# INTRODUCTION

## *Part 1. Historical Introduction*

### 11.1 The two basic concepts of calculus

The remarkable progress that has been made in science and technology during the last Century is due in large part to the development of mathematics. That branch of mathematics known as integral and differential calculus serves as a natural and powerful tool for attacking a variety of problems that arise in physics, astronomy, engineering, chemistry, geology, biology, and other fields including, rather recently, some of the social sciences.

TO give the reader an idea of the many different types of problems that can be treated by the methods of calculus, we list here a few sample questions selected from the exercises that occur in later chapters of this book.

With what speed should a rocket be fired upward SO that it never returns to earth? What is the radius of the smallest circular disk that can cover every isosceles triangle of a given perimeter $L$? What volume of material is removed from a solid sphere of radius 2r if a hole of radius $r$ is drilled through the center ? If a strain of bacteria grows at a rate proportional to the amount present and if the population doubles in one hour, by how much will it increase at the end of two hours? If a ten-pound force stretches an elastic spring one inch, how much work is required to stretch the spring one foot ?

These examples, chosen from various fields, illustrate some of the technical questions that can be answered by more or less routine applications of calculus.

Calculus is more than a technical tool-it is a collection of fascinating and exciting ideas that have interested thinking men for centuries. These ideas have to do with *speed, area, volume, rate of growth, continuity, tangent line*, and other concepts from a variety of fields. Calculus forces us to stop and think carefully about the meanings of these concepts. Another remarkable feature of the subject is its unifying power. Most of these ideas can be formulated SO that they revolve around two rather specialized problems of a geometric nature. W e turn now to a brief description of these problems.

Consider a curve C which lies above a horizontal base line such as that shown in Figure 1.1. We assume this curve has the property that every vertical line intersects it once at most.

The shaded portion of the figure consists of those points which lie below the curve C, above the horizontal base, and between two parallel vertical segments joining C to the base. The first fundamental problem of calculus is this : *To assign a number which measures the area of this shaded region.*

Consider next a line drawn tangent to the curve, as shown in Figure 1.1. The second fundamental problem may be stated as follows: *To assign a number which measures the steepness of this line.*

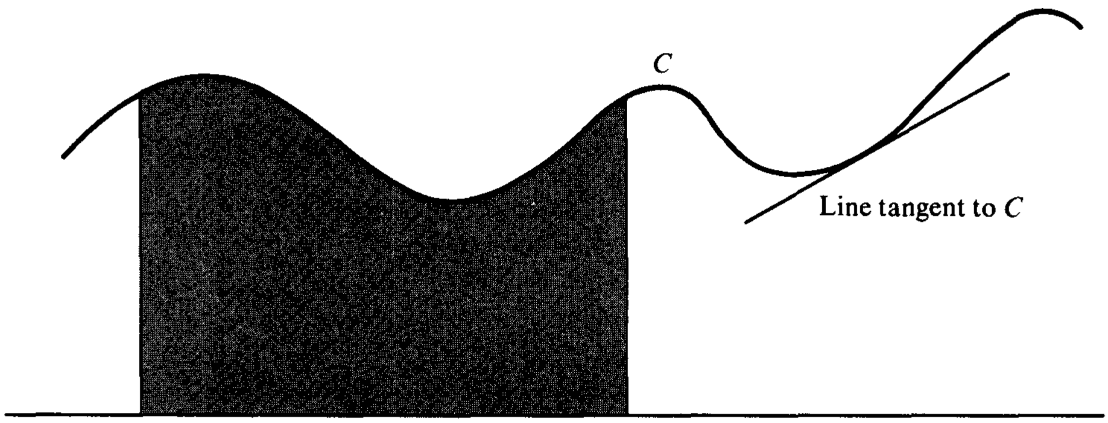



FIGURE 1.1

Basically, calculus has to do with the precise formulation and solution of these two special problems. It enables us to *define* the concepts of area and tangent line and *to calculate* the area of a given region or the steepness of a given tangent line. *Integral calculus* deals with the problem of area and will be discussed in Chapter 1. *Differential calculus* deals with the problem of tangents and will be introduced in Chapter 4.

The study of calculus requires a certain mathematical background. The present chapter deals with fhis background material and is divided into four parts : Part 1 provides historical perspective; Part 2 discusses some notation and terminology from the mathematics of sets; Part 3 deals with the real-number system; Part 4 treats mathematical induction and the summation notation. If the reader is acquainted with these topics, he can proceed directly to the development of integral calculus in Chapter 1. If not, he should become familiar with the material in the unstarred sections of this Introduction before proceeding to Chapter 1.

## I1.2 Historical background

The birth of integral calculus occurred more than 2000 years ago when the Greeks attempted to determine areas by a process which they called the *method ofexhaustion.* The essential ideas of this method are very simple and can be described briefly as follows: Given a region whose area is to be determined, we inscribe in it a polygonal region which approximates the given region and whose area we can easily compute. Then we choose another polygonal region which gives a better approximation, and we continue the process, taking polygons with more and more sides in an attempt to exhaust the given region. The method is illustrated for a semicircular region in Figure 1.2. It was used successfully by Archimedes (287-212 B.C.) to find exact formulas for the area of a circle and a few other special figures.

The development of the method of exhaustion beyond the point to which Archimedes carried it had to wait nearly eighteen centuries until the use of algebraic symbols and techniques became a standard part of mathematics. The elementary algebra that is familiar to most high-school students today was completely unknown in Archimedes' time, and it would have been next to impossible to extend his method to any general class of regions without some convenient way of expressing rather lengthy calculations in a compact and simplified form.

A slow but revolutionary change in the development of mathematical notations began in the 16th Century A.D. The cumbersome system of Roman numerals was gradually displaced by the Hindu-Arabie characters used today, the symbols + and – were introduced for the first time, and the advantages of the decimal notation began to be recognized. During this same period, the brilliant successes of the Italian mathematicians Tartaglia,

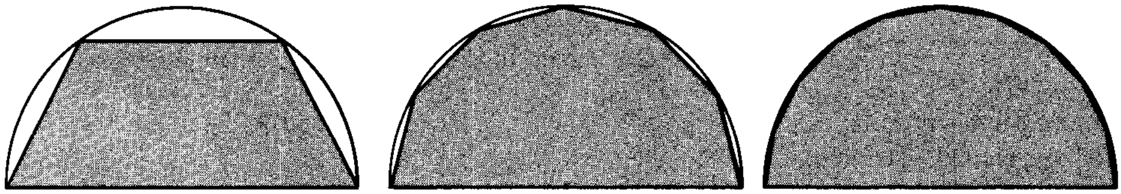


FIGURE 1.2 The method of exhaustion applied to a semicircular region.

Cardano, and Ferrari in finding algebraic solutions of cubic and quartic equations stimulated a great deal of activity in mathematics and encouraged the growth and acceptance of a new and superior algebraic language. With the widespread introduction of well-chosen algebraic symbols, interest was revived in the ancient method of exhaustion and a large number of fragmentary results were discovered in the 16th Century by such pioneers as Cavalieri, Toricelli, Roberval, Fermat, Pascal, and Wallis.

Gradually the method of exhaustion was transformed into the subject now called integral calculus, a new and powerful discipline with a large variety of applications, not only to geometrical problems concerned with areas and volumes but also to problems in other sciences. This branch of mathematics, which retained some of the original features of the method of exhaustion, received its biggest impetus in the 17th Century, largely due to the efforts of Isaac Newton (1642-1727) and Gottfried Leibniz (1646–1716), and its development continued well into the 19th Century before the subject was put on a firm mathematical basis by such men as Augustin-Louis Cauchy (1789-1857) and Bernhard Riemann (1826-1866). Further refinements and extensions of the theory are still being carried out in contemporary mathematics.

### I1.3 The method of exhaustion for the area of a parabolic segment

Before we proceed to a systematic treatment of integral calculus, it will be instructive to apply the method of exhaustion directly to one of the special figures treated by Archimedes himself. The region in question is shown in Figure 1.3 and can be described as follows: If we choose an arbitrary point on the base of this figure and denote its distance from 0 by $x$, then the vertical distance from this point to the curve is $x^2$. In particular, if the length of the base itself is $b$, the altitude of the figure is $b^2$. The vertical distance from x to the curve is called the "ordinate" at x. The curve itself is an example of what is known

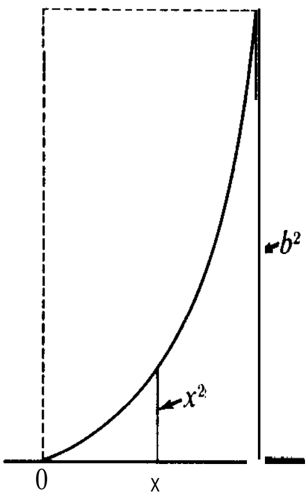



FIGURE 1.3 A parabolic segment.

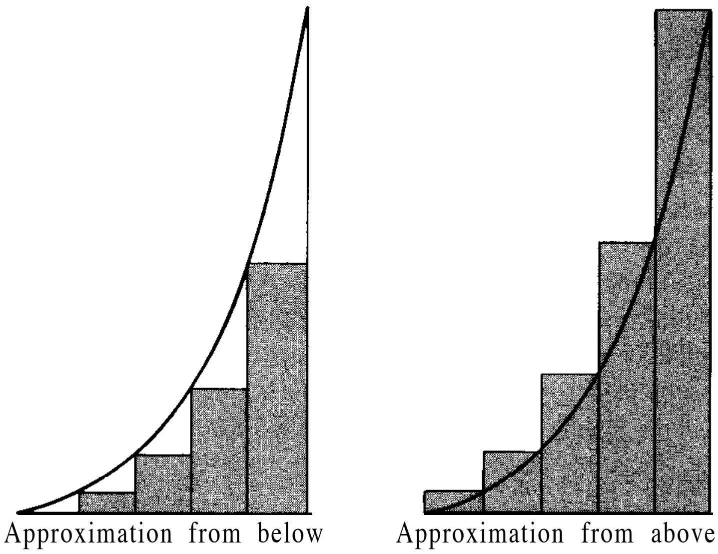



FIGURE 1.4

as a ***parabola.*** The region bounded by it and the two line segments is called a ***parabolic segment.***

This figure may be enclosed in a rectangle of base $b$ and altitude $b^2$, as shown in Figure 1.3. Examination of the figure suggests that the area of the parabolic segment is less than half the area of the rectangle. Archimedes made the surprising discovery that the area of the parabolic segment is exactly *one-third* that of the rectangle; that is to say, $A = b^3/3$, where A denotes the area of the parabolic segment. We shall show presently how to arrive at this result.

It should be pointed out that the parabolic segment in Figure 1.3 is not shown exactly as Archimedes drew it and the details that follow are not exactly the same as those used by him.

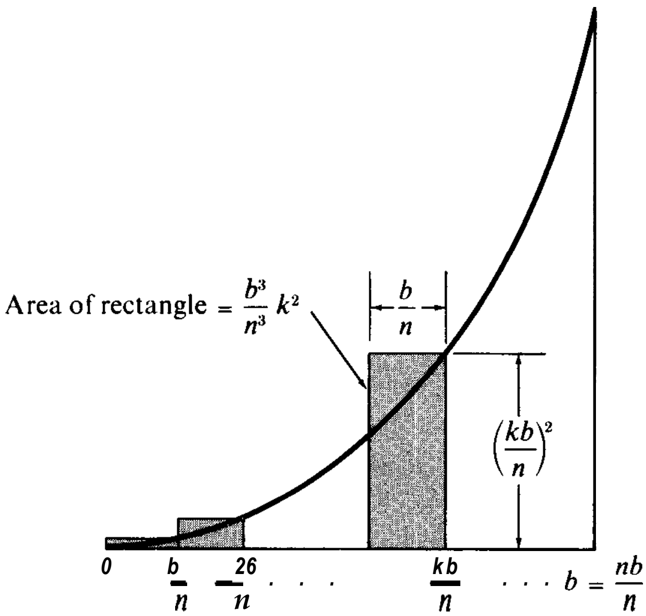



FIGURE 1.5 Calculation of the area of a parabolic segment.

Nevertheless, the essential *ideas* are those of Archimedes; what is presented here is the method of exhaustion in modern notation.

The method is simply this: We slice the figure into a number of strips and obtain two approximations to the region, one from below and one from above, by using two sets of rectangles as illustrated in Figure 1.4. (We use rectangles rather than arbitrary polygons to simplify the computations.) The area of the parabolic segment is larger than the total area of the inner rectangles but smaller than that of the outer rectangles.

If each strip is further subdivided to obtain a new approximation with a larger number of strips, the total area of the inner rectangles *increases,* whereas the total area of the outer rectangles *decreases.* Archimedes realized that an approximation to the area within any desired degree of accuracy could be obtained by simply taking enough strips.

Let us carry out the actual computations that are required in this case. For the sake of simplicity, we subdivide the base into $n$ *equal* parts, each of length $b/n$ (see Figure 1.5). The points of subdivision correspond to the following values of $x$:

$$0, \frac{b}{n}, \frac{2b}{n}, \frac{3b}{n}, \ldots, \frac{(n-1)b}{n}, \frac{nb}{n} = b$$

A typical point of subdivision corresponds to $x = kb/n$, where $k$ takes the successive values $k = 0, 1, 2, 3, \ldots, n$. At each point $kb/n$ we construct the outer rectangle of altitude $(kb/n)^2$ as illustrated in Figure 1.5. The area of this rectangle is the product of its base and altitude and is equal to

$$\left(\frac{b}{n}\right)\left(\frac{kb}{n}\right)^2 = \frac{b^3}{n^3}k^2.$$

Let us denote by $S_n$ the sum of the areas of all the outer rectangles. Then since the $k$th rectangle has area $(b^3/n^3)k^2$, we obtain the formula

$$\text{(I.1)} \qquad S_n = \frac{b^3}{n^3}(1^2 + 2^2 + 3^2 + \cdots + n^2).$$

In the same way we obtain a formula for the sum $s_n$ of all the inner rectangles:

$$\text{(I.2)} \qquad s_n = \frac{b^3}{n^3}[1^2 + 2^2 + 3^2 + \cdots + (n-1)^2].$$

This brings us to a very important stage in the calculation. Notice that the factor multiplying $b^3/n^3$ in Equation (1.1) is the sum of the squares of the first $n$ integers:

$$1^2 + 2^2 + \cdots + n^2.$$

[The corresponding factor in Equation (1.2) is similar except that the sum has only $n - 1$ terms.] For a large value of $n$, the computation of this sum by direct addition of its terms is tedious and inconvenient. Fortunately there is an interesting identity which makes it possible to evaluate this sum in a simpler way, namely,

$$\text{(I.3)} \qquad 1^2 + 2^2 + \cdots + n^2 = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}.$$

This identity is valid for every integer $n \geq 1$ and can be proved as follows: Start with the formula $(k + 1)^3 = k^3 + 3k^2 + 3k + 1$ and rewrite it in the form

$$3k^2 + 3k + 1 = (k + 1)^3 - k^3.$$

Taking $k = 1, 2, \ldots, n - 1$, we get $n - 1$ formulas

$$3 \cdot 1^2 + 3 \cdot 1 + 1 = 2^3 - 1^3$$

$$3 \cdot 2^2 + 3 \cdot 2 + 1 = 3^3 - 2^3$$

$$3(n - 1)^2 + 3(n - 1) + 1 = n^3 - (n - 1)^3.$$

When we add these formulas, all the terms on the right cancel except two and we obtain

$$3[1^2 + 2^2 + \cdots + (n - 1)^2] + 3[1 + 2 + \ldots + (n - 1)] + (n - 1) = n^3 - 1^3.$$

The second sum on the left is the sum of terms in an arithmetic progression and it simplifies to $\frac{1}{2}n(n - 1)$. Therefore this last equation gives us

(I.4) $$1^2 + 2^2 + \cdots + (n - 1)^2 = \frac{n^-}{-} - \frac{n^-}{-} + \frac{n}{-}.$$

Adding $n^2$ to both members, we obtain (1.3).

For our purposes, we do not need the exact expressions given in the right-hand members of (1.3) and (1.4). All we need are the two *inequalities*

(I.5) $$1^2 + 2^2 + \cdots + (n - 1)'' < \frac{n^3}{3} < 1^2 + 2^2 + \ldots + n^2$$

which are valid for every integer $n \geq 1$. These inequalities can de deduced easily as consequences of (1.3) and (I.4), or they can be proved directly by induction. (A proof by induction is given in Section 14.1.)

If we multiply both inequalities in (1.5) by $b^3/n^3$ and make use of (1.1) and (I.2), we obtain

(I.6) $$s_n < \frac{b^3}{3} < S_n$$

for every $n$. The inequalities in (1.6) tell us that $b^3/3$ is a number which lies between $s_n$ and $S_n$ for every $n$. We will now prove that $b^3/3$ is the *only* number which has this property. In other words, we assert that if A is any number which satisfies the inequalities

*(1.7)* $$s_n < A < S_n$$

for every positive integer $n$, then $A = b^3/3$. It is because of this fact that Archimedes concluded that the area of the parabolic segment is $b^3/3$.

To prove that $A = b^3/3$, we use the inequalities in (1.5) once more. Adding $n^2$ to both sides of the leftmost inequality in (I.5), we obtain

$$1^2 + 2^2 + \cdots + n^2 < \frac{n^3}{3} + n^2.$$

Multiplying this by $b^3/n^3$ and using (I.1), we find

$$S_n < \frac{b^3}{3} + \frac{b^3}{n}. \tag{I.8}$$

Similarly, by subtracting $n^2$ from both sides of the rightmost inequality in (1.5) and multiplying by $b^3/n^3$, we are led to the inequaiity

$$\frac{b^3}{3} - \frac{b^3}{n} < s_n. \tag{I.9}$$

Therefore, any number $A$ satisfying (1.7) must also satisfy

$$\frac{b^3}{3} - \frac{b^3}{n} < A < \frac{b^3}{3} + \frac{b^3}{n} \tag{1.10}$$

for every integer $n \geq 1$. Now there are only three possibilities:

$$A > \frac{b^3}{3}, \qquad A < \frac{b^3}{3}, \qquad A = \frac{b^3}{3}.$$

If we show that each of the first two leads to a contradiction, then we must have $A = b^3/3$, since, in the manner of Sherlock Holmes, this exhausts all the possibilities.

Suppose the inequality $A > b^3/3$ were true. From the second inequality in (1.10) we obtain

$$A - \frac{b^3}{3} < \frac{b^3}{n} \tag{1.11}$$

for every integer $n \geq 1$. Since $A - b^3/3$ is positive, we may divide both sides of (1.11) by $A - b^3/3$ and then multiply by $n$ to obtain the equivalent statement

$$n < \frac{b^3}{A - b^3/3}$$

for every $n$. But this inequality is obviously false when $n \geq b^3/(A - b^3/3)$. Hence the inequality $A > b^3/3$ leads to a contradiction. By a similar argument, we can show that the inequality $A < b^3/3$ also leads to a contradiction, and therefore we must have $A = b^3/3$, as asserted.

## *I 1.4 Exercises

1. (a) Modify the region in Figure 1.3 by assuming that the ordinate at each $x$ is $2x^2$ instead of $x^2$. Draw the new figure. Check through the principal steps in the foregoing section and find what effect this has on the calculation of the area. Do the same if the ordinate at each x is (b) $3x^2$, (c) $\frac{1}{4}x^2$, (d) $2x^2 + 1$, (e) $ax^2 + c$.
2. Modify the region in Figure 1.3 by assuming that the ordinate at each x is $x^3$ instead of $x^2$. Draw the new figure.
   (a) Use a construction similar to that illustrated in Figure 1.5 and show that the outer and inner sums $S_n$ and $s_n$ are given by

$$S_n = \frac{b^4}{n^4}(1^3 + 2^3 + \cdots + n^3), \qquad s_n = \frac{b^4}{n^4}[1^3 + 23 + \ldots + (n - 1)^3].$$

   (b) Use the inequalities (which can be proved by mathematical induction; see Section 14.2)

$$1^3 + 2^3 + \cdots + (n - 1)^3 < \frac{n^4}{4} < 1^3 + 2^3 + \ldots + n^3 \tag{1.12}$$

   to show that $s_n < b^4/4 < S_n$ for every n, and prove that $b^4/4$ is the *only* number which lies between $s_n$ and $S_n$ for every n.
   (c) What number takes the place of $b^4/4$ if the ordinate at each x is $ax^3 + c$?
3. The inequalities (1.5) and (1.12) are special cases of the more general inequalities

$$1” + 2” + \ldots + (n - 1)” < \frac{n^{k+1}}{k + 1} < 1^k + 2” + \ldots + n^k \tag{1.13}$$

   that are valid for every integer $n \geq 1$ and every integer $k \geq 1$. Assume the validity of (1.13) and generalize the results of Exercise 2.

## I1.5 A critical analysis of Archimedes' method

From calculations similar to those in Section 1 1.3, Archimedes concluded that the area of the parabolic segment in question is $b^3/3$. This fact was generally accepted as a mathematical theorem for nearly 2000 years before it was realized that one must re-examine the result from a more critical point of view. To understand why anyone would question the validity of Archimedes' conclusion, it is necessary to know something about the important changes that have taken place in the recent history of mathematics.

Every branch of knowledge is a collection of ideas described by means of words and symbols, and one cannot understand these ideas unless one knows the exact meanings of the words and symbols that are used. Certain branches of knowledge, known as *deductive systems,* are different from others in that a number of "undefined" concepts are chosen in advance and all other concepts in the system are defined in terms of these. Certain statements about these undefined concepts are taken as *axioms* or *postulates* and other statements that can be deduced from the axioms are called *theorems*. The most familiar example of a deductive system is the Euclidean theory of elementary geometry that has been studied by well-educated men since the time of the ancient Greeks.

The spirit of early Greek mathematics, with its emphasis on the theoretical and postulational approach to geometry as presented in Euclid's *Elements*, dominated the thinking of mathematicians until the time of the Renaissance. A new and vigorous phase in the development of mathematics began with the advent of algebra in the 16th Century, and the next 300 years witnessed a flood of important discoveries. Conspicuously absent from this period was the logically precise reasoning of the deductive method with its use of axioms, definitions, and theorems. Instead, the pioneers in the 16th, 17th, and 18th centuries resorted to a curious blend of deductive reasoning combined with intuition, pure guesswork, and mysticism, and it is not surprising to find that some of their work was later shown to be incorrect. However, a surprisingly large number of important discoveries emerged from this era, and a great deal of the work has survived the test of history-a tribute to the unusual skil1 and ingenuity of these pioneers.

As the flood of new discoveries began to recede, a new and more critical period emerged. Little by little, mathematicians felt forced to return to the classical ideals of the deductive method in an attempt to put the new mathematics on a firm foundation. This phase of the development, which began early in the 19th Century and has continued to the present day, has resulted in a degree of logical purity and abstraction that has surpassed all the traditions of Greek science. At the same time, it has brought about a clearer understanding of the foundations of not only calculus but of all of mathematics.

There are many ways to develop calculus as a deductive system. One possible approach is to take the real numbers as the undefined objects. Some of the rules governing the operations on real numbers may then be taken as axioms. One such set of axioms is listed in Part 3 of this Introduction. New concepts, such as *integral*, *limit*, *continuity*, *derivative,* must then be defined in terms of real numbers. Properties of these concepts are then deduced as theorems that follow from the axioms.

Looked at as part of the deductive system of calculus, Archimedes' result about the area of a parabolic segment cannot be accepted as a theorem until a satisfactory definition of area is given first. It is not clear whether Archimedes had ever formulated a precise definition of what he meant by area. He seems to have taken it for granted that every region has an area associated with it. On this assumption he then set out to calculate areas of particular regions. In his calculations he made use of certain facts about area that cannot be proved until we know what is *meant* by area. For instance, he assumed that if one region lies inside another, the area of the smaller region cannot exceed that of the larger region. Also, if a region is decomposed into two or more parts, the sum of the areas of the individual parts is equal to the area of the whole region. All these are properties we would like area to possess, and we shall insist that any definition of area should imply these properties. It is quite possible that Archimedes himself may have taken area to be an undefined concept and then used the properties we just mentioned as *axioms* about area.

Today we consider the work of Archimedes as being important not so much because it helps us to compute areas of particular figures, but rather because it suggests a reasonable way to *define* the concept of area for more or less *arbitrary* figures. As it turns out, the method of Archimedes suggests a way to define a much more general concept known as the *integral.* The integral, in turn, is used to compute not only area but also quantities such as arc length, volume, work and others.

If we look ahead and make use of the terminology of integral calculus, the result of the calculation carried out in Section 1 1.3 for the parabolic segment is often stated as follows :

"The integral of $x^2$ from 0 to $b$ is $b^3/3$."

It is written symbolically as

$$\int_0^b x^2\, dx = \frac{b^3}{3},$$

The symbol $\int$ (an elongated $S$) is called an *integral sign*, and it was introduced by Leibniz in 1675. The process which produces the number $b^3/3$ is called *integration*. The numbers 0 and $b$ which are attached to the integral sign are referred to as the *limits of integration*. The symbol $\int_0^b x^2\, dx$ must be regarded as a whole. Its definition will treat it as such, just as the dictionary describes the word "lapidate" without reference to "lap," "id," or "ate."

Leibniz' symbol for the integral was readily accepted by many early mathematicians because they liked to think of integration as a kind of "summation process" which enabled them to add together infinitely many "infinitesimally small quantities." For example, the area of the parabolic segment was conceived of as a sum of infinitely many infinitesimally thin rectangles of height $x^2$ and base $dx$. The integral sign represented the process of adding the areas of all these thin rectangles. This kind of thinking is suggestive and often very helpful, but it is not easy to assign a precise meaning to the idea of an "infinitesimally small quantity." Today the integral is defined in terms of the notion of real number without using ideas like "infinitesimals." This definition is given in Chapter 1.

## I1.6 The approach to calculus to be used in this book

A thorough and complete treatment of either integral or differential calculus depends ultimately on a careful study of the real number system. This study in itself, when carried out in full, is an interesting but somewhat lengthy program that requires a small volume for its complete exposition. The approach in this book is to begin with the real numbers as *undefined objects* and simply to list a number of fundamental properties of real numbers which we shall take as *axioms*. These axioms and some of the simplest theorems that can be deduced from them are discussed in Part 3 of this chapter.

Most of the properties of real numbers discussed here are probably familiar to the reader from his study of elementary algebra. However, there are a few properties of real numbers that do not ordinarily corne into consideration in elementary algebra but which play an important role in the calculus. These properties stem from the so-called *least-upper-bound axiom* (also known as the *completeness* or *continuity axiom)* which is dealt with here in some detail. The reader may wish to study Part 3 before proceeding with the main body of the text, or he may postpone reading this material until later when he reaches those parts of the theory that make use of least-Upper-bound properties. Material in the text that depends on the least-Upper-bound axiom will be clearly indicated.

To develop calculus as a complete, forma1 mathematical theory, it would be necessary to state, in addition to the axioms for the real number system, a list of the various "methods of proof" which would be permitted for the purpose of deducing theorems from the axioms. Every statement in the theory would then have to be justified either as an "established law" (that is, an axiom, a definition, or a previously proved theorem) or as the result of applying one of the acceptable methods of proof to an established law. A program of this sort would be extremely long and tedious and would add very little to a beginner's understanding of the subject. Fortunately, it is not necessary to proceed in this fashion in order to get a good understanding and a good working knowledge of calculus. In this book the subject is introduced in an informal way, and ample use is made of geometric intuition whenever it is convenient to do so. At the same time, the discussion proceeds in a manner that is consistent with modern standards of precision and clarity of thought. All the important theorems of the subject are explicitly stated and rigorously proved.

To avoid interrupting the principal flow of ideas, some of the proofs appear in separate starred sections. For the same reason, some of the chapters are accompanied by supplementary material in which certain important topics related to calculus are dealt with in detail. Some of these are also starred to indicate that they may be omitted or postponed without disrupting the continuity of the presentation. The extent to which the starred sections are taken up or not will depend partly on the reader's background and skill and partly on the depth of his interests. A person who is interested primarily in the basic techniques may skip the starred sections. Those who wish a more thorough course in calculus, including theory as well as technique, should read some of the starred sections.

## *Part 2. Some Basic Concepts of the Theory of Sets*

### 12.1 Introduction to set theory

In discussing any branch of mathematics, be it analysis, algebra, or geometry, it is helpful to use the notation and terminology of set theory. This subject, which was developed by Boole and Cantor† in the latter part of the 19th Century, has had a profound influence on the development of mathematics in the 20th Century. It has unified many seemingly disconnected ideas and has helped to reduce many mathematical concepts to their logical foundations in an elegant and systematic way. A thorough treatment of the theory of sets would require a lengthy discussion which we regard as outside the scope of this book. Fortunately, the basic notions are few in number, and it is possible to develop a working knowledge of the methods and ideas of set theory through an informal discussion. Actually, we shall discuss not so much a new theory as an agreement about the precise terminology that we wish to apply to more or less familiar ideas.

In mathematics, the word "set" is used to represent a collection of objects viewed as a single entity. The collections called to mind by such nouns as "flock," "tribe," "crowd," "team," and "electorate" are all examples of sets. The individual objects in the collection are called *elements* or *members* of the set, and they are said to *belong to* or to be *contained in* the set. The set, in turn, is said to *contain* or be *composed of* its elements.

† George Boole (1815-1864) was an English mathematician and logician. His book, *An Investigation of the Laws of Thought*, published in 1854, marked the creation of the first workable system of symbolic logic. Georg F. L. P. Cantor (1845-1918) and his school created the modern theory of sets during the period 1874-1895.

We shall be interested primarily in sets of mathematical objects: sets of numbers, sets of curves, sets of geometric figures, and so on. In many applications it is convenient to deal with sets in which nothing special is assumed about the nature of the individual objects in the collection. These are called abstract sets. Abstract set theory has been developed to deal with such collections of arbitrary objects, and from this generality the theory derives its power.

### 12.2 Notations for designating sets

Sets usually are denoted by capital letters : $A, B, C, \ldots, X, Y, Z$; elements are designated by lower-case letters: $a, b, c, \ldots, x, y, z$. We use the special notation

$$x \in S$$

to mean that "$x$ is an element of $S$" or "$x$ belongs to $S$." If $x$ does not belong to $S$, we write $x \notin S$. When convenient, we shall designate sets by displaying the elements in braces; for example, the set of positive even integers less than 10 is denoted by the symbol $\{2, 4, 6, 8\}$ whereas the set of all positive even integers is displayed as $\{2, 4, 6, \ldots\}$, the three dots taking the place of "and so on." The dots are used only when the meaning of "and so on" is clear. The method of listing the members of a set within braces is sometimes referred to as *the roster notation.*

The first basic concept that relates one set to another is *equality* of sets:

DEFINITION OF SET EQUALITY. *Two sets $A$ and $B$ are said to be equal (or identical) if they consist of exactly the same elements, in which case we write $A = B$. If one of the sets contains an element not in the other, we say the sets are unequal and we write $A \neq B$.*

EXAMPLE 1. According to this definition, the two sets $\{2, 4, 6, 8\}$ and $\{2, 8, 6, 4\}$ are equal since they both consist of the four integers 2, 4, 6, and 8. Thus, when we use the roster notation to describe a set, the order in which the elements appear is irrelevant.

EXAMPLE 2. The sets $\{2, 4, 6, 8\}$ and $\{2, 2, 4, 4, 6, 8\}$ are equal even though, in the second set, each of the elements 2 and 4 is listed twice. Both sets contain the four elements 2, 4, 6, 8 and no others; therefore, the definition requires that we call these sets equal. This example shows that we do not insist that the objects listed in the roster notation be distinct. A similar example is the set of letters in the word *Mississippi,* which is equal to the set $\{M, i, s, p\}$, consisting of the four distinct letters $M$, $i$, $s$, and $p$.

### 12.3 Subsets

From a given set $S$ we may form new sets, called *subsets* of $S$. For example, the set consisting of those positive integers less than 10 which are divisible by 4 (the set $\{4, 8\}$) is a subset of the set of all even integers less than 10. In general, we have the following definition.

DEFINITION OF A SUBSET. *A set $A$ is said to be a subset of a set $B$, and we write*

$$A \subseteq B,$$

*whenever every element of $A$ also belongs to $B$. We also say that $A$ is contained in $B$ or that $B$ contains $A$. The relation $\subseteq$ is referred to as set inclusion.*

The statement $A \subseteq B$ does not rule out the possibility that $B \subseteq A$. In fact, we may have both $A \subseteq B$ and $B \subseteq A$, but this happens only if $A$ and $B$ have the same elements. In other words,

$$A = B \quad \text{if and only if} \quad A \subseteq B \text{ and } B \subseteq A.$$

This theorem is an immediate consequence of the foregoing definitions of equality and inclusion. If $A \subseteq B$ but $A \neq B$, then we say that $A$ is *a proper subset* of $B$; we indicate this by writing $A \subset B$.

In all our applications of set theory, we have a fixed set $S$ given in advance, and we are concerned only with subsets of this given set. The underlying set $S$ may vary from one application to another ; it will be referred to as the *universal set* of each particular discourse. The notation

$$\{x \mid x \in S \text{ and } x \text{ satisfies } P\}$$

will designate the set of all elements $x$ in $S$ which satisfy the property $P$. When the universal set to which we are referring is understood, we omit the reference to $S$ and write simply $\{x \mid x \text{ satisfies } P\}$. This is read "the set of all $x$ such that $x$ satisfies $P$." Sets designated in this way are said to be described by a defining property. For example, the set of all positive real numbers could be designated as $\{x \mid x > 0\}$; the universal set $S$ in this case is understood to be the set of all real numbers. Similarly, the set of all even positive integers $\{2, 4, 6, \ldots\}$ can be designated as $\{x \mid x \text{ is a positive even integer}\}$. Of course, the letter $x$ is a dummy and may be replaced by any other convenient symbol. Thus, we may write

$$\{x \mid x > 0\} = \{y \mid y > 0\} = \{t \mid t > 0\}$$

and so on.

It is possible for a set to contain no elements whatever. This set is called the *empty set* or the *void set*, and will be denoted by the symbol $\varnothing$. We will consider $\varnothing$ to be a subset of every set. Some people find it helpful to think of a set as analogous to a container (such as a bag or a box) containing certain objects, its elements. The empty set is then analogous to an empty container.

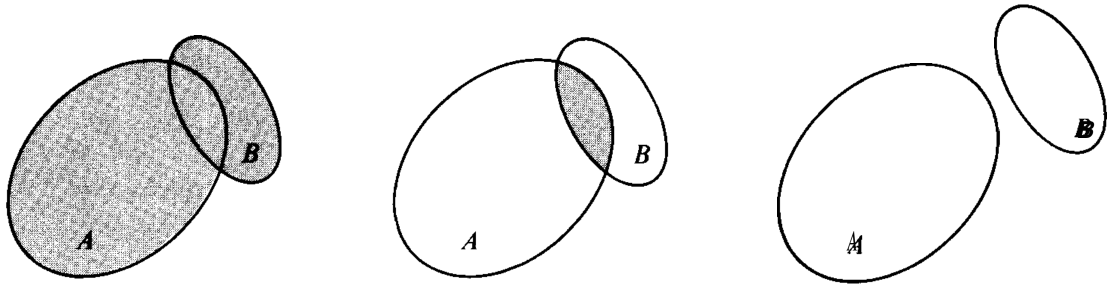
To avoid logical difficulties, we must distinguish between the element $x$ and the set $\{x\}$ whose only element is $x$. (A box with a hat in it is conceptually distinct from the hat itself.) In particular, the empty set $\varnothing$ is not the same as the set $\{\varnothing\}$. In fact, the empty set $\varnothing$ contains no elements, whereas the set $\{\varnothing\}$ has one element, $\varnothing$. (A box which contains an empty box is not empty.) Sets consisting of exactly one element are sometimes called *one-element sets*.

Diagrams often help us visualize relations between sets. For example, we may think of a set $S$ as a region in the plane and each of its elements as a point. Subsets of $S$ may then be thought of as collections of points within $S$. For example, in Figure 1.6(b) the shaded portion is a subset of $A$ and also a subset of $B$. Visual aids of this type, called *Venn diagrams*, are useful for testing the validity of theorems in set theory or for suggesting methods to prove them. Of course, the proofs themselves must rely only on the definitions of the concepts and not on the diagrams.

### 12.4 Unions, intersections, complements

From two given sets $A$ and $B$, we can form a new set called the *union* of $A$ and $B$. This new set is denoted by the symbol

$$A \cup B \quad \text{(read: "}A \text{ union } B\text{")}$$



(a) $A \cup B$ (b) $A \cap B$ (c) $A \cap B = \varnothing$

FIGURE 1.6 Unions and intersections.

and is defined as the set of those elements which are in $A$, in $B$, or in both. That is to say, $A \cup B$ is the set of all elements which belong to at least one of the sets $A$, $B$. An example is illustrated in Figure 1.6(a), where the shaded portion represents $A \cup B$.

Similarly, the *intersection* of $A$ and $B$, denoted by

$$A \cap B \quad (\text{read: ``}A \text{ intersection } B\text{''}),$$

is defined as the set of those elements common to *both* $A$ and $B$. This is illustrated by the shaded portion of Figure 1.6(b). In Figure 1.6(c), the two sets $A$ and $B$ have no elements in common; in this case, their intersection is the empty set $\varnothing$. Two sets $A$ and $B$ are said to be *disjoint* if $A \cap B = \varnothing$.

If $A$ and $B$ are sets, the *difference* $A - B$ (also called the *complement of $B$ relative to $A$*) is defined to be the set of all elements of $A$ which are not in $B$. Thus, by definition,

$$A - B = \{x \mid x \in A \text{ and } x \notin B\}.$$

In Figure 1.6(b) the unshaded portion of $A$ represents $A - B$; the unshaded portion of $B$ represents $B - A$.

The operations of union and intersection have many formal similarities to (as well as differences from) ordinary addition and multiplication of real numbers. For example, since there is no question of order involved in the definitions of union and intersection, it follows that $A \cup B = B \cup A$ and that $A \cap B = B \cap A$. That is to say, union and intersection are *commutative* operations. The definitions are also phrased in such a way that the operations are *associative*:

$$(A \cup B) \cup C = A \cup (B \cup C) \quad \text{and} \quad (A \cap B) \cap C = A \cap (B \cap C).$$

These and other theorems related to the "algebra of sets" are listed as Exercises in Section 1 2.5. One of the best ways for the reader to become familiar with the terminology and notations introduced above is to carry out the proofs of each of these laws. A sample of the type of argument that is needed appears immediately after the Exercises.

The operations of union and intersection can be extended to finite or infinite collections of sets as follows: Let $\mathscr{F}$ be a nonempty class† of sets. The union of all the sets in $\mathscr{F}$ is

† To help simplify the language, we call a collection of sets a *class*. Capital script letters $\mathscr{A}, \mathscr{B}, \mathscr{C}, \ldots$ are used to denote classes. The usual terminology and notation of set theory applies, of course, to classes. Thus, for example, $A \in \mathscr{F}$ means that $A$ is one of the sets in the class $\mathscr{F}$, and $\mathscr{A} \subseteq \mathscr{B}$ means that every set in $\mathscr{A}$ is also in $\mathscr{B}$, and so forth.

defined as the set of those elements which belong to at least one of the sets in $\mathscr{F}$ and is denoted by the symbol

$$\bigcup_{A \in \mathscr{F}} A .$$

If $\mathscr{F}$ is a finite collection of sets, say $\mathscr{F} = \{A, , A,, \ldots , A,\}$, we write

$$\bigcup_{A \in \mathscr{F}} A = \bigcup_{k=1}^{n} A_k = A_1 \; u \; A, \; u \; \ldots \; u \; A, \; .$$

Similarly, the intersection of all the sets in $\mathscr{F}$ is defined to be the set of those elements which belong to every one of the sets in $\mathscr{F}$; it is denoted by the symbol

$$\bigcap_{A \in \mathscr{F}} A .$$

For finite collections (as above), we write

$$\bigcap_{A \in \mathscr{F}} A = \bigcap_{k=1}^{n} A_k = A_1 \cap A_2 \cap \cdots \cap A_n .$$

Unions and intersections have been defined in such a way that the associative laws for these operations are automatically satisfied. Hence, there is no ambiguity when we write $A, \; u \; A_2 \; u \; \ldots u \; A,$ or $A, \cap A_2 \cap \, . \; ' \, . \cap A,$.

## 12.5 Exercises

1. Use the roster notation to designate the following sets of real numbers.

$$A = \{x \quad x^2 - 1 = 0\} . \qquad D = \{x \mid x^3 - 2x^2 + x = 2\} .$$

$$B = \{x \quad (x - 1)^2 = 0\} . \qquad E = \{x \mid (x + 8)^2 = 9^2\} .$$

$$C = \{x \mid x + 8 = 9\} . \qquad F = \{x \quad (x^2 + 16x)^2 = 17^2\} .$$

2. For the sets in Exercise 1, note that $B \subseteq A$. List all the inclusion relations $\subseteq$ that hold among the sets $A, B, C, D, E, F$.
3. Let $A = \{1\}, B = \{1, 2\}$. Discuss the validity of the following statements (prove the ones that are true and explain why the others are not true).
   (a) $A \subset B$. (d) $1 \in A$.
   (b) $A \subseteq B$. (e) $1 \subseteq A$.
   (c) $A \in B$. (f) $1 \subset B$.
4. Solve Exercise 3 if $A = \{1\}$ and $B = \{\{1\}, 1\}$.
5. Given the set $S = \{1, 2, 3, 4)$. Display all subsets of S. There are 16 altogether, counting $\varnothing$ and S.
6. Given the following four sets

$$A = \{1, 2\}, \qquad B = \{\{1\}, \{2\}\}, \qquad C = \{\{1\}, \{1, 2\}\}, \qquad D = \{\{1\}, \{2\}, \{1, 2\}\},$$

discuss the validity of the following statements (prove the ones that are true and explain why the others are not true).

(a) $A = B$. (d) $A \in C$. (g) $B \subset D$.
(b) $A \subseteq B$. (e) $A \subset D$. (h) $B \in D$.
(c) $A \subset C$. (f) $B \subset C$. (i) $A \in D$.

7. Prove the following properties of set equality.
(a) $\{a, a\} = \{a\}$.
(b) $\{a, b\} = \{b, a\}$.
(c) $\{a\} = \{b, c\}$ if and only if $a = b = c$.

Prove the set relations in Exercises 8 through 19. (Sample proofs are given at the end of this section).

8. *Commutative laws:* $A \cup B = B \cup A$, $A \cap B = B \cap A$.
9. *Associative laws:* $A \cup (B \cup C) = (A \cup B) \cup C$, $A \cap (B \cap C) = (A \cap B) \cap C$.
10. *Distributive laws:* $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.
11. $A \cup A = A$, $A \cap A = A$,
12. $A \subseteq A \cup B$, $A \cap B \subseteq A$.
13. $A \cup \varnothing = A$, $A \cap \varnothing = \varnothing$.
14. $A \cup (A \cap B) = A$, $A \cap (A \cup B) = A$.
15. If $A \subseteq C$ and $B \subseteq C$, then $A \cup B \subseteq C$.
16. If $C \subseteq A$ and $C \subseteq B$, then $C \subseteq A \cap B$.
17. (a) If $A \subset B$ and $B \subset C$, prove that $A \subset C$.
(b) If $A \subseteq B$ and $B \subseteq C$, prove that $A \subseteq C$.
(c) What can you conclude if $A \subset B$ and $B \subseteq C$?
(d) If $x \in A$ and $A \subseteq B$, is it necessarily true that $x \in B$?
(e) If $x \in A$ and $A \in B$, is it necessarily true that $x \in B$?
18. $A - (B \cap C) = (A - B) \cup (A - C)$.
19. Let $\mathscr{F}$ be a class of sets. Then

$$B - \bigcup_{A \in \mathscr{F}} A = \bigcap_{A \in \mathscr{F}} (B - A) \quad \text{and} \quad B - \bigcap_{A \in \mathscr{F}} A = \bigcup_{A \in \mathscr{F}} (B - A).$$

20. (a) Prove that one of the following two formulas is always right and the other one is sometimes wrong :

$$\text{(i) } A - (B - C) = (A - B) \cup C,$$

$$\text{(ii) } A - (B \cup C) = (A - B) - C.$$

(b) State an additional necessary and sufficient condition for the formula which is sometimes incorrect to be always right.

*Proof of the commutative law* $A \cup B = B \cup A$. Let $X = A \cup B$, $Y = B \cup A$. To prove that $X = Y$ we prove that $X \subseteq Y$ and $Y \subseteq X$. Suppose that $x \in X$. Then $x$ is in at least one of $A$ or $B$. Hence, $x$ is in at least one of $B$ or $A$; so $x \in Y$. Thus, every element of $X$ is also in $Y$, so $X \subseteq Y$. Similarly, we find that $Y \subseteq X$, so $X = Y$.

*Proof of* $A \cap B \subseteq A$. If $x \in A \cap B$, then $x$ is in both $A$ and $B$. In particular, $x \in A$. Thus, every element of $A \cap B$ is also in $A$; therefore, $A \cap B \subseteq A$.

# *Part 3. A Set of Axioms for the Real-Number System*

## 13.1 Introduction

There are many ways to introduce the real-number system. One popular method is to begin with the positive integers 1, 2, 3, , . . and use them as building blocks to construct a more comprehensive system having the properties desired. Briefly, the idea of this method is to take the positive integers as undefined concepts, state some axioms concerning them, and then use the positive integers to build a larger system consisting of the positive ***rational*** numbers (quotients of positive integers). The positive rational numbers, in turn, may then be used as a basis for constructing the positive ***irrational*** numbers (real numbers like $\sqrt{2}$ and $\pi$ that are not rational). The final step is the introduction of the negative real numbers and zero. The most difficult part of the whole process is the transition from the rational numbers to the irrational numbers.

Although the need for irrational numbers was apparent to the ancient Greeks from their study of geometry, satisfactory methods for constructing irrational numbers from rational numbers were not introduced until late in the 19th Century. At that time, three different theories were outlined by Karl Weierstrass (1815–1897), Georg Cantor (1845–1918), and Richard Dedekind (1831-1916). In 1889, the Italian mathematician Guiseppe Peano (1858-1932) listed five axioms for the positive integers that could be used as the starting point of the whole construction. A detailed account of this construction, beginning with the Peano postulates and using the method of Dedekind to introduce irrational numbers, may be found in a book by E. Landau, ***Foundations** of **Analysis** (**New*** York, Chelsea Publishing Co., 1951).

The point of view we shall adopt here is nonconstructive. We shall start rather far out in the process, taking the real numbers themselves as undefined objects satisfying a number of properties that we use as axioms. That is to say, we shall assume there exists a set **R** of objects, called real numbers, which satisfy the 10 axioms listed in the next few sections. All the properties of real numbers can be deduced from the axioms in the list. When the real numbers are defined by a constructive process, the properties we list as axioms must be proved as theorems.

In the axioms that appear below, lower-case letters $a, b, c, \ldots, x, y, z$ represent arbitrary real numbers unless something is said to the contrary. The axioms fall in a natural way into three groups which we refer to as the ***field axioms,*** the ***order axioms,*** and the ***least-upper-bound axiom*** (also called the ***axiom of continuity*** or the ***completeness axiom).***

## 13.2 The field axioms

Along with the set **R** of real numbers we assume the existence of two operations called ***addition*** and ***multiplication,*** such that for every pair of real numbers $x$ and $y$ we can form the ***sum*** of $x$ and $y$, which is another real number denoted by $x + y$, and the ***product*** of $x$ and $y$, denoted by $xy$ or by $x \cdot y$. It is assumed that the sum $x + y$ and the product $xy$ are ***uniquely determined*** by $x$ and $y$. In other words, given $x$ and $y$, there is exactly one real number $x + y$ and exactly one real number $xy$. We attach no special meanings to the symbols $+$ and $\cdot$ other than those contained in the axioms.

**AXIOM 1. COMMUTATIVE LAWS.** $x + y = y + x, \quad xy = yx.$

**AXIOM 2. ASSOCIATIVE LAWS.** $x + (y + z) = (x + y) + z, \quad x(yz) = (xy)z.$

**AXIOM 3. DISTRIBUTIVE LAW.** $x(y + z) = xy + xz.$

**AXIOM 4. EXISTENCE OF IDENTITY ELEMENTS.** *There exist two distinct real numbers, which we denote by 0 and 1, such that for ecery real x we have* $x + 0 = x$ *and* $1 \cdot x = x$.

**AXIOM 5. EXISTENCE OF NEGATIVES.** *For ecery real number x there is a real number y such that* $x + y = 0$.

**AXIOM 6. EXISTENCE OF RECIPROCALS.** *For every real number* $x \neq 0$ *there is a real number y such that* $xy = 1$.

*Note:* The numbers 0 and 1 in Axioms 5 and 6 are those of Axiom 4.

From the above axioms we can deduce all the usual laws of elementary algebra. The most important of these laws are collected here as a list of theorems. In all these theorems the symbols *a, b, c, d* represent arbitrary real numbers.

**THEOREM I.1. CANCELLATION LAW FOR ADDITION.** *If* $a + b = a + c$, *then* $b = c$. *(In particular, this shows that the number 0 of Axiom 4 is unique.)*

**THEOREM 1.2. POSSIBILITY OF SUBTRACTION.** *Given a and b, there is exactly one x such that* $a + x = b$. *This x is denoted by* $b - a$. *In particular,* $0 - a$ *is written simply* $-a$ *and is called the negative of a.*

**THEOREM 1.3.** $b - a = b + (-a).$

**THEOREM 1.4.** $-(-a) = a.$

**THEOREM 1.5.** $a(b - c) = ab - ac.$

**THEOREM 1.6.** $0 \cdot a = a \cdot 0 = 0.$

**THEOREM 1.7. CANCELLATION LAW FOR MULTIPLICATION.** *If* $ab = ac$ *and* $a \neq 0$, *then* $b = c$. *(In particular, this shows that the number 1 of Axiom 4 is unique.)*

**THEOREM 1.8. POSSIBILITY OF DIVISION.** *Given a and b with* $a \neq 0$, *there is exactly one x such that* $ax = b$. *This x is denoted by* $b/a$ *or* $\frac{b}{a}$ *and is called the quotient of b and a. In particular,* $1/a$ *is also written* $a^{-1}$ *and is called the reciprocal of a.*

**THEOREM 1.9.** *If* $a \neq 0$, *then* $b/a = b \cdot a^{-1}$.

**THEOREM 1.10.** *If* $a \neq 0$, *then* $(a^{-1})^{-1} = a$.

**THEOREM I.11.** *If* $ab = 0$, *then* $a = 0$ *or* $b = 0$.

**THEOREM 1.12.** $(-a)b = -(ah)$ *and* $(-a)(-b) = ab$.

**THEOREM 1.13.** $(a/b) + (c/d) = (ad + bc)/(bd)$ *if* $b \neq 0$ *and* $d \neq 0$.

**THEOREM 1.14.** $(a/b)(c/d) = (ac)/(bd)$ *if* $b \neq 0$ *and* $d \neq 0$.

**THEOREM 1.15.** $(a/b)/(c/d) = (ad)/(bc)$ *if* $b \neq 0$, $c \neq 0$, *and* $d \neq 0$.

To illustrate how these statements may be obtained as consequences of the axioms, we shall present proofs of Theorems 1.1 through 1.4. Those readers who are interested may find it instructive to carry out proofs of the remaining theorems.

*Proof of* 1.1. Given $a + b = a + c$. By Axiom 5, there is a number $y$ such that $y + a = 0$. Since sums are uniquely determined, we have $y + (a + b) = y + (a + c)$. Using the associative law, we obtain $(y + a) + b = (y + a) + c$ or $0 + b = 0 + c$. But by Axiom 4 we have $0 + b = b$ and $0 + c = c$, so that $b = c$. Notice that this theorem shows that there is only one real number having the property of 0 in Axiom 4. In fact, if 0 and 0' both have this property, then $0 + 0' = 0$ and $0 + 0 = 0$. Hence $0 + 0' = 0 + 0$ and, by the cancellation law, $0 = 0'$.

*Proof of* 1.2. Given $a$ and $b$, choose $y$ so that $a + y = 0$ and let $x = y + b$. Then $a + x = a + (y + b) = (a + y) + b = 0 + b = b$. Therefore there is at least one $x$ such that $a + x = b$. But by Theorem 1.1 there is at most one such $x$. Hence there is *exactly* one.

*Proof of* 1.3. Let $x = b - a$ and let $y = b + (-a)$. We wish to prove that $x = y$. Now $x + a = b$ (by the definition of $b - a$) and

$$y + a = [b + (-a)] + a = b + [(-a) + a] = b + 0 = b.$$

Therefore $x + a = y + a$ and hence, by Theorem 1.1, $x = y$.

*Proof of* 1.4. We have $a + (-a) = 0$ by the definition of $-a$. But this equation tells us that $a$ is the negative of $-a$. That is, $a = -(-a)$, as asserted.

## *I 3.3 Exercises

1. Prove Theorems 1.5 through 1.15, using Axioms 1 through 6 and Theorems 1.1 through 1.4.

In Exercises 2 through 10, prove the given statements or establish the given equations. You may use Axioms 1 through 6 and Theorems 1.1 through 1.15.

2. $-0 = 0$.
3. $1^{-1} = 1$.
4. Zero has no reciprocal.
5. $-(a + b) = -a - b$.
6. $-(a - b) = -a + b$.
7. $(a - b) + (b - c) = a - c$.
8. If $a \neq 0$ and $b \neq 0$, then $(ab)^{-1} = a^{-1}b^{-1}$.
9. $-(a/b) = (-a/b) = a/(-b)$ if $b \neq 0$.
10. $(a/b) - (c/d) = (ad - bc)/(bd)$ if $b \neq 0$ and $d \neq 0$.

## I 3.4 The order axioms

This group of axioms has to do with a concept which establishes an ***ordering*** among the real numbers. This ordering enables us to make statements about one real number being larger or smaller than another. We choose to introduce the order properties as a set of axioms about a new undefined concept called *positiveness* and then to define terms like *less than* and *greater than* in terms of positiveness.

We shall assume that there exists a certain subset $\mathbf{R}^+ \subset$ R, called the set of *positive* numbers, which satisfies the following three order axioms :

AXIOM 7. *If $x$ and $y$ are in $\mathbf{R}^+$, so are $x + y$ and $xy$.*

AXIOM 8. *For every real $x \neq 0$, either $x \in \mathbf{R}^+$ or $-x \in \mathbf{R}^+$, but not both.*

AXIOM 9. $0 \notin \mathbf{R}^+$.

Now we can define the symbols $<, >, \leq$, and $\geq$, called, respectively, *less than, greater than, less than or equal to,* and *greater than or equal to,* as follows:

$x < y$ means that $y - x$ is positive;

$y > x$ means that $x < y$;

$x \leq y$ means that either $x < y$ or $x = y$;

$y \geq x$ means that $x \leq y$.

Thus, we have $x > 0$ if and only if $x$ is positive. If $x < 0$, we say that $x$ is *negative;* if $x \geq 0$, we say that $x$ is *nonnegative.* A pair of simultaneous inequalities such as $x < y$, $y < z$ is usually written more briefly as $x < y < z$; similar interpretations are given to the compound inequalities $x \leq y < z$, $x < y \leq z$, and $x \leq y \leq z$.

From the order axioms we can derive all the usual rules for calculating with inequalities. The most important of these are listed here as theorems.

THEOREM I.16. TRICHOTOMY LAW. *For arbitrary real numbers $a$ and $b$, exactly one of the three relations $a < b$, $b < a$, $a = b$ holds.*

THEOREM 1.17. TRANSITIVE LAW. *Zf $a < b$ andb $< c$, then $a < c$.*

THEOREM I.18. *If $a < b$, then $a + c < b + c$.*

THEOREM 1.19. *If $a < b$ and $c > 0$, then $ac < bc$.*

THEOREM 1.20. *If $a \neq 0$, then $a^2 > 0$.*

THEOREM 1.21. $1 > 0$.

THEOREM 1.22. *If $a < b$ and $c < 0$, then $ac > bc$.*

THEOREM 1.23. *If $a < b$, then $-a > -b$. Znparticular, fa $< 0$, then $-a > 0$.*

THEOREM 1.24. *If $ab > 0$, then both $a$ and $b$ are positive or both are negative.*

THEOREM 1.25. *If $a < c$ and $b < d$, then $a + b < c + d$.*

Again, we shall prove only a few of these theorems as samples to indicate how the proofs may be carried out. Proofs of the others are left as exercises.

*Proof of* 1.16. Let $x = b - a$. If $x = 0$, then $b - a = a - b = 0$, and hence, by Axiom 9, we cannot have $a > b$ or $b > a$. If $x \neq 0$, Axiom 8 tells us that either $x > 0$ or $x < 0$, but not both; that is, either $a < b$ or $b < a$, but not both. Therefore, exactly one of the three relations, $a = b$, $a < b$, $b < a$, holds.

*Proof of* 1.17. If $a < b$ and $b < c$, then $b - a > 0$ and $c - b > 0$. By Axiom 7 we may add to obtain $(b - a) + (c - b) > 0$. That is, $c - a > 0$, and hence $a < c$.

*Proof of* 1.18. Let $x = a + c$, $y = b + c$. Then $y - x = b - a$. But $b - a > 0$ since $a < b$. Hence $y - x > 0$, and this means that $x < y$.

*Proof of* 1.19. If $a < b$, then $b - a > 0$. If $c > 0$, then by Axiom 7 we may multiply $c$ by $(b - a)$ to obtain $(b - a)c > 0$. But $(b - a)c = bc - ac$. Hence $bc - ac > 0$, and this means that $ac < bc$, as asserted.

*Proof of* 1.20. If $a > 0$, then $a \cdot a > 0$ by Axiom 7. If $a < 0$, then $-a > 0$, and hence $(-a) \cdot (-a) > 0$ by Axiom 7. In either case we have $a^2 > 0$.

*Proof of* 1.21. Apply Theorem 1.20 with $a = 1$.

### *I 3.5 Exercises

1. Prove Theorems 1.22 through 1.25, using the earlier theorems and Axioms 1 through 9.

In Exercises 2 through 10, prove the given statements or establish the given inequalities. You may use Axioms 1 through 9 and Theorems 1.1 through 1.25.

2. There is no real number $x$ such that $x^2 + 1 = 0$.
3. The sum of two negative numbers is negative.
4. If $a > 0$, then $1/a > 0$; if $a < 0$, then $1/a < 0$.
5. If $0 < a < b$, then $0 < b^{-1} < a^{-1}$.
6. Ifu $\leq b$ and $b \leq c$, then $a \leq c$.
7. Ifu $\leq b$ and $b \leq c$, and $a = c$, thenb $= c$.
8. For all real $a$ and $b$ we have $a^2 + b^2 \geq 0$. If $a$ and $b$ are not both 0, then $a^2 + b^2 > 0$.
9. There is no real number $a$ such that $x \leq a$ for all real $x$.
10. If $x$ has the property that $0 \leq x < h$ for *every* positive real number $h$, then $x = 0$.

### 13.6 Integers and rational numbers

There exist certain subsets of **R** which are distinguished because they have special properties not shared by all real numbers. In this section we shall discuss two such subsets, the *integers* and the *rational numbers.*

To introduce the positive integers we begin with the number 1, whose existence is guaranteed by Axiom 4. The number 1 + 1 is denoted by 2, the number 2 + 1 by 3, and so on. The numbers 1, 2, 3, . . . , obtained in this way by repeated addition of 1 are all positive, and they are called the *positive integers.* Strictly speaking, this description of the positive integers is not entirely complete because we have not explained in detail what we mean by the expressions "and so on," or "repeated addition of 1." Although the intuitive meaning of these expressions may seem clear, in a careful treatment of the real-number system it is necessary to give a more precise definition of the positive integers. There are many ways to do this. One convenient method is to introduce first the notion of an *inductive set.*

DEFINITION OF AN INDUCTIVE SET. *A set of real numbers is called an inductive set if it has the following two properties:*
(a) *The number* 1 *is in the set.*
(b) *For every $x$ in the set, the number $x + 1$ is also in the set.*

For example, **R** is an inductive set. So is the set $\mathbf{R}^+$. Now we shall define the positive integers to be those real numbers which belong to every inductive set.

DEFINITION OF POSITIVE INTEGERS. *A real number is called a positive integer if it belongs to every inductive set.*

Let **P** denote the set of all positive integers. Then **P** is itself an inductive set because (a) it contains 1, and (b) it contains $x + 1$ whenever it contains $x$. Since the members of P belong to every inductive set, we refer to **P** as the *smallest* inductive set. This property of the set **P** forms the logical basis for a type of reasoning that mathematicians call *proof by induction,* a detailed discussion of which is given in Part 4 of this Introduction.

The negatives of the positive integers are called the *negative integers.* The positive integers, together with the negative integers and 0 (zero), form a set **Z** which we call simply the *set of integers.*

In a thorough treatment of the real-number system, it would be necessary at this stage to prove certain theorems about integers. For example, the sum, difference, or product of two integers is an integer, but the quotient of two integers need not be an integer. However, we shall not enter into the details of such proofs.

Quotients of integers $a/b$ (where $b \neq 0$) are called *rational numbers.* The set of rational numbers, denoted by Q, contains **Z** as a subset. The reader should realize that all the field axioms and the order axioms are satisfied by Q. For this reason, we say that the set of rational numbers is an *ordered field.* Real numbers that are not in Q are called *irrational.*

### 13.7 Geometric interpretation of real numbers as points on a line

The reader is undoubtedly familiar with the geometric representation of real numbers by means of points on a straight line. A point is selected to represent 0 and another, to the right of 0, to represent 1, as illustrated in Figure 1.7. This choice determines the scale. If one adopts an appropriate set of axioms for Euclidean geometry, then each real number corresponds to exactly one point on this line and, conversely, each point on the line corresponds to one and only one real number. For this reason the line is often called the *real line* or the *real axis,* and it is customary to use the words *real number* and *point* interchangeably. Thus we often speak of the *point* $x$ rather than the point corresponding to the real number $x$.

The ordering relation among the real numbers has a simple geometric interpretation. If $x < y$, the point $x$ lies to the left of the point $y$, as shown in Figure 1.7. Positive numbers lie to the right of 0 and negative numbers to the left of 0. If $a < b$, a point $x$ satisfies the inequalities $a < x < b$ if and only if $x$ is ***between** a* and *b.*

This device for representing real numbers geometrically is a very worthwhile aid that helps us to discover and understand better certain properties of real numbers. However, the reader should realize that all properties of real numbers that are to be accepted as theorems must be deducible from the axioms without any reference to geometry. This does not mean that one should not make use of geometry in studying properties of real numbers. On the contrary, the geometry often suggests the method of proof of a particular theorem, and sometimes a geometric argument is more illuminating than a purely ***analytic*** proof (one depending entirely on the axioms for the real numbers). In this book, geometric

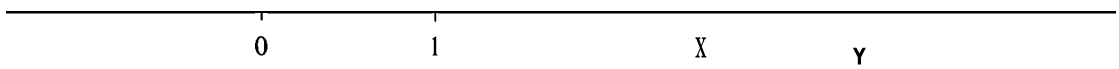


FIGURE 1.7 Real numbers represented geometrically on a line.

arguments are used to a large extent to help motivate or clarify a particular discussion. Nevertheless, the proofs of all the important theorems are presented in analytic form.

## 13.8 Upper bound of a set, maximum element, least upper bound (supremum)

The nine axioms listed above contain all the properties of real numbers usually discussed in elementary algebra. There is another axiom of fundamental importance in calculus that is ordinarily not discussed in elementary algebra courses. This axiom (or some property equivalent to it) is used to establish the existence of irrational numbers.

Irrational numbers arise in elementary algebra when we try to solve certain quadratic equations. For example, it is desirable to have a real number $x$ such that $x^2 = 2$. From the nine axioms above, we cannot prove that such an $x$ exists in **R**, because these nine axioms are also satisfied by Q, and there is no rational number $x$ whose square is 2. (A proof of this statement is outlined in Exercise 11 of Section 1 3.12.) Axiom 10 allows us to introduce irrational numbers in the real-number system, and it gives the real-number system a property of continuity that is a keystone in the logical structure of calculus.

Before we describe Axiom 10, it is convenient to introduce some more terminology and notation. Suppose $S$ is a nonempty set of real numbers and suppose there is a number $B$ such that

$$x \leq B$$

for every $x$ in $S$. Then $S$ is said to be ***bounded above*** by $B$. The number $B$ is called an ***upper bound*** for $S$. We say *an* upper bound because every number greater than $B$ will also be an upper bound. If an upper bound $B$ is also a member of $S$, then $B$ is called the ***largest member*** or the ***maximum element*** of $S$. There can be at most one such $B$. If it exists, we write

$$B = \max S .$$

Thus, $B = \max S$ if $B \in S$ and $x \leq B$ for all $x$ in $S$. A set with no upper bound is said to be *unbounded above.*

The following examples serve to illustrate the meaning of these terms.

EXAMPLE 1. Let S be the set of all positive real numbers. This set is unbounded above. It has no upper bounds and it has no maximum element.

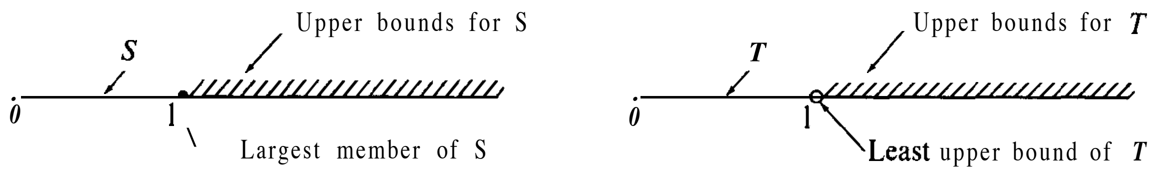
EXAMPLE 2. Let S be the set of all real $x$ satisfying $0 \leq x \leq 1$. This set is bounded above by 1. In fact, 1 is its maximum element.

EXAMPLE 3. Let $T$ be the set of all real x satisfying $0 \leq x < 1$. This is like the set in Example 2 except that the point 1 is not included. This set is bounded above by 1 but it has no maximum element.

Some sets, like the one in Example 3, are bounded above but have no maximum element. For these sets there is a concept which takes the place of the maximum element. This is called the ***least upper bound*** of the set and it is defined as follows:

DEFINITION OF LEAST UPPER BOUND. *A number B is called a least upper bound of a nonempty set S if B has the following two properties:*
(a) *B is an upper boundfor S.*
(b) *No number less than B is an upper boundfor S.*

If S has a maximum element, this maximum is also a least upper bound for S. But if S does not have a maximum element, it may still have a least upper bound. In Example 3 above, the number 1 is a least upper bound for $T$ although $T$ has no maximum element. (See Figure 1.8.)




(a) S has a largest member: max $S = 1$

(b) $T$ has no largest member, but it has a least upper bound: sup $T = 1$

FIGURE 1.8 Upper bounds, maximum element, supremum.

THEOREM 1.26. *Two different numbers cannot be least upper bounds for the same set.*

*Proof.* Suppose that $B$ and C are two least upper bounds for a set S. Property (b) implies that $C \geq B$ since $B$ is a least upper bound; similarly, $B \geq C$ since C is a least upper bound. Hence, we have $B = C$.

This theorem tells us that if there is a least upper bound for a set S, there is *only* one and we may speak of *the* least upper bound.

It is common practice to refer to the least upper bound of a set by the more concise term ***supremum,*** abbreviated ***sup.*** We shall adopt this convention and write

$$B = \sup S$$

to express the fact that $B$ is the least upper bound, or supremum, of S.

### 13.9 The least-Upper-bound axiom (completeness axiom)

Now we are ready to state the least-Upper-bound axiom for the real-number system.

AXIOM 10. *Every nonempty set $S$ of real numbers which is bounded above has a supremum; that is, there is a real number $B$ such that $B = \sup S$.*

We emphasize once more that the supremum of S need not be a member of S. In fact, sup S belongs to S if and only if S has a maximum element, in which case max S = sup S.

Definitions of the terms *lower bound, bounded below, smallest member* (or *minimum element)* may be similarly formulated. The reader should formulate these for himself. If S has a minimum element, we denote it by min S.

A number $L$ is called a *greatest lower bound* (or *infimum*) of S if (a) $L$ is a lower bound for $S$, and (b) no number greater than $L$ is a lower bound for S. The infimum of S, when it exists, is uniquely determined and we denote it by inf S. If $S$ has a minimum element, then $\min S = \inf S$.

Using Axiom 10, we can prove the following.

THEOREM 1.27. *Every nonempty set $S$ that is bounded below has a greatest lower bound; that is, there is a real number $L$ such that $L = \inf S$.*

*Proof.* Let $-S$ denote the set of negatives of numbers in S. Then $-S$ is nonempty and bounded above. Axiom 10 tells us that there is a number $B$ which is a supremum for $-S$. It is easy to verify that $-B = \inf S$.

Let us refer once more to the examples in the foregoing section. In Example 1, the set of all positive real numbers, the number 0 is the infimum of S. This set has no minimum element. In Examples 2 and 3, the number 0 is the minimum element.

In all these examples it was easy to decide whether or not the set $S$ was bounded above or below, and it was also easy to determine the numbers sup $S$ and inf S. The next example shows that it may be difficult to determine whether upper or lower bounds exist.

EXAMPLE 4. Let $S$ be the set of all numbers of the form $(1 + 1/n)^n$, where $n = 1, 2, 3, \ldots$. For example, taking $n = 1$, 2, and 3, we find that the numbers 2, $\frac{9}{4}$, and $\frac{64}{27}$ are in S. Every number in the set is greater than 1, so the set is bounded below and hence has an infimum. With a little effort we can show that 2 is the smallest element of $S$ so $\inf S = \min S = 2$. The set $S$ is also bounded above, although this fact is not as easy to prove. (Try it!) Once we know that $S$ is bounded above, Axiom 10 tells us that there is a number which is the supremum of $S$. In this case it is not easy to determine the value of sup $S$ from the description of $S$. In a later chapter we will learn that sup $S$ is an irrational number approximately equal to 2.718. It is an important number in calculus called the Euler number e.

### 13.10 The Archimedean property of the real-number system

This section contains a number of important properties of the real-number system which are consequences of the least-Upper-bound axiom.

THEOREM 1.28. *The set* **P** *of positive integers* $1, 2, 3, \ldots$ *is unbounded above.*

*Proof.* Assume **P** is bounded above. We shall show that this leads to a contradiction. Since **P** is nonempty, Axiom 10 tells us that **P** has a least upper bound, say $b$. The number $b - 1$, being less than $b$, cannot be an upper bound for **P**. Hence, there is at least one positive integer $n$ such that $n > b - 1$. For this $n$ we have $n + 1 > b$. Since $n + 1$ is in **P**, this contradicts the fact that $b$ is an upper bound for **P**.

As corollaries of Theorem 1.28, we immediately obtain the following consequences:

THEOREM 1.29. *For every real $x$ there exists a positive integer $n$ such that $n > x$.*

*Proof.* If this were not so, some $x$ would be an upper bound for **P**, contradicting Theorem 1.28.

THEOREM 1.30. *If $x > 0$ and if $y$ is an arbitrary real number, there exists a positive integer $n$ such that $nx > y$.*

*Proof.* Apply Theorem 1.29 with $x$ replaced by $y/x$.

The property described in Theorem 1.30 is called the *Archimedean property* of the real-number system. Geometrically it means that any line segment, no matter how long, may be covered by a finite number of line segments of a given positive length, no matter how small. In other words, a small ruler used often enough can measure arbitrarily large distances. Archimedes realized that this was a fundamental property of the straight line and stated it explicitly as one of the axioms of geometry. In the 19th and 20th centuries, non-Archimedean geometries have been constructed in which this axiom is rejected.

From the Archimedean property, we can prove the following theorem, which will be useful in our discussion of integral calculus.

THEOREM 1.31. *If three real numbers $a$, $x$, and $y$ satisfy the inequalities*

$$a \le x \le a + \frac{y}{n} \tag{1.14}$$

*for every integer $n \ge 1$, then $x = a$.*

*Proof.* If $x > a$, Theorem 1.30 tells us that there is a positive integer $n$ satisfying $n(x - a) > y$, contradicting (1.14). Hence we cannot have $x > a$, so we must have $x = a$.

## 13.11 Fundamental properties of the supremum and infimum

This section discusses three fundamental properties of the supremum and infimum that we shall use in our development of calculus. The first property states that any set of numbers with a supremum contains points arbitrarily close to its supremum; similarly, a set with an infimum contains points arbitrarily close to its infimum.

THEOREM 1.32. *Let $h$ be a given positive number and let $S$ be a set of real numbers.*

(a) *If $S$ has a supremum, then for some $x$ in $S$ we have*

$$x > \sup S - h .$$

(b) *If $S$ has an injmum, then for some $x$ in $S$ we have*

$$x < \inf S + h .$$

*Proof of* (a). If we had $x \leq \sup S \quad h$ for *all* $x$ in $S$, then $\sup S - h$ would be an upper bound for $S$ smaller than its least upper bound. Therefore we must have $x > \sup S - h$ for at least one $x$ in $S$. This proves (a). The proof of(b) is similar.

THEOREM 1.33. ADDITIVE PROPERTY. *Given nonempty subsets $A$ and $B$ of* R, *let $C$ denote the set*

$$C = \{a + b \mid a \in A, b \in B\} .$$

(a) *If each of $A$ and $B$ has a supremum, then $C$ has a supremum, and*

$$\sup C = \sup A + \sup B .$$

(b) *If each of $A$ and $B$ has an injmum, then $C$ has an infimum, and*

$$\inf C = \inf A + \inf B .$$

*Proof.* Assume each of $A$ and $B$ has a supremum. If $c \in C$, then $c = a + b$, where $a \in A$ and $b \in B$. Therefore $c \leq \sup A + \sup B$; so $\sup A + \sup B$ is an upper bound for $C$. This shows that $C$ has a supremum and that

$$\sup C \leq \sup A + \sup B .$$

Now let $n$ be any positive integer. By Theorem 1.32 (with $h = 1/n$) there is an $a$ in $A$ and a $b$ in $B$ such that

$$a > \sup A - \frac{1}{n}, \qquad b > \sup B - \frac{1}{n} .$$

Adding these inequalities, we obtain

$$a + b > \sup A + \sup B - \frac{2}{n}, \quad \text{or} \quad \sup A + \sup B < a + b + \frac{2}{n} \leq \sup C + \frac{2}{n},$$

since $a + b \leq \sup C$. Therefore we have shown that

$$\sup C \leq \sup A + \sup B < \sup C + \frac{2}{n}$$

for every integer $n \geq 1$. By Theorem 1.31, we must have sup C = sup A + sup *B*. This proves (a), and the proof of(b) is similar.

THEOREM 1.34. *Given two nonempty subsets $S$ and $T$ of* R *such that*

$$s \leq t$$

*for every $s$ in $S$ and every $t$ in $T$. Then $S$ has a supremum, and $T$ has an infimum, and they satisfy the inequality*

$$\sup S \leq \inf T.$$

*Proof.* Each $t$ in $T$ is an upper bound for S. Therefore S has a supremum which satisfies the inequality $\sup S \leq t$ for all $t$ in $T$. Hence $\sup S$ is a lower bound for $T$, so $T$ has an infimum which cannot be less than sup S. In other words, we have $\sup S \leq \inf T$, as asserted.

## *I 3.12 Exercises

1. If $x$ and $y$ are arbitrary real numbers with $x < y$, prove that there is at least one real $z$ satisfying $x < z < y$.
2. If $x$ is an arbitrary real number, prove that there are integers $m$ and $n$ such that $m < x < n$.
3. If $x > 0$, prove that there is a positive integer $n$ such that $1/n < x$.
4. If $x$ is an arbitrary real number, prove that there is exactly one integer $n$ which satisfies the inequalities $n \leq x < n + 1$. This $n$ is called the greatest integer in $x$ and is denoted by $[x]$. For example, $[5] = 5$, $[\frac{5}{2}] = 2$, $[-\frac{8}{3}] = -3$.
5. If $x$ is an arbitrary real number, prove that there is exactly one integer $n$ which satisfies $x \leq n < x + 1$.
6. If $x$ and $y$ are arbitrary real numbers, $x < y$, prove that there exists at least one rational number $r$ satisfying $x < r < y$, and hence infinitely many. This property is often described by saying that the rational numbers are *dense* in the real-number system.
7. If $x$ is rational, $x \neq 0$, and $y$ irrational, prove that $x + y$, $x - y$, $xy$, $x/y$, and $y/x$ are all irrational.
8. Is the sum or product of two irrational numbers always irrational?
9. If $x$ and $y$ are arbitrary real numbers, $x < y$, prove that there exists at least one irrational number $z$ satisfying $x < z < y$, and hence infinitely many.
10. An integer $n$ is called *even* if $n = 2m$ for some integer $m$, and *odd* if $n + 1$ is even. Prove the following statements :
    (a) An integer cannot be both even and odd.
    (b) Every integer is either even or odd.
    (c) The sum or product of two even integers is even. What can you say about the sum or product of two odd integers?
    (d) If $n^2$ is even, so is $n$. If $a^2 = 2b^2$, where $a$ and $b$ are integers, then both $a$ and $b$ are even.
    (e) Every rational number can be expressed in the form $a/b$, where $a$ and $b$ are integers, at least one of which is odd.
11. Prove that there is no rational number whose square is 2.

[*Hint:* Argue by contradiction. Assume $(a/b)^2 = 2$, where $a$ and $b$ are integers, at least one of which is odd. Use parts of Exercise 10 to deduce a contradiction.]

12. The Archimedean property of the real-number system was deduced as a consequence of the least-Upper-bound axiom. Prove that the set of rational numbers satisfies the Archimedean property but not the least-Upper-bound property. This shows that the Archimedean property does not imply the least-Upper-bound axiom.

## *I 3.13 Existence of square roots of nonnegative real numbers

It was pointed out earlier that the equation $x^2 = 2$ has no solutions among the rational numbers. With the help of Axiom 10, we can prove that the equation $x^2 = a$ has a solution among the *real* numbers if $a \geq 0$. Each such $x$ is called a ***square root*** of $a$.

First, let us see what we can say about square roots without using Axiom 10. Negative numbers cannot have square roots because if $x^2 = a$, then $a$, being a square, must be nonnegative (by Theorem 1.20). Moreover, if $a = 0$, then $x = 0$ is the only square root (by Theorem 1.11). Suppose, then, that $a > 0$. If $x^2 = a$, then $x \neq 0$ and $(-x)^2 = a$, so both $x$ and its negative are square roots. In other words, if $a$ has a square root, then it has two square roots, one positive and one negative. Also, it has *at most two* because if $x^2 = a$ and $y^2 = a$, then $x^2 = y^2$ and $(x - y)(x + y) = 0$, and so, by Theorem 1.11, either $x = y$ or $x = -y$. Thus, if $a$ has a square root, it has *exactly* two.

The existence of at least one square root can be deduced from an important theorem in calculus known as the intermediate-value theorem for continuous functions, but it may be instructive to see how the existence of a square root can be proved directly from Axiom 10.

THEOREM 1.35. *Every nonnegatioe real number $a$ has a unique nonnegative square root.*

*Note:* If $a \geq 0$, we denote its nonnegative square root by $a^{1/2}$ or by $\sqrt{a}$. If $a > 0$, the negative square root is $-a^{1/2}$ or $-\sqrt{a}$.

*Proof.* If $a = 0$, then 0 is the only square root. Assume, then, that $a > 0$. Let S be the set of all positive $x$ such that $x^2 \leq a$. Since $(1 + a)^2 > a$, the number $1 + a$ is an upper bound for S. Also, S is nonempty because the number $a/(1 + a)$ is in $S$; in fact, $a^2 \leq a(1 + a)^2$ and hence $a^2/(1 + a)^2 \leq a$. By Axiom 10, $S$ has a least upper bound which we shall call $b$. Note that $b \geq a/(1 + a)$ so $b > 0$. There are only three possibilities: $b^2 > a$, $b^2 < a$, or $b^2 = a$.

Suppose $b^2 > a$ and let $c = b - (b^2 - a)/(2b) = \frac{1}{2}(b + a/b)$. Then $0 < c < b$ and $c^2 = b^2 - (b^2 - a) + (b^2 - a)^2/(4b^2) = a + (b^2 - a)^2/(4b^2) > a$. Therefore $c^2 > x^2$ for each $x$ in S, and hence $c > x$ for each $x$ in S. This means that c is an upper bound for S. Since $c < b$, we have a contradiction because $b$ was the *least* upper bound for S. Therefore the inequality $b^2 > a$ is impossible.

Suppose $b^2 < a$. Since $b > 0$, we may choose a positive number c such that $c < b$ and such that $c < (a - b^2)/(3b)$. Then we have

$$(b + c)^2 = b^2 + c(2b + c) < b^2 + 3bc < b^2 + (a - b^2) = a$$

Therefore $b + c$ is in S. Since $b + c > b$, this contradicts the fact that $b$ is an upper bound for S. Therefore the inequality $b^2 < a$ is impossible, and the only remaining alternative is $b^2 = a$.

## *I 3.14 Roots of higher order. Rational powers

The least-Upper-bound axiom can also be used to show the existence of roots of higher order. For example, if $n$ is a positive *odd* integer, then for each real $x$ there is exactly one real $y$ such that $y^n = x$. This $y$ is called the nth *root* of $x$ and is denoted by

$$y = x^{1/n} \qquad \text{or} \qquad y = \sqrt[n]{x}\,. \tag{1.15}$$

When $n$ is *even,* the situation is slightly different. In this case, if $x$ is negative, there is no real $y$ such that $y^n = x$ because $y^n \geq 0$ for all real $y$. However, if $x$ is positive, it can be shown that there is one and only one positive $y$ such that $y^n = x$. This $y$ is called *the positive nth root* of $x$ and is denoted by the symbols in (1.15). Since $n$ is even, $(-y)^n = y^n$ and hence each $x > 0$ has two real nth roots, $y$ and $-y$. However, the symbols $x^{1/n}$ and $\sqrt[n]{x}$ are reserved for the *positive nth* root. We do not discuss the proofs of these statements here because they will be deduced later as consequences of the intermediate-value theorem for continuous functions (see Section 3.10).

If $r$ is a positive rational number, say $r = m/n$, where $m$ and $n$ are positive integers, we define $x^r$ to be $(x^m)^{1/n}$, the nth root of $x^m$, whenever this exists. If $x \neq 0$, we define $x^{-r} = 1/x^r$ whenever $x^r$ is defined. From these definitions, it is easy to verify that the usual laws of exponents are valid for rational exponents : $x^r \cdot x^s = x^{r+s}$, $(x^r)^s = x^{rs}$, and $(xy)^r = x^r y^r$.

## *I 3.15 Representation of real numbers by decimals

A real number of the form

$$r = a_0 + \frac{a_1}{10} + \frac{a_2}{10^2} + \cdots + \frac{a_n}{10^n}, \tag{1.16}$$

where $a_0$ is a nonnegative integer and $a_1, a_2, \ldots, a_n$ are integers satisfying $0 \leq a_k \leq 9$, is usually written more briefly as follows:

$$r = a_0.a_1a_2 \cdots a_n\,.$$

This is said to be a *finite decimal representation* of $r$. For example,

$$\frac{1}{2} = \frac{5}{10} = 0.5\,, \qquad \frac{1}{50} = \frac{2}{10^2} = 0.02\,, \qquad \frac{29}{4} = 7 + \frac{2}{10} + \frac{5}{10^2} = 7.25\,.$$

Real numbers like these are necessarily rational and, in fact, they all have the form $r = a/10^n$, where $a$ is an integer. However, not all rational numbers can be expressed with finite decimal representations. For example, if $\frac{1}{3}$ could be so expressed, then we would have $\frac{1}{3} = a/10^n$ or $3a = 10^n$ for some integer $a$. But this is impossible since 3 is not a factor of any power of 10.

Nevertheless, we can approximate an arbitrary real number $x > 0$ to any desired degree of accuracy by a sum of the form (1.16) if we take $n$ large enough. The reason for this may be seen by the following geometric argument: If $x$ is not an integer, then $x$ lies between two consecutive integers, say $a_0 < x < a_0 + 1$. The segment joining $a_0$ and $a_0 + 1$ may be subdivided into ten equal parts. If x is not one of the subdivision points, then x must lie between two consecutive subdivision points. This gives us a pair of inequalities of the form

$$a_0 + \frac{a_1}{10} < x < a_0 + \frac{a_1 + 1}{10},$$

where *a,* is an integer (0 $\leq$ *a,* $\leq$ 9). Next we divide the segment joining *a,* + $a_1/10$ and *a,,* + (*a,* + 1)/10 into ten equal parts (each of length $10^{-2}$) and continue the process. If after a finite number of steps a subdivision point coincides with x, then x is a number of the form (1.16). Otherwise the process continues indefinitely, and it generates an infinite set of integers *a,* , $a_2$ , $a_3$ , . . . . In this case, we say that $x$ has the infinite decimal representation

$$x = a_0.a_1a_2a_3 \cdots.$$

At the nth stage, x satisfies the inequalities

$$a_0 + \frac{a_1}{10} + \cdots + \frac{a_n}{10^n} < x < a_0 + \frac{a_1}{10} + \cdots + \frac{a_n + 1}{10”}.$$

This gives us two approximations to x, one from above and one from below, by finite decimals that differ by $10^{-n}$. Therefore we can achieve any desired degree of accuracy in our approximations by taking $n$ large enough.

When x $= \frac{1}{3}$, it is easy to verify that a, = 0 and a, = 3 for all $n \geq 1$, and hence the corresponding infinite decimal expansion is

$$\tfrac{1}{3} = 0.333 \cdots.$$

Every irrational number has an infinite decimal representation. For example, when $x = \sqrt{2}$ we may calculate by trial and error as many digits in the expansion as we wish. Thus, $\sqrt{2}$ lies between 1.4 and 1.5, because (1 .4)$^2$ < 2 < (1.5)$^2$. Similarly, by squaring and comparing with 2, we find the following further approximations:

$$1.41 < \sqrt{2} < 1.42, \qquad 1.414 < \sqrt{2} < 1.415) \qquad 1.4142 < \sqrt{2} < 1.4143.$$

Note that the foregoing process generates a succession of intervals of lengths $10^{-1}, 10^{-2}, 10^{-3}, \ldots$, each contained in the preceding and each containing the point x. This is an example of what is known as a sequence of *nested intervals,* a concept that is sometimes used as a basis for constructing the irrational numbers from the rational numbers.

Since we shall do very little with decimals in this book, we shall not develop their properties in any further detail except to mention how decimal expansions may be defined analytically with the help of the least-Upper-bound axiom.

If x is a given positive real number, let *a,* denote the largest integer $\leq$ x. Having chosen *a,* , we let *a,* denote the largest integer such that

$$a_0 + \frac{a_1}{10} \leq x.$$

More generally, having chosen $a_1, a_2, \ldots, a_{n-1}$, we let $a_n$ denote the largest integer such that

(1.17) $$a_0 + \frac{a_1}{10} + \frac{a_2}{10^2} + \cdots + \frac{a_n}{10^n} \le x\,.$$

Let $S$ denote the set of all numbers

(1.18) $$a_0 + \frac{a_1}{10} + \frac{a_2}{10^2} + \cdots + \frac{a_n}{10^n}$$

obtained in this way for $n = 0, 1, 2, \ldots$. Then $S$ is nonempty and bounded above, and it is easy to verify that $x$ is actually the least upper bound of $S$. The integers $a_0, a_1, a_2, \ldots$ so obtained may be used to define a decimal expansion of $x$ if we write

$$x = a_0.a_1a_2a_3 \cdots$$

to mean that the $n$th digit $a_n$ is the largest integer satisfying (1.17). For example, if $x = \frac{1}{8}$, we find $a_0 = 0$, $a_1 = 1$, $a_2 = 2$, $a_3 = 5$, and $a_n = 0$ for all $n \ge 4$. Therefore we may write

$$\tfrac{1}{8} = 0.125000 \cdots.$$

If in (1.17) we replace the inequality sign $\le$ by $<$, we obtain a slightly different definition of decimal expansions. The least upper bound of all numbers of the form (1.18) is again $x$, although the integers $a_0, a_1, a_2, \ldots$ need not be the same as those which satisfy (1.17). For example, if this second definition is applied to $x = \frac{1}{8}$, we find $a_0 = 0$, $a_1 = 1$, $a_2 = 2$, $a_3 = 4$, and $a_n = 9$ for all $n \ge 4$. This leads to the infinite decimal representation

$$\tfrac{1}{8} = 0.124999 \cdots.$$

The fact that a real number might have two different decimal representations is merely a reflection of the fact that two different sets of real numbers can have the same supremum.

## *Part 4. Mathematical Induction, Summation Notation, and Related Topics*

### 14.1 An example of a proof by mathematical induction

There is no *largest* integer because when we add 1 to an integer $k$, we obtain $k + 1$, which is larger than $k$. Nevertheless, starting with the number 1, we can reach any positive integer whatever in a finite number of steps, passing successively from $k$ to $k + 1$ at each step. This is the basis for a type of reasoning that mathematicians call ***proof by induction.*** We shall illustrate the use of this method by proving the pair of inequalities used in Section

I1.3 in the computation of the area of a parabolic segment, namely

$$1^2 + 2^2 + \cdots + (n-1)^2 < \frac{n^3}{3} < 1^2 + 2^2 + \cdots + n^2. \tag{1.19}$$

Consider the leftmost inequality first, and let us refer to this formula as $A(n)$ (an assertion involving $n$). It is easy to verify this assertion directly for the first few values of $n$. Thus, for example, when $n$ takes the values 1, 2, and 3, the assertion becomes

$$A(1):\ 0 < \frac{1^3}{3}, \qquad A(2):\ 1^2 < \frac{2^3}{3}, \qquad A(3):\ 1^2 + 2^2 < \frac{3^3}{3},$$

provided we agree to interpret the sum on the left as 0 when $n = 1$.

Our object is to prove that $A(n)$ is true for every positive integer $n$. The procedure is as follows: Assume the assertion has been proved for a particular value of $n$, say for $n = k$. That is, assume we have proved

$$A(k):\ 1^2 + 2^2 + \ldots + (k-1)^2 < \frac{k^3}{3}$$

for a fixed $k \geq 1$. *Now using this,* we shall deduce the corresponding result for $k + 1$:

$$A(k+1):\ 1^2 + 2^2 + \ldots + k^2 < \frac{(k+1)^3}{3}.$$

Start with $A(k)$ and add $k^2$ to both sides. This gives the inequality

$$1^2 + 2^2 + \ldots + k^2 < \frac{k^3}{3} + k^2.$$

To obtain $A(k + 1)$ as a consequence of this, it suffices to show that

$$\frac{k^3}{3} + k^2 < \frac{(k+1)^3}{3}.$$

But this follows at once from the equation

$$\frac{(k+1)^3}{3} = \frac{k^3 + 3k^2 + 3k + 1}{3} = \frac{k^3}{3} + k^2 + k + \frac{1}{3}.$$

Therefore we have shown that $A(k + 1)$ follows from $A(k)$. *Now,* since $A(1)$ has been verified directly, we conclude that $A(2)$ is also true. Knowing that $A(2)$ is true, we conclude that $A(3)$ is true, and so on. Since every integer can be reached in this way, $A(n)$ is true for all positive integersn. This proves the leftmost inequality in (1.19). The rightmost inequality can be proved in the same way.

### 14.2 The principle of mathematical induction

The reader should make certain that he understands the *pattern* of the foregoing proof. First we proved the assertion $A(n)$ for $n = 1$. Next we showed that *if* the assertion is true for a particular integer, *then* it is also true for the next integer. From this, we concluded that the assertion is true for all positive integers.

The idea of induction may be illustrated in many nonmathematical ways. For example, imagine a row of toy soldiers, numbered consecutively, and suppose they are so arranged that if any one of them falls, say the one labeled $k$, it will knock over the next one, labeled $k + 1$. Then anyone can visualize what would happen if soldier number 1 were toppled backward. It is also clear that if a later soldier were knocked over first, say the one labeled $n_1$, then all soldiers behind *him* would fall. This illustrates a slight generalization of the method of induction which can be described in the following way.

*Method of proof by induction.* Let $A(n)$ be an assertion involving an integer $n$. We conclude that $A(n)$ is true for every $n \geq n_1$ if we can perform the following two steps:

(a) Prove that $A(n_1)$ is true.

(b) Let $k$ be an arbitrary but fixed integer $\geq n_1$. Assume that $A(k)$ is true and prove that $A(k + 1)$ is also true.

In actual practice $n_1$ is usually 1. The logical justification for this method of proof is the following theorem about real numbers.

THEOREM 1.36. PRINCIPLE OF MATHEMATICAL INDUCTION. *Let $S$ be a set of positive integers which has the following two properties:*

(a) *The number 1 is in the set $S$.*

(b) *If an integer $k$ is in $S$, then so is $k + 1$.*

*Then every positive integer is in the set $S$.*

*Proof.* Properties (a) and (b) tell us that $S$ is an inductive set. But the positive integers were defined to be exactly those real numbers which belong to every inductive set. (See Section 1 3.6.) Therefore $S$ contains every positive integer.

Whenever we carry out a proof of an assertion A(n) for all $n \geq 1$ by mathematical induction, we are applying Theorem 1.36 to the set $S$ of all the integers for which the assertion is true. If we want to prove that $A(n)$ is true only for $n \geq n_1$, we apply Theorem 1.36 to the set of $n$ for which $A(n + n_1 - 1)$ is true.

### *I 4.3 The well-ordering principle

There is another important property of the positive integers, called the well-ordering principle, that is also used as a basis for proofs by induction. It can be stated as follows.

THEOREM 1.37. WELL-ORDERING PRINCIPLE. *Every nonempty set of positive integers contains a smallest member.*

Note that the well-ordering principle refers to sets of *positive* integers. The theorem is not true for arbitrary sets of integers. For example, the set of all integers has no smallest member .

The well-ordering principle can be deduced from the principle of induction. This is demonstrated in Section 14.5. We conclude this section with an example showing how the well-ordering principle can be used to prove theorems about positive integers.

Let A(n) denote the following assertion:

$$\mathbf{A(n)}\colon\ 1^2 + 2^2 + \cdots + n^2 = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}.$$

Again, we note that A(1) is true, since

$$1^2 = \tfrac{1}{3} + \tfrac{1}{2} + \tfrac{1}{6}.$$

Now there are only two possibilities. We have either

(i) A(n) is true for every positive integer $n$, or

(ii) there is at least one positive integer $n$ for which A(n) is false.

We shall prove that alternative (ii) leads to a contradiction. Assume (ii) holds. Then by the well-ordering principle, there must be a *smallest* positive integer, say $k$, for which $A(k)$ is false. (We apply the well-ordering principle to the set of all positive integers $n$ for which A(n) is false. Statement (ii) says that this set is nonempty.) This $k$ must be greater than 1, because we have verified that A(1) is true. Also, the assertion must be true for $k - 1$, since $k$ was the smallest integer for which $A(k)$ is false; therefore we may write

$$A(k - 1)\colon\ 1^2 + 2^2 + \cdots + (k - 1)^2 = \frac{(k-1)^3}{3} + \frac{(k-1)^2}{2} + \frac{k-1}{6}.$$

Adding $k^2$ to both sides and simplifying the right-hand side, we find

$$1^2 + 2^2 + \cdots + k^2 = \frac{k^3}{3} + \frac{k^2}{2} + \frac{k}{6}.$$

But this equation states that $A(k)$ is true; therefore we have a contradiction, because $k$ is an integer for which $A(k)$ is false. In other words, statement (ii) leads to a contradiction. Therefore (i) holds, and this proves that the identity in question is valid for all values of $n \geq 1$. An immediate consequence of this identity is the rightmost inequality in (1.19).

A proof like this which makes use of the well-ordering principle is also referred to as a proof by induction. Of course, the proof could also be put in the more usual form in which we verify $A(1)$ and then pass from $A(k)$ to $A(k + 1)$.

### 14.4 Exercises

1. Prove the following formulas by induction :

   (a) $1 + 2 + 3 + \cdots + n = n(n + 1)/2$.

   (b) $1 + 3 + 5 + \cdots + (2n - 1) = n^2$.

   (c) $1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2$.

   (d) $1^3 + 2^3 + \cdots + (n - 1)^3 < n^4/4 < 1^3 + 2^3 + \cdots + n^3$.

2. Note that

$$1 = 1,$$
$$1 - 4 = -(1 + 2),$$
$$1 - 4 + 9 = 1 + 2 + 3,$$
$$1 - 4 + 9 - 16 = -(1 + 2 + 3 + 4).$$

Guess the general law suggested and prove it by induction.

3. Note that

$$1 + \tfrac{1}{2} = 2 - \tfrac{1}{2},$$
$$1 + \tfrac{1}{2} + \tfrac{1}{4} = 2 - \tfrac{1}{4},$$
$$1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} = 2 - \tfrac{1}{8}.$$

Guess the general law suggested and prove it by induction.

4. Note that

$$1 - \tfrac{1}{2} = \tfrac{1}{2},$$
$$(1 - \tfrac{1}{2})(1 - \tfrac{1}{3}) = \tfrac{1}{3},$$
$$(1 - \tfrac{1}{2})(1 - \tfrac{1}{3})(1 - \tfrac{1}{4}) = \tfrac{1}{4}.$$

Guess the general law suggested and prove it by induction.

5. Guess a general law which simplifies the product

$$\left(1 - \frac{1}{4}\right)\left(1 - \frac{1}{9}\right)\left(1 - \frac{1}{16}\right) \cdots \left(1 - \frac{1}{n^2}\right)$$

and prove it by induction.

6. Let $A(n)$ denote the statement: $1 + 2 + \cdots + n = \frac{1}{8}(2n + 1)^2$.
   (a) Prove that if $A(k)$ is true for an integer $k$, then $A(k + 1)$ is also true.
   (b) Criticize the statement: "By induction it follows that $A(n)$ is true for all $n$."
   (c) Amend $A(n)$ by changing the equality to an inequality that is true for all positive integers $n$.
7. Let $n_1$ be the smallest positive integer $n$ for which the inequality $(1 + x)^n > 1 + nx + nx^2$ is true for all $x > 0$. Compute $n_1$, and prove that the inequality is true for all integers $n \geq n_1$.
8. Given positive real numbers $a_1, a_2, a_3, \ldots$, such that $a_n \leq ca_{n-1}$ for all $n \geq 2$, where $c$ is a fixed positive number, use induction to prove that $a_n \leq a_1c^{n-1}$ for all $n \geq 1$.
9. Prove the following statement by induction: If a line of unit length is given, then a line of length $\sqrt{n}$ can be constructed with straightedge and compass for each positive integer $n$.
10. Let $b$ denote a fixed positive integer. Prove the following statement by induction: For every integer $n \geq 0$, there exist nonnegative integers $q$ and $r$ such that

$$n = qb + r, \qquad 0 \leq r < b.$$

11. Let $n$ and $d$ denote integers. We say that $d$ is a *divisor* of $n$ if $n = cd$ for some integer $c$. An integer $n$ is called a ***prime*** if $n > 1$ and if the only positive divisors of $n$ are 1 and $n$. Prove, by induction, that every integer $n > 1$ is either a prime or a product of primes.
12. Describe the fallacy in the following "proof" by induction:

*Statement.* Given any collection of $n$ blonde girls. If at least one of the girls has blue eyes, then all $n$ of them have blue eyes.

"*Proof.*" The statement is obviously true when $n = 1$. The step from $k$ to $k + 1$ can be illustrated by going from $n = 3$ to $n = 4$. Assume, therefore, that the statement is true when $n = 3$ and let $G_1$, G,, G,, $G_4$ be four blonde girls, at least one of which, say $G_1$, has blue eyes. Taking $G_1, G_2$, and $G_3$ together and using the fact that the statement is true when $n = 3$, we find that $G_2$ and $G_3$ also have blue eyes. Repeating the process with G,, G,, and $G_4$, we find that $G_4$ has blue eyes. Thus all four have blue eyes. A similar argument allows us to make the step from $k$ to $k + 1$ in general.

*Corollary.* All blonde girls have blue eyes.

***Proof.*** Since there exists at least one blonde girl with blue eyes, we can apply the foregoing result to the collection consisting of all blonde girls.

*Note:* This example is from G. Pólya, who suggests that the reader may want to test the validity of the statement by experiment.

### *I 4.5 Proof of the well-ordering principle

In this section we deduce the well-ordering principle from the principle of induction.

Let $T$ be a nonempty collection of positive integers. We want to prove that $T$ has a smallest member, that is, that there is a positive integer $t_0$ in $T$ such that $t_0 \leq t$ for all $t$ in $T$.

Suppose $T$ has no smallest member. We shall show that this leads to a contradiction. The integer 1 cannot be in $T$ (otherwise it would be the smallest member of T). Let S denote the collection of all positive integers $n$ such that $n < t$ for all $t$ in $T$. *Now* 1 is in $S$ because $1 < t$ for all $t$ in $T$. Next, let $k$ be a positive integer in S. Then $k < t$ for all $t$ in $T$. We shall prove that $k + 1$ is also in $S$. If this were not so, then for some $t_1$ in $T$ we would have $t_1 \leq k + 1$. Since $T$ has no smallest member, there is an integer $t_2$ in $T$ such that $t_2 < t_1$, and hence $t_2 < k + 1$. But this means that $t_2 \leq k$, contradicting the fact that $k < t$ for all $t$ in $T$. Therefore $k + 1$ is in S. By the induction principle, S contains all positive integers. Since Tisnonempty, there is a positive integer $t$ in $T$. But this $t$ must also be in S (since S contains all positive integers). It follows from the definition of S that $t < t$, which is a contradiction. Therefore, the assumption that $T$ has no smallest member leads to a contradiction. It follows that $T$ must have a smallest member, and in turn this proves that the well-ordering principle is a consequence of the principle of induction.

### 14.6 The summation notation

In the calculations for the area of the parabolic segment, we encountered the sum

$$1^2 + 2^2 + 3^2 + \cdots + n^2 . \tag{1.20}$$

Note that a typical term in this sum is of the form $k^2$, and we get all the terms by letting $k$ run through the values $1, 2, 3, \ldots, n$. There is a very useful and convenient notation which enables us to write sums like this in a more compact form. This is called the *summation notation* and it makes use of the Greek letter sigma, $\sum$. Using summation notation, we can write the sum in (1.20) as follows:

$$\sum_{k=1}^{n} k^2.$$

This symbol is read: "The sum of $k^2$ for $k$ running from 1 to $n$." The numbers appearing under and above the sigma tell us the range of values taken by $k$. The letter $k$ itself is referred to as the *index of summation*. Of course, it is not important that we use the letter $k$; any other convenient letter may take its place. For example, instead of $\sum_{k=1}^{n} k^2$ *we* could write $\sum_{i=1}^{n} i^2$, $\sum_{j=1}^{n} j^2$, $\sum_{m=1}^{n} m^2$, etc., all of which are considered as alternative notations for the same thing. The letters $i$, j, $k$, $m$, etc. that are used in this way are called *dummy indices*. It would not be a good idea to use the letter $n$ for the dummy index in this particular example because $n$ is already being used for the number of terms.

More generally, when we want to form the sum of several real numbers, say a, , a,, . . . , $a_n$, we denote such a sum by the symbol

(1.21) $$a, + a_2 + \ldots + a,$$

which, using summation notation, can be written as follows:

(1.22) $$\sum_{k=1}^{n} a_k .$$

For example, we have

$$\sum_{k=1}^{4} a_k = a_1 + a_2 + a_3 + a_4 ,$$

$$\sum_{i=1}^{5} x_i = x_1 + x_2 + x_3 + x_4 + x_5$$

Sometimes it is convenient to begin summations from 0 or from some value of the index beyond 1. For example, we have

$$\sum_{i=0}^{4} x_i = x_0 + x_1 + x_2 + x_3 + x_4 ,$$

$$\sum_{n=2}^{5} n^3 = 2^3 + 3^3 + 4^3 + 5^3.$$

Other uses of the summation notation are illustrated below:

$$\sum_{m=0}^{4} x^{m+1} = x + x^2 + x^3 + x^4 + x^5,$$

$$\sum_{j=1}^{6} 2^{j-1} = 1 + 2 + 2^2 + 2^3 + 2^4 + 2^5.$$

To emphasize once more that the choice of dummy index is unimportant, we note that the last sum may also be written in each of the following forms:

$$\sum_{q=1}^{6} 2^{q-1} = \sum_{r=0}^{5} 2^r = \sum_{n=0}^{5} 2^{5-n} = \sum_{k=1}^{6} 2^{6-k}.$$

*Note:* From a strictly logical standpoint, the symbols in (1.21) and (1.22) do not appear among the primitive symbols for the real-number system. In a more careful treatment, we could define these new symbols in terms of the primitive undefined symbols of our system.

This may be done by a process known as *definition by induction* which, like proof by induction, consists of two parts:

(a) We define

$$\sum_{k=1}^{1} a_k = a_1 .$$

(b) Assuming that we have defined $\sum_{k=1}^{n} a_k$ for a fixed $n \geq 1$, we further define

$$\sum_{k=1}^{n+1} a_k = \left(\sum_{k=1}^{n} a_k\right) + a_{n+1} .$$

To illustrate, we may take $n = 1$ in (b) and use (a) to obtain

$$\sum_{k=1}^{2} a_k = \sum_{k=1}^{1} a_k + a_2 = a_1 + a_2 .$$

Now, having defined $\sum_{k=1}^{2} a_k$, we can use (b) again with $n = 2$ to obtain

$$\sum_{k=1}^{3} a_k = \sum_{k=1}^{2} a_k + a_3 = (a_1 + a_2) + a_3 .$$

By the associative law for addition (Axiom 2), the sum $(a_1 + a_2) + a_3$ is the same as *a,* + $(a_2 +$ *a,)*, and therefore there is no danger of confusion if we drop the parentheses and simply write *a,* + $a_2 + a_3$ for $\sum_{k=1}^{3} a_k$ . Similarly, we have

$$\sum_{k=1}^{4} a_k = \sum_{k=1}^{3} a_k + a_4 = (a_1 + a_2 + a_3) + a_4 .$$

In this case we can *proue* that the sum $(a_1 + a_2 + a_3) + a_4$ is the same as $(a_1 + a_2) + (a_3 + a_4)$ or a, + $(a_2 + a_3 +$ *a,)*, and therefore the parentheses can be dropped again without danger of ambiguity, and we agree to write

$$\sum_{k=1}^{4} a_k = a, + a_2 + a_3 + a_4 .$$

Continuing in this way, we find that (a) and (b) together give us a complete definition of the symbol in (1.22). The notation in (1.21) is considered to be merely an alternative way of writing (1.22). It is justified by a general associative law for addition which we shall not attempt to state or to prove here.

The reader should notice that *definition by induction* and *proof by induction* involve the same underlying idea. A definition by induction is also called a *recursiue definition.*

## 14.7 Exercises

1. Find the numerical values of the following sums :

(a) $\sum_{k=1}^{4} k$, (c) $\sum_{r=0}^{3} 2^{2r+1}$, (e) $\sum_{i=0}^{5} (2i + 1)$,

(b) $\sum_{n=2}^{5} 2^{n-2}$, (d) $\sum_{n=1}^{4} n^n$, (f) $\sum_{k=1}^{5} \frac{1}{k(k+1)}$.

2. Establish the following properties of the summation notation:

$$\text{(a)} \sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k \qquad \text{(additive property).}$$

$$\text{(b)} \sum_{k=1}^{n} (ca_k) = c \sum_{k=1}^{n} a_k \qquad \text{(homogeneous property).}$$

$$\text{(c)} \sum_{k=1}^{n} (a_k - a_{k-1}) = a_n - a_0 \qquad \text{(telescoping property).}$$

Use the properties in Exercise 2 whenever possible to derive the formulas in Exercises 3 through 8.

3. $\sum_{k=1}^{n} 1 = n$. (This means $\sum_{k=1}^{n} a_k$, where each $a_k = 1$.)

4. $\sum_{k=1}^{n} (2k - 1) = n^2$. [*Hint:* $2k - 1 = k^2 - (k - 1)^2$.]

5. $\sum_{k=1}^{n} k = \frac{n^2}{2} + \frac{n}{2}$. [*Hint:* Use Exercises 3 and 4.]

6. $\sum_{k=1}^{n} k^2 = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}$. [*Hint:* $k^3 - (k - 1)^3 = 3k^2 - 3k + 1$.]

7. $\sum_{k=1}^{n} k^3 = \frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4}$.

8. (a) $\sum_{k=0}^{n} x^k = \dfrac{1 - x^{n+1}}{1 - x}$ if $x \neq 1$. *Note:* $x^0$ is defined to be 1.

[*Hint:* Apply Exercise 2 to $(1 - x) \sum_{k=0}^{n} x^k$.]

(b) What is the sum equal to when $x = 1$?

9. Prove, by induction, that the sum $\sum_{k=1}^{2n} (-1)^k(2k + 1)$ is proportional to $n$, and find the constant of proportionality.

10. (a) Give a reasonable definition of the symbol $\sum_{k=m}^{m+n} a_k$.
(b) Prove, by induction, that for $n \geq 1$ we have

$$\sum_{k=n+1}^{2n} \frac{1}{k} = \sum_{m=1}^{2n} \frac{(-1)^{m+1}}{m}.$$

11. Determine whether each of the following statements is true or false. In each case give a reason for your decision.

(a) $\sum_{n=0}^{100} n^4 = \sum_{n=1}^{100} n^4$.

(b) $\sum_{j=0}^{100} 2 = 200$.

(c) $\sum_{k=0}^{100} (2 + k) = 2 + \sum_{k=0}^{100} k$.

(d) $\sum_{i=1}^{100} (i + 1)^2 = \sum_{i=0}^{99} i^2$.

(e) $\sum_{k=1}^{100} k^3 = \left(\sum_{k=1}^{100} k\right) \cdot \left(\sum_{k=1}^{100} k^2\right)$.

(f) $\sum_{k=0}^{100} k^3 = \left(\sum_{k=0}^{100} k\right)^3$.

12. Guess and prove a general rule which simplifies the sum

$$\sum_{k=1}^{n} \frac{1}{k(k+1)}.$$

13. Prove that $2(\sqrt{n+1} - \sqrt{n}) < \frac{1}{\sqrt{n}} < 2(\sqrt{n} - \sqrt{n-1})$ if $n \geq 1$. Then use this to prove that

$$2\sqrt{m} - 2 < \sum_{n=1}^{m} \frac{1}{\sqrt{n}} < 2\sqrt{m} - 1$$

if $m \geq 2$. In particular, when $m = 10^6$, the sum lies between 1998 and 1999.

### 14.8 Absolute values and the triangle inequality

Calculations with inequalities arise quite frequently in calculus. They are of particular importance in dealing with the notion of *absolute value.* If $x$ is a real number, the absolute value of $x$ is a nonnegative real number denoted by $|x|$ and defined as follows:

$$|x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x \leq 0. \end{cases}$$

Note that $-|x| \leq x \leq |x|$. When real numbers are represented geometrically on a real axis, the number $|x|$ is called the *distance* of $x$ from 0. If $a > 0$ and if a point $x$ lies between $-a$ and $a$, then $|x|$ is nearer to 0 than $a$ is. The analytic statement of this fact is given by the following theorem.

THEOREM 1.38. *If $a \geq 0$, then $|x| \leq a$ if and only if $-a \leq x \leq a$.*

*Proof.* There are two statements to prove: first, that the inequality $|x| \leq a$ implies the two inequalities $-a \leq x \leq a$ and, conversely, that $-a \leq x \leq a$ implies $|x| \leq a$.

Suppose $|x| \leq a$. Then we also have $-a \leq -|x|$. But either $x = |x|$ or $x = -|x|$ and hence $-a \leq -|x| \leq x \leq |x| \leq a$. This proves the first statement.

To prove the converse, assume $-a \leq x \leq a$. Then if $x \geq 0$, we have $|x| = x \leq a$, whereas if $x \leq 0$, we have $|x| = -x \leq a$. In either case we have $|x| \leq a$, and this completes the proof.

Figure 1.9 illustrates the geometrical significance of this theorem.

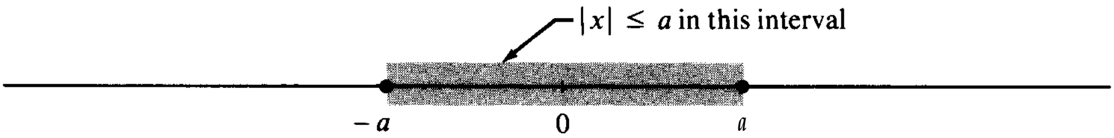



FIGURE 1.9 Geometrical significance of Theorem 1.38.

As a consequence of Theorem 1.38, it is easy to derive an important inequality which states that the absolute value of a sum of two real numbers cannot exceed the sum of their absolute values.

THEOREM 1.39. *For arbitrary real numbers $x$ and $y$, we have*

$$|x + y| \le |x| + |y| .$$

*Note:* This property is called the *triangle inequality,* because when it is generalized to vectors it states that the length of any side of a triangle is less than or equal to the sum of the lengths of the other two sides.

*Proof.* Adding the inequalities $-|x| \le x \le |x|$ and $-|y| \le y \le |y|$, we obtain

$$-(|x| + |y|) \le x + y \le |x| + |y| ,$$

and hence, by Theorem 1.38, we conclude that $|x + y| \le |x| + |y|$.

If we take $x = a - c$ and $y = c - b$, then $x + y = a - b$ and the triangle inequality becomes

$$|a - b| \le |a - c| + |b - c| .$$

This form of the triangle inequality is often used in practice.

Using mathematical induction, we may extend the triangle inequality as follows:

THEOREM 1.40. *For arbitrary real numbers $a_1, a_2, \ldots, a_n$ we have*

$$\left|\sum_{k=1}^{n} a_k\right| \le \sum_{k=1}^{n} |a_k| .$$

*Proof.* When $n = 1$ the inequality is trivial, and when $n = 2$ it is the triangle inequality. Assume, then, that it is true for $n$ real numbers. Then for $n + 1$ real numbers $a_1, a_2, \ldots, a_{n+1}$, we have

$$\left|\sum_{k=1}^{n+1} a_k\right| = \left|\sum_{k=1}^{n} a_k + a_{n+1}\right| \le \left|\sum_{k=1}^{n} a_k\right| + |a_{n+1}| \le \sum_{k=1}^{n} |a_k| + |a_{n+1}| = \sum_{k=1}^{n+1} |a_k| .$$

Hence the theorem is true for $n + 1$ numbers if it is true for $n$. By induction, it is true for every positive integer $n$.

The next theorem describes an important inequality that we shall use later in connection with our study of vector algebra.

THEOREM 1.41. THE CAUCHY-SCHWARZ INEQUALITY. *If $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ are arbitrary real numbers, we have*

$$\left(\sum_{k=1}^{n} a_k b_k\right)^2 \le \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right) . \tag{1.23}$$

*The equality sign holds if and only if there is a real number $x$ such that $a_k x + b_k = 0$ for each $k = 1, 2, \ldots, n$.*

*Proof.* We have $\sum_{k=1}^{n} (a_k x + b_k)^2 \geq 0$ for every real x because a sum of squares can never be negative. This may be written in the form

$$Ax^2 + 2Bx + C \geq 0\,, \tag{1.24}$$

where

$$A = \sum_{k=1}^{n} a_k^2\,, \qquad B = \sum_{k=1}^{n} a_k b_k\,, \qquad C = \sum_{k=1}^{n} b_k^2\,.$$

We wish to prove that $B^2 \leq AC$. If $A = 0$, then each $a_k = 0$, so $B = 0$ and the result is trivial. If $A \neq 0$, we may complete the square and write

$$Ax^2 + 2Bx + C = A\left(x + \frac{B}{A}\right)^2 + \frac{AC - B^2}{A}\,.$$

The right side has its smallest value when $x = -B/A$. Putting $x = -B/A$ in (I.24), we obtain $B^2 \leq AC$. This proves (1.23). The reader should verify that the equality sign holds if and only if there is an x such that $a_k x + b_k = 0$ for each $k$.

## 14.9 Exercises

1. Prove each of the following properties of absolute values.
   (a) $|x| = 0$ if and only if $x = 0$.
   (b) $|-x| = |x|$.
   (c) $|x - y| = |y - x|$.
   (d) $|x|^2 = x^2$.
   (e) $|x| = \sqrt{x^2}$.
   (f) $|xy| = |x|\,|y|$.
   (g) $|x/y| = |x|/|y|$ if $y \neq 0$.
   (h) $|x - y| \leq |x| + |y|$.
   (i) $|x| - |y| \leq |x - y|$.
   (j) $\big||x| - |y|\big| \leq |x - y|$.
2. Each inequality $(a_i)$, listed below, is equivalent to exactly one inequality $(b_j)$. For example, $|x| < 3$ if and only if $-3 < x < 3$, and hence $(a_1)$ is equivalent to $(b_2)$. Determine all equivalent pairs.
   $(a_1)$ $|x| < 3$.
   $(a_2)$ $|x - 1| < 3$.
   $(a_3)$ $|3 - 2x| < 1$.
   $(a_4)$ $|1 + 2x| \leq 1$.
   $(a_5)$ $|x - 1| > 2$.
   $(a_6)$ $|x + 2| \geq 5$.
   $(a_7)$ $|5 - x^{-1}| < 1$.
   $(a_8)$ $|x - 5| < |x + 1|$.
   $(a_9)$ $|x^2 - 2| \leq 1$.
   $(a_{10})$ $x < x^2 - 12 < 4x$.
   $(b_1)$ $4 < x < 6$.
   $(b_2)$ $-3 < x < 3$.
   $(b_3)$ $x > 3$ or $x < -1$.
   $(b_4)$ $x > 2$.
   $(b_5)$ $-2 < x < 4$.
   $(b_6)$ $-\sqrt{3} \leq x \leq -1$ or $1 \leq x \leq \sqrt{3}$.
   $(b_7)$ $1 < x < 2$.
   $(b_8)$ $x \leq -7$ or $x \geq 3$.
   $(b_9)$ $\frac{1}{6} < x < \frac{1}{4}$.
   $(b_{10})$ $-1 \leq x \leq 0$.
3. Determine whether each of the following is true or false. In each case give a reason for your decision.
   (a) $x < 5$ implies $|x| < 5$.
   (b) $|x - 5| < 2$ implies $3 < x < 7$.
   (c) $|1 + 3x| \leq 1$ implies $x \geq -\frac{2}{3}$.
   (d) There is no real x for which $|x - 1| = |x - 2|$.
   (e) For every $x > 0$ there is a $y > 0$ such that $|2x + y| = 5$.
4. Show that the equality sign holds in the Cauchy-Schwarz inequality if and only if there is a real number x such that $a_k x + b_k = 0$ for every $k = 1, 2, \ldots, n$.

## *I 4.10 Miscellaneous exercises involving induction

In this section we assemble a number of miscellaneous facts whose proofs are good exercises in the use of mathematical induction. Some of these exercises may serve as a basis for supplementary classroom discussion.

*Factorials and binomial coefficients.* The symbol $n!$ (read "$n$ factorial") may be defined by induction as follows: $0! = 1$, $n! = (n-1)!\,n$ if $n \geq 1$. Note that $n! = 1 \cdot 2 \cdot 3 \cdots n$.

If $0 \leq k \leq n$, the *binomial coefficient* $\binom{n}{k}$ is defined as follows:

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}\,.$$

*Note:* Sometimes ${}_nC_k$ is written for $\binom{n}{k}$. These numbers appear as coefficients in the binomial theorem. (See Exercise 4 below.)

1. Compute the values of the following binomial coefficients :
(a) $\binom{5}{3}$, (b) $\binom{7}{0}$, (c) $\binom{7}{1}$, (d) $\binom{7}{2}$, (e) $\binom{17}{14}$, (f) $\binom{0}{0}$.
2. (a) Show that $\binom{n}{k} = \binom{n}{n-k}$. (c) Find $k$, given that $\binom{14}{k} = \binom{14}{k-4}$.
(b) Find $n$, given that $\binom{n}{10} = \binom{n}{7}$. (d) Is there a $k$ such that $\binom{12}{k} = \binom{12}{k-3}$?
3. Prove that $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$. This is called the *law of Pascal's triangle* and it provides a rapid way of computing binomial coefficients successively. Pascal's triangle is illustrated here for $n \leq 6$.

$$\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}$$

4. Use induction to prove the binomial theorem

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}\,.$$

Then use the theorem to derive the formulas

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n \quad \text{and} \quad \sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0, \quad \text{if } n > 0.$$

*The product notation.* The product of $n$ real numbers $a_1, a_2, \ldots, a_n$ is denoted by the symbol $\prod_{k=1}^{n} a_k$, which may be defined by induction. The symbol $a_1 a_2 \ldots a_n$ is an alternative notation for this product. Note that

$$n! = \prod_{k=1}^{n} k.$$

5. Give a definition by induction for the product $\prod_{k=1}^{n} a_k$.

Prove the following properties of products by induction:

6. $\prod_{k=1}^{n} (a_k b_k) = \left(\prod_{k=1}^{n} a_k\right)\left(\prod_{k=1}^{n} b_k\right)$ (multiplicative property).

An important special case is the relation $\prod_{k=1}^{n} (ca_k) = c^n \prod_{k=1}^{n} a_k$.

7. $\prod_{k=1}^{n} \frac{a_k}{a_{k-1}} = \frac{a_n}{a_0}$ if each $a_k \neq 0$ (telescoping property).

8. If $x \neq 1$, show that

$$\prod_{k=1}^{n} (1 + x^{2^{k-1}}) = \frac{1 - x^{2^n}}{1 - x}.$$

What is the value of the product when $x = 1$?

9. If $a_k < b_k$ for each $k = 1, 2, \ldots, n$, it is easy to prove by induction that $\sum_{k=1}^{n} a_k < \sum_{k=1}^{n} b_k$. Discuss the corresponding inequality for products:

$$\prod_{k=1}^{n} a_k < \prod_{k=1}^{n} b_k.$$

*Some special inequalities*

10. If $x > 1$, prove by induction that $x^n > x$ for every integer $n \geq 2$. If $0 < x < 1$, prove that $x^n < x$ for every integer $n \geq 2$.
11. Determine all positive integers $n$ for which $2^n < n!$.
12. (a) Use the binomial theorem to prove that for $n$ a positive integer we have

$$\left(1 + \frac{1}{n}\right)^n = 1 + \sum_{k=1}^{n}\left\{\frac{1}{k!}\prod_{r=0}^{k-1}\left(1 - \frac{r}{n}\right)\right\}.$$

(b) If $n > 1$, use part (a) and Exercise 11 to deduce the inequalities

$$2 < \left(1 + \frac{1}{n}\right)^n < 1 + \sum_{k=1}^{n} \frac{1}{k!} < 3.$$

13. (a) Let $p$ be a positive integer. Prove that

$$b^p - a^p = (b - a)(b^{p-1} + b^{p-2}a + b^{p-3}a^2 + \cdots + ba^{p-2} + a^{p-1}).$$

[*Hint:* Use the telescoping property for sums.]

(b) Let $p$ and $n$ denote positive integers. Use part (a) to show that

$$n^p < \frac{(n + 1)^{p+1} - n^{p+1}}{p + 1} < (n + 1)^p$$

(c) Use induction to prove that

$$\sum_{k=1}^{n-1} k^p < \frac{n^{p+1}}{p+1} < \sum_{k=1}^{n} k^p .$$

Part (b) will assist in making the inductive step from $n$ to $n + 1$.

14. Let $a_1, \ldots, a_n$ be $n$ real numhers, all having the same sign and all greater than -1. Use induction to prove that

$$(1 + a_1)(1 + a_2) \cdots (1 + a_n) \geq 1 + a_1 + a_2 + \cdots + a_n .$$

In particular, when $a_1 = a_2 = \cdots = a_n = x$, where $x > -1$, this yields

(1.25) $$(1 + x)^n \geq 1 + nx \qquad \textit{(Bernoulli's inequality).}$$

Show that when $n > 1$ the equality sign holds in (1.25) only for $x = 0$.

15. If $n \geq 2$, prove that $n!/n^n \leq (\frac{1}{2})^k$, where $k$ is the greatest integer $\leq n/2$.

16. The numbers 1, 2, 3, 5, 8, 13, 21, . . . , in which each term after the second is the sum of its two predecessors, are called *Fibonucci numbers*. They may be defined by induction as follows :

$$a_1 = 1, \qquad a_2 = 2, \qquad a_{n+1} = a_n + a_{n-1} \quad \text{if } n \geq 2.$$

Prove that

$$a_n \leq \left(\frac{1 + \sqrt{5}}{2}\right)^n$$

for every $n \geq 1$.

*Znequulities relating di'rent types of uveruges.* Let $x_1, x_2, \ldots, x_n$ be $n$ positive real numbers. If $p$ is a nonzero integer, the ***pth-power mean*** $M_p$ of the $n$ numbers is defined as follows :

$$M_p = \left(\frac{x_1^p + \cdots + x_n^p}{n}\right)^{1/p}.$$

The number $M_1$ is also called the *urithmetic mean,* $M_2$ the *root mean square,* and $M_{-1}$ the *hurmonic mean.*

17. Ifp > 0, prove that $M_p < M_{2p}$ when $x_1, x_2, \ldots, x_n$ are not all equal.

[*Hint:* Apply the Cauchy-Schwarz inequality with $a_k = x_k^p$ and $b_k = 1$.]

18. Use the result of Exercise 17 to prove that

$$a^4 + b^4 + c^4 \geq \tfrac{64}{3}$$

if $a^2 + b^2 + c^2 = 8$ and $a > 0$, $b > 0$, $c > 0$.

19. Let $a_1, \ldots, a_n$ be $n$ positive real numbers whose product is equal to 1. Prove that $a_1 + \cdots + a_n \geq n$ and that the equality sign holds only if every $a_k = 1$.

[*Hint:* Consider two cases: (a) All $a_k = 1$; (b) not all $a_k = 1$. Use induction. In case (b) notice that if $a_1a_2 \cdots a_{n+1} = 1$, then at least one factor, say $a_1$, exceeds 1 and at least one factor, say $a_{n+1}$, is less than 1. Let $b_1 = a_1a_{n+1}$ and apply the induction hypothesis to the product $b_1a_2 \cdots a_n$, using the fact that $(a_1 - 1)(a_{n+1} - 1) < 0$.]

20. The *geometric mean* G of $n$ positive real numbers $x_1, \ldots, x_n$ is defined by the formula $G = (x_1x_2 \cdots x_n)^{1/n}$.
(a) Let $M_p$ denote the pth power mean. Prove that $G \le M_1$ and that $G = M_1$ only when $x_1 = x_2 = \cdots = x_n$.
(b) Let $p$ and $q$ be integers, $q < 0 < p$. From part (a) deduce that $M_q < G < M_p$ when $x_1, x_2, \ldots, x_n$ are not all equal.

21. Use the result of Exercise 20 to prove the following statement : If $a$, $b$, and $c$ are positive real numbers such that $abc = 8$, then $a + b + c \ge 6$ and $ab + ac + bc \ge 12$.

22. If $x_1, \ldots, x_n$ are positive numbers and if $y_k = 1/x_k$, prove that

$$\left(\sum_{k=1}^{n} x_k\right)\left(\sum_{k=1}^{n} y_k\right) \ge n^2 .$$

23. If $a$, $b$, and $c$ are positive and if $a + b + c = 1$, prove that $(1 - a)(1 - b)(1 - c) \ge 8abc$.