It is possible that an eigenvector v of the delay equation $(*)$ $x'(t) + Bx'(t-h) = A_0x(t) + A_1x(t-h)$ simultaneously satisfies $(sI - A_0)v = 0 = (sB - A_1)v$ for its associated eigenvalue s. In this case we use the generalized eigenvalue approach demonstrated in Examples 4.1 and 4.2 to find the value of $z = e^{-sh}$.

V. Conclusion

In this note, we have given a technique for determining the imaginary axis eigenvalues of a neutral or retarded matrix delay system. These eigenvalues were proven to be contained in the set of imaginary axis generalized eigenvalues of a matrix pair which is formed from the system coefficient matrices. In the case of a retarded system, the generalized eigenvalue problem is just an eigenvalue problem. After an inversion of a constant matrix, this is also true for neutral systems which are nonsingular.

Via an equivalence involving the theory of matrix polynomials, delay system imaginary axis eigenvalues were also shown to be limited to the set of s-values for which a certain second degree matrix polynomial becomes singular at s. When the above matrix polynomial is converted to an operator polynomial in s, and then evaluated at s, the Kernel contains a rank one product matrix formed from the system eigenvector associated with the pure imaginary eigenvalue. This is conveniently expressed in terms of Kronecker products, and a means of recovering the eigenvector from the Kernel which works in most practical cases was demonstrated.

The motivation for the construction of the matrix pair for the method comes from the theory of quadratic functionals for delay differential equations. It is possible that the ideas in this note can be carried over to many other types of delay and Volterra equations of interest in control, and any investigations demonstrating this would certainly be worthwhile.

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Sliding Mode Observer for Nonlinear Uncertain Systems

Yi Xiong and Mehrdad Saif

***Abstract*—A new sliding mode observer for a class of nonlinear uncertain systems is proposed in this article. The proposed sliding mode observer works under much less conservative conditions than previous nonlinear unknown input observers. Also, a functional version of the observer is proposed in certain cases where it may not be possible to design an observer capable of estimating the entire state of the system.**

***Index Terms*—Nonlinear observer, sliding mode observer, unknown input observer.**

I. Introduction

State observation is an important problem for both linear and nonlinear systems. One nonlinear observer design approach deals with nonlinear systems for which observers with linearizable error dynamics can be designed (see, e.g., [2], [13], [14], [16], [25]). In several works (e.g., [4], [7], [9], [15]) systems which are composed of a linear unforced part and a nonlinear state dependent controlled part are considered. In [10], an observer is given for a class of nonlinear systems which are not necessarily control affine. However, the gain of the proposed observer is not easily computable. Recently [6], [11] proposed observers based on some ideas from the high gain approach whose gain could easily be designed.

In [8], the approach of designing an extended the sliding mode observer using equivalent control for linear systems [19] was extended to nonlinear systems of the form

$$\begin{aligned} \dot{x} &= f(x) + B(x)u \\ y &= h(x). \end{aligned} \tag{1}$$

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This extension was also applied to nonlinear systems in triangular input form in [1], output and output derivative injection form in [5]. In papers [17] and [21], a framework similar to a Luenberger observer were used by appending a switching function with constant or time-varying gains.

The main contribution of this article is to develop a systematic sliding mode observer (SMO) design approach for nonlinear systems subject to unknown inputs. Previous SMO designs using equivalent control method did not consider the system uncertainties. As we pointed out in [22], estimation error is generally unavoidable if these SMO are applied directly for systems with unknown input. A class of SMO proposed by [20] and [22] can provide unknown input insensitive estimation for linear systems with unknown input. Reference [24] considered the nonlinear unknown input observer (NUIO) design for nonlinear systems which can be transformed into output injection form. This article also sheds extra light on the NUIO design problem.

II. Main Results

We consider multivariable nonlinear systems described in state space form by equations of the following form:

$$\begin{aligned} \dot{x} &= f(x) + B(x, u) + \sum_{i=1}^{m} g_i(x) d_i(x, u, t) \\ y_1 &= h_1(x) \\ \cdots & \quad \cdots \\ y_p &= h_p(x) \end{aligned} \tag{2}$$

in which $x \in M$, a C^∞ connected manifold of dimension n, $f(x), B(x, u), G(x) = [g_1(x), \ldots, g_m(x)]$ are smooth vector fields on M, and $h_i(x), i = 1, \ldots, p$ are smooth functions from M to R. The term $d(x, u, t)$ represents the uncertainty due to modeling error or component/actuator faults, namely the unknown input. In what follows, local coordinates are generally used. When global properties are considered, notions are simplified by assuming that M accepts a global coordinate system.

Assumption 1: We assume that $p \geq m$, and the first m outputs have a vector relative degree $\{q_1, \ldots, q_m\}$ corresponding to $G(x)$ at each point $x_0 \in M$. Stated differently, this means

$$L_{g_j} L_f^k h_i(x) = 0$$

for all $j = 1, \ldots, m$, for all $k < q_i - 1$, for all $i = 1, \ldots, m$, and for all x in M. Further, the $m \times m$ matrix

$$A(x) = \begin{pmatrix} L_{g_1} L_f^{q_1 - 1} h_1(x) & \cdots & L_{g_m} L_f^{q_1 - 1} h_1(x) \\ L_{g_1} L_f^{q_2 - 1} h_2(x) & \cdots & L_{g_m} L_f^{q_2 - 1} h_2(x) \\ \cdots & \cdots & \cdots \\ L_{g_1} L_f^{q_m - 1} h_m(x) & \cdots & L_{g_m} L_f^{q_m - 1} h_m(x) \end{pmatrix}$$

is nonsingular at each point $x_0 \in M$.

Assumption 2: The distribution spanned by the vector fields $g_1(x), \ldots, g_m(x)$ is involutive.

Lemma 1: Given the system (2), if assumptions 1–2 are valid, then

$$q_1 + \cdots + q_m \leq n.$$

For $i = 1, \ldots, m$ set

$$\begin{aligned} \phi_1^i &= h_i(x) \\ \phi_2^i &= L_f h_i(x) \\ \cdots & \quad \cdots \\ \phi_{q_i}^i &= L_f^{q_i - 1} h_i(x) \end{aligned}$$

if $q = q_1 + \cdots + q_m$ is strictly less than n, it is always possible to find $n - q$ more functions $\phi_{q+1}(x), \ldots, \phi_n(x)$ such that the mapping

$$\Phi(x) = col(\phi_1^1(x), \ldots, \phi_{q_1}^1(x), \ldots, \phi_{q_m}^m(x), \phi_{q+1}(x), \ldots, \phi_n(x))$$

has a Jacobian matrix which is nonlinear at each $x_0 \in M$ and therefore qualifies as a local coordinate transformation in M. Moreover, based on Assumption 2, it is always possible to choose $\phi_{q+1}(x), \ldots, \phi_n(x)$ in such a way that

$$L g_j \phi_i(x) = 0 \tag{3}$$

for all $i = q + 1, \ldots, n$, $j = 1, \ldots, m$, and all x in M.

Set

$$x_d^i = \begin{pmatrix} x_1^i \\ x_2^i \\ \cdots \\ x_{q_i}^i \end{pmatrix} = \begin{pmatrix} \phi_1^i(x) \\ \phi_2^i(x) \\ \cdots \\ \phi_{q_i}^i(x) \end{pmatrix}$$

for $i = 1, \ldots, m$, and $x_d = ((x_d^1)^T, \ldots, (x_d^m)^T)$

$$x_o = \begin{pmatrix} x_o^1 \\ x_o^2 \\ \cdots \\ x_o^{n-q} \end{pmatrix} = \begin{pmatrix} \phi_{q+1}(x) \\ \phi_{q+2}(x) \\ \cdots \\ \phi_n(x) \end{pmatrix}$$

the equations under new coordinates can be written as

$$\begin{aligned} \dot{x}_1^i &= x_2^i + b_1^i(x_d, x_o, u) \\ \cdots & \quad \cdots \\ \dot{x}_{q_i - 1}^i &= x_{q_i}^i + b_{q_i - 1}^i(x_d, x_o, u) \\ \dot{x}_{q_i}^i &= a_i(x_d, x_o) + b_{q_i}^i(x_d, x_o, u) + \sum_{j=1}^{m} c_{ij}(x_d, x_o) d_j \\ y_i &= x_1^i \end{aligned} \tag{4}$$

for $i = 1, \ldots, m$, and

$$\begin{aligned} \dot{x}_o &= q(x_d, x_o) + p(x_d, x_o, u) \\ y_{m+1} &= h_{m+1}(x_d, x_o) \\ \cdots & \quad \cdots \\ y_p &= h_p(x_d, x_o) \end{aligned} \tag{5}$$

where

$$\begin{aligned} a_i(x_d, x_o) &= L_f^{q_i} h_i(\Phi^{-1}(x_d, x_o)) \\ c_{ij} &= L_{g_j} L_f^{q_i - 1} h_i(\Phi^{-1}(x_d, x_o)) \end{aligned}$$

and

$$b_k^i(x_d, x_o, u) = \frac{\partial (L_f^{k-2} h_i)}{\partial x} B(\Phi^{-1}(x_d, x_o), u).$$

The above lemma is a trivial extension of [12, Prop. 5.1.2]. The nonlinear transformation $\Phi(x)$ decomposes the system (2) into two subsystems. Of the two subsystems, only x_d subsystem is effected by unknown inputs. Of course, this is not a complete decomposition because it is relies on Assumption 1 and 2.

Next, we shall discuss the observer design for system (4)–(5).

A. Observability of Subsystems

It is well known that the observability of nonlinear systems is associated with inputs. The observability of nonlinear subsystem (4)–(5) is of interest here.

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Inputs which make a nonlinear system unobservable are called "bad inputs" or "not universal inputs." For known input signals, one can avoid applying those "bad inputs." However, the unknown inputs which may be the result of a fault or certain other disturbances are beyond ones control. Therefore, observability for all unknown inputs is in general necessary in order to design a nonlinear unknown input observer, unless it can be guaranteed *a priori* that the unknown inputs do not belong to the class of bad inputs. Based on the work in [10], we have following lemma that shows the conditions such that the observability of x_d subsystem is independent of unknown inputs.

Lemma 2: Let $\overline{x_d^i} = \{x_d^1, \ldots, x_d^{i-1}, x_d^{i+1}, \ldots, x_d^m\}$. Assume each x_d^i subsystem in (4) has its input term in the following form

$$\begin{aligned} b_1^i(x_d, x_o, u) &= b_1^i(x_1^i, x_2^i, \overline{x_d^i}, x_o, u) \\ b_2^i(x_d, x_o, u) &= b_2^i(x_1^i, x_2^i, x_3^i, \overline{x_d^i}, x_o, u) \\ &\cdots \quad \cdots \\ b_{q_i-1}^i(x_d, x_o, u) &= b_{q_i-1}^i(x_d, x_o, u) \\ b_{q_i}^i(x_d, x_o, u) &= b_{q_i}^i(x_d, x_o, u) \end{aligned} \tag{6}$$

and the functions

$$1 + \frac{\partial b_j^i}{\partial x_{j+1}^i} \neq 0, \; j = 1, \ldots, q_i - 1$$

and state x_o, $\overline{x_d^i}$ is considered as inputs for x_d^i subsystem. Then x_d^i subsystem is uniformly observable for all inputs u, d, $\overline{x_d^i}$ and x_o.

An open problem which does not effect the development here is whether x_o subsystem is observable if and only if the original system (2) is observable.

B. Sliding Mode Observer for Subsystem With Unknown Inputs

We shall first build an unknown input independent observer for x_d subsystem.

Theorem 1: If each x_d^i subsystem in (4) has its input term in the following form:

$$\begin{aligned} b_1^i(x_d, x_o, u) &= b_1^i(y, u) \\ b_2^i(x_d, x_o, u) &= b_2^i(x_2^i, y, u) \\ &\cdots \quad \cdots \\ b_{q_i-1}^i(x_d, x_o, u) &= b_{q_i-1}^i(x_2^i, \ldots, x_{q_i-1}^i, y, u) \\ b_{q_i}^i(x_d, x_o, u) &= b_{q_i}^i(x_d, x_o, u) \end{aligned} \tag{7}$$

there exists a choice of λ_j^i, $i = 1, \ldots, m$, $j = 1, \ldots, q_i$ such that following sliding mode observer can be built to estimate states

$$\begin{aligned} \dot{\hat{x}}_1^i &= \hat{x}_2^i + b_1^i(y, u) + \lambda_1^i \operatorname{sign}(y_i - \hat{x}_1^i) \\ \dot{\hat{x}}_2^i &= \hat{x}_3^i + b_2^i(\hat{x}_2^i, y, u) + \lambda_2^i \operatorname{sign}(\overline{e_2^i}) \\ &\cdots \quad \cdots \\ \dot{\hat{x}}_{q_i-1}^i &= \hat{x}_{q_i}^i + b_{q_i-1}^i(\hat{x}_2^i, \ldots, \hat{x}_{q_i-1}^i, y, u) \\ &\quad + \lambda_{q_i-1}^i \operatorname{sign}(\overline{e_{q_i-1}^i}) \\ \dot{\hat{x}}_{q_i}^i &= a_i(\hat{x}_d, \hat{x}_o) + b_{q_i}^i(\hat{x}_d, \hat{x}_o, u) + \lambda_{q_i}^i \operatorname{sign}(\overline{e_{q_i}^i}) \end{aligned} \tag{8}$$

where $\overline{e_j^i}$, $j = 1, \ldots, q_i$ are given by

$$\overline{e_j^i} = (\lambda_{j-1}^i \operatorname{sign}(\overline{e_{j-1}^i}))_{eq} \tag{9}$$

for $j = 2, \ldots, q_i$, and $\overline{e_1^i} = e_1^i = y_i - \hat{x}_1^i$ can be obtained directly.

Proof: From (4), (7) and (8), we obtain the following observation error dynamics of $e_j^i = x_j^i - \hat{x}_j^i$, $j = 1, \ldots, q_i$,

$$\begin{aligned} \dot{e}_1^i &= e_2^i - \lambda_1^i \operatorname{sign}(e_1^i) \\ \dot{e}_2^i &= e_3^i + b_2^i(x_2^i, y, u) - b_2^i(\hat{x}_2^i, y, u) - \lambda_2^i \operatorname{sign}(\overline{e_2^i}) \\ &\cdots \quad \cdots \\ \dot{e}_{q_i-1}^i &= e_{q_i}^i + b_{q_i-1}^i(x_2^i, \ldots, x_{q_i-1}^i, y, u) \\ &\quad - b_{q_i-1}^i(\hat{x}_2^i, \ldots, \hat{x}_{q_i-1}^i, y, u) \\ &\quad - \lambda_{q_i-1}^i \operatorname{sign}(\overline{e_{q_i-1}^i}) \\ \dot{e}_{q_i}^i &= a_i(x_d, x_o) + b_{q_i}^i(x_d, x_o, u) + \sum_{j=1}^m c_{ij}(x_d, x_o) d_j \\ &\quad - a_i(\hat{x}_d, \hat{x}_o) - b_{q_i}^i(\hat{x}_d, \hat{x}_o, u) - \lambda_{q_i}^i \operatorname{sign}(\overline{e_{q_i}^i}). \end{aligned} \tag{10}$$

As the known and unknown inputs are bounded, the state does not go to infinity in finite time. Consequently, the observation error is also bounded. By choosing $\lambda_1^i > \sup |e_2^i(t)|$, e_1^i goes to zero in finite time t_1. Moreover, after t_1, we have

$$\overline{e_2^i} = (\lambda_1^i \operatorname{sign}(e_1))_{eq} = e_2^i.$$

Therefore, after t_1, the second error equation becomes

$$\dot{e}_2^i = e_3^i + b_2^i(x_2^i, y, u) - b_2^i(\hat{x}_2^i, y, u) - \lambda_2^i \operatorname{sign}(e_2^i).$$

If $\lambda_2^i > |e_3^i + b_2^i(x_2^i, y, u) - b_2^i(\hat{x}_2^i, y, u)|$, e_2^i goes to zero in finite time $t_2 > t_1$. Therefore, after $t > t_2$

$$\begin{aligned} \overline{e_3^i} &= (\lambda_2^i \operatorname{sign}(e_2))_{eq} = e_3^i \\ \dot{e}_3^i &= e_4^i + b_3^i(x_2^i, x_3^i, y, u) - b_3^i(\hat{x}_2^i, \hat{x}_3^i, y, u) - \lambda_3^i \operatorname{sign}(e_3^i). \end{aligned}$$

We run the procedure up to step q_i, thus after t_{q_i-1}, we have

$$\begin{aligned} \dot{e}_{q_i}^i = a_i(x_d, x_o) + b_{q_i}^i(x_d, x_o, u) + \sum_{j=1}^m c_{ij}(x_d, x_o) d_j \\ - a_i(\hat{x}_d, \hat{x}_o) - b_{q_i}^i(\hat{x}_d, \hat{x}_o, u) - \lambda_{q_i}^i \operatorname{sign}(e_{q_i}^i). \end{aligned}$$

Let $\lambda_{q_i}^i > |a_i(x_d, x_o) + b_{q_i}^i(x_d, x_o, u) + \sum_{j=1}^m c_{ij}(x_d, x_o) d_j - a_i(\hat{x}_d, \hat{x}_o) - b_{q_i}^i(\hat{x}_d, \hat{x}_o, u)|$, e_{q_i} converges to zero in finite time $t_{q_i} > t_{q_i-1}$. □

The equivalent control signal $(v)_{eq}$ for signal v is calculated by low pass filtering the signal v with anti-peaking structure[8]. This antipeaking structure is based on the principle that the observation error information is not used before reaching the sliding manifold linked with this information. Moreover, we reach the manifold one by one in a sort of a recursive fashion. As a result of this, obtain a subdynamics of dimension one is encountered and consequently the peaking phenomena is avoided. More precisely

$$\overline{e_j^i} = (\lambda_{j-1}^i \operatorname{sign}(\overline{e_{j-1}^i}))_{eq} = 0$$

before $\overline{e_{j-1}^i}$ reaching its sliding manifold.

At this point, several important remarks are in order:

Remark 1: Note that the transformation in Lemma 3 decomposed the nonlinear system (2) into two interconnected subsystem with states x_d and x_o. Of these two subsystems, the unknown inputs only appear in one, namely the x_d. The observer proposed in the above discussion will provide state estimates for this subsystem. On the other hand, the x_o subsystem is free of the unknown inputs, and any applicable existing observer can be used to estimate its state $\hat{x}_o$. This estimate is needed

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in observer (8), for estimating the state of x_d. On the other hand, if x_o subsystem is unobservable, or if it is too complex to build an observer for it, $\hat{x}_o$ will not approximate real value of x_o. However, as long as both $\hat{x}_o$ and x_o are bounded, observer (8) will always converge as long as gain $\lambda_{q_i}^i$ is selected large enough.

It is, therefore, possible to let $\hat{x}_o = 0$ and still estimate part of the state. The reason for this desirable property is that x_o is only confined to the last equation. In such a case, we are essentially building a functional sliding mode observer. It is well known, at least in the linear realm, that it is generally possible to design functional observers under less restrictive conditions than full state observers. Such observers can be useful for control purposes and more so for fault diagnosis applications. In fault diagnosis applications observers are essentially used as residual generators. The estimation error of the observer is used as a residual that is used to draw conclusions about the health of the system. In such applications, a full estimate of the actual state may not be necessary and a functional observer may accomplish the fault detection task under less restrictive existence conditions, [23].

Remark 2: Note also that if $q_1 + \cdots + q_m = n$, then the x_o subsystem will actually disappear, and all states can be estimated.

Remark 3: The system form (7) is more general than the output injection or triangular input form, but more conservative than the form (6), which assures uniform observability.

Remark 4: It should be noted that if $q_i = 1,\ i = 1, \ldots, m$, the above theorem does not put any special constraint on the input term. $q_i = 1,\ i = 1, \ldots, m$ means all states of subsystem x_d are measurable. An observer for such a system may be unnecessary for controller synthesis, but will be useful in fault detection and isolation (FDI) applications.

Remark 5: The m observers for $x_d^i (i = 1, \ldots, m)$ subsystems can run in parallel, although in each observer, e_j^i converges to zero if all the e_k^i with $k < j$ have already converged to zero. If we allow states x_d^i to be estimated after x_d^k with $k < i$ have been estimated correctly, then the input term can be in a more general form

$$\begin{aligned}
b_1^i(x_d, x_o, u) &= b_1^i(y_d^i, x_d^1, \ldots, x_d^{i-1}, u)\\
b_2^i(x_d, x_o, u) &= b_2^i(x_2^i, y_d^i, x_d^1, \ldots, x_d^{i-1}, u)\\
&\cdots \qquad \cdots\\
b_{q_i-1}^i(x_d, x_o, u) &= b_{q_i-1}^i(x_2^i, \ldots, x_{q_i-1}^i, y_d^i, x_d^1, \ldots, x_d^{i-1}, u)\\
b_{q_i}^i(x_d, x_o, u) &= b_{q_i}^i(x_d, x_o, u)
\end{aligned}$$

where $y_d^i = [y_i, y_{i+1}, \ldots, y_m]$. Further, if x_o subsystem is independent of x_d and can be estimated correctly by certain observer, then it is not a problem for x_o to appear in all input terms of x_d subsystem.

Remark 6: The NUIO design method reported in [18], transforms the system (2) by $z = T(x)$, where

$$\frac{\partial T(x)}{\partial x} G(x) = 0 \tag{11}$$

such that z is actually an unknown input free subsystem. Then an observer for the reduced order z subsystem is built. Equation (11) is the same as condition (3) of Lemma 1. According to the Frobenius theorem [12], (11) is solvable if and only if $G(x)$ is involutive. Therefore, although involutivity of $G(x)$ is a very strong assumption, we cannot avoid it if total unknown input decoupling is required.

In NUIO design, the original state is obtained through $x = \Phi(z, y^*)$, where y^* is obtained from a nonlinear transformation $y^* = S(y)$. It stated that the inverse function $x = \Phi(z, y^*)$ exists, if and only if

$$\operatorname{rank}\begin{pmatrix} \dfrac{\partial T(x)}{\partial x} \\ \dfrac{\partial S(y)}{\partial x} \end{pmatrix} = n;\ \lim_{\|x\|\to\infty} \|(T(x) \quad S(y))^T\| = \infty. \tag{12}$$

However, the question as to under what conditions there exists $S(y)$ that satisfy conditions (12) is still unresolved. In [18] it is stated that $p > m$ is a necessary condition. Here we show that this is not true at least in theory by analyzing the existence of NUIO using the transformation given in Lemma 1. Note that the relative degree $q_i = 1 (i = 1, \ldots, m)$ means all states of x_d subsystem are measurable. Therefore, the following conditions are sufficient for the existence of a NUIO:

1) $p_i = 1,\ i = 1, \ldots, m$;
2) x_o subsystem is detectable.

Above conditions may be satisfied even if $p = m$. On the other hand, if any relative degree $q_k > 1 (1 \le k \le m)$, it is expected that no NUIO exists. Under this case, the states $x_2^k, \ldots, x_{q_k}^k$ will not be able to derive through a nonlinear transform on y and z, the states of unknown input free subsystem. Therefore, the proposed SMO works under much less conservative conditions than NUIO.

Remark 7: The basic ideas presented in this article, can be used to design robust (functional) sliding mode observers for linear systems. In that case, a transformation based on the special coordinate basis (SCB) theory can be used to decompose the systems into parts with and without the unknown inputs. From there, similar equivalent control bases sliding mode observers can be designed, [22].

III. Illustrative Example

Example 1: A three-phase current motor model [3] is described by following nonlinear equations:

$$\dot{x} = \begin{bmatrix} x_2 \\ -A_1 x_2 - A_2 x_3 \sin x_1 - A_3 \sin 2x_1 \\ -D_1 x_3 + D_2 \cos x_1 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$

where $x = [x_1, x_2, x_3]^T$, and x_1, x_2 and x_3 denote the motor's rotor angle, speed deviation and field flux linkage, respectively. The known inputs are u_1 (nominal mechanical power input) and u_2 (field voltage) and unknown inputs are

$$d_1 = \Delta A_1 x_2 + \Delta A_2 x_3 \sin x_1 + \Delta A_3 \sin 2x_1$$

which represents uncertainties of parameters A_1, A_2 and A_3, and

$$d_2 = \Delta D_1 x_3 + \Delta D_2 \cos x_1$$

which represents uncertainties of parameters D_1, and D_2. All changes are induced by the operating temperature, or component incipient faults. The unknown inputs d_1 or d_2 may be small enough to be neglected in different operation conditions.

Remark 8: It is not always necessary to design the proposed sliding mode observer based on the transformed system model as it is not always easy to find the suitable transformation. For nonlinear systems with linear output equation, we suggest to check the possibility of designing an observer based on the original system equation. In this example we shall illustrate this intuitive design procedure.

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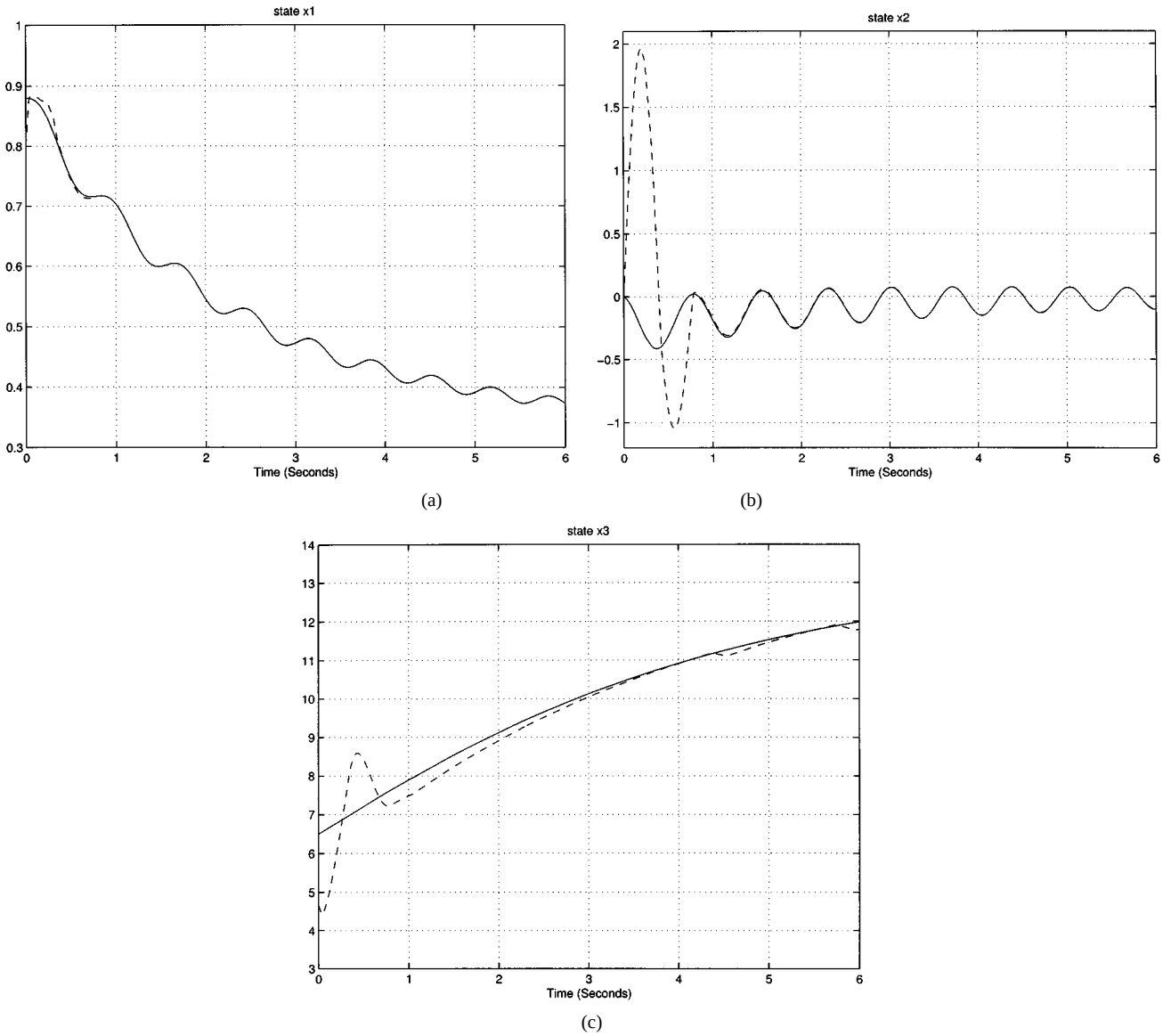

Fig. 1. Simulation results showing actual versus estimated states.

In the following discussion to illustrate the robust SMO design, we shall consider several scenarios and the corresponding observer design for each case. However, for the sake of brevity, simulation results based on only one case are illustrated.

Case 1: Both d_1 and d_2 are Nonzero: In this case, it is noted that $x_1,\ x_2$ is a sub-system in the triangular form of (4) if x_1 is measured, namely $y_1 = x_1$, which will make x_2 to be observed. If we have $y_2 = x_3$, then all states can be estimated by following observer:

$$\begin{aligned}\dot{\hat{x}}_1 &= \hat{x}_2 + \lambda_1 \operatorname{sign}(y_1 - \hat{x}_1)\\ \dot{\hat{x}}_2 &= -A_1\hat{x}_2 - A_2 y_2 \sin y_1 - A_3 \sin 2y_1\\ &\quad + \lambda_2 \operatorname{sign}((\lambda_1 \operatorname{sign}(y_1 - \hat{x}_1)_{eq})\\ \dot{\hat{x}}_3 &= -D_1\hat{x}_3 + D_2 \cos y_1 + \lambda_1 \operatorname{sign}(y_2 - \hat{x}_3).\end{aligned}$$

Case 2: d_1 is Nonzero, d_2 is Zero: In this case, x_3 is an unknown input free subsystem. Fortunately, it is also detectable because $D_1 > 0$. We can build the following observer:

$$\begin{aligned}\dot{\hat{x}}_1 &= \hat{x}_2 + \lambda_1 \operatorname{sign}(y_1 - \hat{x}_1)\\ \dot{\hat{x}}_2 &= -A_1\hat{x}_2 - A_2\hat{x}_3 \sin y_1 - A_3 \sin 2y_1\\ &\quad + \lambda_2 \operatorname{sign}((\lambda_1 \operatorname{sign}(y_1 - \hat{x}_1)_{eq})\\ \dot{\hat{x}}_3 &= -D_1\hat{x}_3 + D_2 \cos y_1.\end{aligned}$$

Case 3: d_1 is Zero, d_2 is Nonzero: In this case, with only one measurement for x_1, the system can be transformed into triangular form and all states can be estimated. Note $g(x) = [0 \quad 0 \quad 1]^T$, it is easy to show that the relative degree of output $y = h(x) = x_1$ corresponding to $g(x)$ is $r_1 = 3$ at point $x_1 \neq k\pi$. The transformation is given by

$$\begin{aligned}\xi_1 &= \phi_1(x) = h_1(x) = x_1\\ \xi_2 &= \phi_2(x) = L_f h_1(x) = x_2\\ \xi_3 &= \phi_3(x) = L_f^2 h_1(x) = f_2(x).\end{aligned}$$

The above transformation is valid at any point other than $x_1 \neq k\pi$, and its inverse transformation is

$$\begin{aligned}x_1 &= \xi_1\\ x_2 &= \xi_2\\ x_3 &= z(\xi) = \frac{\xi_3 + A_1\xi_2 + A_3 \sin 2\xi_1}{-A_2 \sin \xi_1}.\end{aligned}$$

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The nonlinear system under the new coordinates is now described by

$$\begin{aligned}\dot{\xi}_1 &= \xi_2\\ \dot{\xi}_2 &= \xi_3\\ \dot{\xi}_3 &= -A_1\xi_3 - \xi_2(A_2 z(\xi)\ \cos\ \xi_1 + 2A_3\ \cos\ 2\xi_1)\\ &\quad - A_2\ \sin\ \xi_1(-D_1 z(\xi) + D_2\ \cos\ \xi_1)\\ y_1 &= \xi_1.\end{aligned}$$

Then a SMO in the form of (8) can be designed. In fact a SMO can be designed in this case without using the above complicated transformation, in which case the observer is given by

$$\begin{aligned}\dot{\hat{x}}_1 &= \hat{x}_2 + \lambda_1\ \mathrm{sign}(y_1 - \hat{x}_1)\\ \dot{\hat{x}}_2 &= -A_1\hat{x}_2 - A_2\hat{x}_3\ \sin\ y_1 - A_3\ \sin\ 2y_1\\ &\quad + \lambda_2\ \mathrm{sign}((\lambda_1\ \mathrm{sign}(y_1 - \hat{x}_1)_{eq})\\ \dot{\hat{x}}_3 &= -D_1\hat{x}_3 + D_2\ \cos\ y_1 + \lambda_3\ \mathrm{sign}(\overline{e_3})\end{aligned}$$

note the equivalent control signal based on the second equation is

$$(\lambda_2\ \mathrm{sign}(\overline{e_2}))_{eq} = -A_2 e_3\ \sin\ y_1$$

thus

$$\overline{e_3} = \frac{(\lambda_2\ \mathrm{sign}(\overline{e_2}))_{eq}}{-A_2\ \sin\ y_1} = e_3.$$

Obviously, it is true only if $\sin\ y_1 \neq 0$.

For simulation purposes, the parameters in the model have the following values: $A_1 = 0.2703$, $A_2 = 12.01$, $A_3 = -48.04$, $D_1 = 0.3222$, $D_2 = 1.9$, and $\Delta D_2 = 0.6$. The control input $u_1 = 36.19$, $u_2 = 1.9333$. The gain $\lambda_1 = \lambda_2 = \lambda_3 = 200$. The initial state is assumed to be $x_0 = \{0.88, 0.0, 6.5\}$, and initial value of observer is $\xi_0 = \{0.0, 20\}$ and $\eta_0 = 0.8$. The simulation results for case 3 are shown in Fig. 1. Based on the figures, it is clear that the observer's estimate quickly converge to the true state of the system in spite of the considered modeling uncertainties.

IV. Conclusion

A robust sliding mode observer for a class of uncertain nonlinear systems was proposed in this article. The uncertainties in the nonlinear systems are assumed to be structured and representable through unknown input terms to the system. The sliding mode observer design which uses equivalent control concepts is based on a coordinate transformation on the nonlinear system which renders a decomposition of the system into two subsystems. The main advantage of the proposed observer over previously proposed ones is that it can be designed under less restrictive existence conditions. Applications of the proposed observer to model based fault detection is a subject of future research.

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