} \right) \qquad (14.47)$$

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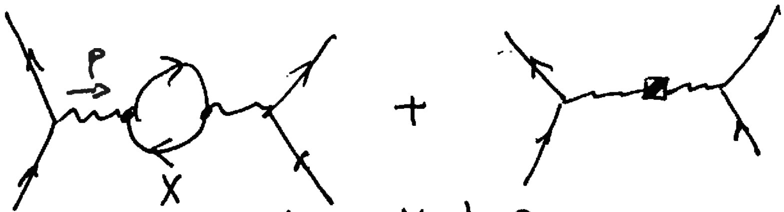
$\Longrightarrow$ to 14.20A

14.3.d Further corrections (from other loops!)

i) Vacuum Polarization Correction

We also have



that is of order $e_R^4 Q_e Q_\mu Q_x^2$

Logarithm in the loop

$p^2 = Q^2$

From TQC1, we know that $\Pi^{\mu\nu} = \Pi(Q^2)\left[Q^2 g^{\mu\nu} - p^\mu p^\nu\right]$

and we have evaluated $\Pi(Q^2)$, that for $Q^2 >> m_x$

$$\Pi(Q^2) = \frac{e_R^2}{12\pi^2} \ln\left(\frac{m_x^2}{-Q^2}\right) + \text{regular terms as } \frac{m}{Q} \to 0 \qquad (14.48)$$

Adding this contribution to the tree level result corresponds to substitut the charge e_R by e_{ff} given by

$$e^2_{eff}(Q^2) = e_R^2 \left[1 + \frac{e_R^2}{12\pi^2} \ln\left(\frac{-Q^2}{m_x^2}\right) \right] \qquad (14.49)$$

Apparently, (14.48) is infrared divergent for $m_\gamma \to 0$. However,

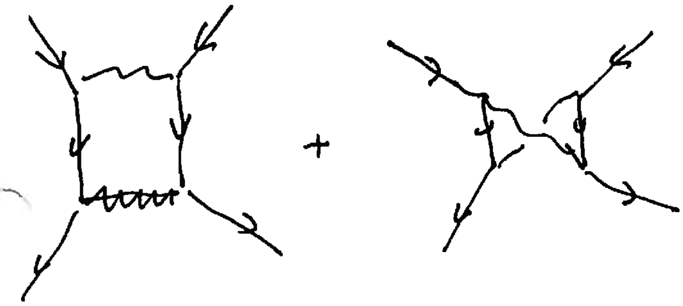
the electric charge $\overset{\text{must be}}{\cancel{\text{to}}}$ defined by a measurement at $\overset{an}{a}$ scale $(-Q_2^2)$

and (14.49) implies that

$$e_{eff}^2(-Q_1)^2 = e_{eff}^2(-Q_2^2) + \frac{e_R^4}{12\pi^2}\ln\left(\frac{-Q_1^2}{-Q_2^2}\right) \qquad (14.50)$$

ii) Box diagrams

This process $e^-e^+\to\mu^+\mu^-$ is also modified by



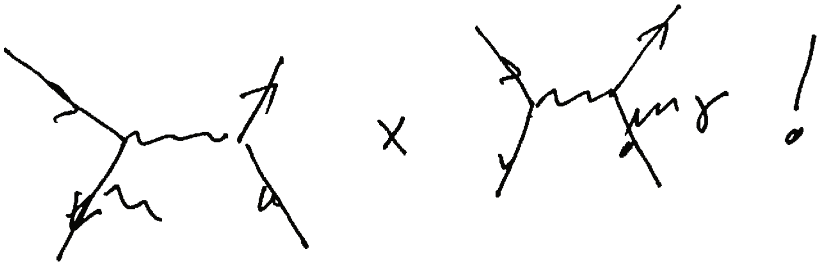
that is order $e_R^4 Q_e^2 Q_\mu^2$

This process is UV finite since it's proportional to

$$\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2\,k^2\,\not k\,\not k} \sim \int\frac{d^4k}{k^6}$$

However, it is IR divergent with its divergence being cancele

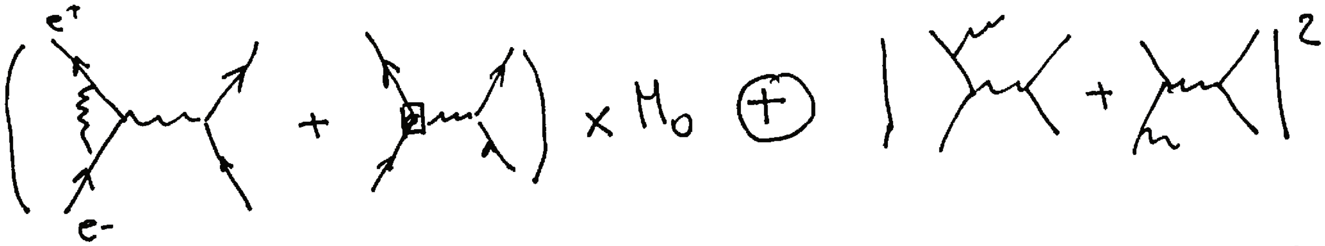
by the interference term



$$\int d^4k\,\frac{1}{k^2}\frac{1}{k^2}\frac{1}{k\cdot p}$$

iii) Initial state corrections

Let's consider the contributions ~~[illegible]~~ of order $Q_e^{4} Q_\mu^2 e_R^{6}$



This is completely analogous to wht we analyzed before $(Q_e^2 Q_\mu^4 e_R^6)$ when we integrate over all possible photon final states, i.e, the IR singularities ~~[illegible]~~ cancel out. However, the IR divergent ~~[illegible]~~ region is different and the physical interpretation is very different!

The initial state IR radiation originates from the propagator

$$\frac{1}{(P_e-P_\gamma)^2-m_e^2} \overset{m_e=0}{=} \frac{-1}{2P_e\cdot P_\gamma} = \frac{1}{2E_eE_\gamma(1-\cos\theta_{e\gamma})} \qquad (14.51)$$

In page (14.20A) we difined the exclusive cross sections $\sigma_{2\to2}$ and $\sigma_{2\to3}$ that take into account the photon-muon separation angle. However, ~~the~~ the cut $\theta_{\mu\gamma} > \theta_{cut}$ does not avoid the IR divergence in (14.51).

The solution to this problem is to assume that the initial state as a probabilty $(f_i(x,Q))$ of finding $i = e, \gamma$ in the initial state. so, the remaining IR divergence is absorved by it! For instance,

$$f_\gamma(z) = \frac{e_R^2}{8\pi^2}\left[\frac{1+(1-z)^2}{z}\right]\ln\frac{Q^2}{m_e}$$

with $z = E_\gamma/Q$.

References

We follow the approach in H. Schwartz book on chapter 20, ~~[illegible]~~ To learn ~~mom~~ more see Peskin-Schroeder sections 6.4 and 6.5.