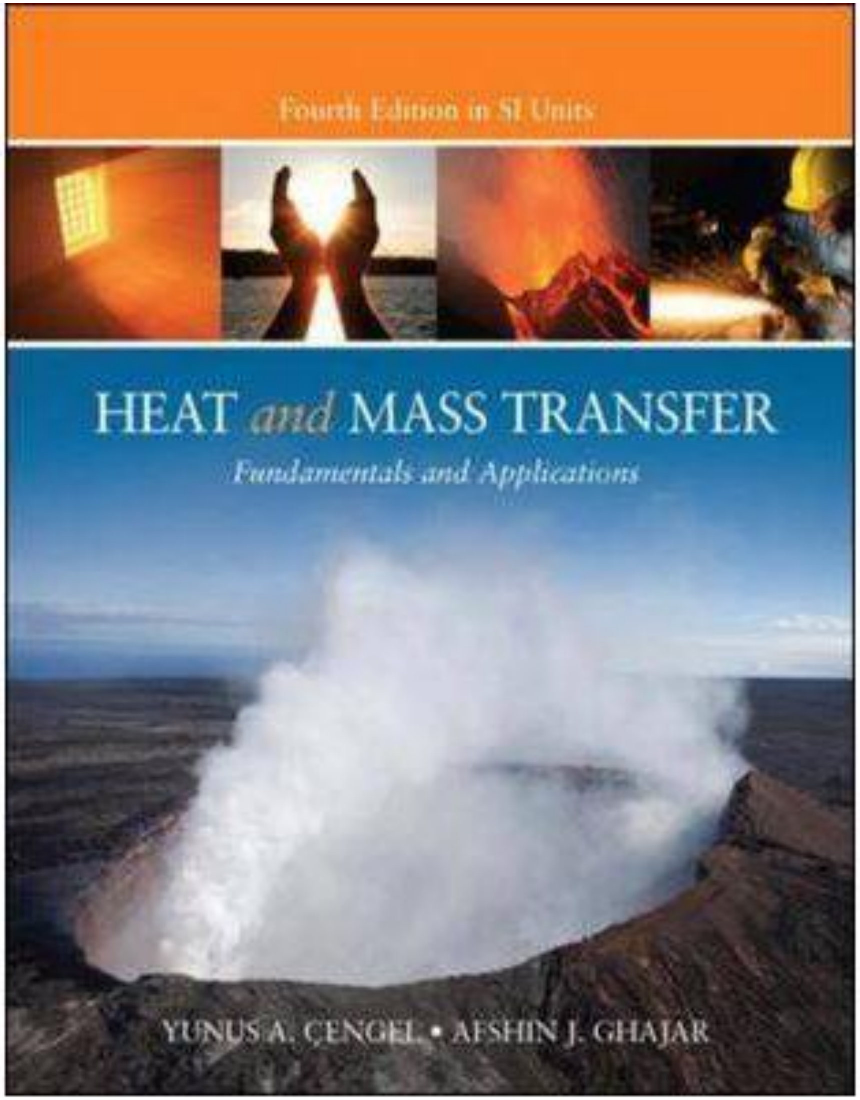
MAP 2320 – MÉTODOS NUMÉRICOS EM EQUAÇÕES DIFERENCIAIS II

2º Semestre - 2019

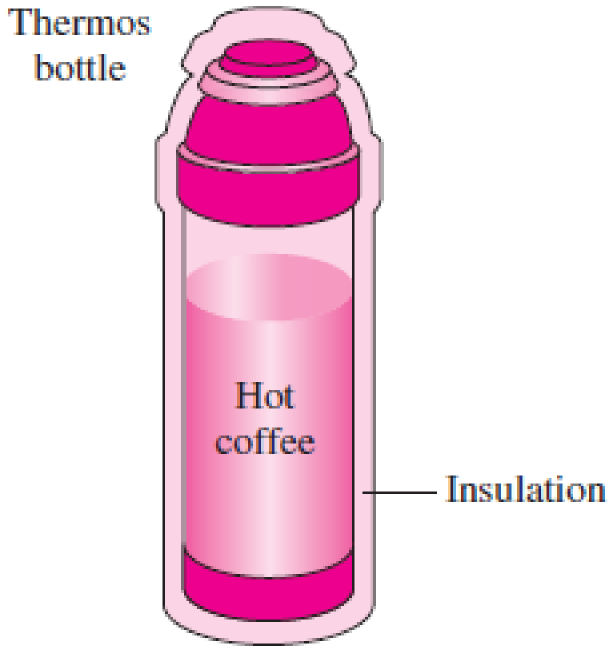
Prof. Dr. Luis Carlos de Castro Santos

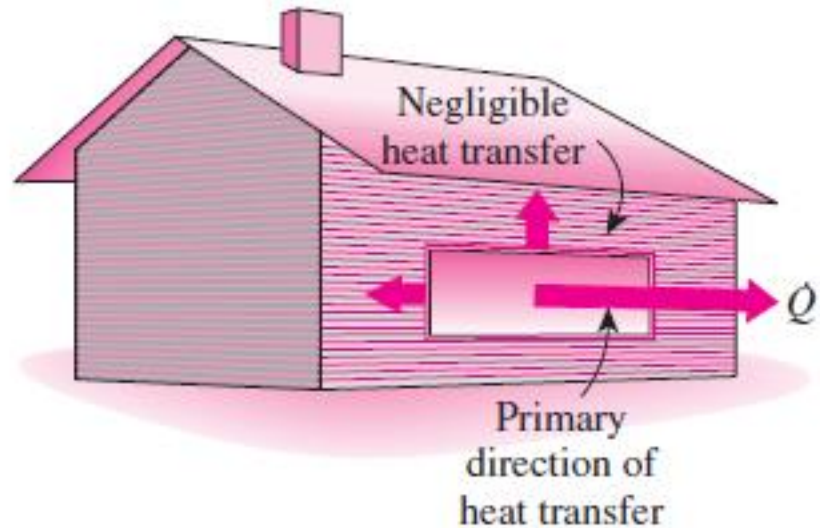
lsantos@ime.usp.br

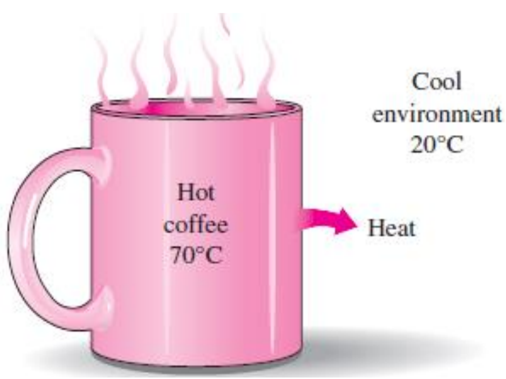


Heat and Mass Transfer (SI Unit)

By (author) Yunus A. Cengel , By (author) Afshin J. Ghajar







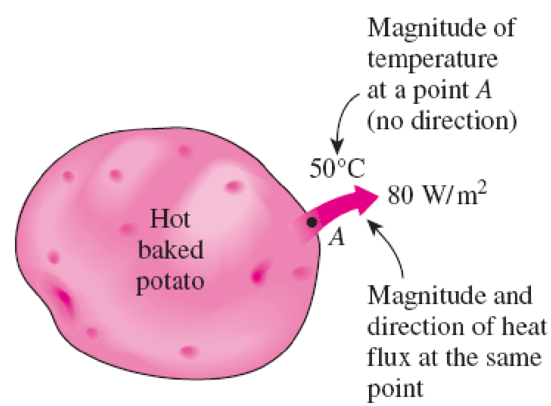


FIGURE 2–1
Heat transfer has direction as well as magnitude, and thus it is a *vector* quantity.

- The rate of heat conduction through a medium in a specified direction (say, in the x-direction) is expressed by **Fourier's law of heat conduction** for one-dimensional heat conduction as:

$$\dot{Q}_{\text{cond}} = -kA\frac{dT}{dx} \qquad \text{(W)}$$

Heat is conducted in the direction of decreasing temperature, and thus the temperature gradient is negative when heat is conducted in the positive x -direction.

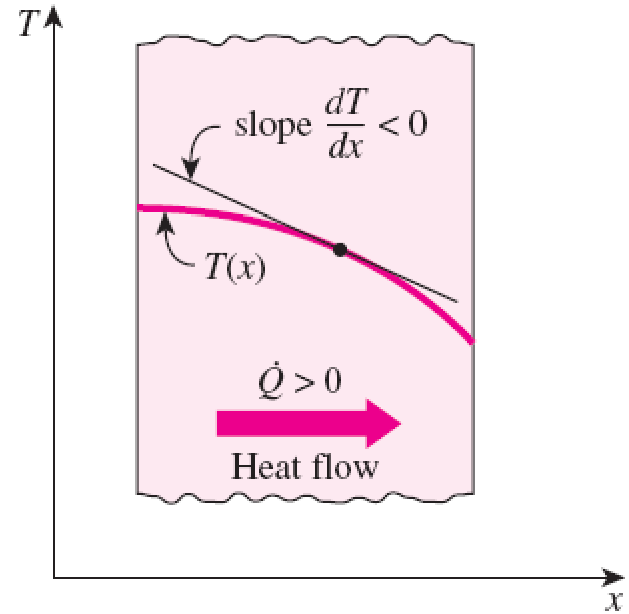


FIGURE 2–7

The temperature gradient dT/dx is simply the slope of the temperature curve on a T-x diagram.

- The heat flux vector at a point *P* on the surface of the figure must be perpendicular to the surface, and it must point in the direction of decreasing temperature
- If *n* is the normal of the isothermal surface at point *P*, the rate of heat conduction at that point can be expressed by Fourier's law as

$$\dot{Q}_n = -kA\frac{\partial T}{\partial n} \qquad \text{(W)}$$

$$\vec{\dot{Q}}_n = \dot{Q}_x \vec{i} + \dot{Q}_y \vec{j} + \dot{Q}_z \vec{k}$$

$$\dot{Q}_x = -kA_x \frac{\partial T}{\partial x}, \qquad \dot{Q}_y = -kA_y \frac{\partial T}{\partial y},$$

$$\dot{Q}_z = -kA_z \frac{\partial T}{\partial z}$$

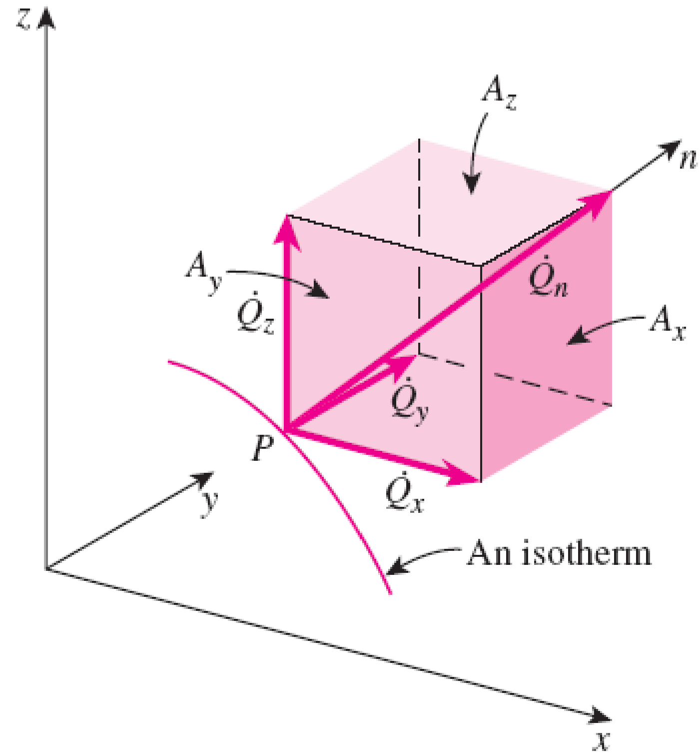


FIGURE 2–8
The heat transfer vector is always normal to an isothermal surface and can be resolved into its components like any other vector.

Heat Conduction Equation in a Large Plane Wall

$$\begin{pmatrix}\text{Rate of heat}\\\text{conduction}\\\text{at } x\end{pmatrix} - \begin{pmatrix}\text{Rate of heat}\\\text{conduction}\\\text{at } x+\Delta x\end{pmatrix} + \begin{pmatrix}\text{Rate of heat}\\\text{generation}\\\text{inside the}\\\text{element}\end{pmatrix} = \begin{pmatrix}\text{Rate of change}\\\text{of the energy}\\\text{content of the}\\\text{element}\end{pmatrix}$$

$$\dot{Q}_x - \dot{Q}_{x+\Delta x} + \dot{E}_{\text{gen, element}} = \frac{\Delta E_{\text{element}}}{\Delta t} \qquad \textbf{(2-6)}$$

$$\Delta E_{\text{element}} = E_{t+\Delta t} - E_t = mc(T_{t+\Delta t} - T_t) = \rho c A \Delta x (T_{t+\Delta t} - T_t)$$

$$\dot{E}_{\text{gen, element}} = \dot{e}_{\text{gen}} V_{\text{element}} = \dot{e}_{\text{gen}} A \Delta x$$

Substituting into Eq. 2–6, we get

$$\dot{Q}_x - \dot{Q}_{x+\Delta x} + \dot{e}_{\text{gen}} A \Delta x = \rho c A \Delta x \frac{T_{t+\Delta t} - T_t}{\Delta t}$$

Dividing by $A\Delta x$ gives

$$-\frac{1}{A}\frac{\dot{Q}_{x+\Delta x} - \dot{Q}_x}{\Delta x} + \dot{e}_{\text{gen}} = \rho c \frac{T_{t+\Delta t} - T_t}{\Delta t}$$

Taking the limit as $\Delta x \to 0$ and $\Delta t \to 0$ yields

$$\frac{1}{A}\frac{\partial}{\partial x}\left(kA\frac{\partial T}{\partial x}\right) + \dot{e}_{\text{gen}} = \rho c \frac{\partial T}{\partial t}$$

$$\lim_{\Delta x \to 0} \frac{\dot{Q}_{x+\Delta x} - \dot{Q}_x}{\Delta x} = \frac{\partial \dot{Q}}{\partial x} = \frac{\partial}{\partial x}\left(-kA\frac{\partial T}{\partial x}\right)$$

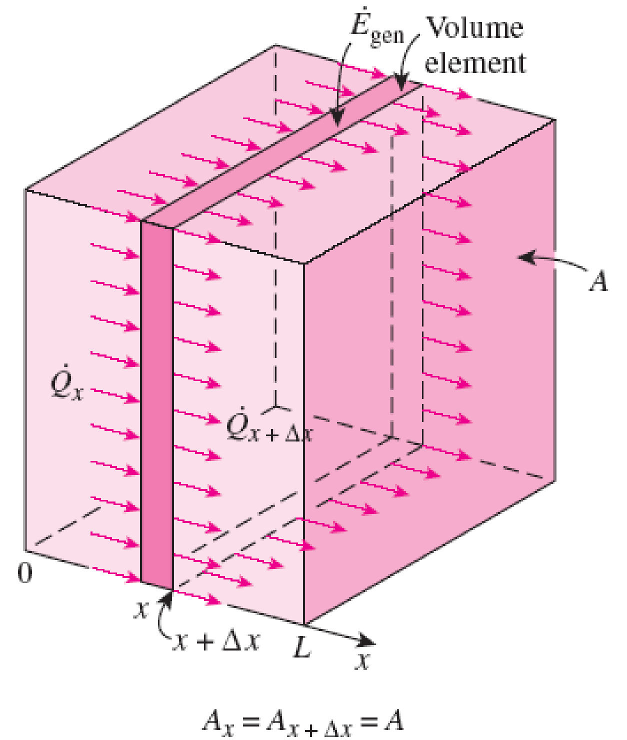


FIGURE 2–12

One-dimensional heat conduction through a volume element in a large plane wall.

Variable conductivity:
$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\dot{e}_{\text{gen}}=\rho c\frac{\partial T}{\partial t}$$

Constant conductivity:
$$\frac{\partial^2 T}{\partial x^2}+\frac{\dot{e}_{\text{gen}}}{k}=\frac{1}{\alpha}\frac{\partial T}{\partial t}$$

(1) *Steady-state:*
$(\partial/\partial t=0)$
$$\frac{d^2 T}{dx^2}+\frac{\dot{e}_{\text{gen}}}{k}=0$$

(2) *Transient, no heat generation:*
$(\dot{e}_{\text{gen}}=0)$
$$\frac{\partial^2 T}{\partial x^2}=\frac{1}{\alpha}\frac{\partial T}{\partial t}$$

(3) *Steady-state, no heat generation:*
$(\partial/\partial t=0$ and $\dot{e}_{\text{gen}}=0)$
$$\frac{d^2 T}{dx^2}=0$$

General, one-dimensional:

No generation — Steady-state

$$\frac{\partial^2 T}{\partial x^2}+\cancelto{0}{\frac{\dot{e}_{\text{gen}}}{k}}=\frac{1}{\alpha}\cancelto{0}{\frac{\partial T}{\partial t}}$$

Steady, one-dimensional:

$$\frac{d^2 T}{dx^2}=0$$

The simplification of the one-dimensional heat conduction equation in a plane wall for the case of constant conductivity for steady conduction with no heat generation.

Heat Conduction Equation in a Long Cylinder

$$\begin{pmatrix}\text{Rate of heat}\\ \text{conduction}\\ \text{at } r\end{pmatrix} - \begin{pmatrix}\text{Rate of heat}\\ \text{conduction}\\ \text{at } r+\Delta r\end{pmatrix} + \begin{pmatrix}\text{Rate of heat}\\ \text{generation}\\ \text{inside the}\\ \text{element}\end{pmatrix} = \begin{pmatrix}\text{Rate of change}\\ \text{of the energy}\\ \text{content of the}\\ \text{element}\end{pmatrix}$$

$$\dot{Q}_r - \dot{Q}_{r+\Delta r} + \dot{E}_{\text{gen, element}} = \frac{\Delta E_{\text{element}}}{\Delta t}$$

$$\Delta E_{\text{element}} = E_{t+\Delta t} - E_t = mc(T_{t+\Delta t} - T_t) = \rho c A \Delta r (T_{t+\Delta t} - T_t)$$

$$\dot{E}_{\text{gen, element}} = \dot{e}_{\text{gen}} V_{\text{element}} = \dot{e}_{\text{gen}} A \Delta r$$

$$\dot{Q}_r - \dot{Q}_{r+\Delta r} + \dot{e}_{\text{gen}} A \Delta r = \rho c A \Delta r \frac{T_{t+\Delta t} - T_t}{\Delta t}$$

$$-\frac{1}{A}\frac{\dot{Q}_{r+\Delta r} - \dot{Q}_r}{\Delta r} + \dot{e}_{\text{gen}} = \rho c \frac{T_{t+\Delta t} - T_t}{\Delta t}$$

Taking the limit as $\Delta r \to 0$ and $\Delta t \to 0$ yields

$$\frac{1}{A}\frac{\partial}{\partial r}\left(kA\frac{\partial T}{\partial r}\right) + \dot{e}_{\text{gen}} = \rho c \frac{\partial T}{\partial t}$$

$$\lim_{\Delta r \to 0} \frac{\dot{Q}_{r+\Delta r} - \dot{Q}_r}{\Delta r} = \frac{\partial \dot{Q}}{\partial r} = \frac{\partial}{\partial r}\left(-kA\frac{\partial T}{\partial r}\right)$$

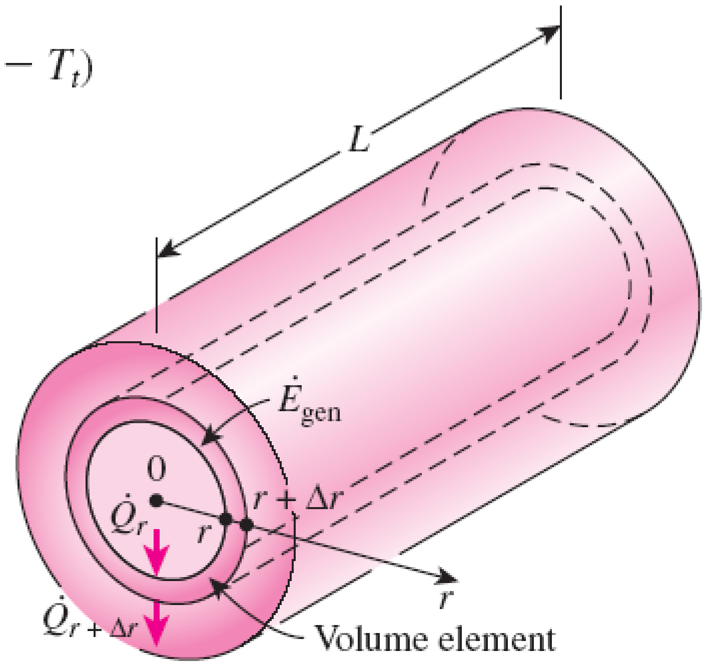


FIGURE 2–14

One-dimensional heat conduction through a volume element in a long cylinder.

Variable conductivity:
$$\frac{1}{r}\frac{\partial}{\partial r}\left(rk\frac{\partial T}{\partial r}\right)+\dot{e}_{\text{gen}}=\rho c\frac{\partial T}{\partial t}$$

Constant conductivity:
$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+\frac{\dot{e}_{\text{gen}}}{k}=\frac{1}{\alpha}\frac{\partial T}{\partial t}$$

(1) *Steady-state:* $(\partial/\partial t=0)$
$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right)+\frac{\dot{e}_{\text{gen}}}{k}=0$$

(2) *Transient, no heat generation:* $(\dot{e}_{\text{gen}}=0)$
$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)=\frac{1}{\alpha}\frac{\partial T}{\partial t}$$

(3) *Steady-state, no heat generation:* $(\partial/\partial t=0$ and $\dot{e}_{\text{gen}}=0)$
$$\frac{d}{dr}\left(r\frac{dT}{dr}\right)=0$$

(*a*) The form that is ready to integrate
$$\frac{d}{dr}\left(r\frac{dT}{dr}\right)=0$$

(*b*) The equivalent alternative form
$$r\frac{d^2T}{dr^2}+\frac{dT}{dr}=0$$

Two equivalent forms of the differential equation for the one-dimensional steady heat conduction in a cylinder with no heat generation.

Heat Conduction Equation in a Sphere

Variable conductivity:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 k \frac{\partial T}{\partial r}\right) + \dot{e}_{gen} = \rho c \frac{\partial T}{\partial t}$$

Constant conductivity:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right) + \frac{\dot{e}_{gen}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

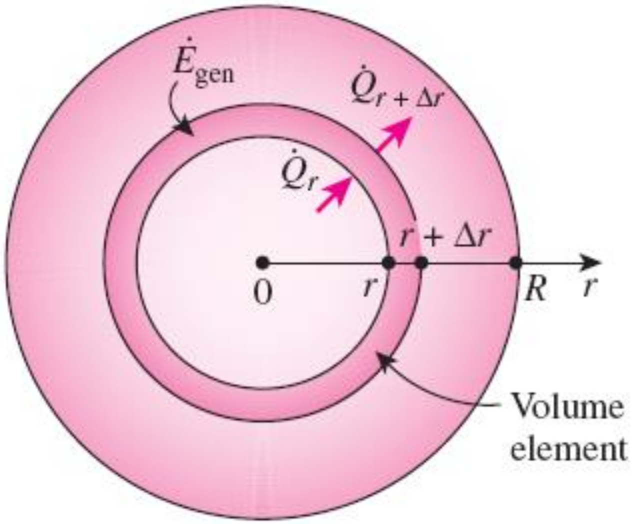


FIGURE 2–16

One-dimensional heat conduction through a volume element in a sphere.

(1) *Steady-state:* $(\partial/\partial t = 0)$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dT}{dr}\right) + \frac{\dot{e}_{gen}}{k} = 0$$

(2) *Transient, no heat generation:* $(\dot{e}_{gen} = 0)$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right) = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

(3) *Steady-state, no heat generation:* $(\partial/\partial t = 0$ and $\dot{e}_{gen} = 0)$

$$\frac{d}{dr}\left(r^2 \frac{dT}{dr}\right) = 0 \quad \text{or} \quad r\frac{d^2T}{dr^2} + 2\frac{dT}{dr} = 0$$

Combined One-Dimensional Heat Conduction Equation

An examination of the one-dimensional transient heat conduction equations for the plane wall, cylinder, and sphere reveals that all three equations can be expressed in a compact form as

$$\frac{1}{r^n}\frac{\partial}{\partial r}\left(r^n k \frac{\partial T}{\partial r}\right) + \dot{e}_{\text{gen}} = \rho c \frac{\partial T}{\partial t}$$

$n = 0$ for a plane wall

$n = 1$ for a cylinder

$n = 2$ for a sphere

In the case of a plane wall, it is customary to replace the variable *r* by *x*.

This equation can be simplified for steady-state or no heat generation cases as described before.

BOUNDARY AND INITIAL CONDITIONS

The description of a heat transfer problem in a medium is not complete without a full description of the thermal conditions at the bounding surfaces of the medium.

Boundary conditions: The *mathematical expressions* of the thermal conditions at the boundaries.

The temperature at any point on the wall at a specified time depends on the condition of the geometry at the beginning of the heat conduction process.

Such a condition, which is usually specified at time ***t* = 0**, is called the **initial condition**, which is a mathematical expression for the temperature distribution of the medium initially.

$$T(x, y, z, 0) = f(x, y, z)$$

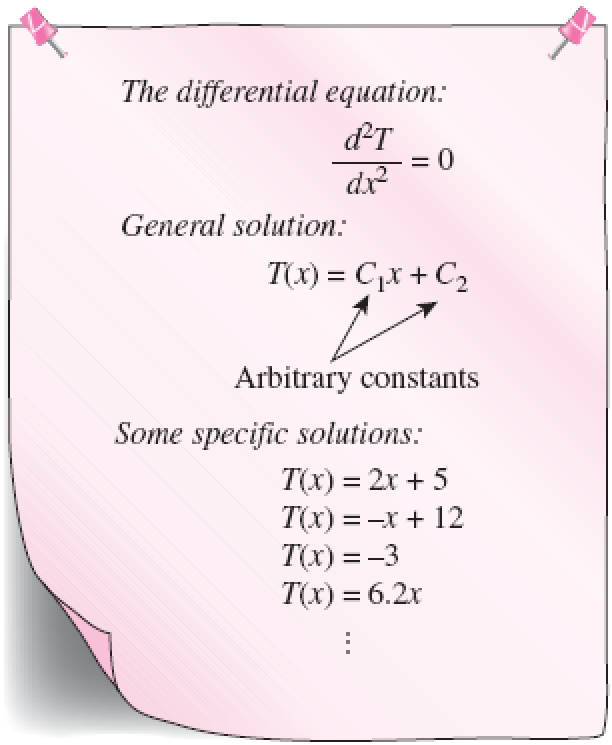


FIGURE 2–25

The general solution of a typical differential equation involves arbitrary constants, and thus an infinite number of solutions.

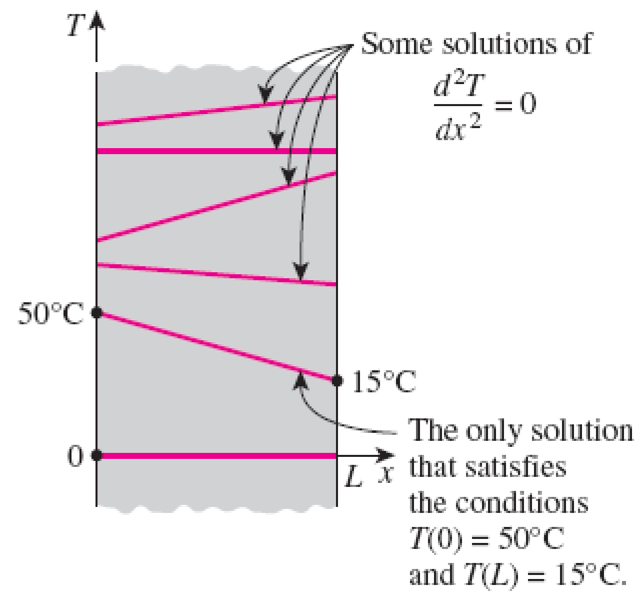


FIGURE 2–26

To describe a heat transfer problem completely, two boundary conditions must be given for each direction along which heat transfer is significant.

Boundary Conditions

- Specified Temperature Boundary Condition
- Specified Heat Flux Boundary Condition
- Convection Boundary Condition
- Radiation Boundary Condition
- Interface Boundary Conditions
- Generalized Boundary Conditions

1 Specified Temperature Boundary Condition

The *temperature* of an exposed surface can usually be measured directly and easily.

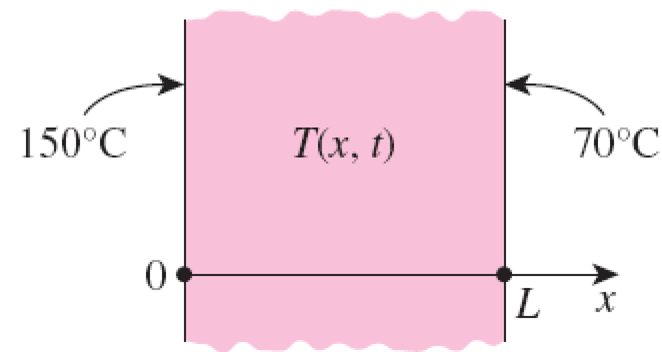
Therefore, one of the easiest ways to specify the thermal conditions on a surface is to specify the temperature.

For one-dimensional heat transfer through a plane wall of thickness *L*, for example, the specified temperature boundary conditions can be expressed as

$$T(0, t) = T_1$$
$$T(L, t) = T_2$$

where T_1 and T_2 are the specified temperatures at surfaces at $x = 0$ and $x = L$, respectively.

The specified temperatures can be constant, which is the case for steady heat conduction, or may vary with time.



$$T(0, t) = 150°C$$
$$T(L, t) = 70°C$$

FIGURE 2–27

Specified temperature boundary conditions on both surfaces of a plane wall.

2 Specified Heat Flux Boundary Condition

The heat flux in the positive *x*-direction anywhere in the medium, including the boundaries, can be expressed by

$$\dot{q} = -k\frac{\partial T}{\partial x} = \left(\begin{array}{c}\text{Heat flux in the}\\ \text{positive } x - \text{direction}\end{array}\right) \qquad (\text{W/m}^2)$$

For a plate of thickness *L* subjected to heat flux of 50 W/m^2 into the medium from both sides, for example, the specified heat flux boundary conditions can be expressed as

$$-k\frac{\partial T(0,t)}{\partial x} = 50 \qquad \text{and} \qquad -k\frac{\partial T(L,t)}{\partial x} = -50$$

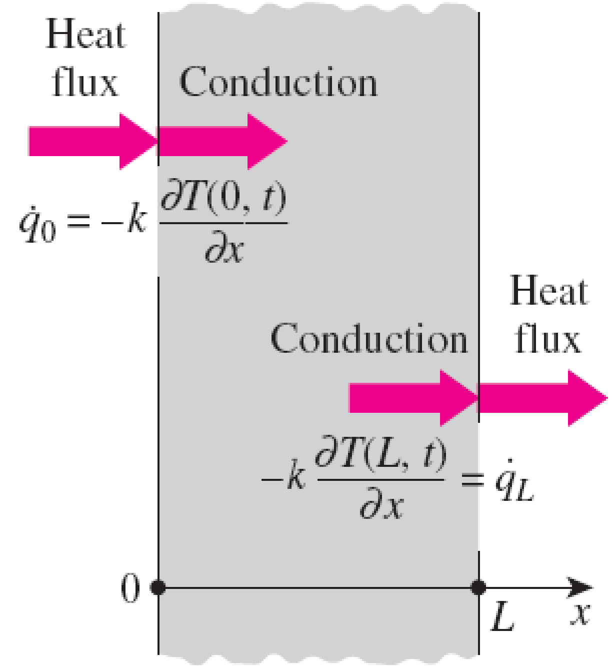


FIGURE 2–28

Specified heat flux boundary conditions on both surfaces of a plane wall.

Special Case: Insulated Boundary

A well-insulated surface can be modeled as a surface with a specified heat flux of zero. Then the boundary condition on a perfectly insulated surface (at *x* = 0, for example) can be expressed as

$$k\frac{\partial T(0, t)}{\partial x} = 0 \qquad \text{or} \qquad \frac{\partial T(0, t)}{\partial x} = 0$$

On an insulated surface, the first derivative of temperature with respect to the space variable (the temperature gradient) in the direction normal to the insulated surface is zero.

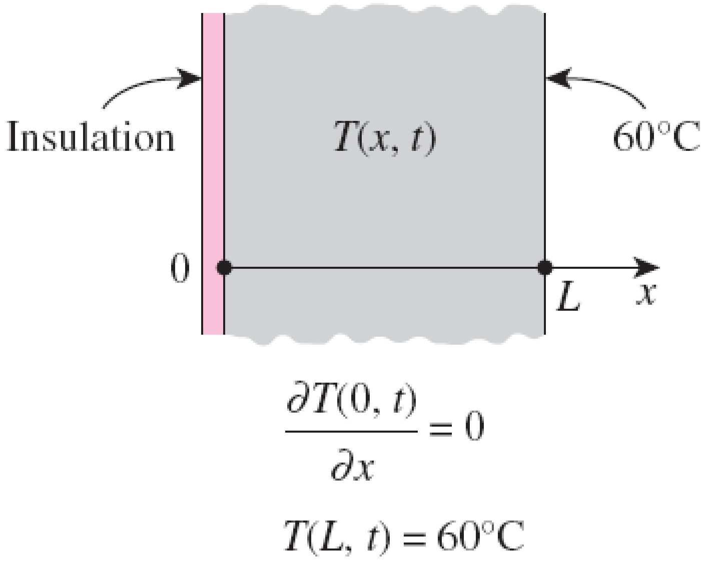


FIGURE 2–29

A plane wall with insulation and specified temperature boundary conditions.

Equações Diferenciais Parciais: Uma Introdução (Versão Preliminar)

Reginaldo J. Santos
Departamento de Matemática-ICEx
Universidade Federal de Minas Gerais
http://www.mat.ufmg.br/~regi

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3

Equação do Calor em uma Barra

Neste capítulo estudaremos a equação do calor unidimensional usando o método de separação de variáveis e as séries de Fourier.

Pode-se mostrar que a temperatura em uma barra homogênea, isolada dos lados, em função da posição e do tempo, $u(x,t)$, satisfaz a equação diferencial parcial

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

chamada **equação do calor em uma barra**. Aqui $\alpha > 0$ é uma constante que depende do material que compõe a barra é chamada de **difusividade térmica**.

3.1 Extremidades a Temperaturas Fixas

Vamos determinar a temperatura em função da posição e do tempo, $u(x,t)$ em uma barra isolada dos lados, de comprimento L, sendo conhecidos a distribuição de temperatura inicial, $f(x)$, e as temperaturas nas extremidades, T_1 e T_2, que são mantidas constantes com o tempo, ou seja, vamos resolver o problema de valor inicial e de fronteira (PVIF)

$$\left\{\begin{array}{l} \dfrac{\partial u}{\partial t} = \alpha^2 \dfrac{\partial^2 u}{\partial x^2} \\ u(x,0) = f(x),\ 0 < x < L \\ u(0,t) = T_1,\ u(L,t) = T_2 \end{array}\right.$$

Vamos inicialmente resolver o problema com $T_1 = T_2 = 0$, que chamamos de **condições de fronteira homogêneas.**

3.1.1 Condições de Fronteira Homogêneas

$$\begin{cases} \dfrac{\partial u}{\partial t} = \alpha^2 \dfrac{\partial^2 u}{\partial x^2} \\ u(x,0) = f(x),\ 0 < x < L \\ u(0,t) = 0,\ u(L,t) = 0 \end{cases}$$

Vamos usar um método chamado **separação de variáveis**. Vamos procurar uma solução na forma de um produto de uma função de x por uma função de t, ou seja,

$$u(x,t) = X(x)T(t).$$

Calculando-se as derivadas parciais temos que

$$\frac{\partial u}{\partial t} = X(x)T'(t) \quad \text{e} \quad \frac{\partial^2 u}{\partial x^2} = X''(x)T(t).$$

Substituindo-se na equação diferencial obtemos

$$X(x)T'(t) = \alpha^2 X''(x)T(t).$$

Dividindo-se por $\alpha^2 X(x)T(t)$ obtemos

$$\frac{X''(x)}{X(x)} = \frac{1}{\alpha^2}\frac{T'(t)}{T(t)}$$

O primeiro membro depende apenas de x, enquanto o segundo depende apenas de t. Isto só é possível se eles forem iguais a uma constante, ou seja,

$$\frac{X''(x)}{X(x)} = \frac{1}{\alpha^2}\frac{T'(t)}{T(t)} = \lambda.$$

Obtemos então duas equações diferenciais ordinárias com condições de fronteira:

$$\begin{cases} X''(x) - \lambda X(x) = 0, \quad X(0) = 0, \; X(L) = 0 & (3.1) \\ T'(t) - \alpha^2 \lambda T(t) = 0 & (3.2) \end{cases}$$

As condições $X(0) = X(L) = 0$ decorrem do fato de que a temperatura nas extremidades da barra é mantida igual a zero, ou seja,

$$0 = u(0, t) = X(0)T(t) \quad \text{e} \quad 0 = u(L, t) = X(L)T(t).$$


Lembrete!

A equação $X''(x) - \lambda X(x) = 0$ (a sua equação característica é $r^2 - \lambda = 0$) pode ter como soluções,

Se $\lambda > 0:\; X(x) = c_1 e^{\sqrt{\lambda}\,x} + c_2 e^{-\sqrt{\lambda}\,x}.$

Se $\lambda = 0:\; X(x) = c_1 + c_2 x.$

Se $\lambda < 0:\; X(x) = c_1 \operatorname{sen}(\sqrt{-\lambda}\,x) + c_2 \cos(\sqrt{-\lambda}\,x).$

As condições de fronteira $X(0) = 0$ e $X(L) = 0$ implicam que

Se $\lambda > 0$:

Substituindo-se $x = 0$ e $X = 0$ na solução geral de $X'' - \lambda X = 0$,

$$X(x) = c_1 e^{\sqrt{\lambda}\,x} + c_2 e^{-\sqrt{\lambda}\,x},$$

obtemos que $0 = c_1 + c_2$, ou seja, $c_2 = -c_1$. Logo

$$X(x) = c_1 (e^{\sqrt{\lambda}\,x} - e^{-\sqrt{\lambda}\,x}).$$

Agora substituindo-se $x = L$ e $X = 0$ obtemos que $c_1(e^{\sqrt{\lambda}\,L} - e^{-\sqrt{\lambda}\,L}) = 0$. Logo, se $c_1 \neq 0$, então

$$e^{\sqrt{\lambda}\,L} = e^{-\sqrt{\lambda}\,L}$$

o que só é possível se $\lambda = 0$, que não é o caso.

Se $\lambda = 0$:

Substituindo-se $x = 0$ e $X = 0$ na solução geral de $X'' - \lambda X = 0$,

$$X(x) = c_1 + c_2 x,$$

obtemos que $c_1 = 0$. Logo

$$X(x) = c_2 x.$$

Agora substituindo-se $x = L$ e $X = 0$ obtemos $c_2 L = 0$. Logo, também $c_2 = 0$.

Se $\lambda < 0$:

Substituindo-se $x = 0$ e $X = 0$ na solução geral de $X'' - \lambda X = 0$,

$$X(x) = c_1 \operatorname{sen}(\sqrt{-\lambda}x) + c_2 \cos(\sqrt{-\lambda}x),$$

obtemos que $c_2 = 0$. Logo

$$X(x) = c_1 \operatorname{sen}(\sqrt{-\lambda}x). \tag{3.3}$$

Agora substituindo-se $x = L$ e $X = 0$ em $X(x) = c_1 \operatorname{sen}(\sqrt{-\lambda}x)$, obtemos

$$c_1 \operatorname{sen}(\sqrt{-\lambda}L) = 0.$$

Logo se $c_1 \neq 0$, então $\sqrt{-\lambda}L = n\pi$, para $n = 1, 2, 3, \ldots$

Portanto as condições de fronteira $X(0) = 0$ e $X(L) = 0$ implicam que (3.1) tem solução não identicamente nula somente se $\lambda < 0$ e mais que isso λ tem que ter valores dados por

$$\lambda = -\frac{n^2\pi^2}{L^2}, \; n = 1, 2, 3, \ldots$$

Substituindo-se estes valores de λ em (3.3) concluímos que o problema de valores de fronteira (3.1) tem soluções fundamentais

$$X_n(x) = \operatorname{sen} \frac{n\pi x}{L}, \; \text{para } n = 1, 2, 3, \ldots.$$

Substituindo-se $\lambda = -\frac{n^2\pi^2}{L^2}$ na equação diferencial (3.2) obtemos

$$T'(t) + \frac{\alpha^2 n^2 \pi^2}{L^2} T(t) = 0,$$

que tem solução fundamental

$$T_n(t) = e^{-\frac{\alpha^2 n^2 \pi^2}{L^2} t}, \text{ para } n = 1, 2, 3, \ldots$$

Logo o problema

$$\begin{cases} \dfrac{\partial u}{\partial t} = \alpha^2 \dfrac{\partial^2 u}{\partial x^2} \\ u(0,t) = 0,\ u(L,t) = 0. \end{cases}$$

tem soluções **soluções fundamentais**

$$u_n(x,t) = X_n(x)T_n(t) = \operatorname{sen} \frac{n\pi x}{L} e^{-\frac{\alpha^2 n^2 \pi^2}{L^2} t} \quad \text{para } n = 1, 2, 3, \ldots$$

Combinações lineares das soluções fundamentais são também solução (verifique!),

$$u(x,t) = \sum_{n=1}^{N} c_n u_n(x,t) = \sum_{n=1}^{N} c_n \operatorname{sen} \frac{n\pi x}{L} e^{-\frac{\alpha^2 n^2 \pi^2}{L^2} t}.$$

Mas uma solução deste tipo não necessariamente satisfaz a condição inicial

$$u(x,0) = f(x),$$

para uma função $f(x)$ mais geral.
Vamos supor que a solução do problema de valor inicial e de fronteira possa ser escrita como uma série da forma

$$u(x,t) = \sum_{n=1}^{\infty} c_n u_n(x,t) = \sum_{n=1}^{\infty} c_n \operatorname{sen} \frac{n\pi x}{L} e^{-\frac{\alpha^2 n^2 \pi^2}{L^2} t}. \tag{3.4}$$

Para satisfazer a condição inicial $u(x,0) = f(x)$, temos que impor a condição

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} c_n \operatorname{sen} \frac{n\pi x}{L}.$$

Esta é a série de Fourier de senos de $f(x)$. Assim, pelo Corolário 2.5 na página 184, se a função $f : [0, L] \to \mathbb{R}$ é contínua por partes tal que a sua derivada f' também seja contínua por partes, então os coeficientes da série são dados por

$$c_n = \frac{2}{L} \int_0^L f(x) \operatorname{sen} \frac{n\pi x}{L} dx, \; n = 1, 2, 3 \ldots \tag{3.5}$$

Exemplo 3.1. Vamos considerar uma barra de 40 cm de comprimento, isolada nos lados, com coeficiente $\alpha = 1$, com as extremidades mantidas a temperatura de 0° C e tal que a temperatura inicial é dada por

$$f(x) = \begin{cases} x, & \text{se } 0 \leq x < 20 \\ 40 - x, & \text{se } 20 \leq x \leq 40 \end{cases}$$

Temos que resolver o problema de valor inicial e de fronteira

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} \\ u(x,0) = f(x), \; 0 < x < 40 \\ u(0,t) = 0, \; u(40,t) = 0 \end{cases}$$

A solução é então

$$u(x,t) = \sum_{n=1}^{\infty} c_n \operatorname{sen} \frac{n\pi x}{40} e^{-\frac{n^2\pi^2}{1600}t}$$

em que c_n são os coeficientes da série de senos de $f(x)$, ou seja, usando a tabela na página 202, multiplicando por 2 os valores obtemos:

$$\begin{aligned}
c_n = \;&= \frac{1}{20}\int_0^{40} f(x) \operatorname{sen} \frac{n\pi x}{40}\, dx \\
&= 2\left(b_n(f^{(1)}_{0,1/2}, 40) + 40 b_n(f^{(0)}_{1/2,1}, 40) - b_n(f^{(1)}_{1/2,1}, 40)\right) \\
&= \frac{80}{n^2\pi^2}\left(-s\cos s + \operatorname{sen} s\right)\Big|_0^{n\pi/2} - \frac{80}{n\pi}\cos s\Big|_{n\pi/2}^{n\pi} - \frac{80}{n^2\pi^2}\left(-s\cos s + \operatorname{sen} s\right)\Big|_{n\pi/2}^{n\pi} \\
&= \frac{160}{n^2\pi^2}\left(-\frac{n\pi}{2}\cos\frac{n\pi}{2} + \operatorname{sen}\frac{n\pi}{2}\right) + \frac{80}{n\pi}\cos\frac{n\pi}{2} \\
&= \frac{160 \operatorname{sen}\frac{n\pi}{2}}{n^2\pi^2}, \; n = 1, 2, 3 \ldots
\end{aligned}$$

Entretanto coeficientes de índice par são nulos:

$$c_{2k} = 0$$

$$c_{2k+1} = \frac{160(-1)^k}{(2k+1)^2\pi^2}.$$

Portanto a solução do problema é

$$\begin{aligned} u(x,t) &= \frac{160}{\pi^2}\sum_{n=1}^{\infty}\frac{\operatorname{sen}\frac{n\pi}{2}}{n^2}\operatorname{sen}\frac{n\pi x}{40}e^{-\frac{n^2\pi^2}{1600}t} \\ &= \frac{160}{\pi^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}\operatorname{sen}\frac{(2n+1)\pi x}{40}e^{-\frac{(2n+1)^2\pi^2}{1600}t} \end{aligned}$$

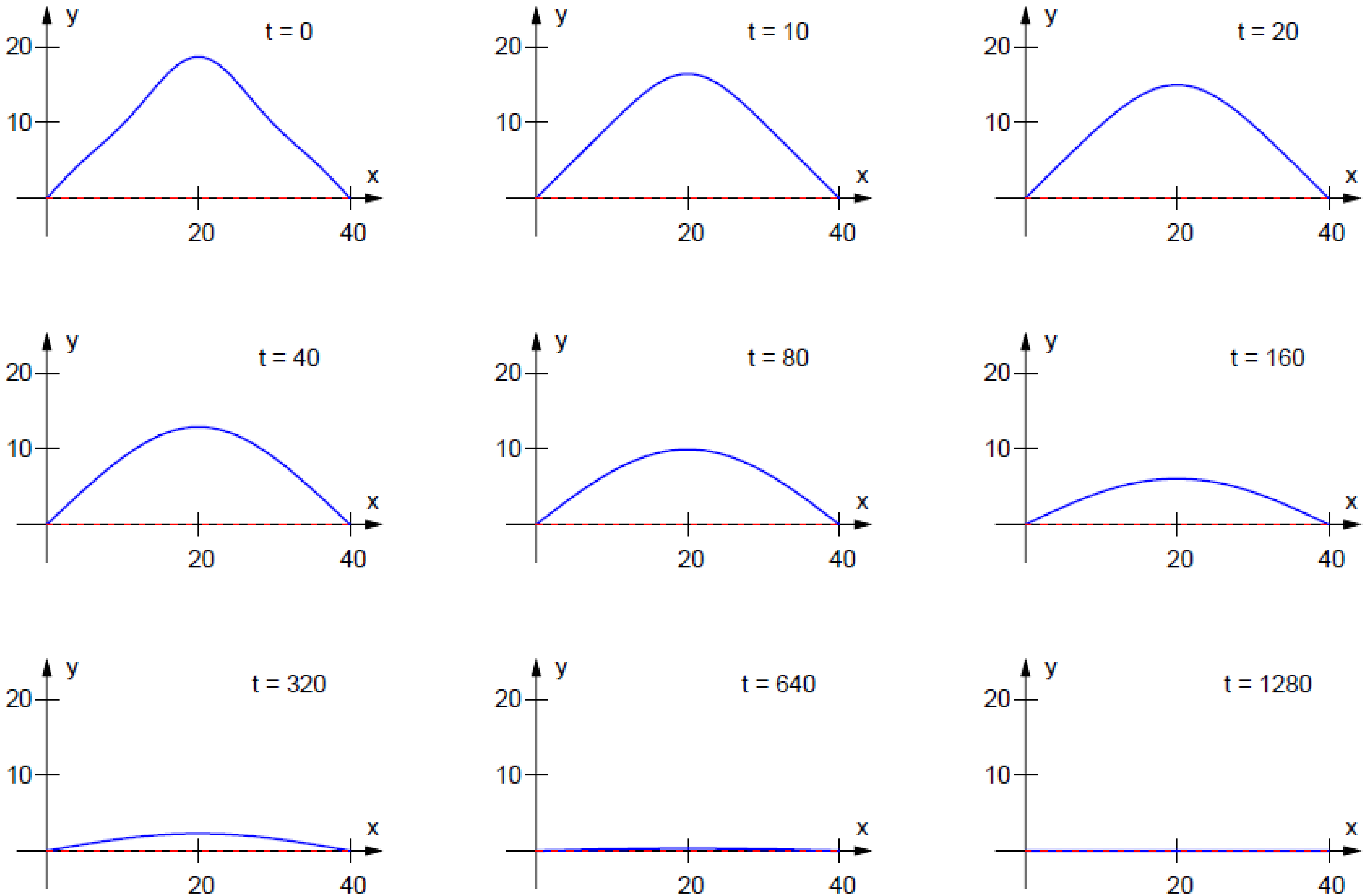

Figura 3.1 – Solução, $u(x,t)$, do PVIF do Exemplo 3.1 tomando apenas 3 termos não nulos da série.

MAP 2320 – MÉTODOS NUMÉRICOS EM EQUAÇÕES DIFERENCIAIS II

2º Semestre - 2019

Roteiro do curso

- Introdução
- Séries de Fourier
- Método de Diferenças Finitas
- **Equação do calor transiente (parabólica)**
- Equação de Poisson (elíptica)
- Equação da onda (hiperbólica)