Sliding Mode Technique and Sliding Mode Observer

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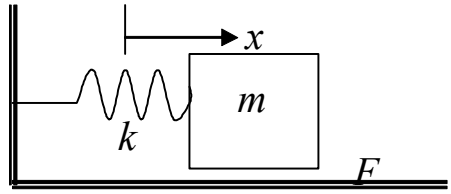


1.

Why do we use sliding mode technique

(1) It is robust respect to bounded uncertainty

(2) It is model-free

Sliding-mode

[1] V.I.Utkin, *Sliding Modes and Their Application in Variable Structure Systems*, MIR Publishers, Moscow, Soviet Union, 1978.

[2] V.I.Utkin, *Sliding Modes in Optimization and Control*, Springer-Verlag, 1992.

There exist discontinuous dynamic

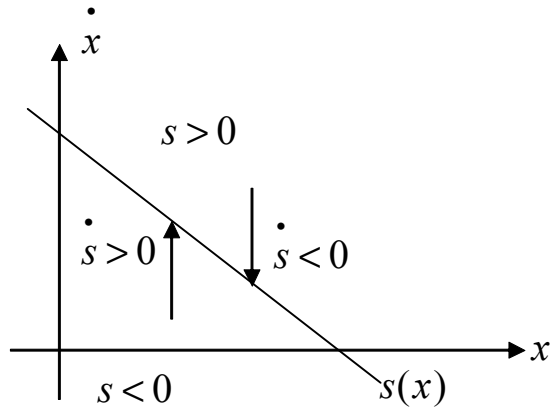
$$m\ddot{x} + kx + F = 0$$

where F is friction as

$$F = \begin{cases} \alpha\dot{x} & \dot{x} \leq 0 \\ -\alpha\dot{x} & \dot{x} < 0 \end{cases}$$

Sliding-mode control,

$$u = \begin{cases} u^{+} & s(x) \geq 0 \\ u^{-} & s(x) < 0 \end{cases}$$



2.

where $s(x) = 0$ is sliding surface. How to design $s(x)$ is key problem of sliding-mode control.

Example 1 *If the tracking error satisfies*

$$e = \left(x - x^d\right) + \lambda\left(\dot{x} - \dot{x}^d\right)$$

We define sliding surface as $s\left(x\right) = 0$, $s\left(x\right) = e$. *If the trajectory convergence to* $s(x)$, $e \to 0$. *So the condition* $s\dot{s} < 0$ *can make the trajectory go to* $s = 0$.

Stability of sliding surface $s = 0$. The dynamic of sliding surface satisfies

$$\dot{s} = -K\,\mathrm{sign}\left(s\right), \quad K \in R^{n\times n}$$

Consider Lyapunov function

$$V = \frac{1}{2}s^T P s, \quad P = P^T > 0$$

The derivative of V is

$$\dot{V} = s^T P \dot{s} = -s^T PK \text{sign}\,(s) = -\left[s_1, s_2 \cdots s_n\right] L \begin{bmatrix} \text{sign}\,(s_1) \\ \text{sign}\,(s_2) \\ \vdots \\ \text{sign}\,(s_n) \end{bmatrix}$$

$$= -\sum_{i=1}^{m} |s_i| \left(l_{ii} + \sum_{j=1, j\neq i}^{m} l_{ij} \text{sign}\,(s_i s_j) \right)$$

where $x_i \text{sign}(x_i) = |x_i|$, $x_i \text{sign}(x_j) = |x_i| \text{sign}(x_i x_j)$, l_{ij} are the elements of the matrix $L = PK$. If

$$l_{ii} > \sum_{j=1, j\neq i}^{m} l_{ij} \tag{1}$$

then we conclude that $\dot{V} < 0$, $s = 0$ becomes stable.

We consider a affine nonlinear system

$$\dot{x} = f(x) + g(x)u \tag{2}$$

equivalent control u_{eq}

$$u_{eq} = -g^{-1} f, \quad 0 < |u_{eq}| < F$$

Sliding mode control u as

$$u = -F \text{sign}\,(s) = \begin{cases} F & s \geq 0 \\ -F & s < 0 \end{cases}$$

We define a dynamic of sliding surface as

$$\dot{s} = f + gu = -gu_{eq} + gu = g\left(u - u_{eq}\right) \tag{3}$$

Consider Lyapunov function

$$V = \frac{1}{2}s^T P s, \quad P = P^T > 0$$

$$\begin{aligned}
&\dot{V} = s^T P g\left(u - u_{eq}\right) = s^T P g\left(F\text{sign}\left(s\right) - u_{eq}\right) \\
&= Pg\left(-Fs^T\text{sign}\left(s\right) - s^T u_{eq}\right) \\
&= Pg\left(-F\left|s\right| - \left|s\right|\frac{u_{eq}}{\text{sign}(s)}\right) \\
&\leq \left\|Pg\right\|\left|s\right|\left(-F - \frac{u_{eq}}{\text{sign}(s)}\right)
\end{aligned}$$

Since $0 < \left|u_{eq}\right| < F$, $\dot{V} < 0$.

In the trajectory tracking case, force the system to trajectory

$$\frac{d}{dt}x_t^* = \varphi(x_t^*)$$

Define $e = x - x^*$. Let

$$\begin{aligned}
&s = Ke, \quad K = K^T > 0 \\
&\dot{s} = K\dot{e} = K\dot{x} + K\dot{x}_t^* = Kf + Kgu - \varphi(x_t^*) \\
&= \left[Kf - \varphi(x_t^*)\right] + Kgu
\end{aligned}$$

equivalent control u_{eq}

$$u_{eq} = -\left(Kg\right)^{-1}\left[Kf - \varphi(x_t^*)\right], \quad 0 < \left|u_{eq}\right| < F$$

$$u = -F sign\,(s) = \begin{cases} F & s \geq 0 \\ -F & s < 0 \end{cases}$$

$$\lim_{t \to \infty} e_t = 0$$

Sliding mode observer

For bounded unknown nonlinear system

$$\begin{aligned} \dot{x} &= f(x, u) \\ y &= Cx \end{aligned}$$

where C is known, $f(x, u)$ is bounded unknown.

$$f(x, u) - Ax = -P^{-1} C^T h(x, u)$$

where $h\,(x_t, u_t)$ is bounded as

$$\overline{h} > \|h\,(x, u)\|$$

Transforms the system into the normal form

$$\dot{x} = Ax + F(x, u), \quad y = Cx \tag{4}$$

where $F(x, u) = f(x, u) - Ax$, A is a special matrix such that the pair (A, C) is observable. Let us construct the sliding mode observer as

$$\begin{aligned} \dot{\widehat{x}} &= A\widehat{x} + S\,(\widehat{x}, e) - Ke \\ \widehat{y} &= C\widehat{x} \end{aligned} \tag{5}$$

where e is output error defined as

$$e = \widehat{y} - y = C\Delta = C\,(\widehat{x}_t - x_t)$$

where Δ_t is observer error, $S\left(\widehat{x}_t, e_t\right)$ is selected

$$S\left(\widehat{x}_t, e\right) = -\rho \frac{P^{-1}C^T C\overline{\Delta}}{\left\|C\overline{\Delta}\right\|} = -\rho P^{-1}C^T \text{sign}(e) \tag{6}$$

where P is a positive defined matrix, ρ is a positive constant, they will be determined after. Clearly the sliding mode observer is not depended on the nonlinear plant , only output y is needed. The derivative of observer error is

$$\begin{aligned} \dot{\Delta} &= A\Delta + S\left(\widehat{x}, e\right) - KC\Delta - F(x,u) \\ &= \left(A - KC\right)\Delta + S\left(\widehat{x}, e\right) - F(x,u) \\ &= A_0\Delta + S\left(\widehat{x}, e\right) - F(x,u) \end{aligned} \tag{7}$$

where $A_0 = A - KC$. Because (A, C) is observable, there exists K such that A_0 is stable. So the following Lyapunov equation has a positive solution P

$$A_0^T P + PA_0 = -Q, \quad Q = Q^T > 0$$

for a positive define matrix Q.

Theorem 1 *If f is bounded, the observer gain satisfies*

$$\rho > \overline{h}$$

then the error between the sliding mode observer and the nonlinear system is asymptotically stable

$$\lim_{t\to\infty} \Delta = 0 \tag{8}$$

Proof. **Proof.** Let consider the following Lyapunov function candidate:

$$V_t = \Delta^T P \Delta$$

Calculate its derivative

$$\dot{V}_t = \Delta^T \left(A_0^T P + P A_0 \right) \Delta + 2\Delta^T P \left[S\left(\widehat{x}, e\right) - F(x,u) \right].$$

Using **A1**

$$F(x_t, u_t) = -P^{-1} C^T h(x,u), \quad \|h(x,u)\| < \rho.$$

If we select $S\left(\overline{x}_t, e_t\right)$ as (6), we have

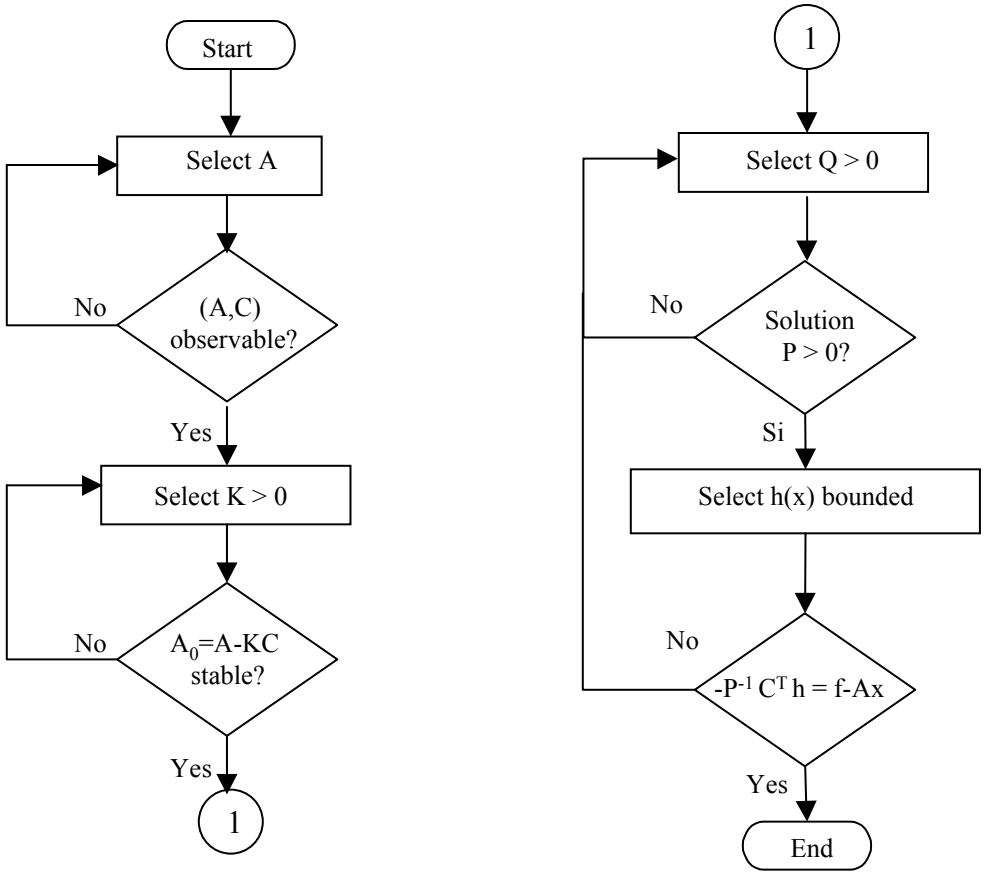
$$\begin{aligned}
\dot{V}_t &= -\Delta^T Q \Delta + 2\Delta^T C^T h(x,u) - 2\tfrac{\Delta^T C^T C \Delta_t}{\|C\Delta\|}\rho \\
&= -\Delta^T Q \Delta + 2\Delta^T C^T h(x,u) - 2\left\|C\Delta\right\|\rho \\
&\leq -\Delta^T Q \Delta + 2\left\|C\Delta\right\|\left(\|h(x,u)\| - \rho\right) \leq 0
\end{aligned}$$

Since $\dot{V}_t \leq 0$, $\Delta \in L_\infty$. From the error equation (7) we also conclude that $\dot{\Delta} \in L_\infty$. Since $\dot{V}_t \leq -\Delta^T Q \Delta$ and V_t is bounded process, Δ is quadratically integrable and bounded $\Delta \in L_2$. Using Barbalat's Lemma we obtain that the observer error Δ is asymptotically stable, so $\lim_{t\to\infty} \Delta = 0$. ■

The design procedure is

Example 2 *The dynamics of an $n-$link robot manipulator may be expressed in the Lagrange form* [**?**]

$$M(q)\ddot{q} + C\left(q, \dot{q}\right)\dot{q} + G(q) + F_d\left(\dot{q}\right) = \tau \tag{9}$$


Start
Select A
No
(A,C) observable?
Yes
Select K > 0
No
A_0=A-KC stable?
Yes
1
1
Select Q > 0
No
Solution P > 0?
Si
Select h(x) bounded
No
$-P^{-1} C^T h = f-Ax$
Yes
End

3.

One-link robot manipulator

$$\dot{x} = f(x, \tau), \quad y = Dx \tag{10}$$

where $x_t = \left[q, \dot{q}\right] \in \Re^2$, $x_1 = q$, $x_2 = \dot{q}$,

$$f(x_t, \tau) = \begin{bmatrix} \dot{q} \\ M^{-1}(q)\left[\tau - C\left(q, \dot{q}\right)\dot{q} - G(q) - F_d\left(\dot{q}\right)\right] \end{bmatrix}$$

$D = [1, 0]$. *Transform into the normal form:*

$$\dot{x} = Ax + F(x, u), \quad y = Dx$$

we can select $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ *to make the pair* (A, D) *observable; If we select* $K = \begin{bmatrix} 2 \\ 2 \end{bmatrix} > 0$, *so* $A_0 = A - KD = \begin{bmatrix} -2 & 1 \\ -1 & 0 \end{bmatrix}$ *is stable; if we use* $Q = \begin{bmatrix} 3 & 1 \\ 8 & 3 \end{bmatrix}$, *the solution of Lyapunov equation*

$$A_0 P + P A_0^T = -Q$$

is positive define $P = \begin{bmatrix} 1.5 & -0.25 \\ 3.25 & 0 \end{bmatrix}$, $\rho = 15$. *Velocity observer,* $e = \widehat{y} - y$

$$\begin{aligned} \dot{\widehat{x}} &= A\widehat{x} - \rho P^{-1} D^T sign(e) - Ke \\ \widehat{y} &= C\widehat{x} \end{aligned}$$