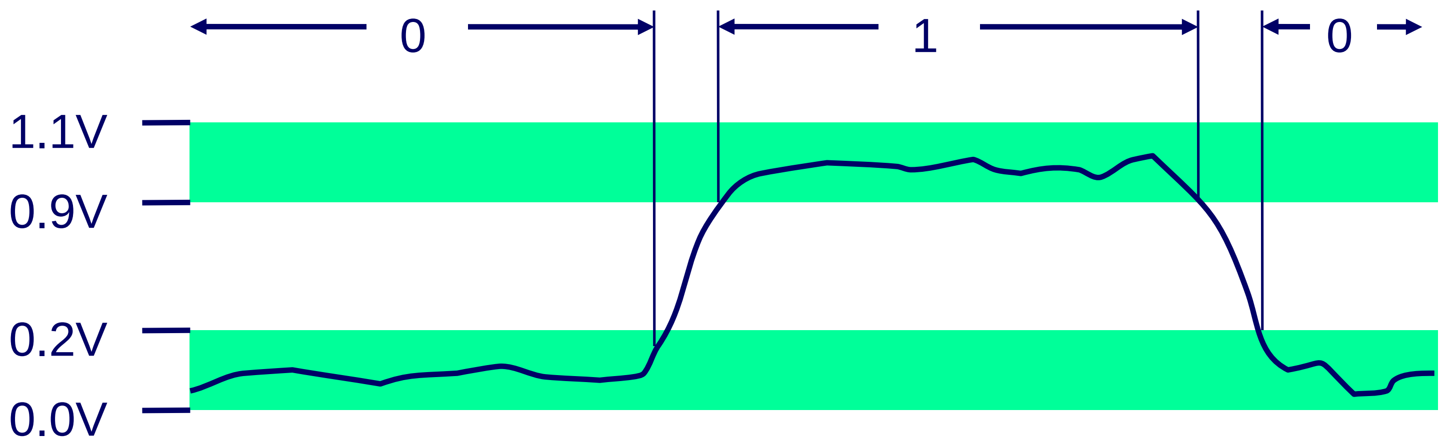
Bits, Bytes, and Integers

Today: Bits, Bytes, and Integers

- **Representing information as bits**
- **Bit-level manipulations**
- **Integers**
  - Representation: unsigned and signed
  - Conversion, casting
  - Expanding, truncating
  - Addition, negation, multiplication, shifting
  - Summary
- **Representations in memory, pointers, strings**

Everything is bits

- **Each bit is 0 or 1**
- **By encoding/interpreting sets of bits in various ways**
  - Computers determine what to do (instructions)
  - … and represent and manipulate numbers, sets, strings, etc…
- **Why bits? Electronic Implementation**
  - Easy to store with bistable elements
  - Reliably transmitted on noisy and inaccurate wires



For example, can count in binary

- **Base 2 Number Representation**
  - Represent 15213_{10} as 11101101101101_2
  - Represent 1.20_{10} as $1.0011001100110011[0011]..._2$
  - Represent 1.5213×10^4 as $1.1101101101101_2 \times 2^{13}$

Encoding Byte Values

- **Byte = 8 bits**
  - Binary 00000000_2 to 11111111_2
  - Decimal: 0_{10} to 255_{10}
  - Hexadecimal 00_{16} to FF_{16}
    - Base 16 number representation
    - Use characters ‘0’ to ‘9’ and ‘A’ to ‘F’
    - Write $FA1D37B_{16}$ in C as
      - 0xFA1D37B
      - 0xfa1d37b

Hex	Decimal	Binary
0	0	0000
1	1	0001
2	2	0010
3	3	0011
4	4	0100
5	5	0101
6	6	0110
7	7	0111
8	8	1000
9	9	1001
A	10	1010
B	11	1011
C	12	1100
D	13	1101
E	14	1110
F	15	1111

Example Data Representations

C Data Type	Typical 32-bit	Typical 64-bit	x86-64
`char`	1	1	1
`short`	2	2	2
`int`	4	4	4
`long`	4	8	8
`float`	4	4	4
`double`	8	8	8
`long double`	–	–	10/16
pointer	4	8	8

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Boolean Algebra

- **Developed by George Boole in 19th Century**
  - Algebraic representation of logic
    - Encode "True" as 1 and "False" as 0

And

- **A&B = 1 when both A=1 and B=1**

&	0	1
0	0	0
1	0	1

Or

- **A|B = 1 when either A=1 or B=1**

|	0	1
0	0	1
1	1	1

Not

- **~A = 1 when A=0**

~	
0	1
1	0

Exclusive-Or (Xor)

- **A^B = 1 when either A=1 or B=1, but not both**

^	0	1
0	0	1
1	1	0

General Boolean Algebras

- **Operate on Bit Vectors**
  - Operations applied bitwise

```
   01101001      01101001      01101001
 & 01010101    | 01010101    ^ 01010101    ~ 01010101
   01000001      01111101      00111100      10101010
```

- **All of the Properties of Boolean Algebra Apply**

Example: Representing & Manipulating Sets

- **Representation**
  - Width w bit vector represents subsets of {0, …, w–1}
  - $a_j = 1$ if $j \in A$

    - 01101001 { 0, 3, 5, 6 }
    - *76543210*

    - 01010101 { 0, 2, 4, 6 }
    - *76543210*
- **Operations**

&	Intersection	01000001	{ 0, 6 }
|	Union	01111101	{ 0, 2, 3, 4, 5, 6 }
^	Symmetric difference	00111100	{ 2, 3, 4, 5 }
~	Complement	10101010	{ 1, 3, 5, 7 }

Bit-Level Operations in C

- **Operations &, |, ~, ^ Available in C**
  - Apply to any "integral" data type
    - long, int, short, char, unsigned
  - View arguments as bit vectors
  - Arguments applied bit-wise
- **Examples (Char data type)**
  - ~0x41 → 0xBE
    - $\sim 01000001_2 \rightarrow 10111110_2$
  - ~0x00 → 0xFF
    - $\sim 00000000_2 \rightarrow 11111111_2$
  - 0x69 & 0x55 → 0x41
    - $01101001_2\ \&\ 01010101_2 \rightarrow 01000001_2$
  - 0x69 | 0x55 → 0x7D
    - $01101001_2 \mid 01010101_2 \rightarrow 01111101_2$

Contrast: Logic Operations in C

- **Contrast to Logical Operators**
  - &&, ||, !
    - View 0 as “False”
    - Anything nonzero as “True”
    - Always return 0 or 1
    - Early termination
- **Examples (char data type)**
  - !0x41 → 0x00
  - !0x00 → 0x01
  - !!0x41 → 0x01
  - 0x69 && 0x55 → 0x01
  - 0x69 || 0x55 → 0x01
  - p && *p (avoids null pointer access)

Contrast: Logic Operations in C

- **Contrast to Logical Operators**
  - &&, ||, !
    - View 0 as “Fal
    - Anything nonze
    - Alway
    - Early
- **Example**
  - !0x41 →
  - !0x00 →
  - !!0x41 →

  - 0x69 && 0x55 → 0x01
  - 0x69 || 0x55 → 0x01
  - p && *p   (avoids null pointer access)

Watch out for && vs. & (and || vs. |)… one of the more common oopsies in C programming

Shift Operations

- **Left Shift: `x << y`**
  - Shift bit-vector **`x`** left **`y`** positions
    - – Throw away extra bits on left
    - Fill with 0's on right
- **Right Shift: `x >> y`**
  - Shift bit-vector **`x`** right **`y`** positions
    - Throw away extra bits on right
  - Logical shift
    - Fill with 0's on left
  - Arithmetic shift
    - Replicate most significant bit on left
- **Undefined Behavior**
  - Shift amount < 0 or ≥ word size

Argument `x`	01100010
<< 3	00010*000*
Log. >> 2	*00*011000
Arith. >> 2	*00*011000

Argument `x`	10100010
<< 3	00010*000*
Log. >> 2	*00*101000
Arith. >> 2	*11*101000

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Encoding Integers

Unsigned

$$B2U(X) = \sum_{i=0}^{w-1} x_i \cdot 2^i$$

Two's Complement

$$B2T(X) = -x_{w-1} \cdot 2^{w-1} + \sum_{i=0}^{w-2} x_i \cdot 2^i$$

Sign Bit

```
short int x =  15213;
short int y = -15213;
```

- **C `short` 2 bytes long**

	Decimal	Hex	Binary
x	15213	**3B 6D**	**00111011 01101101**
y	-15213	**C4 93**	**11000100 10010011**

- **Sign Bit**
  - For 2's complement, most significant bit indicates sign
    - 0 for nonnegative
    - 1 for negative

Two-complement Encoding Example (Cont.)

```
x =      15213: 00111011 01101101
y =     -15213: 11000100 10010011
```

Weight	15213		-15213	
1	1	1	1	1
2	0	0	1	2
4	1	4	0	0
8	1	8	0	0
16	0	0	1	16
32	1	32	0	0
64	1	64	0	0
128	0	0	1	128
256	1	256	0	0
512	1	512	0	0
1024	0	0	1	1024
2048	1	2048	0	0
4096	1	4096	0	0
8192	1	8192	0	0
16384	0	0	1	16384
-32768	0	0	1	-32768
Sum		**15213**		**-15213**

Numeric Ranges

- **Unsigned Values**
  - *UMin* = 0
    000…0
  - *UMax* = $2^w - 1$
    111…1

- **Two's Complement Values**
  - *TMin* = -2^{w-1}
    100…0
  - *TMax* = $2^{w-1} - 1$
    011…1

- **Other Values**
  - Minus 1
    111…1

Values for *W* = 16

	Decimal	Hex	Binary
`UMax`	65535	`FF FF`	`11111111 11111111`
`TMax`	32767	`7F FF`	`01111111 11111111`
`TMin`	-32768	`80 00`	`10000000 00000000`
`-1`	-1	`FF FF`	`11111111 11111111`
`0`	0	`00 00`	`00000000 00000000`

Values for Different Word Sizes

	W			
	8	**16**	**32**	**64**
UMax	255	65,535	4,294,967,295	18,446,744,073,709,551,615
TMax	127	32,767	2,147,483,647	9,223,372,036,854,775,807
TMin	-128	-32,768	-2,147,483,648	-9,223,372,036,854,775,808

- **Observations**
  - $| TMin | \quad = \quad TMax + 1$
    - Asymmetric range
  - $UMax \quad = \quad 2 * TMax + 1$
- **C Programming**
  - #include <limits.h>
  - Declares constants, e.g.,
    - ULONG_MAX
    - LONG_MAX
    - LONG_MIN
  - Values platform specific

Unsigned & Signed Numeric Values

X	B2U(X)	B2T(X)
0000	0	0
0001	1	1
0010	2	2
0011	3	3
0100	4	4
0101	5	5
0110	6	6
0111	7	7
1000	8	–8
1001	9	–7
1010	10	–6
1011	11	–5
1100	12	–4
1101	13	–3
1110	14	–2
1111	15	–1

- **Equivalence**
  - Same encodings for nonnegative values
- **Uniqueness**
  - Every bit pattern represents unique integer value
  - Each representable integer has unique bit encoding
- **$\Rightarrow$ Can Invert Mappings**
  - U2B(x) = B2U^{-1}(x)
    - Bit pattern for unsigned integer
  - T2B(x) = B2T^{-1}(x)
    - Bit pattern for two's comp integer

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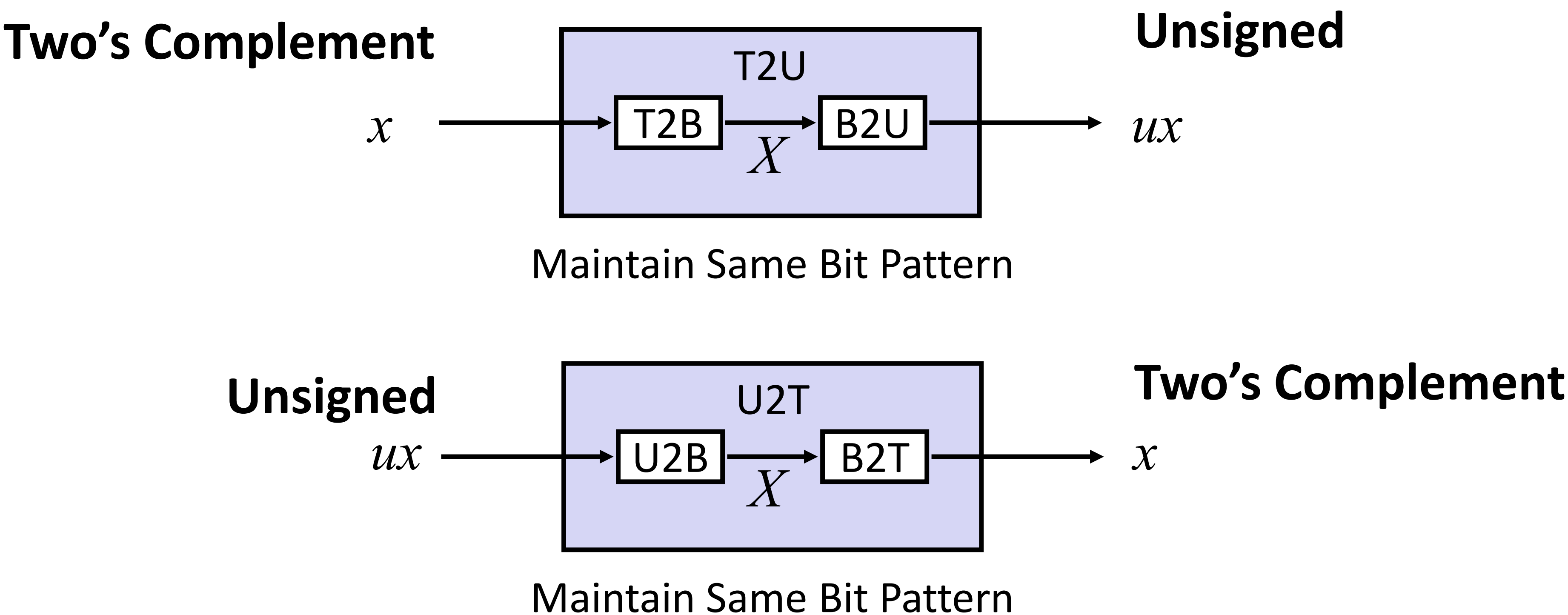
Mapping Between Signed & Unsigned



- **Mappings between unsigned and two's complement numbers:**
  Keep bit representations and reinterpret

Mapping Signed ↔ Unsigned

Bits	Signed	Unsigned
0000	0	0
0001	1	1
0010	2	2
0011	3	3
0100	4	4
0101	5	5
0110	6	6
0111	7	7
1000	-8	8
1001	-7	9
1010	-6	10
1011	-5	11
1100	-4	12
1101	-3	13
1110	-2	14
1111	-1	15

T2U →

← U2T

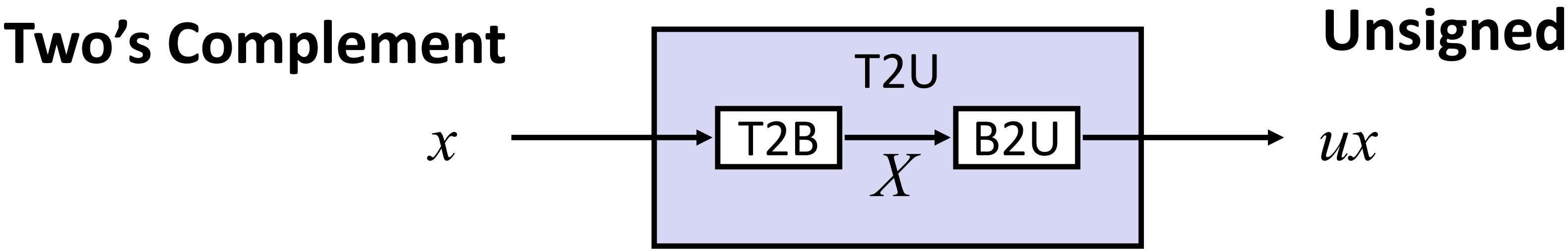
Mapping Signed ↔ Unsigned

Bits	Signed	Unsigned
0000	0	0
0001	1	1
0010	2	2
0011	3	3
0100	4	4
0101	5	5
0110	6	6
0111	7	7
1000	-8	8
1001	-7	9
1010	-6	10
1011	-5	11
1100	-4	12
1101	-3	13
1110	-2	14
1111	-1	15

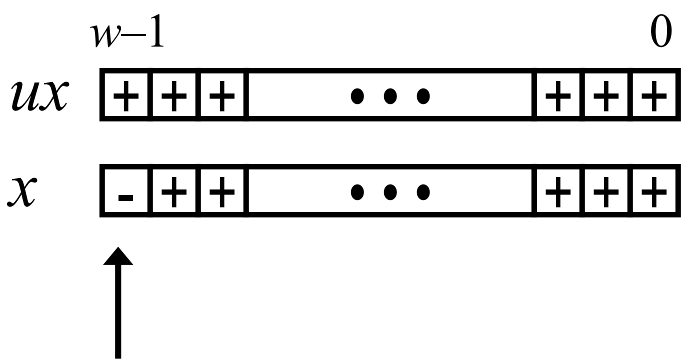
=

+/- 16

Relation between Signed & Unsigned



Maintain Same Bit Pattern



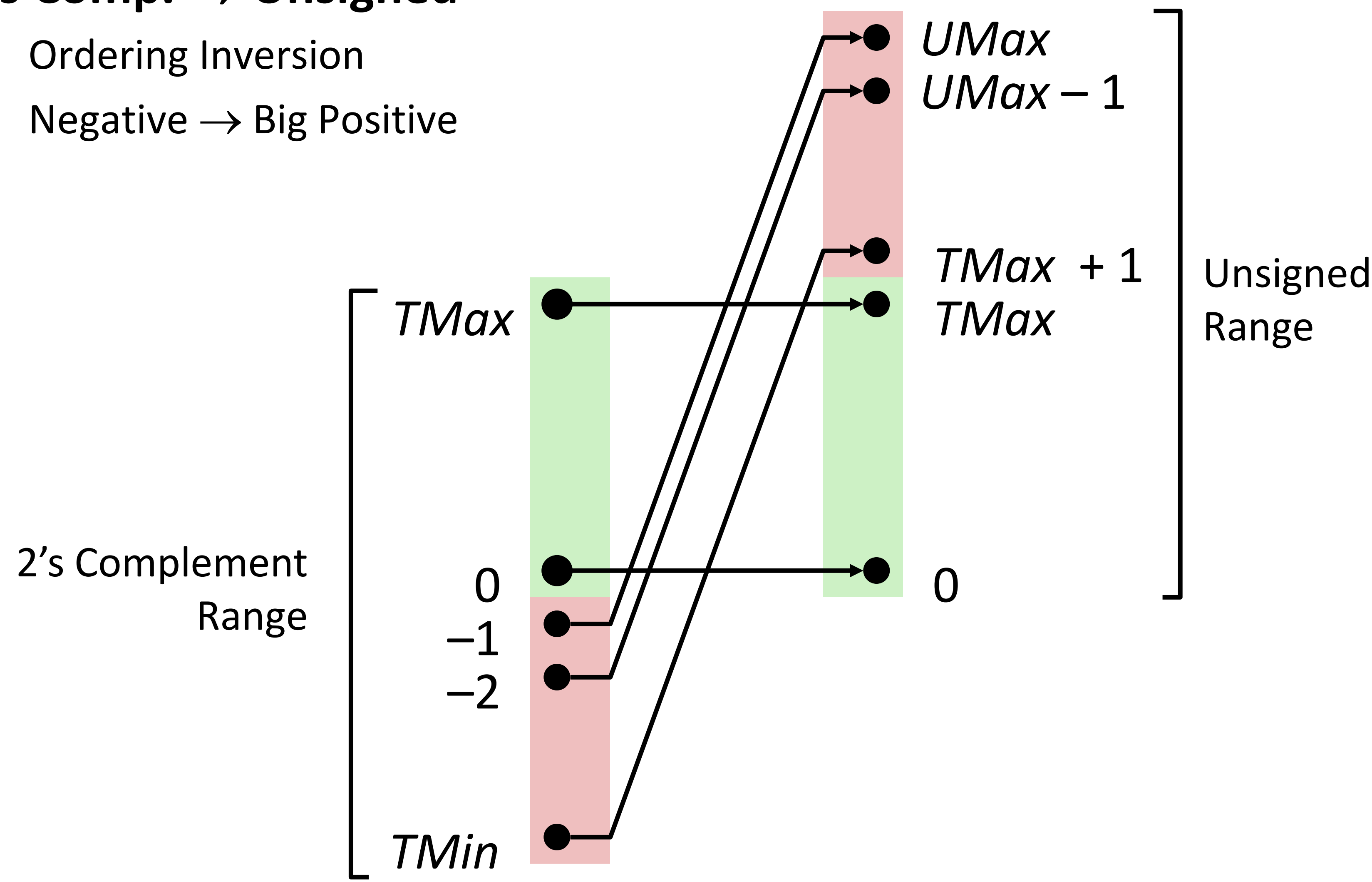
Large negative weight

becomes

Large positive weight

Conversion Visualized

- **2's Comp. → Unsigned**
  - Ordering Inversion
  - Negative → Big Positive



Signed vs. Unsigned in C

- **Constants**
  - By default are considered to be signed integers
  - Unsigned if have “U” as suffix

    ```
    0U, 4294967259U
    ```
- **Casting**
  - Explicit casting between signed & unsigned same as U2T and T2U

    ```
    int tx, ty;
    unsigned ux, uy;
    tx = (int) ux;
    uy = (unsigned) ty;
    ```
  - Implicit casting also occurs via assignments and procedure calls

    ```
    tx = ux;
    uy = ty;
    ```

Casting Surprises

- **Expression Evaluation**
  - If there is a mix of unsigned and signed in single expression, ***signed values implicitly cast to unsigned***
  - Including comparison operations **<, >, ==, <=, >=**
  - Examples for $W = 32$: **TMIN = -2,147,483,648 , TMAX = 2,147,483,647**

Constant$_1$	Constant$_2$	Relation	Evaluation
0	0U	**==**	**unsigned**
-1	0	**<**	**signed**
-1	0U	**>**	**unsigned**
2147483647	-2147483647-1	**>**	**signed**
2147483647U	-2147483647-1	**<**	**unsigned**
-1	-2	**>**	**signed**
(unsigned)-1	-2	**>**	**unsigned**
2147483647	2147483648U	**<**	**unsigned**
2147483647	(int) 2147483648U	**>**	**signed**

Summary
Casting Signed ↔ Unsigned: Basic Rules

- **Bit pattern is maintained**
- **But reinterpreted**
- **Can have unexpected effects: adding or subtracting 2^w**

- **Expression containing signed and unsigned int**
  - `int` is cast to `unsigned`!!

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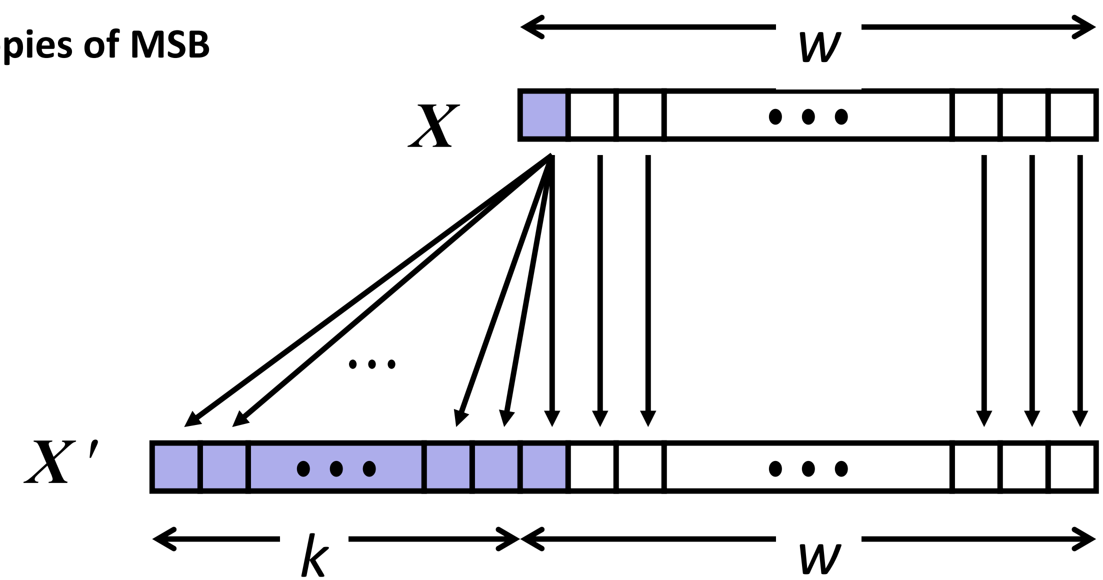
Sign Extension

- **Task:**
  - Given *w*-bit signed integer *x*
  - Convert it to *w*+*k*-bit integer with same value
- **Rule:**
  - Make *k* copies of sign bit:
  - $X' = \underbrace{x_{w-1},\ldots, x_{w-1}}_{\textbf{\textit{k} copies of MSB}}, x_{w-1}, x_{w-2},\ldots, x_0$



Sign Extension Example

```
short int x =  15213;
int       ix = (int) x;
short int y = -15213;
int       iy = (int) y;
```

	Decimal	Hex	Binary
x	15213	3B 6D	00111011 01101101
ix	15213	00 00 3B 6D	00000000 00000000 00111011 01101101
y	-15213	C4 93	11000100 10010011
iy	-15213	FF FF C4 93	11111111 11111111 11000100 10010011

- Converting from smaller to larger integer data type
- C automatically performs sign extension

Summary:
Expanding, Truncating: Basic Rules

- **Expanding (e.g., short int to int)**
  - Unsigned: zeros added
  - Signed: sign extension
  - Both yield expected result

- **Truncating (e.g., unsigned to unsigned short)**
  - Unsigned/signed: bits are truncated
  - Result reinterpreted
  - Unsigned: mod operation
  - Signed: similar to mod
  - For small numbers yields expected behavior

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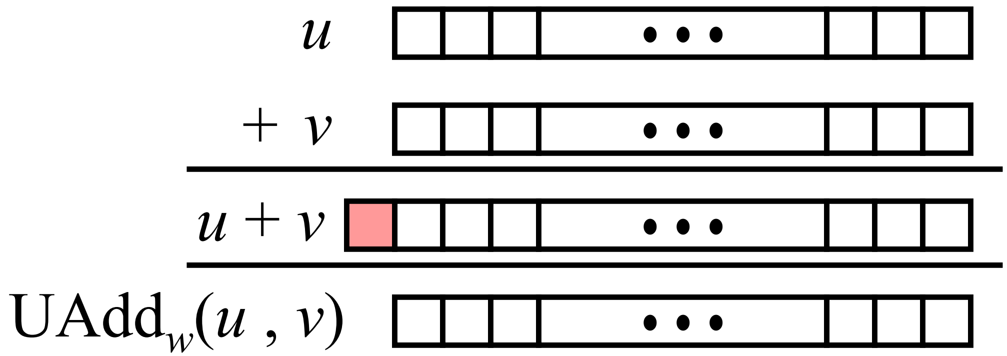
Unsigned Addition

Operands: *w* bits

True Sum: *w*+1 bits

Discard Carry: *w* bits



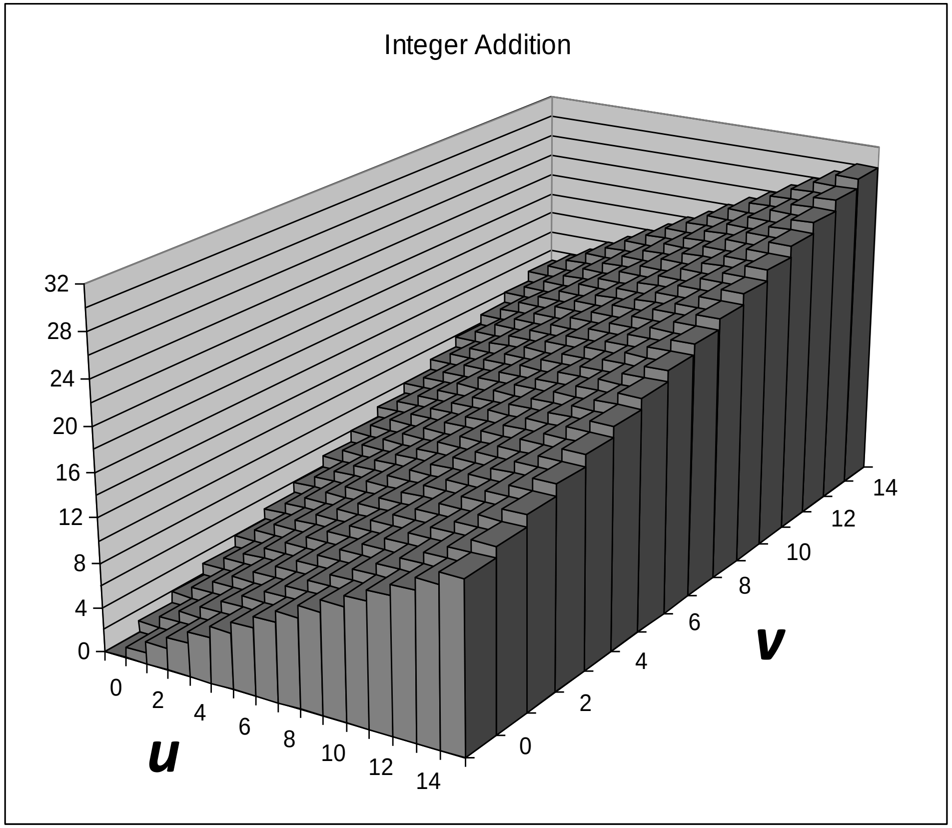
- **Standard Addition Function**
  - Ignores carry output
- **Implements Modular Arithmetic**

$s \quad = \quad \mathrm{UAdd}_w(u\ , v) \quad = \quad u + v \mod 2^w$

Visualizing (Mathematical) Integer Addition

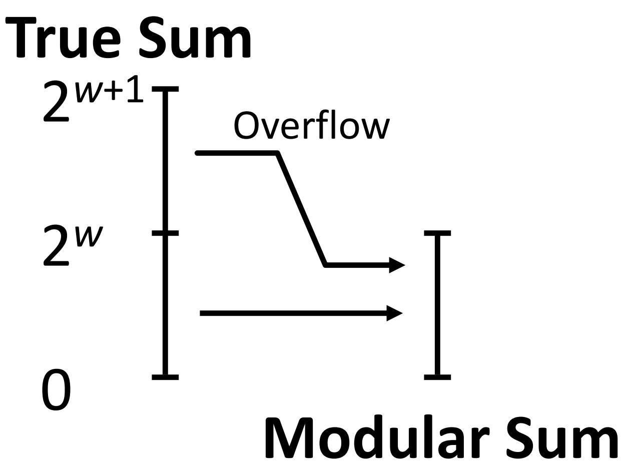
- **Integer Addition**
  - 4-bit integers *u*, *v*
  - Compute true sum $\text{Add}_4(u, v)$
  - Values increase linearly with *u* and *v*
  - Forms planar surface

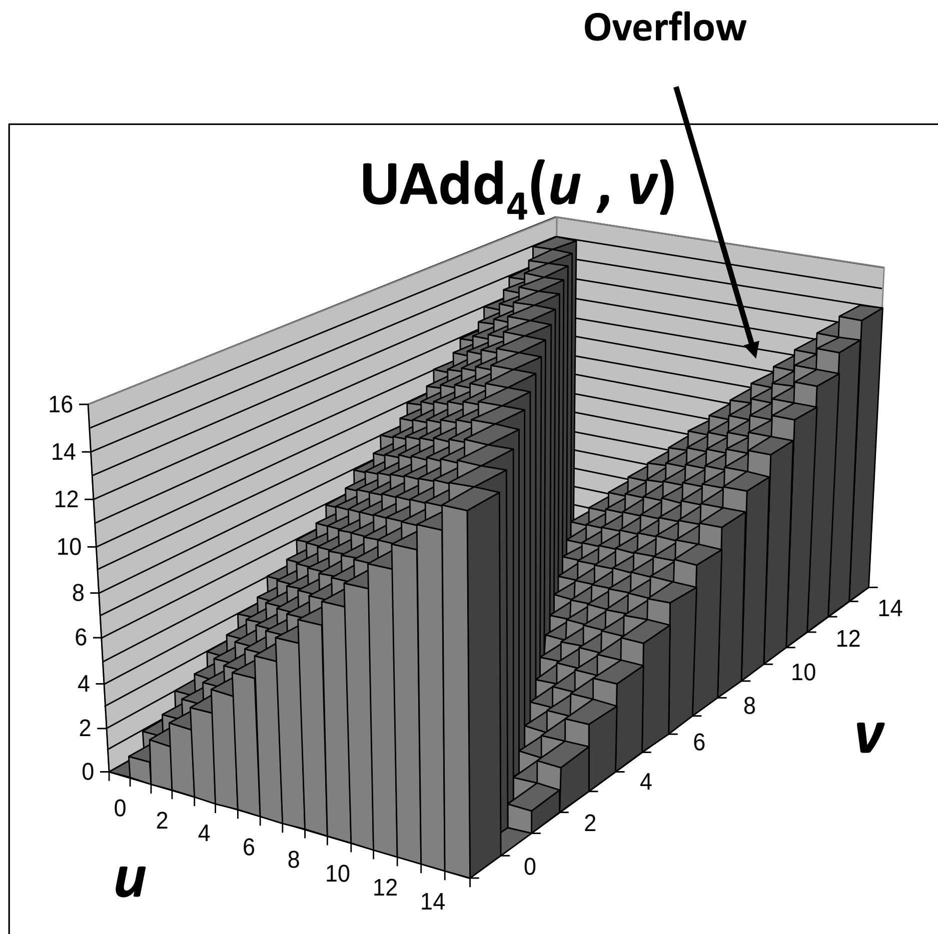
$\text{Add}_4(u, v)$



Visualizing Unsigned Addition

- **Wraps Around**
  - If true sum ≥ 2^w
  - At most once



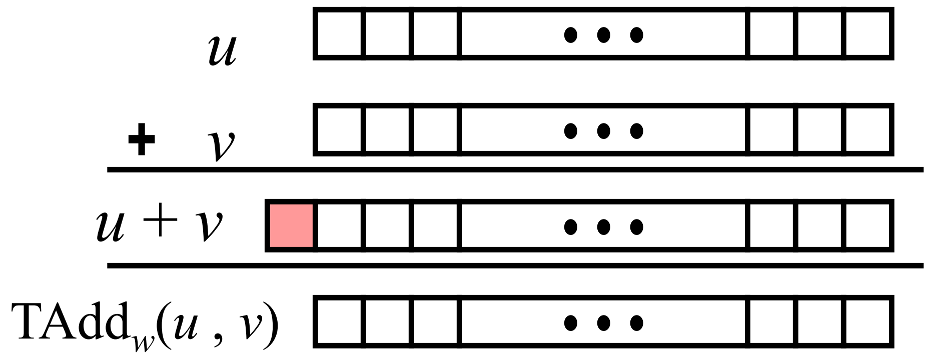


Two’s Complement Addition

Operands: *w* bits

True Sum: *w*+1 bits

Discard Carry: *w* bits



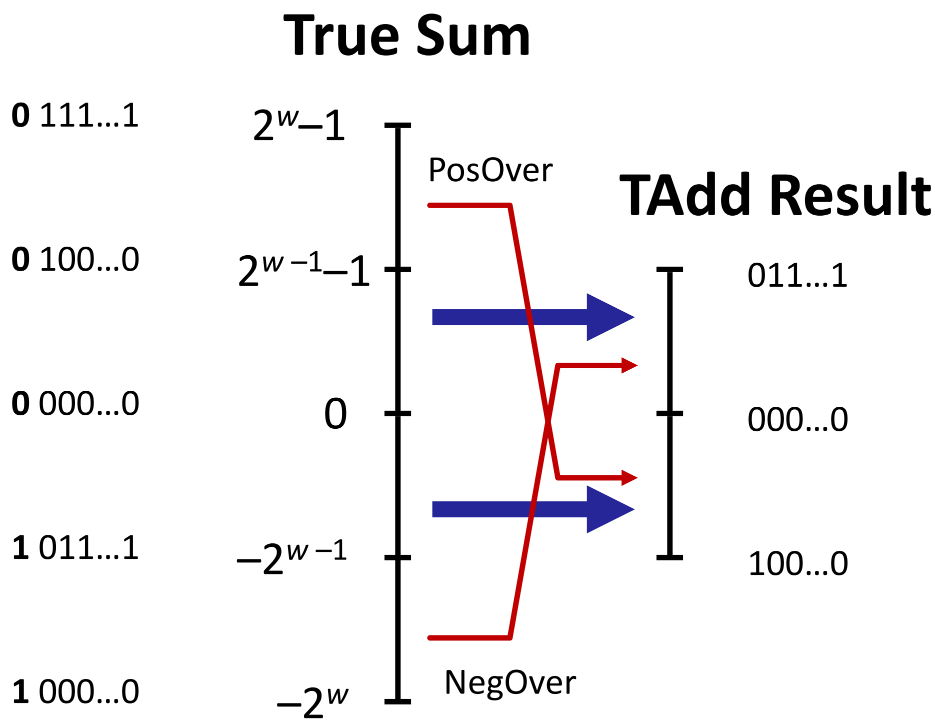
- **TAdd and UAdd have Identical Bit-Level Behavior**
  - Signed vs. unsigned addition in C:

```
int s, t, u, v;
s = (int) ((unsigned) u + (unsigned) v);
t = u + v
```

  - Will give `s == t`

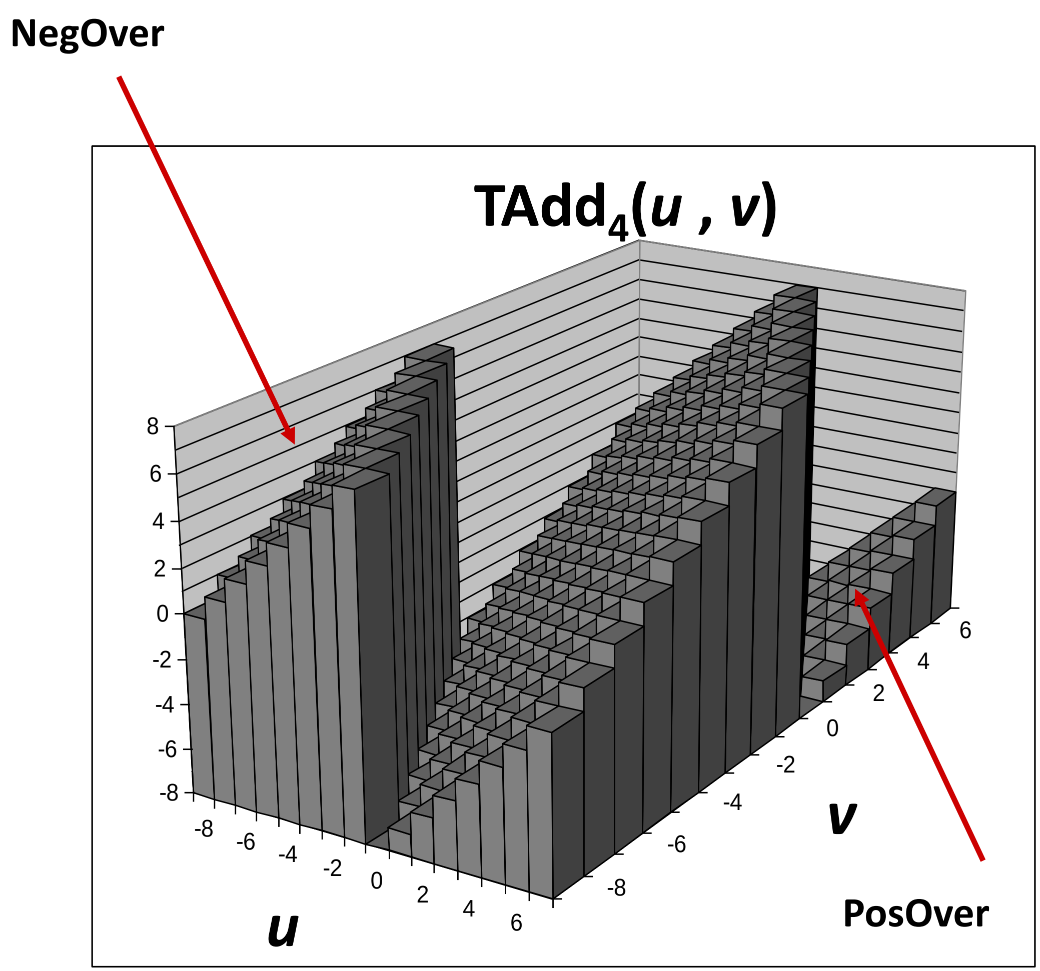
TAdd Overflow

- **Functionality**
  - True sum requires w+1 bits
  - Drop off MSB
  - Treat remaining bits as 2's comp. integer



Visualizing 2's Complement Addition

- **Values**
  - 4-bit two's comp.
  - Range from -8 to +7
- **Wraps Around**
  - If sum $\geq 2^{w-1}$
    - Becomes negative
    - At most once
  - If sum $< -2^{w-1}$
    - Becomes positive
    - At most once



Multiplication

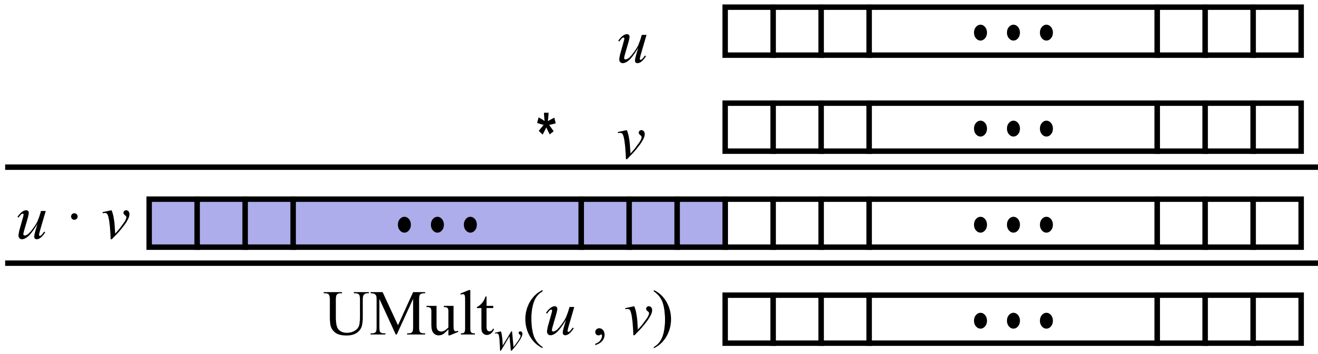
- **Goal: Computing Product of *w*-bit numbers *x*, *y***
  - Either signed or unsigned
- **But, exact results can be bigger than *w* bits**
  - Unsigned: up to 2*w* bits
    - Result range: $0 \leq x * y \leq (2^{w} - 1)^{2} = 2^{2w} - 2^{w+1} + 1$
  - Two's complement min (negative): Up to 2*w*-1 bits
    - Result range: $x * y \geq (-2^{w-1}) * (2^{w-1} - 1) = -2^{2w-2} + 2^{w-1}$
  - Two's complement max (positive): Up to 2*w* bits, but only for $(TMin_w)^2$
    - Result range: $x * y \leq (-2^{w-1})^{2} = 2^{2w-2}$
- **So, maintaining exact results…**
  - would need to keep expanding word size with each product computed
  - is done in software, if needed
    - e.g., by "arbitrary precision" arithmetic packages

Unsigned Multiplication in C

Operands: *w* bits

True Product: 2**w* bits

Discard *w* bits: *w* bits



- **Standard Multiplication Function**
  - Ignores high order *w* bits
- **Implements Modular Arithmetic**

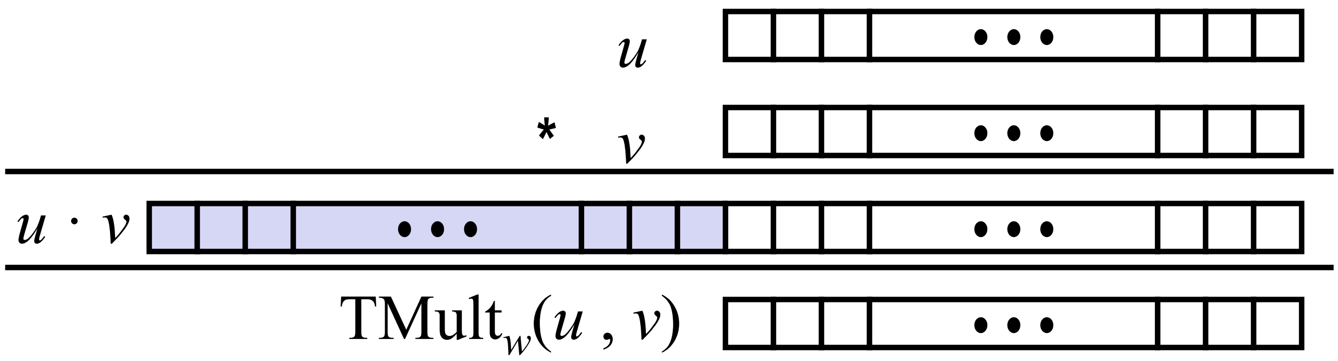
$\text{UMult}_w(u, v) = u \cdot v \bmod 2^w$

Signed Multiplication in C

Operands: *w* bits

True Product: 2**w* bits

Discard *w* bits: *w* bits



- **Standard Multiplication Function**
  - Ignores high order *w* bits
  - Some of which are different for signed vs. unsigned multiplication
  - Lower bits are the same

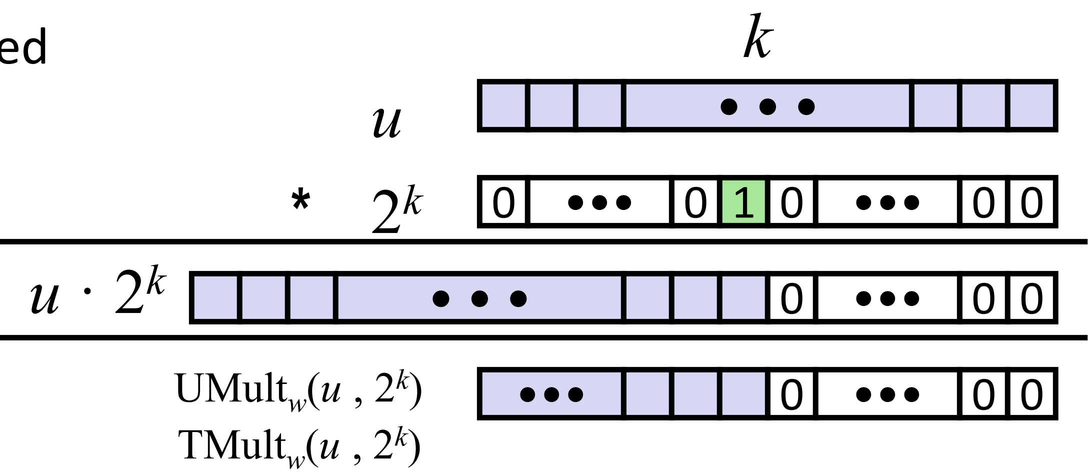
Power-of-2 Multiply with Shift

- **Operation**
  - **`u << k`** gives **`u * 2`**^{k}
  - Both signed and unsigned

Operands: *w* bits

True Product: *w*+*k* bits

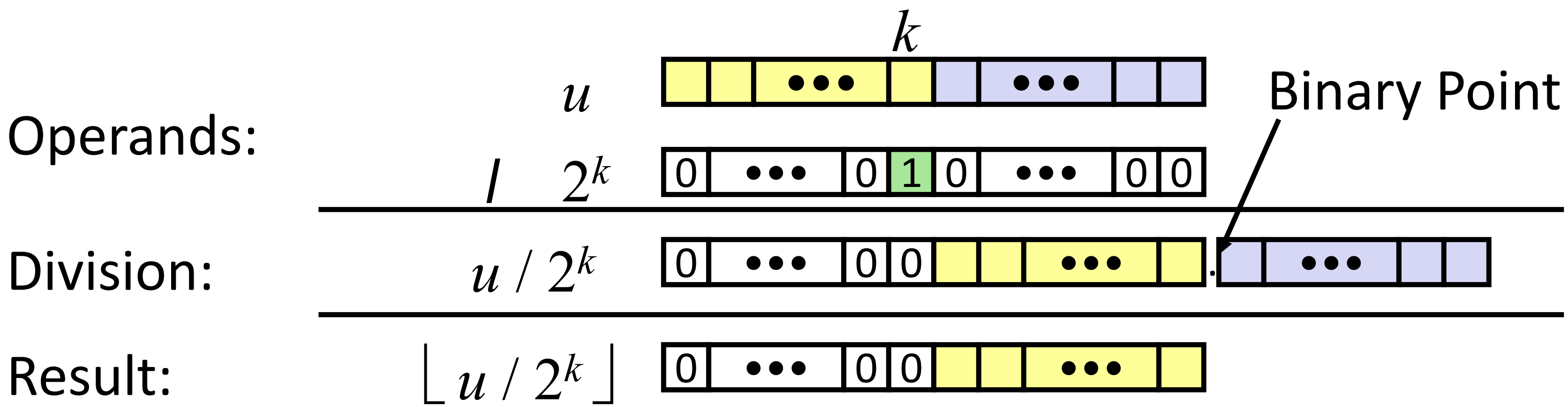
Discard *k* bits: *w* bits



- **Examples**
  - **`u << 3          ==   u * 8`**
  - **`(u << 5) – (u << 3)==    u * 24`**
  - Most machines shift and add faster than multiply
    - Compiler generates this code automatically

Unsigned Power-of-2 Divide with Shift

- **Quotient of Unsigned by Power of 2**
  - `u >> k` gives $\lfloor$ `u / 2`^{k} $\rfloor$
  - Uses logical shift



	Division	Computed	Hex	Binary
`x`	15213	15213	`3B 6D`	`00111011 01101101`
`x >> 1`	7606.5	7606	`1D B6`	`00011101 10110110`
`x >> 4`	950.8125	950	`03 B6`	`00000011 10110110`
`x >> 8`	59.4257813	59	`00 3B`	`00000000 00111011`

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Arithmetic: Basic Rules

- **Addition:**
  - Unsigned/signed: Normal addition followed by truncate, same operation on bit level
  - Unsigned: addition mod 2^w
    - Mathematical addition + possible subtraction of 2^w
  - Signed: modified addition mod 2^w (result in proper range)
    - Mathematical addition + possible addition or subtraction of 2^w

- **Multiplication:**
  - Unsigned/signed: Normal multiplication followed by truncate, same operation on bit level
  - Unsigned: multiplication mod 2^w
  - Signed: modified multiplication mod 2^w (result in proper range)

Why Should I Use Unsigned?

- ***Don't*** **use without understanding implications**
  - Easy to make mistakes

```
unsigned i;
for (i = cnt-2; i >= 0; i--)
  a[i] += a[i+1];
```

  - Can be very subtle

```
#define DELTA sizeof(int)
int i;
for (i = CNT; i-DELTA >= 0; i-= DELTA)
  . . .
```

Counting Down with Unsigned

- **Proper way to use unsigned as loop index**

```
unsigned i;
for (i = cnt-2; i < cnt; i--)
  a[i] += a[i+1];
```

- **See Robert Seacord, *Secure Coding in C and C++***
  - C Standard guarantees that unsigned addition will behave like modular arithmetic
    - 0 – 1 → *UMax*

- **Even better**

```
size_t i;
for (i = cnt-2; i < cnt; i--)
  a[i] += a[i+1];
```

  - Data type `size_t` defined as unsigned value with length = word size
  - Code will work even if `cnt` = *UMax*
  - What if `cnt` is signed and < 0?

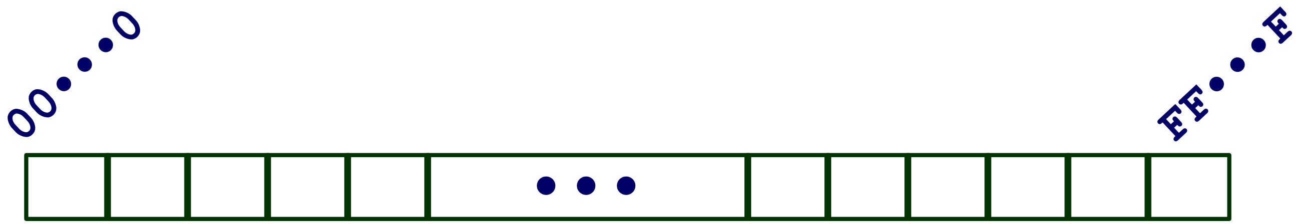
Why Should I Use Unsigned? (cont.)

- ***Do*** **Use When Performing Modular Arithmetic**
  - Multiprecision arithmetic
- ***Do*** **Use When Using Bits to Represent Sets**
  - Logical right shift, no sign extension

Today: Bits, Bytes, and Integers

- Representing information as bits
- Bit-level manipulations
- Integers
  - Representation: unsigned and signed
  - Conversion, casting
  - Expanding, truncating
  - Addition, negation, multiplication, shifting
  - Summary
- **Representations in memory, pointers, strings**

Byte-Oriented Memory Organization



- **Programs refer to data by address**
  - Conceptually, envision it as a very large array of bytes
    - In reality, it’s not, but can think of it that way
  - An address is like an index into that array
    - and, a pointer variable stores an address

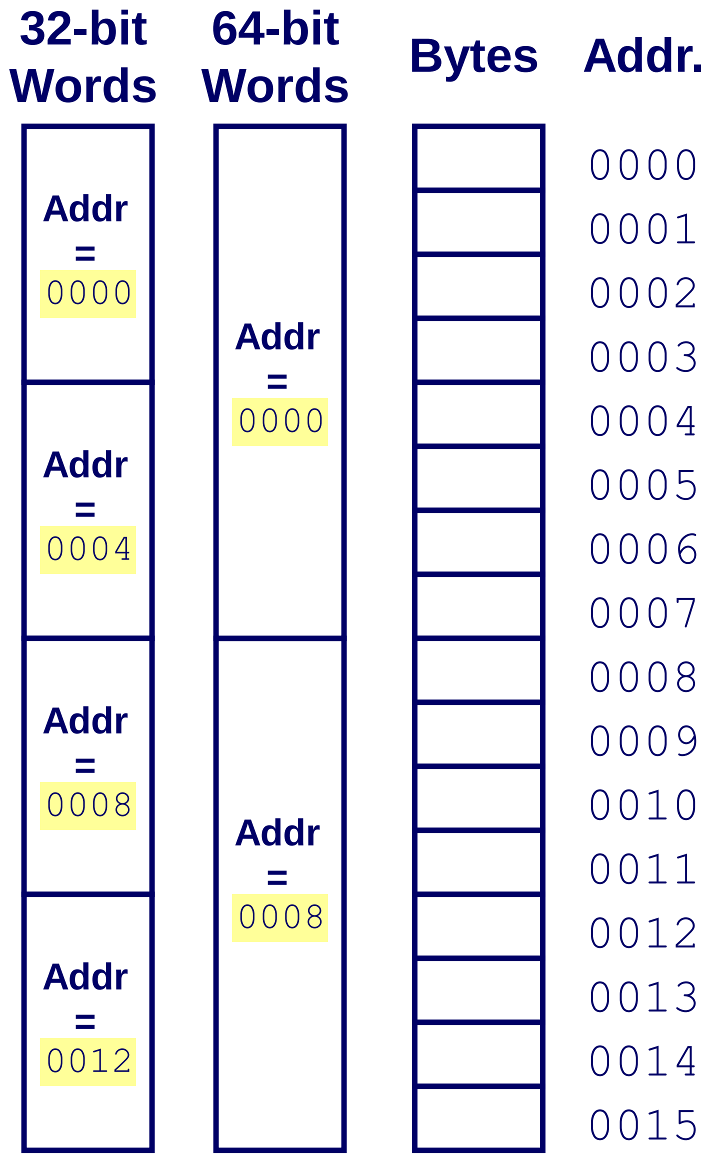
- **Note: system provides private address spaces to each “process”**
  - Think of a process as a program being executed
  - So, a program can clobber its own data, but not that of others

Machine Words

- **Any given computer has a "Word Size"**
  - Nominal size of integer-valued data
    - and of addresses
  - Until recently, most machines used 32 bits (4 bytes) as word size
    - Limits addresses to 4GB (2^{32} bytes)
  - Increasingly, machines have 64-bit word size
    - Potentially, could have 18 EB (exabytes) of addressable memory
    - That's 18.4 X 10^{18}
  - Machines still support multiple data formats
    - Fractions or multiples of word size
    - Always integral number of bytes

Word-Oriented Memory Organization

- **Addresses Specify Byte Locations**
  - Address of first byte in word
  - Addresses of successive words differ by 4 (32-bit) or 8 (64-bit)



Example Data Representations

C Data Type	Typical 32-bit	Typical 64-bit	x86-64
`char`	1	1	1
`short`	2	2	2
`int`	4	4	4
`long`	4	8	8
`float`	4	4	4
`double`	8	8	8
`long double`	–	–	10/16
pointer	4	8	8

Byte Ordering

- **So, how are the bytes within a multi-byte word ordered in memory?**
- **Conventions**
  - Big Endian: Sun, PPC Mac, Internet
    - Least significant byte has highest address
  - Little Endian: x86, ARM processors running Android, iOS, and Windows
    - Least significant byte has lowest address

Byte Ordering Example

- **Example**
  - Variable x has 4-byte value of 0x01234567
  - Address given by &x is 0x100

Big Endian

		0x100	0x101	0x102	0x103		
		01	23	45	67		

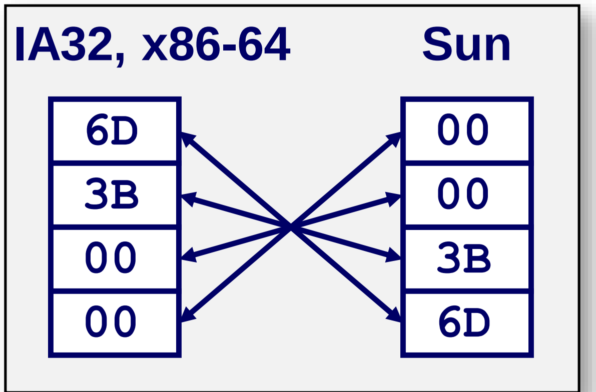
Little Endian

		0x100	0x101	0x102	0x103		
		67	45	23	01		

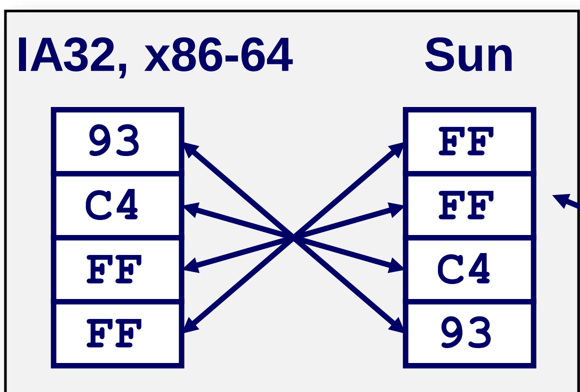
Representing Integers

Decimal:	`15213`
Binary:	`0011 1011 0110 1101`
Hex:	`3 B 6 D`

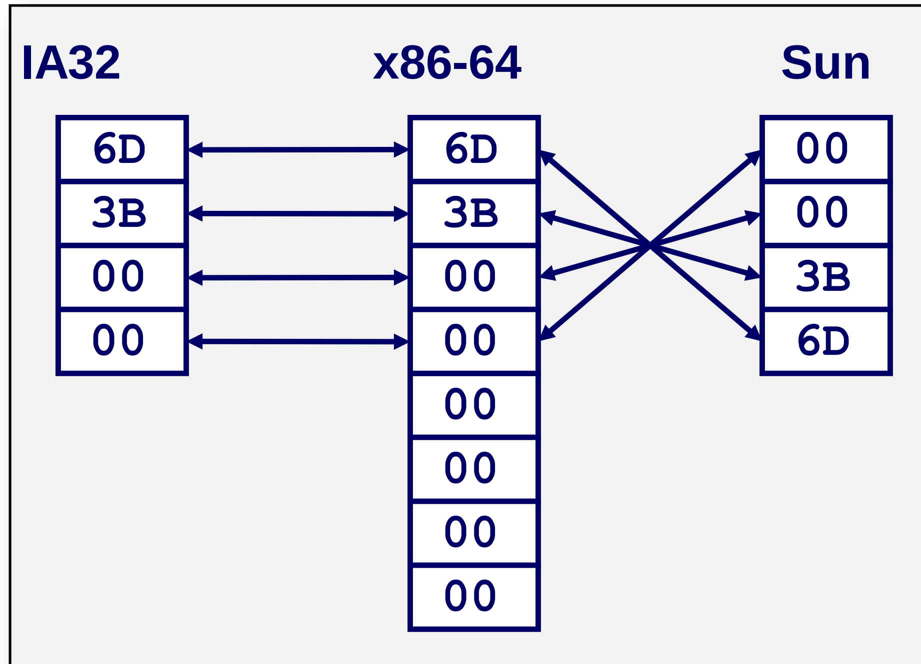
`int A = 15213;`



`int B = -15213;`



`long int C = 15213;`



Two's complement representation

Examining Data Representations

- **Code to Print Byte Representation of Data**
  - Casting pointer to unsigned char * allows treatment as a byte array

```
typedef unsigned char *pointer;

void show_bytes(pointer start, size_t len){
  size_t i;
  for (i = 0; i < len; i++)
    printf("%p\t0x%.2x\n",start+i, start[i]);
  printf("\n");
}
```

Printf directives:

%p: Print pointer

%x: Print Hexadecimal

show_bytes Execution Example

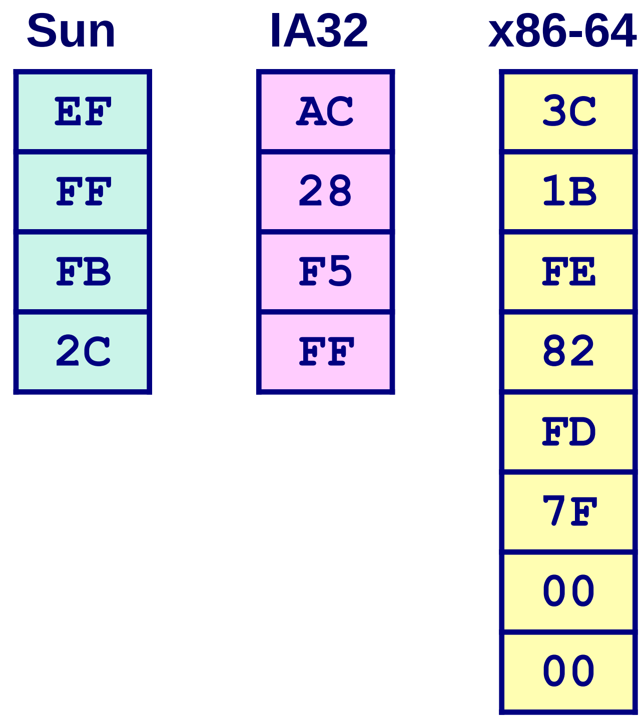
```
int a = 15213;
printf("int a = 15213;\n");
show_bytes((pointer) &a, sizeof(int));
```

Result (Linux x86-64):

```
int a = 15213;
0x7fffb7f71dbc    6d
0x7fffb7f71dbd    3b
0x7fffb7f71dbe    00
0x7fffb7f71dbf    00
```

Representing Pointers

```
int B = -15213;
int *P = &B;
```



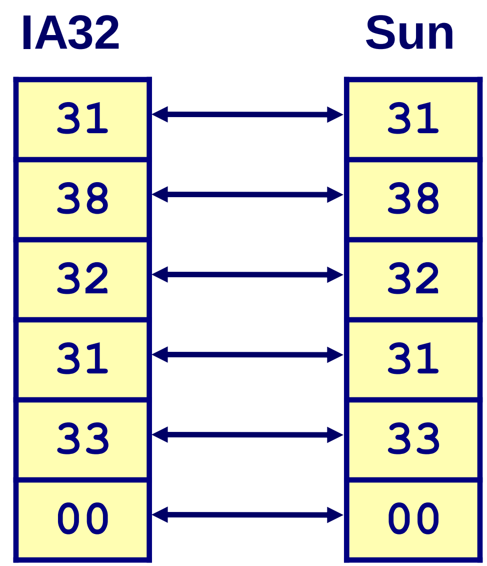
Different compilers & machines assign different locations to objects

Even get different results each time run program

Representing Strings

```
char S[6] = "18213";
```

- **Strings in C**
  - Represented by array of characters
  - Each character encoded in ASCII format
    - Standard 7-bit encoding of character set
    - Character "0" has code 0x30
      - Digit *i* has code 0x30+*i*
  - String should be null-terminated
    - Final character = 0
- **Compatibility**
  - Byte ordering not an issue



Integer C Puzzles

- `x < 0` □□ `((x*2) < 0)`
- `ux >= 0`
- `x & 7 == 7` □□ `(x<<30) < 0`
- `ux > -1`
- `x > y` □□ `-x < -y`
- `x * x >= 0`
- `x > 0 && y > 0` □□ `x + y > 0`
- `x >= 0` □□ `-x <= 0`
- `x <= 0` □□ `-x >= 0`
- `(x|-x)>>31 == -1`
- `ux >> 3 == ux/8`
- `x >> 3 == x/8`
- `x & (x-1) != 0`

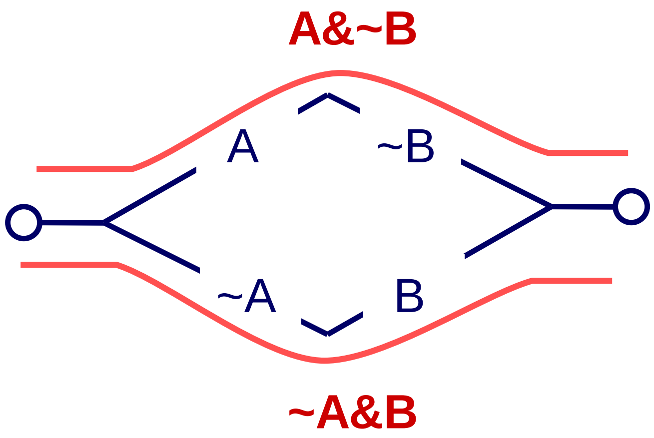
Initialization

```
int x = foo();

int y = bar();

unsigned ux = x;

unsigned uy = y;
```

Bonus extras

Application of Boolean Algebra

- **Applied to Digital Systems by Claude Shannon**
  - 1937 MIT Master's Thesis
  - Reason about networks of relay switches
    - Encode closed switch as 1, open switch as 0



Connection when

A&~B | ~A&B

= A^B

Binary Number Property

Claim

$$1 + 1 + 2 + 4 + 8 + \ldots + 2^{w-1} = 2^w$$

$$1 + \sum_{i=0}^{w-1} 2^i = 2^w$$

- **w = 0:**
  - $1 = 2^0$
- **Assume true for w-1:**
  - $\underbrace{1 + 1 + 2 + 4 + 8 + \ldots + 2^{w-1}}_{=\ 2^w} + 2^w = 2^w + 2^w = 2^{w+1}$

Code Security Example

```
/* Kernel memory region holding user-accessible data */
#define KSIZE 1024
char kbuf[KSIZE];

/* Copy at most maxlen bytes from kernel region to user buffer */
int copy_from_kernel(void *user_dest, int maxlen) {
    /* Byte count len is minimum of buffer size and maxlen */
    int len = KSIZE < maxlen ? KSIZE : maxlen;
    memcpy(user_dest, kbuf, len);
    return len;
}
```

- **Similar to code found in FreeBSD's implementation of getpeername**
- **There are legions of smart people trying to find vulnerabilities in programs**

Typical Usage

```
/* Kernel memory region holding user-accessible data */
#define KSIZE 1024
char kbuf[KSIZE];

/* Copy at most maxlen bytes from kernel region to user buffer */
int copy_from_kernel(void *user_dest, int maxlen) {
    /* Byte count len is minimum of buffer size and maxlen */
    int len = KSIZE < maxlen ? KSIZE : maxlen;
    memcpy(user_dest, kbuf, len);
    return len;
}
```

```
#define MSIZE 528

void getstuff() {
    char mybuf[MSIZE];
    copy_from_kernel(mybuf, MSIZE);
    printf("%s\n", mybuf);
}
```

Malicious Usage

```
/* Declaration of library function memcpy */
void *memcpy(void *dest, void *src, size_t n);
```

```
/* Kernel memory region holding user-accessible data */
#define KSIZE 1024
char kbuf[KSIZE];

/* Copy at most maxlen bytes from kernel region to user buffer */
int copy_from_kernel(void *user_dest, int maxlen) {
    /* Byte count len is minimum of buffer size and maxlen */
    int len = KSIZE < maxlen ? KSIZE : maxlen;
    memcpy(user_dest, kbuf, len);
    return len;
}
```

```
#define MSIZE 528

void getstuff() {
    char mybuf[MSIZE];
    copy_from_kernel(mybuf, -MSIZE);
    . . .
}
```

Mathematical Properties

- **Modular Addition Forms an *Abelian Group***
  - **Closed** under addition

    $0 \leq \mathrm{UAdd}_w(u, v) \leq 2^w - 1$
  - **Commutative**

    $\mathrm{UAdd}_w(u, v) = \mathrm{UAdd}_w(v, u)$
  - **Associative**

    $\mathrm{UAdd}_w(t, \mathrm{UAdd}_w(u, v)) = \mathrm{UAdd}_w(\mathrm{UAdd}_w(t, u), v)$
  - **0** is additive identity

    $\mathrm{UAdd}_w(u, 0) = u$
  - Every element has additive **inverse**
    - Let $\mathrm{UComp}_w(u) = 2^w - u$

      $\mathrm{UAdd}_w(u, \mathrm{UComp}_w(u)) = 0$

Mathematical Properties of TAdd

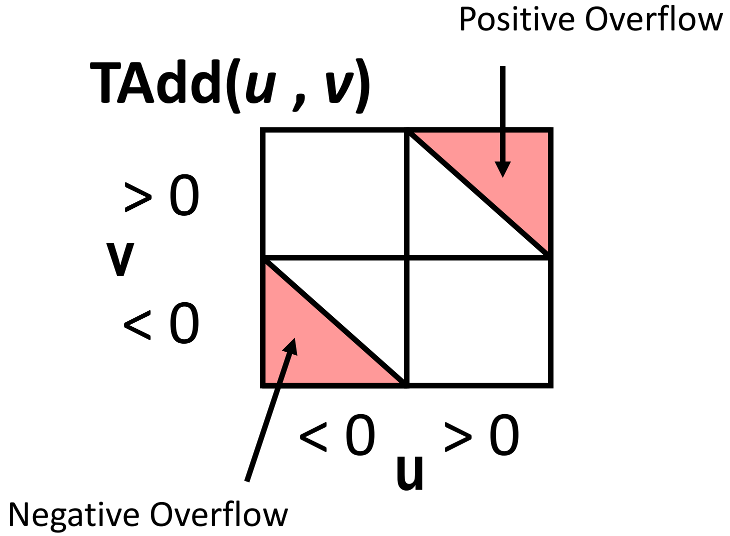
- **Isomorphic Group to unsigneds with UAdd**
  - $\text{TAdd}_w(u, v) = \text{U2T}(\text{UAdd}_w(\text{T2U}(u), \text{T2U}(v)))$
    - Since both have identical bit patterns

- **Two's Complement Under TAdd Forms a Group**
  - Closed, Commutative, Associative, 0 is additive identity
  - Every element has additive inverse

$$TComp_w(u) = \begin{cases} -u & u \neq TMin_w \\ TMin_w & u = TMin_w \end{cases}$$

Characterizing TAdd

- **Functionality**
  - True sum requires *w*+1 bits
  - Drop off MSB
  - Treat remaining bits as 2's comp. integer



$$TAdd_w(u,v) = \begin{cases} u+v+2^w & u+v < TMin_w \quad \textbf{(NegOver)} \\ u+v & TMin_w \le u+v \le TMax_w \\ u+v-2^w & TMax_w < u+v \quad \textbf{(PosOver)} \end{cases}$$

Negation: Complement & Increment

- **Claim: Following Holds for 2's Complement**

  `~x + 1 == -x`

- **Complement**
  - Observation: **`~x + x == 1111…111 == -1`**

	x	1	0	0	1	1	1	0	1
+	~x	0	1	1	0	0	0	1	0
	-1	1	1	1	1	1	1	1	1

- **Complete Proof?**

Complement & Increment Examples

x = 15213

	Decimal	Hex	Binary
x	15213	3B 6D	00111011 01101101
~x	-15214	C4 92	11000100 10010010
~x+1	-15213	C4 93	11000100 10010011
y	-15213	C4 93	11000100 10010011

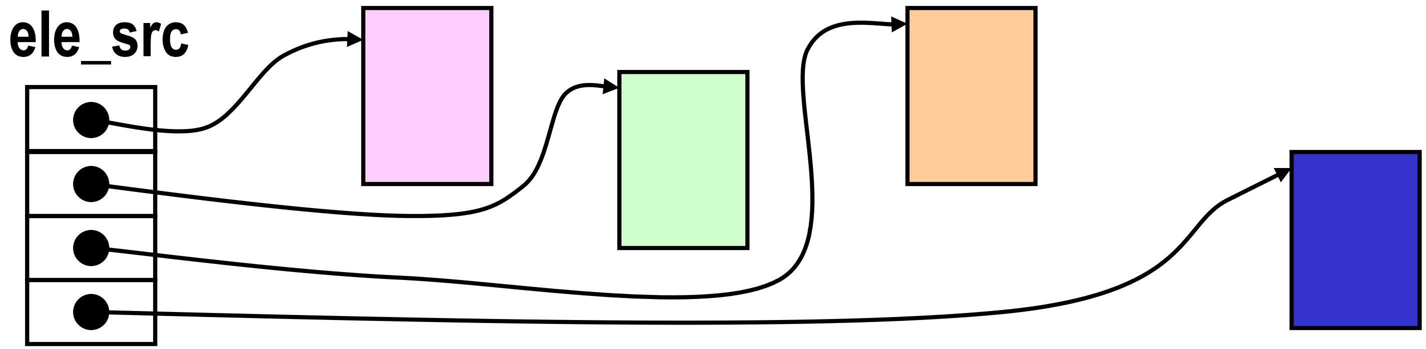
x = 0

	Decimal	Hex	Binary
0	0	00 00	00000000 00000000
~0	-1	FF FF	11111111 11111111
~0+1	0	00 00	00000000 00000000

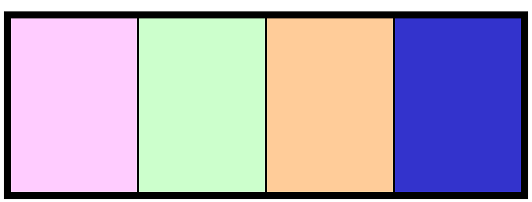
Code Security Example #2

- **SUN XDR library**
  - Widely used library for transferring data between machines

```
void* copy_elements(void *ele_src[], int ele_cnt, size_t ele_size);
```



malloc(ele_cnt * ele_size)



XDR Code

```
void* copy_elements(void *ele_src[], int ele_cnt, size_t ele_size) {
    /*
     * Allocate buffer for ele_cnt objects, each of ele_size bytes
     * and copy from locations designated by ele_src
     */
    void *result = malloc(ele_cnt * ele_size);
    if (result == NULL)
        /* malloc failed */
        return NULL;
    void *next = result;
    int i;
    for (i = 0; i < ele_cnt; i++) {
        /* Copy object i to destination */
        memcpy(next, ele_src[i], ele_size);
        /* Move pointer to next memory region */
        next += ele_size;
    }
    return result;
}
```

XDR Vulnerability

malloc(ele_cnt * ele_size)

- **What if:**
  - `ele_cnt` = $2^{20} + 1$
  - `ele_size` = 4096 = 2^{12}
  - Allocation = ??

- **How can I make this function secure?**

Compiled Multiplication Code

C Function

```
long mul12(long x)
{
  return x*12;
}
```

Compiled Arithmetic Operations

```
  leaq (%rax,%rax,2), %rax
  salq $2, %rax
```

Explanation

```
  t <- x+x*2
  return t << 2;
```

- C compiler automatically generates shift/add code when multiplying by constant

Compiled Unsigned Division Code

C Function

```
unsigned long udiv8
      (unsigned long x)
{
  return x/8;
}
```

Compiled Arithmetic Operations

```
  shrq $3, %rax
```

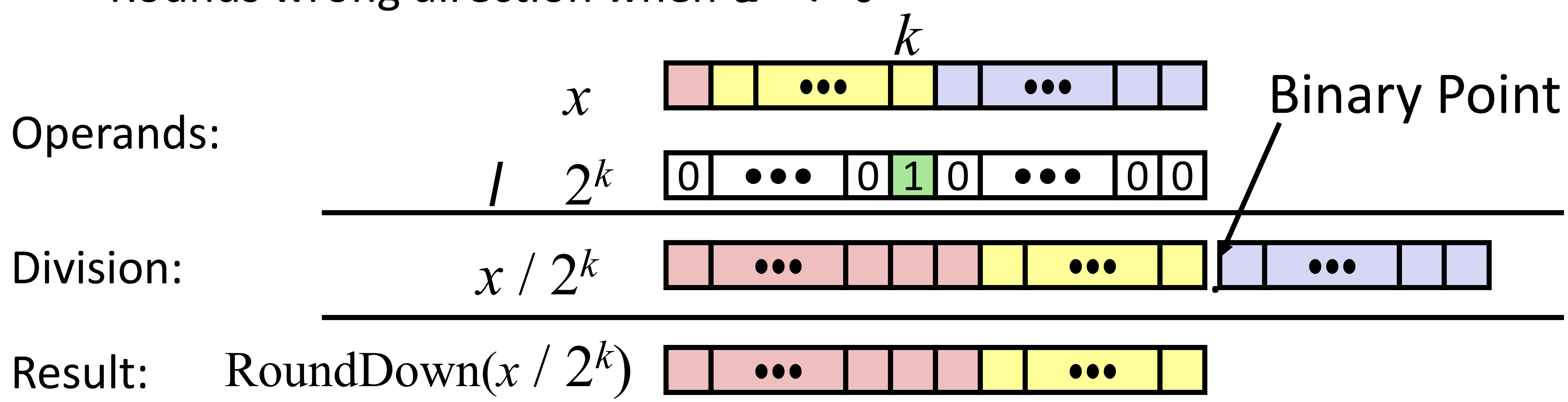
Explanation

```
  # Logical shift
  return x >> 3;
```

- **Uses logical shift for unsigned**
- **For Java Users**
  - Logical shift written as >>>

Signed Power-of-2 Divide with Shift

- **Quotient of Signed by Power of 2**
  - `x >> k` gives ⌊ `x / ` 2^k ⌋
  - Uses arithmetic shift
  - Rounds wrong direction when `u < 0`

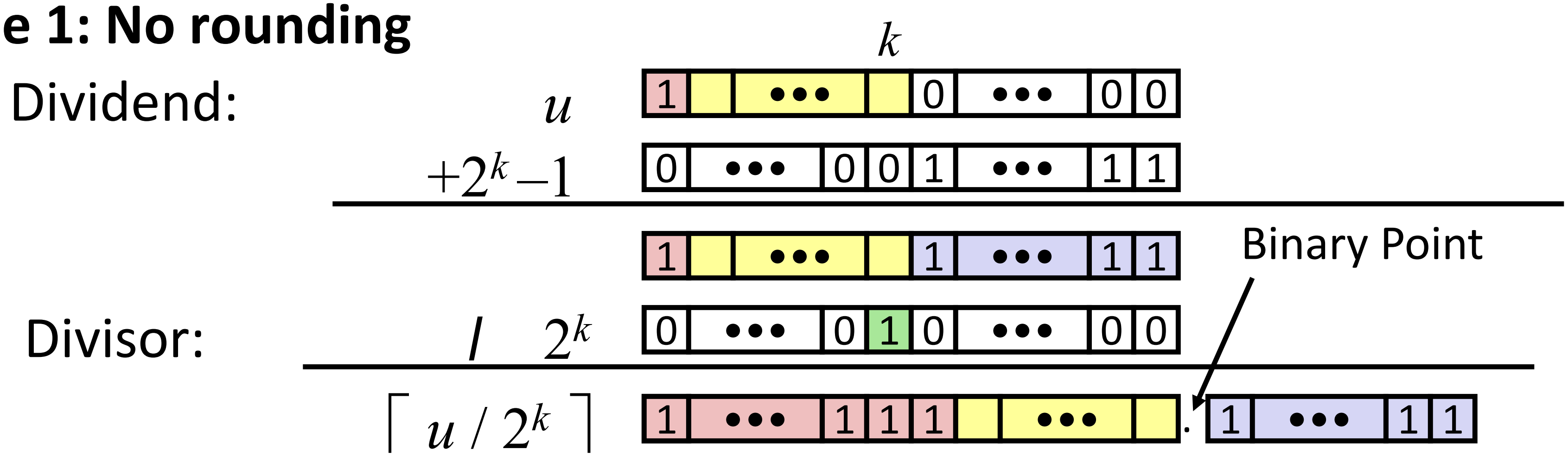


	Division	Computed	Hex	Binary
`y`	-15213	-15213	`C4 93`	`11000100 10010011`
`y >> 1`	-7606.5	-7607	`E2 49`	`11100010 01001001`
`y >> 4`	-950.8125	-951	`FC 49`	`11111100 01001001`
`y >> 8`	-59.4257813	-60	`FF C4`	`11111111 11000100`

Correct Power-of-2 Divide

- **Quotient of Negative Number by Power of 2**
  - Want $\lceil$ **`x / 2`**^{k} $\rceil$ (Round Toward 0)
  - Compute as $\lfloor$ **`(x+2`**^{k}**`-1) / 2`**^{k} $\rfloor$
    - In C: **`(x + (1<<k)-1) >> k`**
    - Biases dividend toward 0

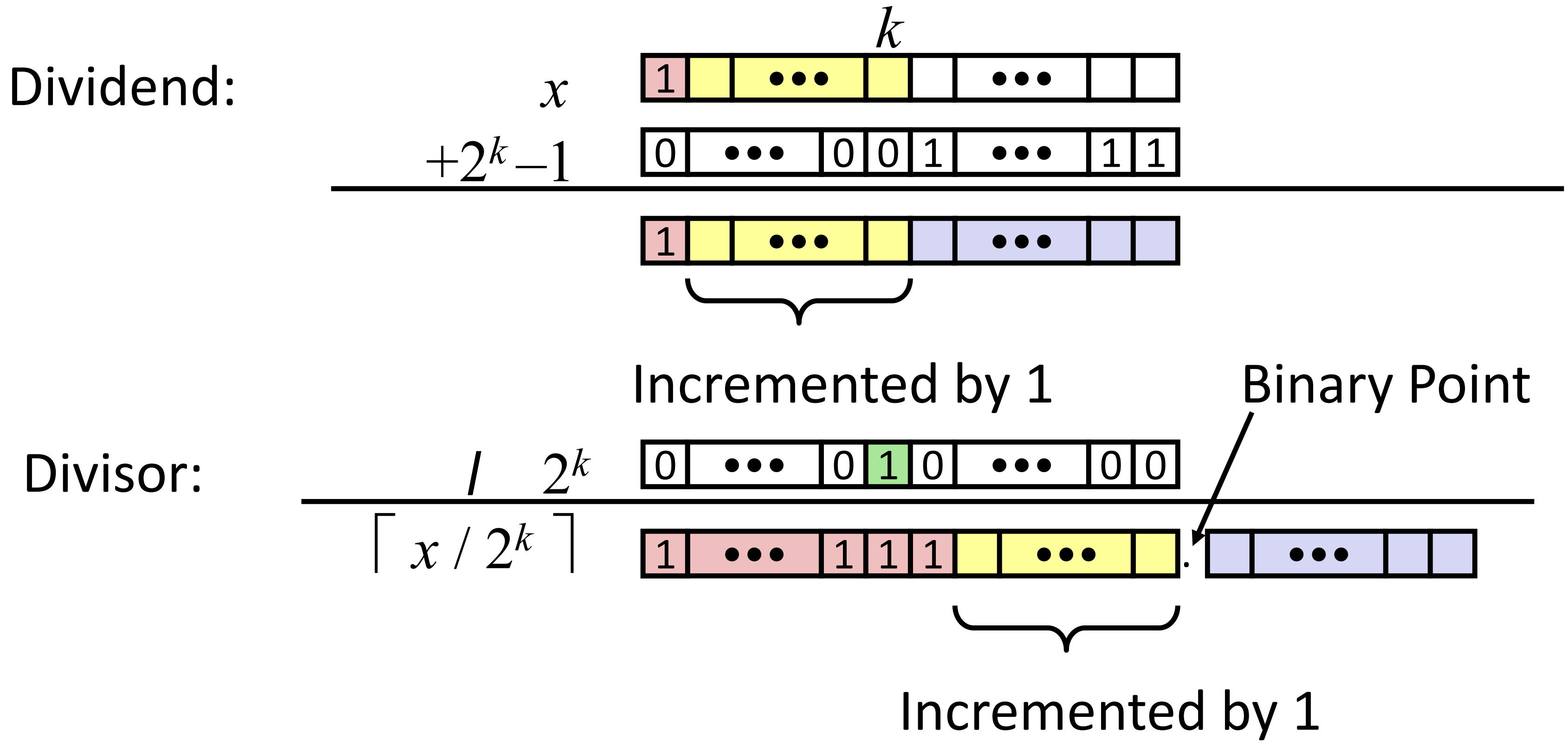
Case 1: No rounding



Biasing has no effect

Correct Power-of-2 Divide (Cont.)

Case 2: Rounding



Biasing adds 1 to final result

Compiled Signed Division Code

C Function

```
long idiv8(long x)
{
  return x/8;
}
```

Compiled Arithmetic Operations

```
  testq %rax, %rax
  js   L4
L3:
  sarq $3, %rax
  ret
L4:
  addq $7, %rax
  jmp  L3
```

Explanation

```
  if x < 0
    x += 7;
  # Arithmetic shift
  return x >> 3;
```

- **Uses arithmetic shift for int**
- **For Java Users**
  - Arith. shift written as >>

Arithmetic: Basic Rules

- **Unsigned ints, 2's complement ints are isomorphic rings: isomorphism = casting**

- **Left shift**
  - Unsigned/signed: multiplication by 2^k
  - Always logical shift

- **Right shift**
  - Unsigned: logical shift, div (division + round to zero) by 2^k
  - Signed: arithmetic shift
    - Positive numbers: div (division + round to zero) by 2^k
    - Negative numbers: div (division + round away from zero) by 2^k
      Use biasing to fix

Properties of Unsigned Arithmetic

- **Unsigned Multiplication with Addition Forms Commutative Ring**
  - Addition is commutative group
  - Closed under multiplication

    $0 \leq \text{UMult}_w(u\,,\,v) \leq 2^w - 1$
  - Multiplication Commutative

    $\text{UMult}_w(u\,,\,v) = \text{UMult}_w(v\,,\,u)$
  - Multiplication is Associative

    $\text{UMult}_w(t, \text{UMult}_w(u\,,\,v)) = \text{UMult}_w(\text{UMult}_w(t, u\,), v)$
  - 1 is multiplicative identity

    $\text{UMult}_w(u\,,\,1) = u$
  - Multiplication distributes over addtion

    $\text{UMult}_w(t, \text{UAdd}_w(u\,,\,v)) = \text{UAdd}_w(\text{UMult}_w(t, u\,), \text{UMult}_w(t, v))$

Properties of Two's Comp. Arithmetic

- **Isomorphic Algebras**
  - Unsigned multiplication and addition
    - Truncating to *w* bits
  - Two's complement multiplication and addition
    - Truncating to *w* bits
- **Both Form Rings**
  - Isomorphic to ring of integers mod 2^w
- **Comparison to (Mathematical) Integer Arithmetic**
  - Both are rings
  - Integers obey ordering properties, e.g.,

    $u > 0 \quad \Rightarrow \quad u + v > v$

    $u > 0, v > 0 \quad \Rightarrow \quad u \cdot v > 0$
  - These properties are not obeyed by two's comp. arithmetic

    TMax `+ 1` `==` *TMin*

    `15213 * 30426 == -10030` (16-bit words)

Reading Byte-Reversed Listings

- **Disassembly**
  - Text representation of binary machine code
  - Generated by program that reads the machine code
- **Example Fragment**

Address	Instruction Code	Assembly Rendition	
`8048365:`	`5b`	`pop`	`%ebx`
`8048366:`	`81 c3 ab 12 00 00`	`add`	`$0x12ab,%ebx`
`804836c:`	`83 bb 28 00 00 00 00`	`cmpl`	`$0x0,0x28(%ebx)`

- **Deciphering Numbers**
  - Value: 0x12ab
  - Pad to 32 bits: 0x000012ab
  - Split into bytes: 00 00 12 ab
  - Reverse: ab 12 00 00