MARCILIO 2016/II

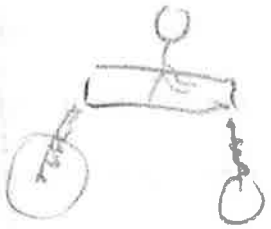
Free body diagrams
- externally applied forces
- reactive forces
- inertia forces

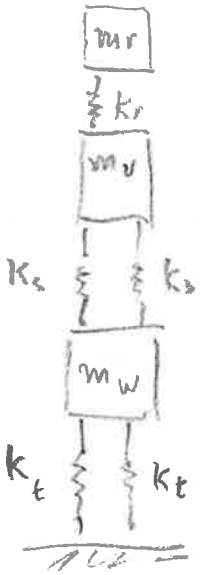
Governing equations
- Newton's second law
- d'Alembert's principle
- principle of conservation of energy

Solving
- standard methods
- numerical methods (FEM + time integr)
- Laplace transform
- matrix methods

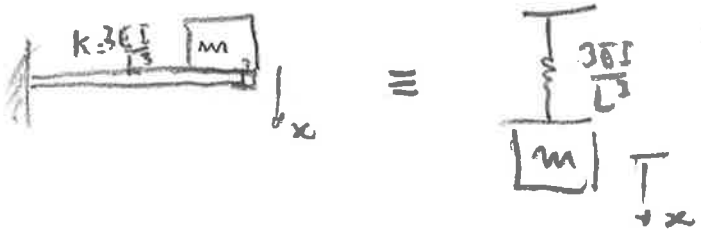
displac. +
velo. + acceler.

Example - Motorcycle rider



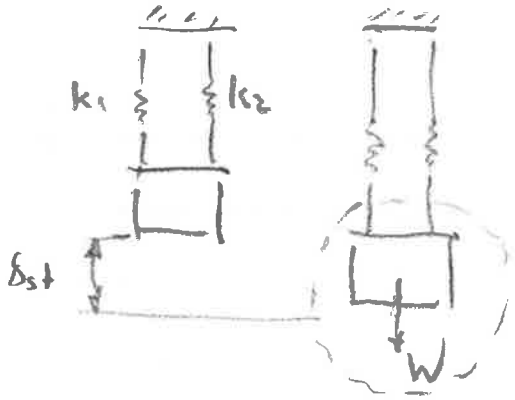


Impact of tubes



mass (density) not considered

Springs in Parallel





$\Sigma F = 0 \quad \uparrow +$

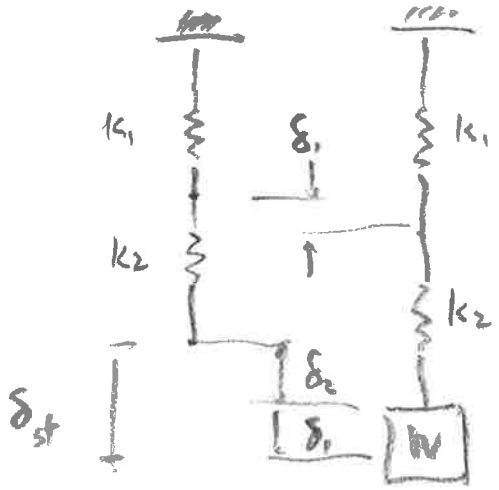
$k_1 \delta_{st} + k_2 \delta_{st} - W = 0$

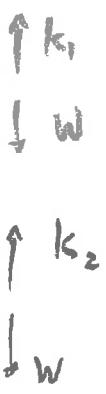
$(k_1 + k_2)\,\delta_{st} = W$

$W = k_{eq}\,\delta_{st}$

$k_{eq} = k_1 + k_2$

Springs in Series





$W = k_1 \delta_1$

$W = k_2 \delta_2$

$k_1 \delta_1 = k_2 \delta_2 = k_{eq}\,\delta_{eq}$

$= k_{eq}(\delta_1 + \delta_2)$

$\delta_{st} = \delta_1 + \delta_2$

$\delta_{st} = \dfrac{k_{eq}}{k_1}\delta_{st} + \dfrac{k_{eq}}{k_2}\delta_{st}$

$\dfrac{1}{k_{eq}} = \dfrac{1}{k_1} + \dfrac{1}{k_2}$ $\quad \boxed{(\text{capacitores})^{-1} = k}$

$\delta_1 = \dfrac{k_{eq}\,\delta_{st}}{k_1}$

$\delta_2 = \dfrac{k_{eq}\,\delta_{st}}{k_2}$



$$\frac{F}{\Delta L} = K = \frac{F}{\varepsilon L} = \frac{F}{\frac{\sigma}{E} L} = \frac{\sigma A}{\sigma L} E = \frac{EA}{L}$$



$$K = \frac{F}{\delta}$$

$M'' = 0$

$M' = Q$

$M = Qx + C$

$M_{x=0} = FL = Q \cdot 0 + C_1$

$C_1 = FL$

$M = Qx + FL$

$Q = -F$

$M = F(L-x)$

$EIk = F(L-x)$

$-EIw'' = F(L-x)$

$-EIw' = F\left(Lx - \frac{x^2}{2}\right) + C_2$

$-EIw' = Fx\left(L - \frac{x}{2}\right) + C_2$

$x = 0 \quad w' = 0, \quad C_2 = 0$

$-EIw = F\left(\frac{Lx^2}{2} - \frac{x^3}{6}\right) + C_3$

$x = 0, w = 0, C_3 = 0$

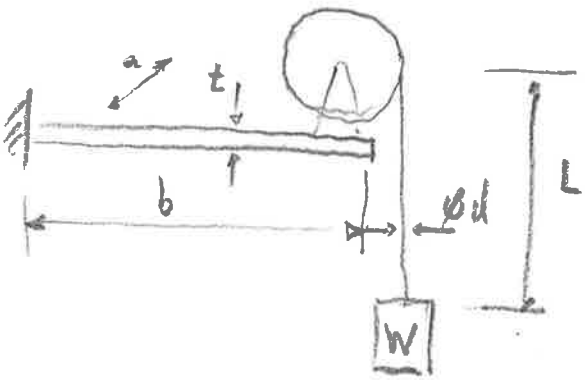
$$\frac{F}{w} = \frac{-EI}{x^2\left(\frac{L}{2} - \frac{x}{6}\right)}$$

$x = L$

$$\frac{F}{w} = \frac{-EI\,3}{L^2 L}$$

$$-\frac{F}{w} = \frac{3EI}{L^3}$$

Ex. 1.4 Obtenha a rigidez equivalente



$$r = \frac{k_{beam}}{k_{rope}} = \frac{3E\cancel{a}t^3}{12b^3}\,\frac{4L}{\pi d^2 E}$$

$t = d \quad b = L$

$$r = \frac{\cancel{3}\,a\,t^{\cancel{3}}\,\cancel{4}L}{\cancel{12}\,t^{\cancel{3}}\,\pi\,t^{\cancel{2}}} = \frac{aL}{\pi t^2}$$

$a = 10t \simeq 3\pi$

$$r = \frac{3L}{t}$$

$$k_{beam} = \frac{3EI}{b^3} = \frac{3Eat^3}{b^3\,12}$$

Inspira

$$k_{rope} = \frac{AE}{L} = \frac{\pi d^2}{4}\,\frac{E}{L}$$

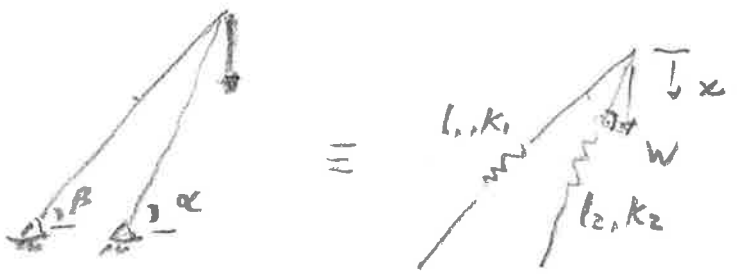
Springs in series $\quad \frac{1}{k_{eq}} = \frac{1}{k_b} + \frac{1}{k_r} \Rightarrow k_{eq} = \frac{E}{4}\,\frac{\pi a t^3 d^2}{\pi d^2 b^3 + 4Lat^3}$

Exercício: checar dimensionalidade

Ex. 1.5

Em classe

K_{eq} na direção vertical



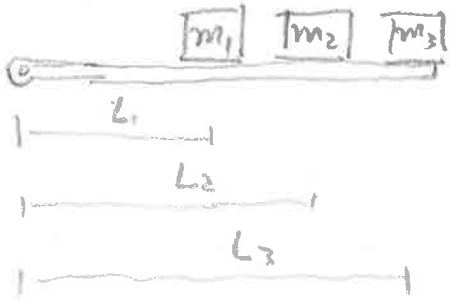
$$U = \frac{1}{2}k_1 x_1^2 + \frac{1}{2}k_2 x_2^2 = \frac{1}{2}k_{eq}\,x^2$$

$k_2 \Rightarrow x_2 = x\cos\alpha \qquad k_1 \Rightarrow x_1 = x\cos(90^\circ - \beta)$

$$k_1 = \frac{A_1E_1}{L_1} \qquad k_2 = \frac{A_2E_2}{L_2}$$

$$\frac{A_1E_1}{L_1}\,x^2\cos^2(90^\circ-\beta) + \frac{A_2E_2}{L_2}\,x^2\cos^2\alpha = k_{eq}\,x^2$$

Equivalent mass (translational)



choosing L_1 to position the equiv mass

$$\dot{x}_2 = \frac{L_2}{L_1}\dot{x}_1 \qquad \dot{x}_3 = \frac{L_3}{L_1}\dot{x}_1$$

$$\dot{x}_1 = \dot{x}_{eq}$$

$$\frac{1}{2} m_{eq} \dot{x}_{eq}^2 = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 + \frac{1}{2} m_3 \dot{x}_3^2$$

$$m_{eq} = m_1 + \left(\frac{L_2}{L_1}\right)^2 m_2 + \left(\frac{L_3}{L_1}\right)^2 m_3$$

Equivalent mass (translational and rotational)



case I - single equivalent translational mass

$$\frac{1}{2} m \dot{x}^2 + \frac{1}{2} J_0 \dot{\theta}^2 = \frac{1}{2} m_{eq} \dot{x}_{eq}^2$$

$$\dot{\theta} = \frac{\dot{x}}{R} \qquad \dot{x}_{eq} = \dot{x}$$

$$m_{eq} = m + \frac{J_0}{R^2}$$

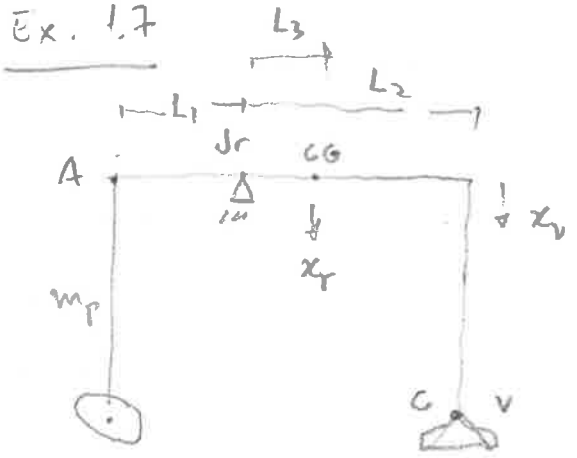
Equivalent rotational mass

$$\dot{\theta}_{eq} = \dot{\theta} \qquad \dot{x} = \dot{\theta} R$$

$$\frac{1}{2} J_{eq} \dot{\theta}^2 = \frac{1}{2} m (R\dot{\theta})^2 + \frac{1}{2} J_0 \dot{\theta}^2$$

$$J_{eq} = J_0 + m R^2$$

Ex. 1.7



Equivalent mass in Point A

$$\frac{1}{2} m_{eq} \dot{x}_{eq_A}^2 = \frac{1}{2} m_p \dot{x}_A^2 + \frac{1}{2} m_v \dot{x}_v^2 + \frac{1}{2} J_r \dot{\theta}_r^2 + \frac{1}{2} m_r \dot{x}_r^2$$

$$\theta_r = \frac{x}{L_1} \qquad x_v = \theta_r L_2 = x \frac{L_2}{L_1} \qquad x_r = \theta_r L_3 = x \frac{L_3}{L_1}$$

$$m_{eq} = m_p + \frac{J_r}{L_1^2} + m_v \frac{L_2^2}{L_1^2} + m_r \frac{L_3^2}{L_1^2}$$

donne

Equiv mass in C

$$m_{eq} = m_v + \frac{J_r}{L_2^2} + m_r \frac{L_1^2}{L_2^2} + m_r \frac{L_3^2}{L_2^2}$$

Resumo da Aula 1

- Métodos de soluções
- Análise / Modelagem / Síntese
- Rigidez e massa equivalentes
- Exercícios

1) Válvula

$$\frac{1}{2} m_{eq} \dot{x}_{eq_A}^2 = \frac{1}{2} m_p \dot{x}_A^2 + \frac{1}{2} m_v \dot{x}_v^2 + \frac{1}{2} m_r \dot{x}_r^2 + \frac{1}{2} J_r \dot{\theta}_r^2$$

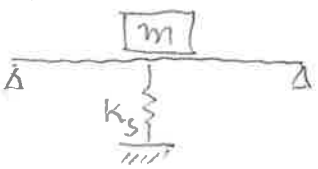
$$\dot{\theta}_r L_1 \qquad \dot{\theta}_r L_1 \qquad \dot{\theta}_r L_2 \qquad \dot{\theta}_r L_3$$

$$m \dot{\theta}^2 L_1^2 = m_p \dot{\theta}^2 L_1^2 + m_v \dot{\theta}^2 L_2^2 + m_r \dot{\theta}^2 L_3^2 + J_r \dot{\theta}^2$$

$$m = m_p + m_v \frac{L_2^2}{L_1^2} + m_r \frac{L_3^2}{L_1^2} + \frac{J_r}{L_1^2}$$

Ex. 1.10

2)



$$k_b = \frac{48 E I}{L^3}$$

k em paralelo $k = k_b + k_s$

$$F = K w$$

$$w_b = \frac{F}{K_b} \qquad w = \frac{1}{3} w_b = \frac{F}{K}$$

$$K = \frac{3F}{w_b} = \frac{3F}{F} k_b = 3 K_b = \frac{144 E I}{L^3}$$

$$K = K_b + K_s$$

$$k_s = K - K_b = 3 k_b - K_b = 2 k_b = \frac{96 E I}{L^3}$$

3) Ex. 1.22 $\qquad k = k_s + k_a$

4) Ex. 1.28

$$\frac{1}{2} J \dot{\theta}^2_{input} = \frac{1}{2} J_{motor} \dot{\theta}^2_{input} + \frac{1}{2} J_{load} \dot{\theta}^2_{load}$$

$$+ \frac{1}{2} \sum_{k=1}^{2N} J_k \dot{\theta}^2_k$$

$$J = (J_{motor} + J_1) + (J_2 + J_3)\left(\frac{n_1}{n_2}\right)^2 + (J_4 + J_5)\left(\frac{n_1}{n_2}\frac{n_3}{n_4}\right)^2 +$$

$$\cdots + (J_{2N} + J_{load})\left(\frac{n_1}{n_2}\frac{n_3}{n_4}\cdots\frac{n_{2N-1}}{n_{2N}}\right)^2$$

i = input

$J_{motor}, J_1 \qquad J_2, J_3 \qquad J_4, J_5 \qquad J_{2N}, J_{load}$

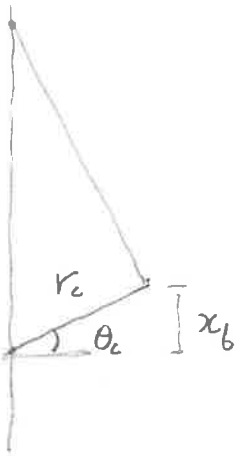
$\dot{\theta}_i \qquad \dot{\theta}_i \frac{n_1}{n_2} \qquad \dot{\theta}_i \frac{n_1}{n_2}\frac{n_3}{n_4} \qquad \dot{\theta}_i \frac{n_1}{n_2}\frac{n_3}{n_4}\cdots\frac{n_{2N-1}}{n_{2N}}$

5) Ex. 1.30

$$\frac{1}{2} m_{eq} \dot{x}^2 = \frac{1}{2} m_{eq} \dot{x}_h^2 = \frac{1}{2} m_h \dot{x}_h^2 + \frac{1}{2} J_b \dot{\theta}_b^2 + \frac{1}{2} J_c \dot{\theta}_c^2$$

$$\theta_b = \frac{x_h}{L_3} \qquad \dot{\theta}_b = \frac{\dot{x}_h}{L_3}$$

$$\theta_c \simeq \frac{x_b}{r_c} \qquad \dot{\theta}_c = \frac{\dot{x}_b}{r_c}$$



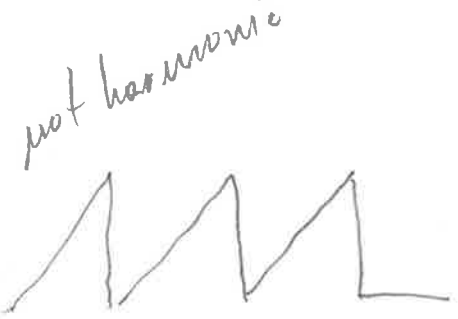
$$x_b = x_h \frac{L_2}{L_3} \longrightarrow \dot{\theta}_c = \frac{\dot{x}_h}{r_c} \frac{L_2}{L_3}$$

$$m_{eq} = m_h + J_b \left(\frac{\dot{x}_h}{L_3} \right)^2 \frac{1}{\dot{x}_h^2} + J_c \left(\frac{\dot{x}_h}{r_c} \frac{L_2}{L_3} \right)^2 \frac{1}{\dot{x}_h^2}$$

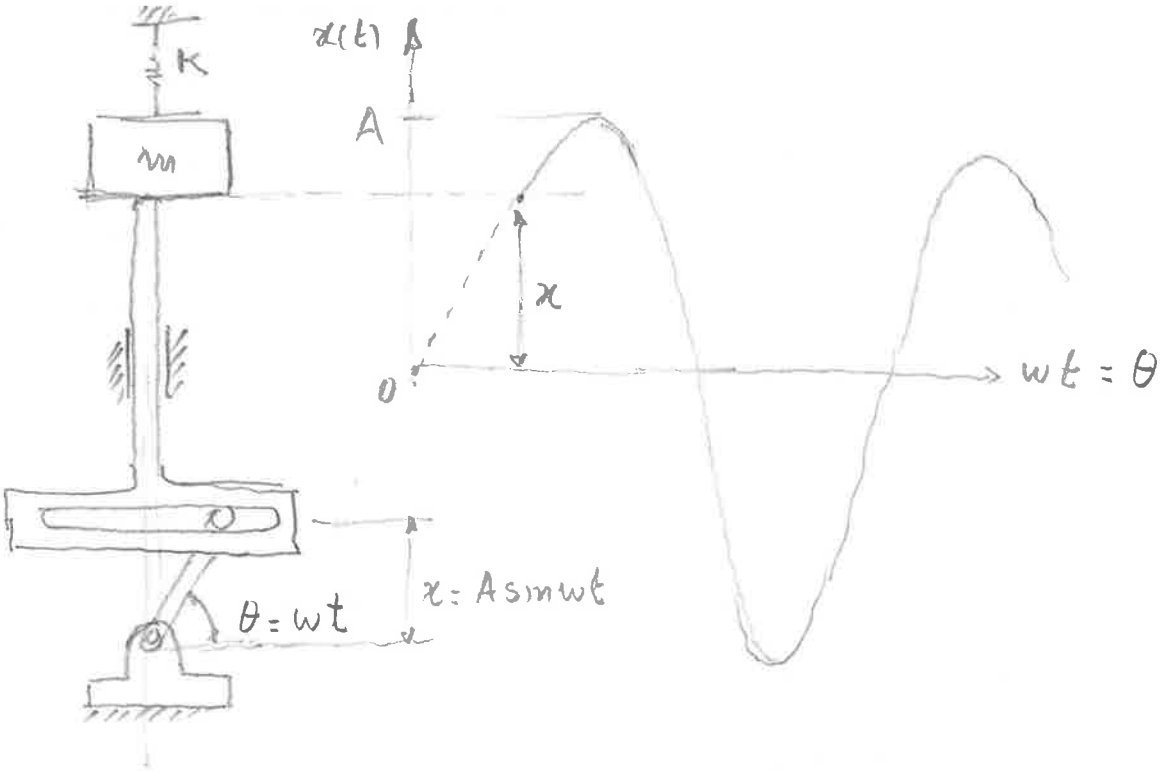
$$m_{eq} = m_h + \frac{J_b}{L_3^2} + J_c \left(\frac{L_2}{r_c L_3} \right)^2$$

Harmonic motion

Periodic motion → eg



↳ Harmonic motion



DEMO ZEGUA

$x = A \sin \omega t$

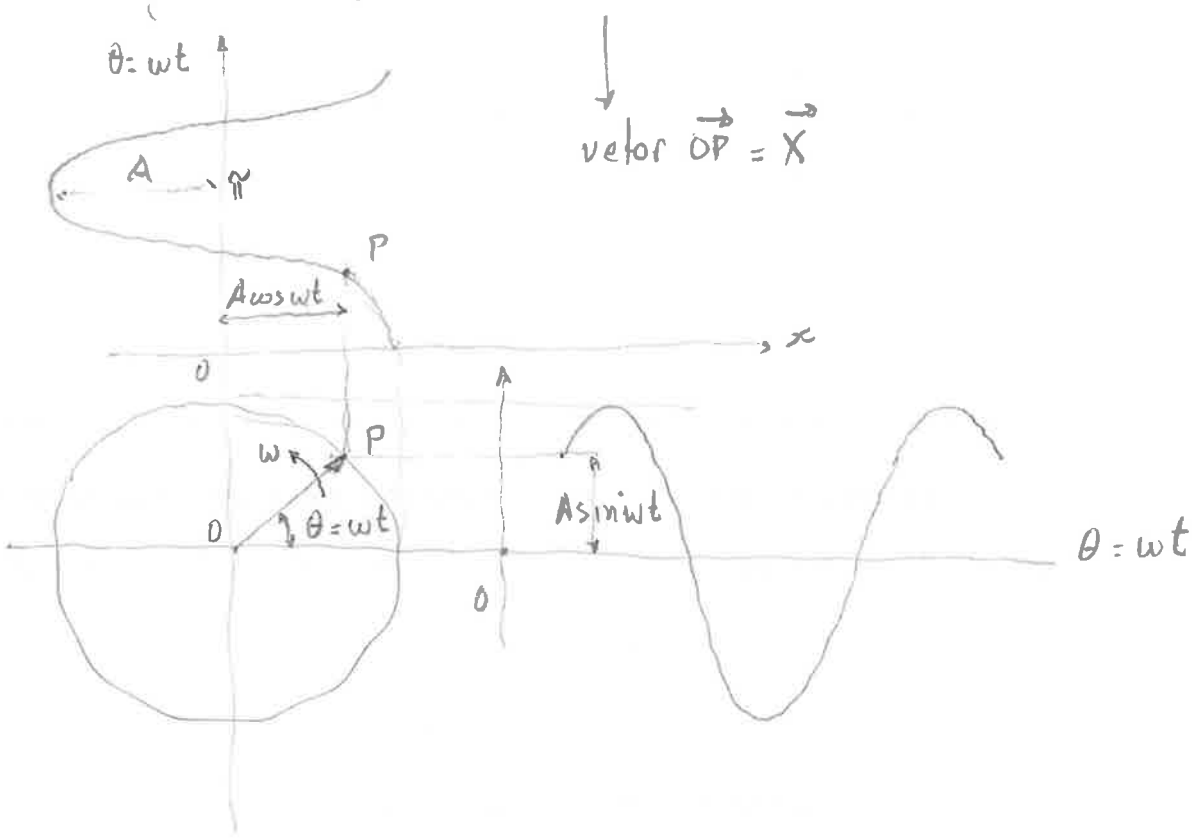
$\dot{x} = \omega A \cos \omega t$

$\ddot{x} = -\omega^2 A \sin \omega t = -\omega^2 x$

Simple harmonic motion → aceleração proporcional ao deslocamento

$x = A \cos \omega t$ → também é movimento harmónico simples

Representação vetorial do movimento harmônico

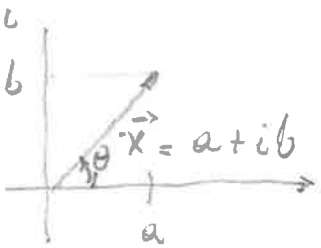


vetor $\vec{OP} = \vec{X}$

$y = A \cos \omega t$ $\quad x = A \sin \omega t$

magnitude do vetor A

Representação por números complexos



A - valor absoluto ou módulo

θ - argumento

$\vec{X} = A \cos\theta + i A \sin\theta$ $\quad \theta = \tan^{-1} \frac{b}{a}$

$\vec{X} = A (\cos\theta + i \operatorname{sen}\theta)$

Função f pode ser representada por uma série

McLaurin Series

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \dots$$

$$f(a) = c_0$$

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \dots$$

$$f'(a) = c_1$$

$$\vdots$$

$$f''(a) = 2c_2$$

$$f'''(a) = 3!\,c_3$$

$$f^{(n)}(a) = n!\,c_n \quad \rightarrow \quad c_n = \frac{f^{(n)}(a)}{n!}$$

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3$$

$x-a = x,\ a = 0$

$f(x) = \sin x \qquad f(0) = 0$

$f'(x) = \cos x \qquad f'(0) = 1$

$f''(x) = -\sin x \qquad f''(0) = 0$

$f'''(x) = -\cos x \qquad f'''(0) = -1$

$$f(x) = \sin x = x - \frac{1}{3!}x^3 + \frac{x^5}{5!} - \frac{x^7}{7!}$$

$$\sin\theta = \theta - \frac{1}{3!}\theta^3 + \frac{\theta^5}{5!} - \frac{\theta^7}{7!}$$

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!}$$

$$\vec{X} = A(\cos\theta + i\,\text{sen}\,\theta)$$

$$\frac{\vec{X}}{A} = 1 - \frac{\theta^2}{2} + \frac{\theta^4}{4!} - \cdots + i\left(\theta - \frac{1}{3!}\theta^3 + \frac{\theta^5}{5!} - \cdots\right)$$

$$= 1 - \frac{\theta^2}{2} + \frac{\theta^4}{4!} + i\theta + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^5}{5!}$$

$$= 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \cdots = e^{i\theta}$$

$$\vec{X} = A e^{i\theta}$$

$$\boxed{e^{i\pi} + 1 = 0}$$

↳ 5 important ctes

$$\vec{X} = A e^{i\omega t}$$

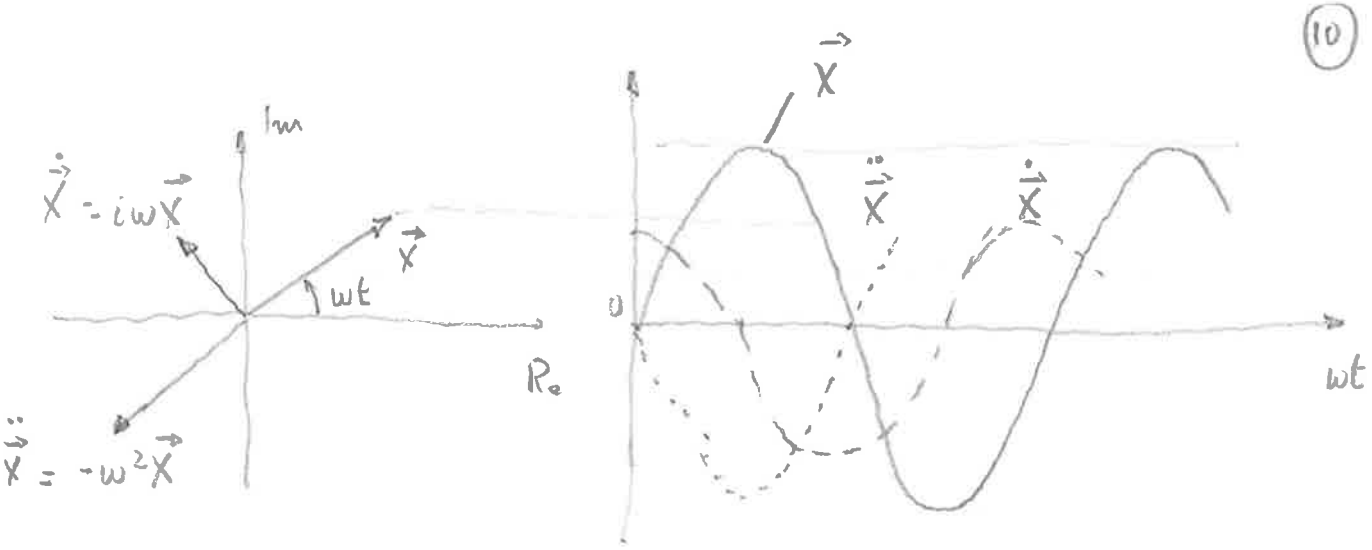
$$\dot{\vec{X}} = i\omega A e^{i\omega t} = i\omega \vec{X}$$

$$\ddot{\vec{X}} = -\omega^2 A e^{i\omega t} = -\omega^2 \vec{X}$$

Assume $\vec{X} = A\cos\omega t$

$$\dot{\vec{X}} = -\omega A \sin\omega t = \omega A(\cos\omega t + \pi/2)$$ ↳ leads $\vec{X}$ by $\pi/2$

$$\ddot{\vec{X}} = -\omega^2 A\cos\omega t = \omega^2 A\cos(\omega t + \pi)$$ ↳ leads $\vec{X}$ by π



Definições

Amplitude

Ciclo

Período $T = \frac{2\pi}{\omega}$

Frequência $\omega = 2\pi f \Rightarrow f = \frac{1}{T} = \frac{\omega}{2\pi}$

Frequência Natural

Ângulo de fase

Batimento

$x_1 = A \sin \omega t \qquad x_2 = A \sin[(\omega + \delta\omega)t]$

$x = x_1 + x_2$

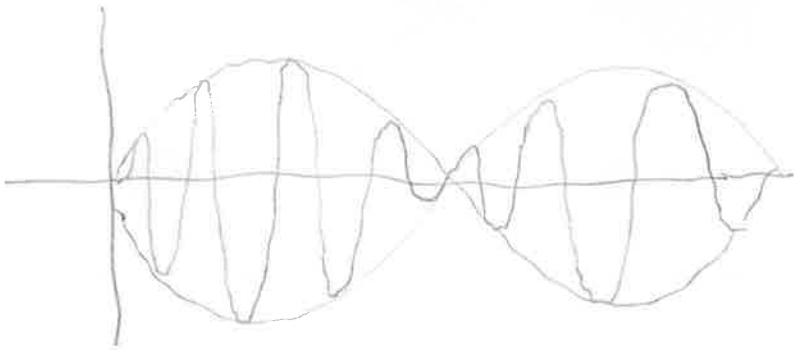
$x = A\left[\sin \omega t + \sin(\omega t + \delta\omega t)\right]$

$\sin A + \sin B = 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2}$

$$x = A\left[2 \sin \frac{\omega t + \omega t + \delta\omega t}{2} \cos \frac{\omega t - \omega t - \delta\omega t}{2} \right]$$

$$x = 2A \sin\left(\omega t + \frac{\delta\omega}{2} t\right) \cos \frac{\delta\omega}{2} t$$

$\delta\omega \rightarrow$ beat frequency



Beating demo: { regua
beat.m

Lissajous

Suppose a horizontal and vertical motion caused by an engine in a structure described by

$$x(t) = X \cos \omega t$$

$$y(t) = Y \cos(\omega t + \phi)$$

$$\frac{x^2}{X^2} = \cos^2 \omega t \qquad \frac{y^2}{Y^2} = \cos^2(\omega t + \phi)$$

$$2\frac{xy}{XY}\cos\phi = 2\cos\omega t \cos(\omega t + \phi)\cos\phi$$

$$\frac{x^2}{X^2} + \frac{y^2}{Y^2} - 2\frac{xy}{XY}\cos\phi = \cos^2\omega t + \cos^2(\omega t + \phi) - 2\cos\omega t\cos(\omega t + \phi)\cos\phi$$

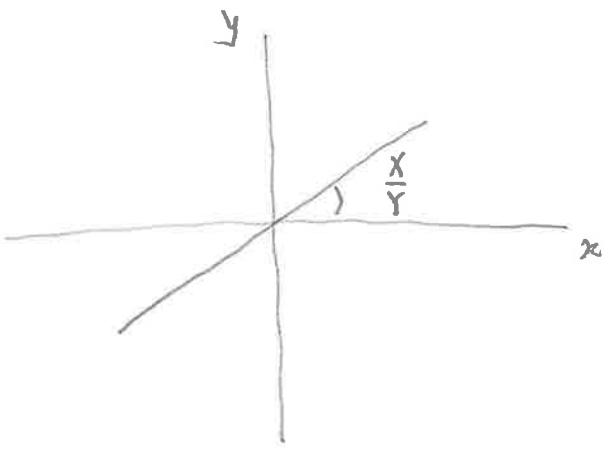
$$\cos^2\alpha = \frac{1}{2}(1 + \cos 2\alpha)$$

$$\frac{x^2}{X^2} + \frac{y^2}{Y^2} - 2\frac{xy}{XY}\cos\phi = \sin^2\phi$$

$$\phi = 0$$

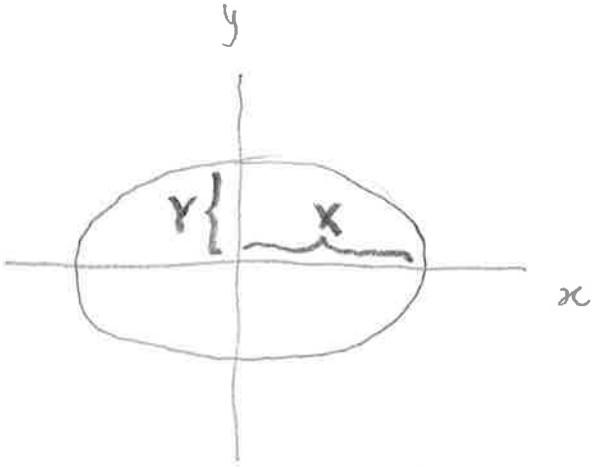
$$\frac{x^2}{X^2} + \frac{y^2}{Y^2} - \frac{2xy}{XY} = 0 \qquad \left(\frac{x}{X} - \frac{y}{Y}\right)^2 = 0$$

$$x = \pm \frac{X}{Y} y$$



$$\phi = \frac{\pi}{2}$$

$$\frac{x^2}{X^2} + \frac{y^2}{Y^2} = 1$$



Demo: Lissajous, two channels wave generator + scope barbante

Applet shows also when

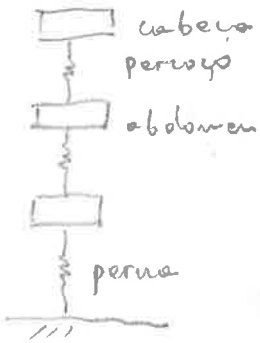
$$x = X \cos \omega t$$

$$y = Y \cos(n \omega t + \phi)$$

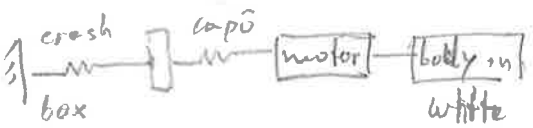
Destacar o que foi importante!

Exercícios

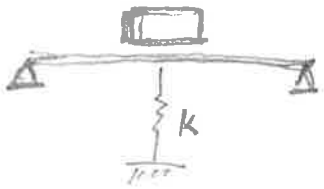
1.1



1.5



1.10



$$K_b = \frac{48EI}{L^3} \qquad k_{eq} = k_b + k$$

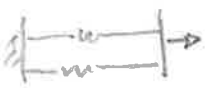
$$x_0 = \frac{mg}{k_b}$$

$$x_{eq} = \frac{mg}{k_b + K} = \frac{1}{3} x_0$$

$$\frac{mg}{k_b+k} = \frac{1}{3}\frac{mg}{k_b} \qquad \frac{k_b+k}{k_b}\frac{1}{3} = \frac{1}{}$$

$$\frac{k}{k_b} = 2 \qquad K = 2K_b = \frac{2 \times 48 \times 2.06\times10^{11}}{2^3} \frac{1.2\ 0.1^3}{12} = 247\,\frac{MN}{m}$$

1.22



$$K = \frac{EA}{L} \qquad k_{eq} = k_1 + k_2$$

$$k_{eq} = \frac{E_1 A_1}{L} + \frac{E_2 A_2}{L} = \frac{A}{L}(E_1 + E_2)$$

1.28 $$\frac{1}{2} J_{eq} \dot{\theta}_i^2 = \frac{1}{2} J_{motor} \dot{\theta}_i^2 + \frac{1}{2} \sum_{k=1}^{2N} J_k \dot{\theta}_k^2 + \frac{1}{2} J_{load} \dot{\theta}_{load}^2$$

$$J_{eq} = (J_{motor} + J_1) + (J_2 + J_3)\left(\frac{n_1}{n_2}\right)^2 + (J_4 + J_5)\left(\frac{n_1}{n_2}\frac{n_3}{n_4}\right)^2$$

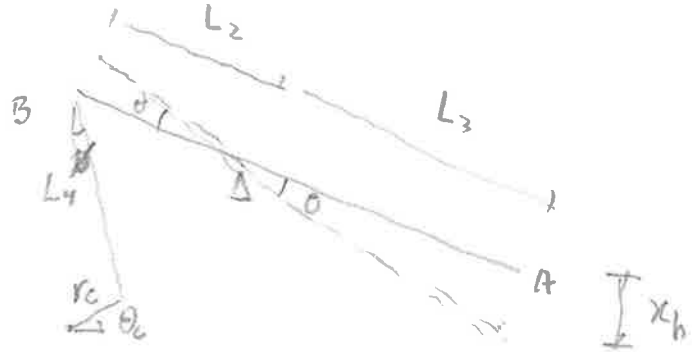
$$+ \cdots + (J_{2N} + J_{load})\left(\frac{n_1}{n_2}\frac{n_3}{n_4}\cdots\frac{n_{2N-1}}{n_{2N}}\right)^2$$

J_{motor}, J_1 $\quad J_2, J_3$ $\quad J_4, J_5$ $\quad J_{2N}, J_{load}$

$\dot{\theta}_i$ $\quad \dot{\theta}_i \frac{n_1}{n_2}$ $\quad \dot{\theta}_i \frac{n_1}{n_2}\frac{n_3}{n_4}$ $\quad \dot{\theta}_i \frac{n_1}{n_2}\frac{n_3}{n_4} \cdots \frac{n_{2N-1}}{n_{2N}}$

$\dot{\theta}_i$ → input angular velocity

1.30



$$\theta = \frac{x_h}{L_3}$$

$$x_B = \theta L_2 = x_h \frac{L_2}{L_3} \cong r_c \sin\theta_c = r_c \theta_c$$

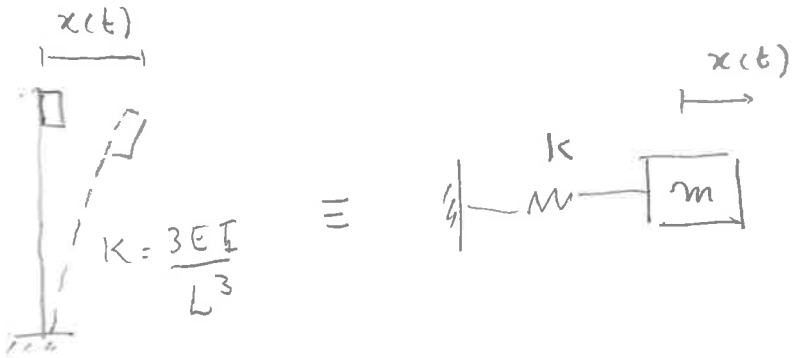
$$\theta_c = \frac{x_B}{r_c} = \frac{x_h L_2}{r_c L_3}$$

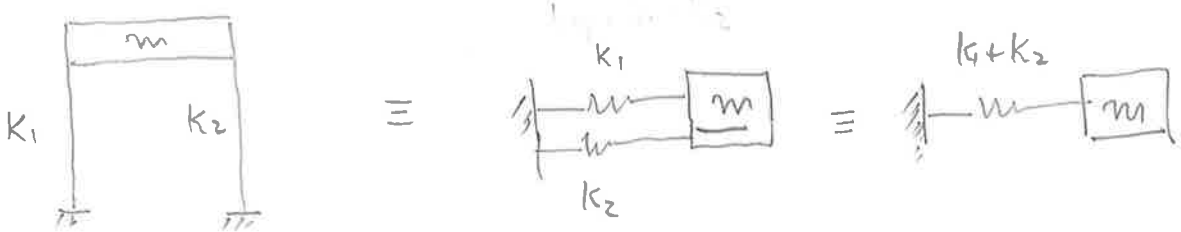
$$T = \frac{1}{2} m_h \dot{x}_h^2 + \frac{1}{2} J_b \dot{\theta}_b^2 + \frac{1}{2} J_c \dot{\theta}_c^2 = \frac{1}{2} m_{eq} \dot{x}_h^2$$

$$m_{eq} = m_h + \frac{J_b}{L_3^2} + J_c \left(\frac{L_2}{L_3 r_c}\right)^2$$

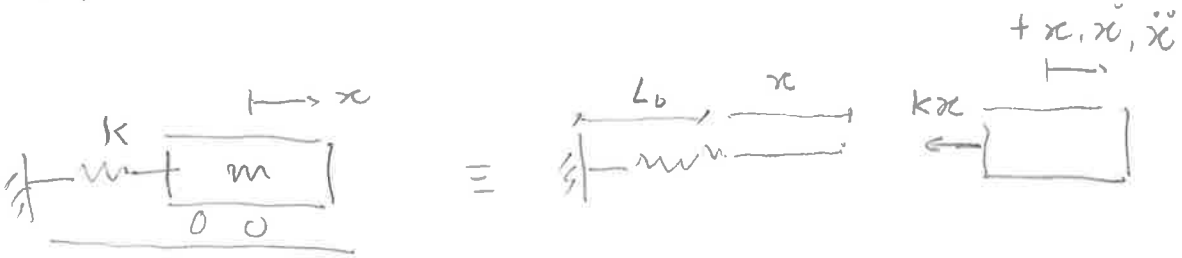
Free Vibration

Equation of motion by Newton's Law





Newton $\quad \Sigma F = ma \qquad \Sigma M = J\ddot{\theta}$



$$\Sigma F = m\ddot{x}$$

$$-Kx = m\ddot{x}$$

$$m\ddot{x} + kx = 0$$

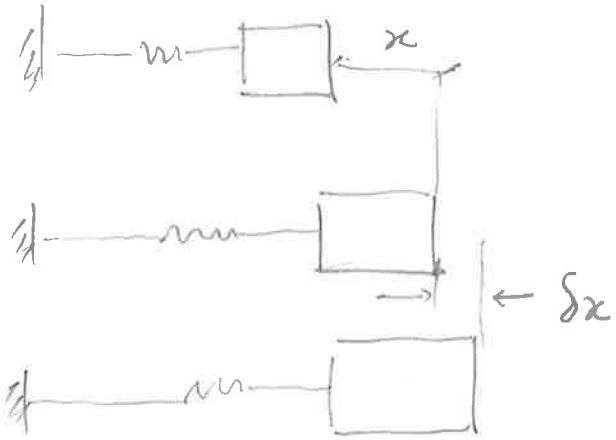
D'Alembert $\quad F - m\ddot{x} = 0$ ← equilíbrio

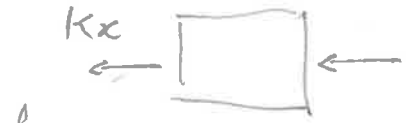
força de inércia

equilíbrio dinâmico ↑ considerado como força (fictícia)

Deslocamento Virtual

Princípio dos Trabalhos Virtuais





força reativa — força de inércia

deslocamento virtual → compatível c/ as restrições do sistema

trabalho virtual da força da mola $-kx\,\delta x$

" " das forças de inércia $-m\ddot{x}\,\delta x$

Σ trabalho virtual = zero

$$-kx\,\delta x - m\ddot{x}\,\delta x = 0$$

$$m\ddot{x} + kx = 0$$

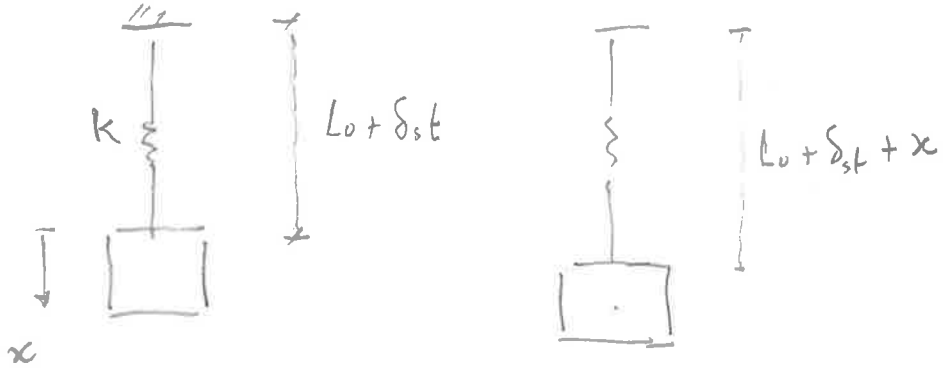
Conservação de energia

Para sistemas conservativos $T + U = cte$

$$\frac{d}{dt}(T+U) = 0 \rightarrow \frac{d}{dt}\left(\frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2\right)$$

$$m\ddot{x} + kx = 0$$

Gravidade



$$k\delta_{st} = G = mg$$

Newton's Law:

$$m\ddot{x} = -k(x + \delta_{st}) + G$$

$$m\ddot{x} = -kx - k\delta_{st} + G$$

$$m\ddot{x} = -kx - G + G$$

$$m\ddot{x} + kx = 0$$

Se a mossa se move verticalmente podemos desconsiderar seu peso (ou a gravidade) conquanto x seja referenciado à posição inicial de equilíbrio estático

Solução da equação de equilíbrio

$$m\ddot{x} + kx = 0$$

Por inspeção

$$x(t) = Ce^{st}$$

$$\dot{x} = sCe^{st}$$

$$\ddot{x} = s^2 Ce^{st}$$

$$ms^2 Ce^{st} + kCe^{st} = 0$$

$$ms^2 + k = 0$$

$$s = \pm\sqrt{-\frac{k}{m}} = \pm i\omega_n$$

destacar

$$\omega_n = \sqrt{\frac{k}{m}}$$

$$x = C_1 e^{i\omega_n t} + C_2 e^{-i\omega_n t}$$

$$e^{i\theta} = \cos\theta + i\,\text{sen}\,\theta$$

$$x = C_1(\cos\omega_n t + i\,\text{sen}\,\omega_n t) + C_2[\cos(-\omega_n t) + i\,\text{sen}(-\omega_n t)]$$

$$x = (C_1 + C_2)\cos\omega_n t + (C_1 i - C_2 i)\,\text{sen}\,\omega_n t$$

$$x = A_1 \cos\omega_n t + A_2\,\text{sen}\,\omega_n t$$

$t = 0 \quad x = x_0 \quad , \dot{x} = \dot{x}_0$

$x_0 = A_1 + A_2 0 \qquad A_1 = x_0$

$\dot{x}_0 = \omega_n A_1 \operatorname{sen} 0 + \omega_n A_2 \cos \omega_n 0 \qquad A_2 = \dfrac{\dot{x}_0}{\omega_n}$

$$x(t) = x_0 \cos \omega_n t + \frac{\dot{x}_0}{\omega_n} \operatorname{sen} \omega_n t$$

Chamemos $\quad A_1 = A \cos \phi \qquad A_2 = A \operatorname{sen} \phi$

$= x_0 \qquad = \dfrac{\dot{x}_0}{\omega_n}$

$$A_1^2 + A_2^2 = A^2 \cos^2 \phi + A^2 \operatorname{sen}^2 \phi = A^2 = x_0^2 + \frac{\dot{x}_0^2}{\omega_n^2}$$

$$A = \sqrt{x_0^2 + \frac{\dot{x}_0^2}{\omega_n^2}} = \text{amplitude}$$

$$\phi = \tan^{-1} \frac{A_2}{A_1} = \tan^{-1} \frac{\dot{x}_0}{\omega_n x_0} = \text{ângulo de fase}$$

$x = A_1 \cos \omega_n t + A_2 \operatorname{sen} \omega_n t$

$x = A \cos \phi \cos \omega_n t + A \operatorname{sen} \phi \operatorname{sen} \omega_n t$

$x = A(\cos \phi \cos \omega_n t + \operatorname{sen} \phi \operatorname{sen} \omega_n t) = A \cos(\omega_n t - \phi)$

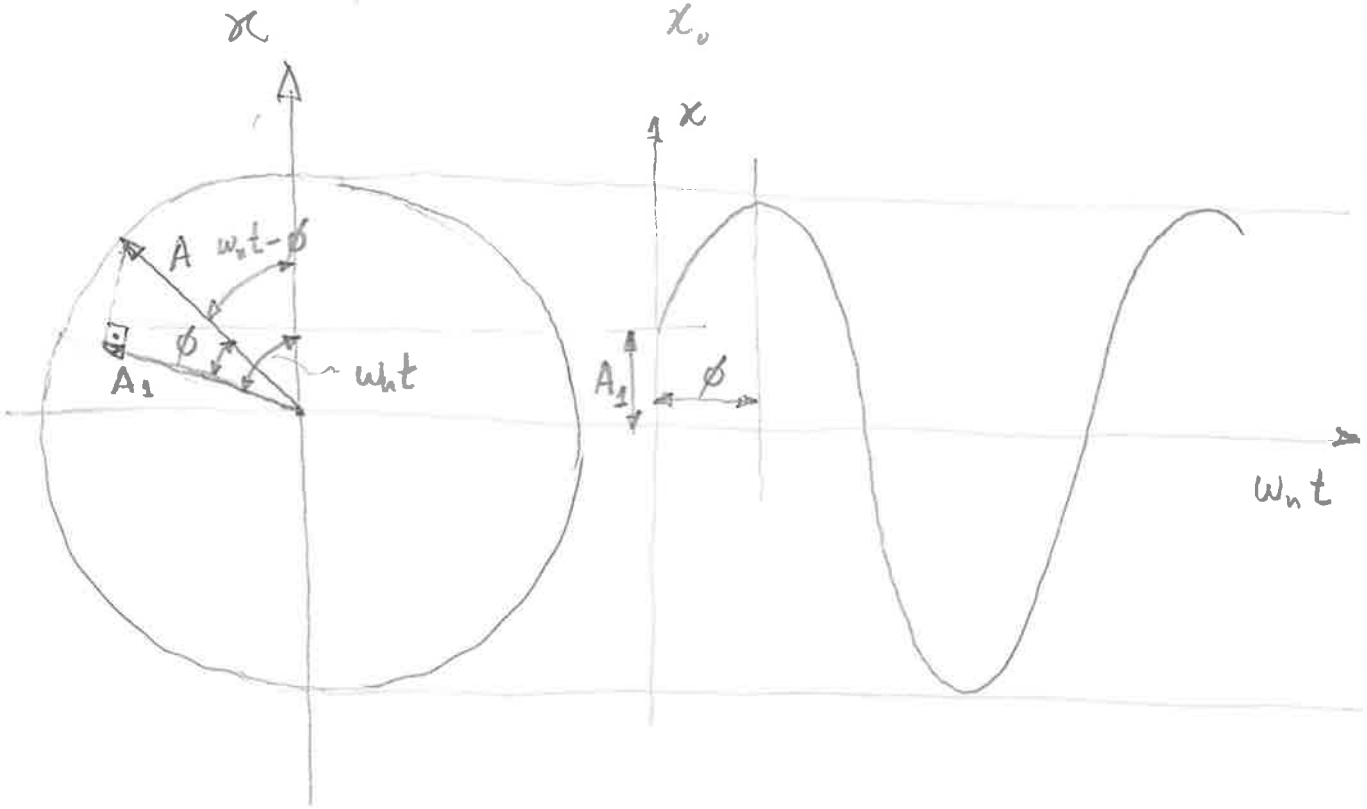
Fazendo agora

$$A_1 = A_0 \sin\phi_0 \qquad A_2 = A_0 \cos\phi_0$$

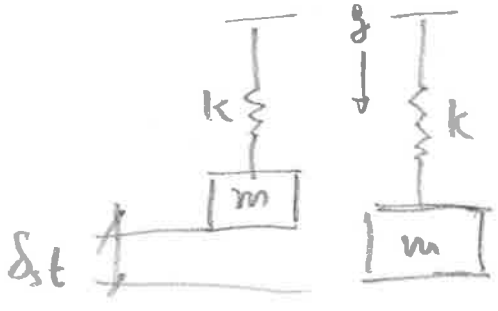
$$x(t) = A_0 \sin(\omega_n t + \phi_0)$$

$$A_0 = A = \sqrt{x_0^2 + \left(\frac{\dot{x}_0}{\omega_n}\right)^2}$$

$$\phi_0 = \tan^{-1}\frac{x_0\omega_n}{\dot{x}_0}$$



Medição da frequência natural com uma régua (sem medir a massa)



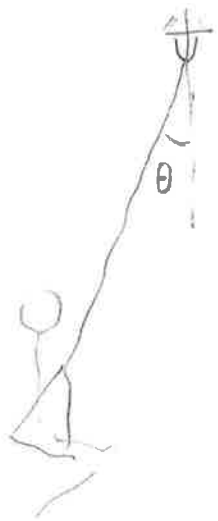
$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{mg/\delta_{st}}{m}} = \sqrt{\frac{g}{\delta_{st}}}$$

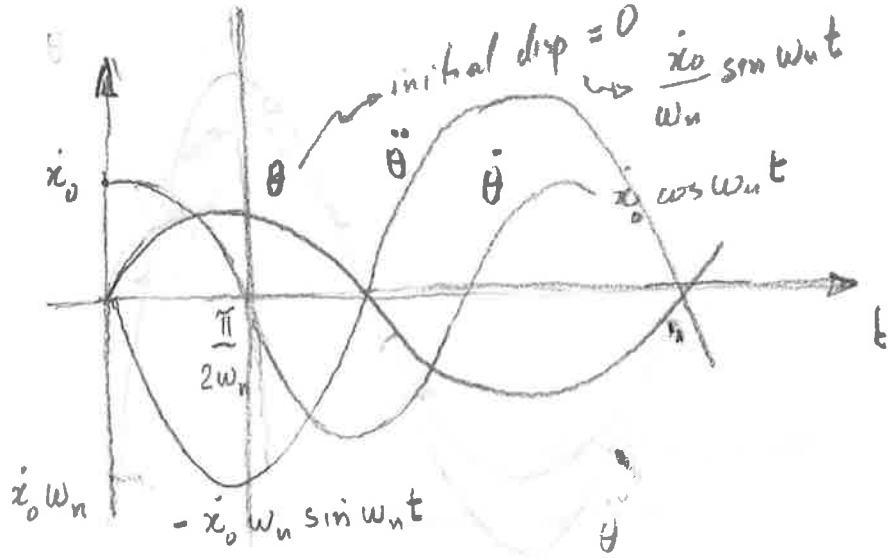
$$f_n = \frac{\omega_n}{2\pi}$$

$$x(t) = A\cos(\omega t - \phi)$$

leads the displ by $\pi/2$ ← $\dot{x}(t) = -\omega_n A \sin(\omega t - \phi) = \omega_n A \cos(\omega_n t - \phi + \pi/2)$

leads the disp by π ← $\ddot{x}(t) = -\omega_n^2 A \cos(\omega t - \phi) = \omega_n^2 A \cos(\omega_n t - \phi + \pi)$





- máxima amplitude
- velocidade zero
- máxima aceleração negativa

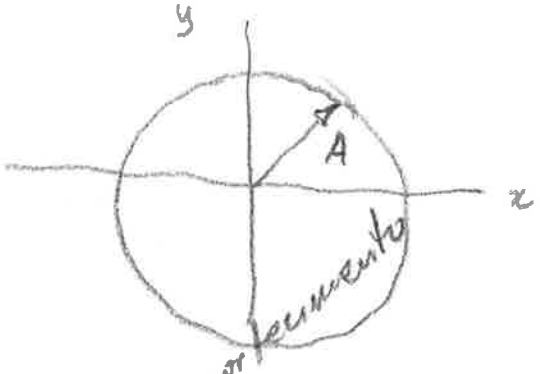
Plano de fase

$$x = A\cos(\omega_n t - \phi) \qquad \dot{x} = \omega_n A \sin(\omega_n t - \phi)$$

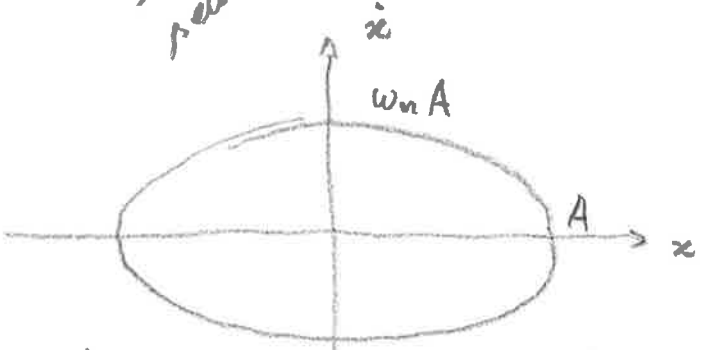
$$\cos(\omega_n t - \phi) = \frac{x}{A} \qquad \sin(\omega_n t - \phi) = \frac{\dot{x}}{\omega_n}\frac{1}{A} = y\frac{1}{A}$$

$$\sin^2(\omega_n t - \phi) + \cos^2(\omega_n t - \phi) = 1$$

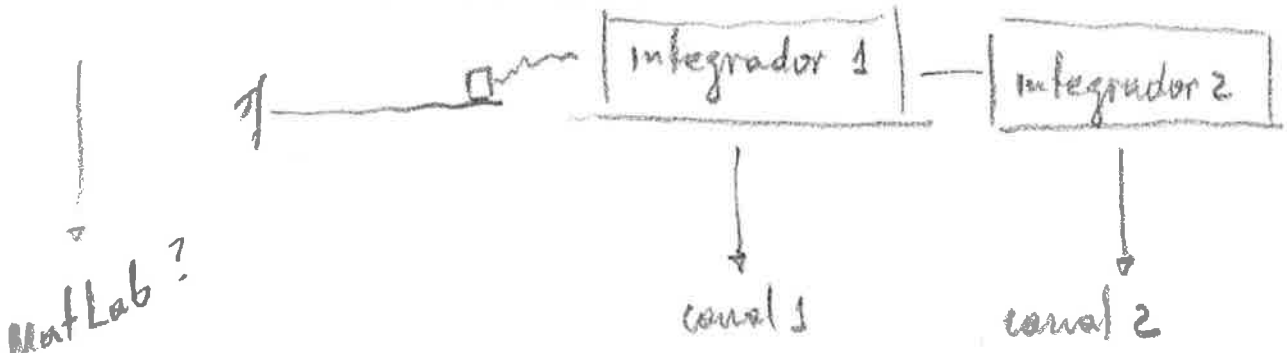
$$\frac{x^2}{A^2} + \frac{y^2}{A^2} = 1$$



$$\frac{x^2}{A^2} + \frac{\dot{x}^2}{\omega_n^2 A^2} = 1$$



Experimento c/ acelerômetro na ponta da régua



Exercício: Frequência natural de turbina eólica com duas pás

Modelo: pás como barra de comprimento L e massa m

$$I = \frac{1}{3} m L^2$$

momento de inércia das pás em torno do eixo

$$I_T = 2I = \frac{2}{3} m L^2$$

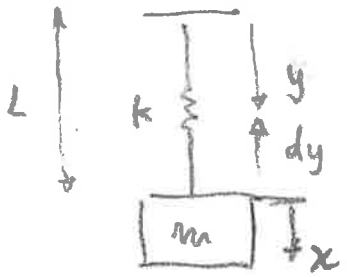
$J_{eixo} = \frac{\pi d^4}{32}$ comprimento do eixo L_2

Rigidez torsional $k = \frac{GJ}{L_2}$

$$\omega_n = \sqrt{\frac{k}{I}} = \sqrt{\frac{GJ}{L_2 \frac{2}{3} m L^2}} = \sqrt{\frac{3\pi d^4 G}{64 m L^2 L_2}}$$

Filme centro de percussão

Efeito da massa da mola na frequência natural



Energia cinética da mola

$$dT_s = \frac{1}{2}\left(\frac{m_s}{L} dy\right)\left(\frac{y}{L}\dot{x}\right)^2$$

$$\frac{1}{2} m \dot{x}^2 + \int_0^L \frac{1}{2} \frac{m_s}{L} dy \times \frac{y^2}{L^2} \dot{x}^2 = \frac{1}{2} k x^2$$

$$\frac{1}{2} m \dot{x}^2 + \frac{1}{2} \frac{m_s}{3} \dot{x}^2 = \frac{1}{2} k x^2$$

$x = X \cos \omega_n t$

$T_{max} = U_{max}$ → max implies that $\cos \omega_n t = 1$

$\sin \omega_n t = 1$

$$\omega_n = \sqrt{\frac{k}{m + \frac{m_s}{3}}}$$

Exercícios

Example 2.2 p. 111

Example 2.6 p. 123

Natural frequency of a spring

Volume: $\frac{\pi d^2}{4} \pi D n$

$$f_{res} \cong \frac{d}{9 D^2 n} \sqrt{\frac{G}{\rho}}$$



PVC

$E \cong 3\,GPa$

$G \cong 1\,GPa$

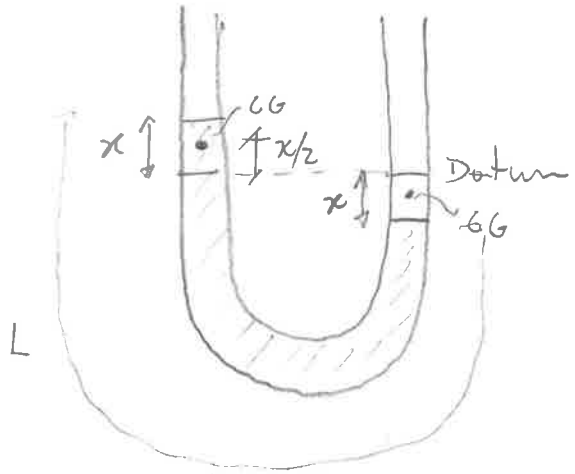
$G = \frac{E}{2(1+\nu)}$

$\nu = 0.4$

$\rho = 1400 \frac{kg}{m^3}$

Use example 2.6 para comentar sobre frequência natural de líquidos e faça o experimento com copo de vinho.

Calcule a frequência natural de um manômetro



Método da energia (Método de Rayleigh)

Energia potencial

$U = mgh$ $\quad$ γ = peso específico

$= \underbrace{A\gamma x}_{\text{peso}} \frac{x}{2} + A\gamma x \frac{x}{2} = A\gamma x^2$

$T = \frac{1}{2} m_{TOTAL} \dot{x}^2$

$T = \frac{1}{2} \frac{A\gamma L}{g} \dot{x}^2$

$x = X \cos \omega_n t$

$U_{max} = A\gamma X^2$ $\qquad$ $T_{max} = \frac{1}{2} \frac{A\gamma L}{g} (-\omega_n^2 X^2)$

$U_{max} = T_{max} \longrightarrow \omega_n = \sqrt{\frac{2g}{L}}$

Revisão aula anterior

- gravidade pode ser desconsiderada se variável x tem como referência posição inicial de equilíbrio estático

- equilíbrio sistema massa-mola



$$m\ddot{x} + kx = 0$$

$$x = Ce^{st}$$

$$s = \pm i\omega_n$$

$$\omega_n = \sqrt{\frac{k}{m}}$$

$$e^{j\theta} = \cos\theta + j\,\text{sen}\,\theta$$

$$x = A_1 \cos\omega_n t + A_2 \,\text{sen}\,\omega_n t$$

$t = 0 \rightarrow x = x_0 \;,\; \dot{x} = \dot{x}_0$

$$A = \sqrt{x_0^2 + \frac{\dot{x}_0^2}{\omega_n^2}} = A_0$$

$$x = x_0 \cos\omega_n t + \frac{\dot{x}_0}{\omega_n}\,\text{sen}\,\omega_n t$$

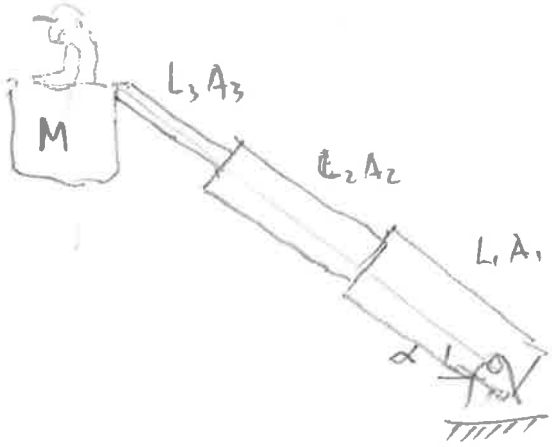
$$\phi = \tan^{-1}\frac{\dot{x}_0}{\omega_n x_0}$$

$$x = A\cos(\omega_n t - \phi)$$

$$\phi_0 = \tan^{-1}\frac{x_0\omega_n}{\dot{x}_0}$$

$$x = A_0\,\text{sen}(\omega_n t + \phi_0)$$

Example 2.2 p. 111



Frequência natural na direção vertical

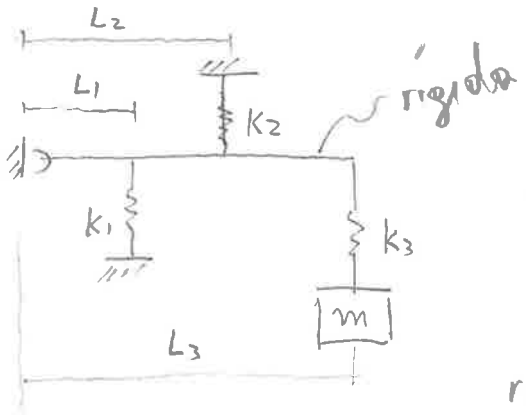
Rigidez em série

$$\frac{1}{k_e} = \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} \qquad k_e = \frac{k_1 k_2 + k_1 k_3 + k_2 k_3}{k_1 k_2 k_3}$$

$$k = k_e \operatorname{sen}\alpha$$

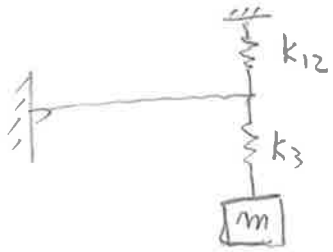
$$\omega_n = \sqrt{\frac{k}{M}}$$

Exercício 2.7 R&V



Calcular frequência natural

reduzir para



em série

$$\omega_n = \sqrt{\frac{k}{m}}$$

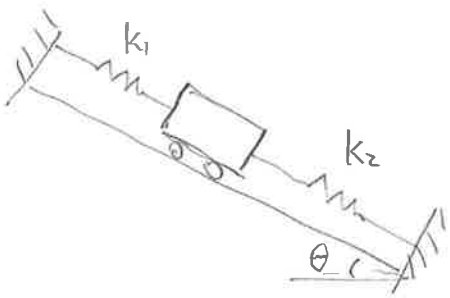
$$\frac{1}{2} k_{12} (\theta L_3)^2 = \frac{1}{2} k_1 (\theta L_1)^2 + \frac{1}{2} k_2 (\theta L_2)^2$$

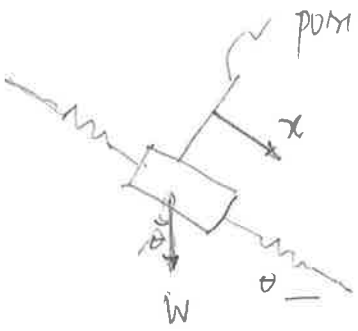
$$k_{12} = k_2 \left(\frac{L_2}{L_3}\right)^2 + k_1 \left(\frac{L_1}{L_3}\right)^2$$

$$\frac{1}{k} = \frac{1}{k_{12}} + \frac{1}{k_3} \qquad k = \frac{k_{12} k_3}{k_{12} + k_3} \qquad \rightarrow \omega_n = \sqrt{\frac{k}{m}}$$

Exercício 2.9

Calcular a frequência natural do sistema abaixo





$$m\ddot{x} = -k_1(x+\delta_{st}) - k_2(x+\delta_{st}) + W\sin\theta$$

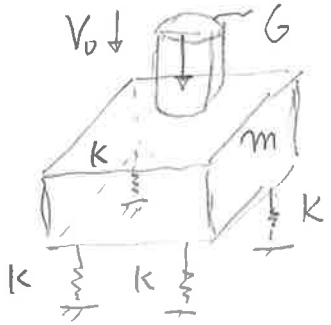
$$W\sin\theta = (k_1+k_2)\,\delta_{st}$$

$$m\ddot{x} = -k_1 x - k_2 x - (k_1+k_2)\,\delta_{st} + (k_1+k_2)\,\delta_{st}$$

$$m\ddot{x} + (k_1+k_2)\,x = 0$$

$$\omega_n = \sqrt{\frac{k_1+k_2}{m}} \quad \longleftarrow \text{não depende de } g$$

EXERCÍCIO 2.16



$k_{eq} = 4k$

Obtenha o movimento do sistema após impacto

$$x_0 = -\frac{mg}{4k}$$

Conservação de momentum

massa de impacto permanece em contato

$$(G+m)\,\dot{x}_0 = GV \qquad \dot{x}_0 = \frac{GV}{G+m}$$

$$\omega_n = \sqrt{\frac{4k}{G+m}}$$

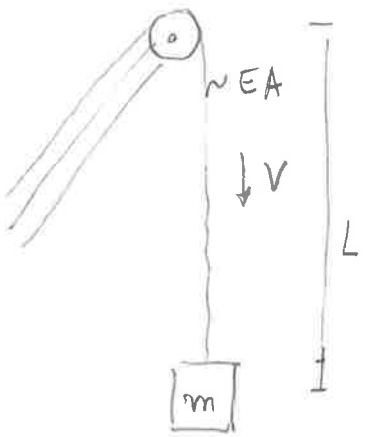
$$x(t) = A_0 \sin(\omega_n t + \phi_0)$$

$$A_0 = \sqrt{x_0^2 + \left(\frac{\dot{x}_0}{\omega_n}\right)^2} = \sqrt{\frac{m^2 g^2}{16k^2} + \frac{G^2 V^2}{(G+m)\,4k}}$$

$$\phi_0 = \tan^{-1}\frac{x_0 \omega_n}{\dot{x}_0} = \tan^{-1} - \frac{g\sqrt{G+m}}{V\sqrt{4k}}$$

Exercício 2.18

Qual o período e amplitude quando o cabo é travado



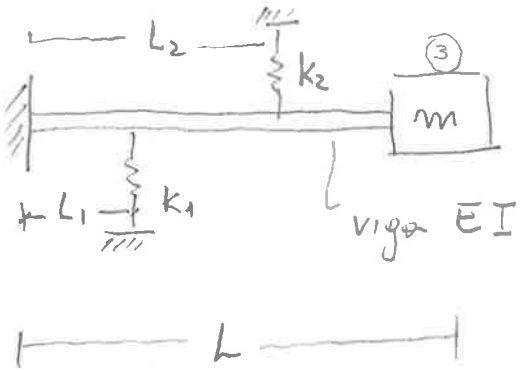
$$k = \frac{EA}{L} \qquad \omega_n = \sqrt{\frac{k}{m}}$$

$$x_0 = 0 \;,\; \dot{x}_0 = V$$

$$A = \frac{\dot{x}_0}{\omega_n} = \frac{V\sqrt{m}}{\sqrt{k}}$$

Exercícios para casa AULA 3

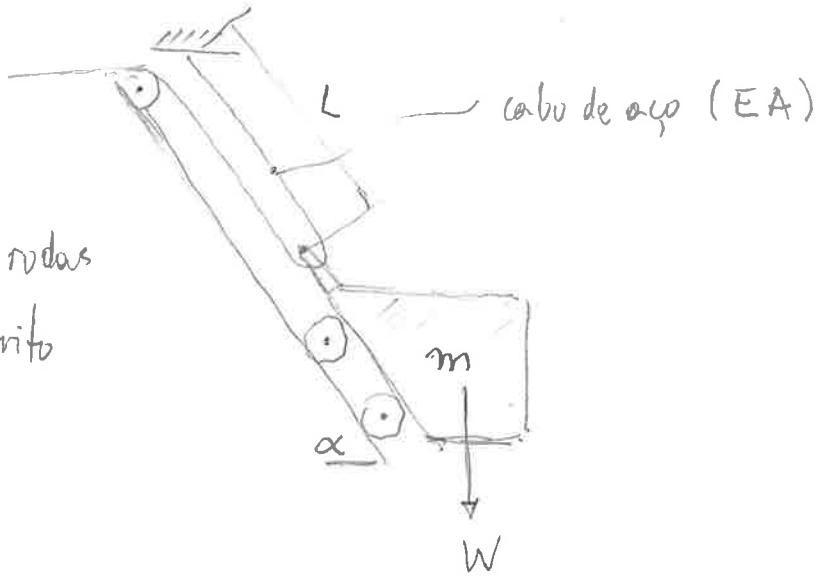
1. Calcule ω_n com e sem k_1 e k_2



Flecha da viga é dada por $w = \frac{Fx^2}{6EI}(3L-x)$

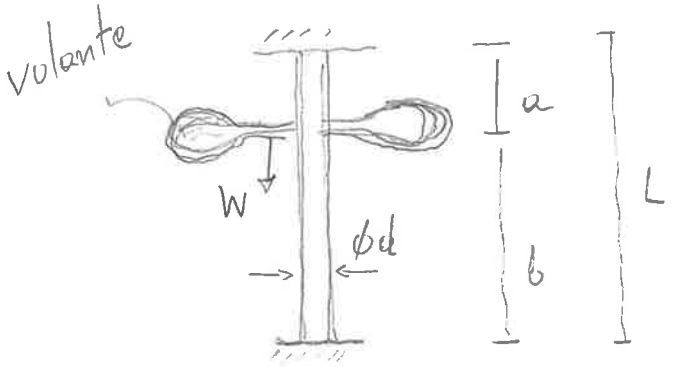
Dica: Calcule k_{eq} em ③

2. Calcule ω_n para o caso abaixo.



polias e rodas sem atrito

Exercício 2.35



Calcular as frequências naturais axial, transversal e torsional

W é suportado por a e b

$$\Delta_a = \frac{F}{K} = \frac{W_a}{EA/a} \qquad \Delta_b = \frac{W_b}{EA/b}$$

$$W = W_a + W_b$$

$$\Delta_a = \Delta_b$$

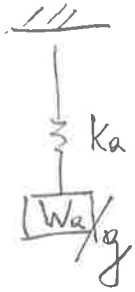
$$W_a\, a = W_b\, b$$

$$W_a\, a = (W - W_a)\, b$$

$$W_a = \frac{Wb}{a+b} = \frac{Wb}{L} = W\,\frac{(L-a)}{L}$$

$$\Delta_a = \frac{W_a}{EA/a} = \frac{W(L-a)\,a}{EAL}$$

modelo massa - mola



$$w_n = \sqrt{\frac{EA/a\ L}{\frac{W(L-a)}{g}}} = \sqrt{\frac{EA\,L\,g}{W(L-a)a}}$$

Vibração transversal

$$K = \frac{3EIL^3}{a^3b^3} \qquad w = \sqrt{\frac{3EIL^3g}{Wa^3(L-a)^3}} \qquad I = \frac{\pi d^4}{64}$$

tabelas



Vibração torsional

$$M_t = \frac{GJ\theta}{L_{eq}} = GJ\left(\frac{\theta}{a} + \frac{\theta}{b}\right)$$

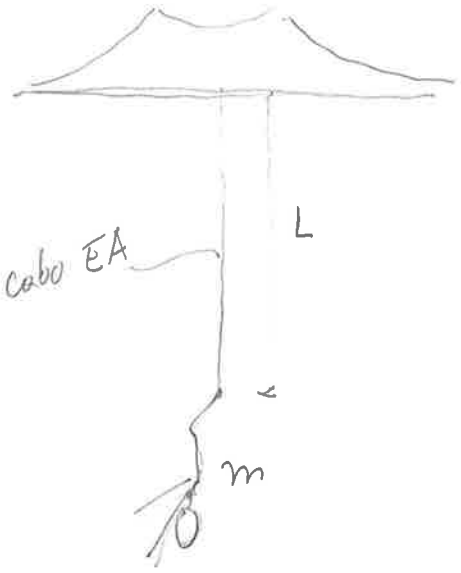
momento polar de inércia

$$K_t = \frac{M_t}{\theta} = GJ\left(\frac{1}{a} + \frac{1}{b}\right) \qquad J = \frac{\pi d^4}{32}$$

$$w_n = \sqrt{\frac{K_t}{J_0}} = \sqrt{\frac{GJ\left(\frac{1}{a} + \frac{1}{b}\right)}{J_0}}$$

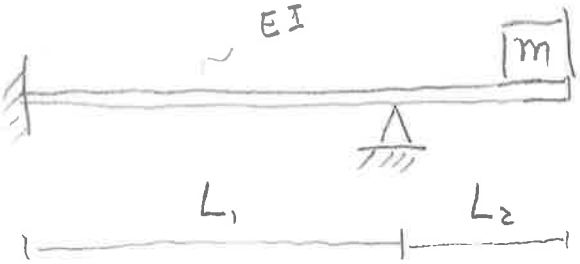
momento polar de massa

3.

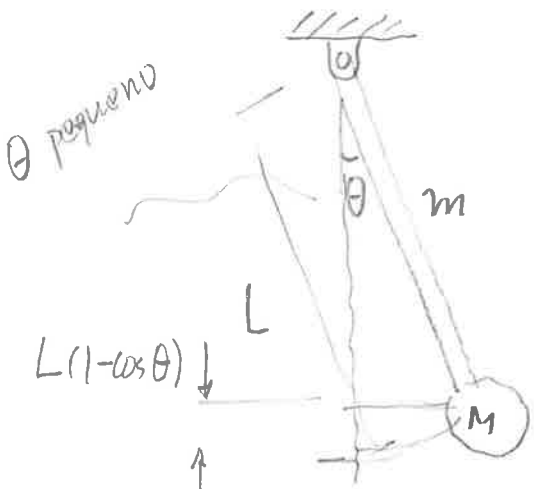


Determine o movimento de vibração após o salto.

4. Calcule ω_n para a viga abaixo, modelando-a com um grau de liberdade.



5. Calcule a frequência natural do pêndulo abaixo considerando também a massa da haste.



Dica: Calcule $T = T_M + T_m$

$U = U_M + U_m$

e faça $\frac{d}{dt}(T+U) = 0$

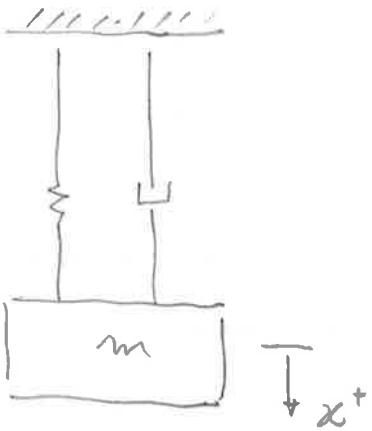
Resp. $\omega_n = \sqrt{\frac{(M+\frac{m}{2})\,g}{(M+\frac{m}{3})\,L}}$

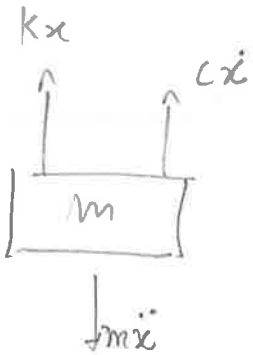
Free vibration with damping

$F = -c\dot{x}$

c → constante de amortecimento

$-c\dot{x}$ → se opõe à velocidade





$\Sigma F = ma \quad +\downarrow$

$-kx - c\dot{x} = m\ddot{x}$

$m\ddot{x} + c\dot{x} + kx = 0$

$x = Ce^{st} \qquad \dot{x} = Cse^{st} \qquad \ddot{x} = Cs^2e^{st}$

$mCs^2e^{st} + cCse^{st} + kCe^{st} = 0$

$ms^2 + cs + k = 0$

$$s_{1,2} = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m} = \frac{-c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}$$

$$x_1 = C_1 e^{s_1 t} \quad , \quad x_2 = C_2 e^{s_2 t}$$

$$x = x_1 + x_2$$

$$x(t) = C_1 e^{\left\{ \frac{-c}{2m} + \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} \right\} t} + C_2 e^{\left\{ \frac{-c}{2m} - \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} \right\} t}$$

Amortecimento crítico, c_c

Definido como $\sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} = 0$

$$c_c = 2m\sqrt{\frac{k}{m}} = 2\sqrt{km} = 2m\omega_n$$

Razão de amortecimento

$$\xi = \frac{c}{c_c}$$

$$\frac{c}{2m} = \frac{c}{c_c}\,\frac{c_c}{2m} = \xi\,\frac{2m\omega_n}{2m} = \xi\omega_n$$

$$\frac{-c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} = -\xi\omega_n \pm \sqrt{(\xi\omega_n)^2 - \omega_n^2}$$

$$= -\xi\omega_n \pm \sqrt{\xi^2 - 1}\;\omega_n$$

$$x(t) = C_1 e^{(-\xi + \sqrt{\xi^2 - 1})\omega_n t} + C_2 e^{(-\xi - \sqrt{\xi^2 - 1})\omega_n t}$$

Para $\xi = 0$ — vibração não amortecida

Para $\xi \neq 0$ { subamortecido
criticamente amortecido
sobreamortecido

Subamortecido $\xi < 1$

$$s_1 = (-\xi + i\sqrt{1-\xi^2})\omega_n \qquad s_2 = (-\xi - i\sqrt{1-\xi^2})\omega_n$$

$$x(t) = C_1 e^{(-\xi + i\sqrt{1-\xi^2})\omega_n t} + C_2 e^{(-\xi - i\sqrt{1-\xi^2})\omega_n t}$$

$$x(t) = e^{-\xi\omega_n t} \left\{ C_1 e^{i\sqrt{1-\xi^2}\,\omega_n t} + C_2 e^{-i\sqrt{1-\xi^2}\,\omega_n t} \right\}$$

$$x(t) = X e^{-\xi\omega_n t} \sin\left(\sqrt{1-\xi^2}\,\omega_n t + \phi\right) = X_0 e^{-\xi\omega_n t} \cos\left(\sqrt{1-\xi^2}\,\omega_n t - \phi_0\right)$$

$$x(t) = e^{-\xi\omega_n t} \left\{ (C_1 + C_2)\cos\sqrt{1-\xi^2}\,\omega_n t + i(C_1 - C_2)\sin\sqrt{1-\xi^2}\,\omega_n t \right\}$$

$$x(t) = e^{-\xi\omega_n t} \left\{ C_1' \cos\sqrt{1-\xi^2}\,\omega_n t + C_2' \sin\sqrt{1-\xi^2}\,\omega_n t \right\}$$

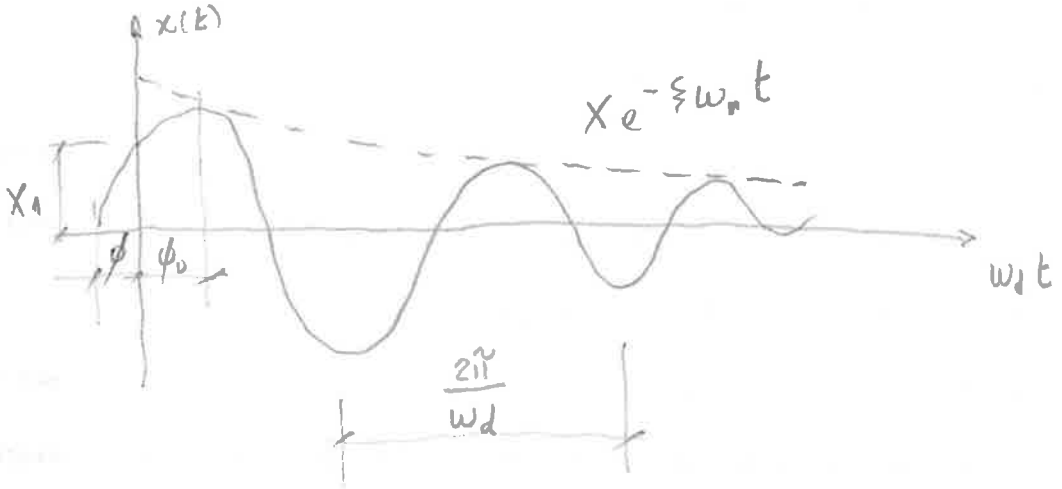
$$x(0) = x_0 \quad , \quad \dot{x}(0) = \dot{x}_0$$

$$C_1' = x_0 \qquad C_2' = \frac{\dot{x}_0 + \xi\omega_0 x_0}{\sqrt{1-\xi^2}\,\omega_n}$$

$$x(t) = e^{-\xi\omega_n t}\left\{ x_0 \cos\sqrt{1-\xi^2}\,\omega_n t + \frac{\dot{x}_0 + \xi\omega_n x_0}{\sqrt{1-\xi^2}\,\omega_n} \sin\sqrt{1-\xi^2}\,\omega_n t \right\}$$

$$X = X_0 = \sqrt{C_1'^2 + C_2'^2} \qquad \phi = \tan^{-1} C_1'/C_2' \qquad \phi_0 = \tan^{-1} -C_2'/C_1'$$

$\sqrt{1-\xi^2}\,\omega_n = \omega_d$ → frequência natural amortecida



Criticamente amortecido, $\xi = 1$, $C = C_c$, $\frac{C}{2m} = \sqrt{\frac{k}{m}} = \omega_n$

$$s_1 = s_2 = \frac{-C_c}{2m} = -\omega_n$$

$$x(t) = e^{-\xi\omega_n t}\left\{ x_0 \cos\omega_d t + \frac{\dot{x}_0 + \xi\omega_n x_0}{\omega_d} \sin\omega_d t \right\}$$

$\omega_d = \sqrt{1-\xi^2}\,\omega_n$, $\xi = 1$, $\omega_d \to 0$

$$x(t) = e^{-\omega_n t}\left\{ x_0 + (\dot{x}_0 + \omega_n x_0)\left(\frac{\sin\omega_d t}{\omega_d}\right) \right\}$$

$$x(t) = e^{-\omega_n t}\left\{ x_0 + (\dot{x}_0 + \omega_n x_0)\, t \right\}$$

$\lim_{\omega_d \to 0} \frac{\sin\omega_d t}{\omega_d} = \frac{\omega_d t}{\omega_d} = t$

Sobreamortecido

$\xi > 1 \qquad c > c_c \qquad \sqrt{\xi^2 - 1} > 0$

$$s_{1,2} = -\xi \pm \sqrt{\xi^2 - 1}$$

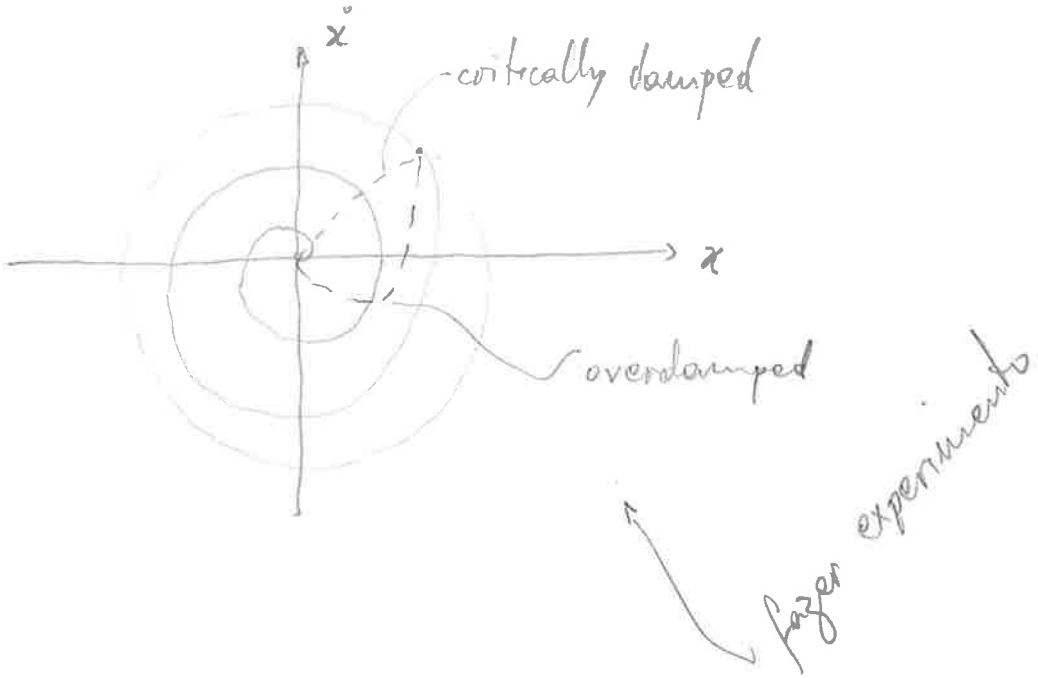
$$x(t) = C_1 e^{(-\xi + \sqrt{\xi^2-1})\omega_n t} + C_2 e^{(-\xi - \sqrt{\xi^2-1})\omega_n t}$$

$$C_1 = \frac{x_0 \omega_n (\xi + \sqrt{\xi^2 - 1}) + \dot{x}_0}{2\omega_n \sqrt{\xi^2 - 1}}$$

$$C_2 = \frac{-x_0 \omega_n (\xi - \sqrt{\xi^2 - 1}) - \dot{x}_0}{2\omega_n \sqrt{\xi^2 - 1}}$$

Movimento aperiódico

Espaço de estado



Decremento logarítmico

Indica a taxa de decaimento da amplitude

$$x_1 = X_0 e^{-\xi \omega_n t_1} \cos(\omega_d t_1 - \phi_0)$$

$$x_2 = X_0 e^{-\xi \omega_n t_2} \cos(\omega_d t_2 - \phi_0)$$

$$\tau_d = \frac{2\pi}{\omega_d}$$

$$x_2 = X_0 e^{-\xi \omega_n (t_1 + \tau_d)} \cos(\omega_d t_1 + 2\pi - \phi_0)$$

$$\frac{x_1}{x_2} = \frac{e^{-\xi \omega_n t_1}}{e^{-\xi \omega_n (t_1 + \tau)}} = e^{\xi \omega_n \tau_d}$$

$$\delta = \ln \frac{x_1}{x_2} = \xi \omega_n \tau_d = \xi \omega_n \frac{2\pi}{\sqrt{1-\xi^2}\,\omega_n}$$

$$\delta = \frac{2\pi\xi}{\sqrt{1-\xi^2}} \simeq 2\pi\xi \quad \text{for } \xi << 1$$

$$\delta = \frac{1}{m} \ln \frac{x_1}{x_{m+1}}$$

Exercício para entregar via e-mail dia 3/10

Use uma massa e mola quaisquer e filme sua vibração. A partir do Tracker ou ImageJ obtenha
- freq natural amortecida, fração de amort, decrem logarit, freq natural

Excitação harmônica

Vibração forçada → Energia é aplicada ao sistema

↓

força ou deslocamento prescritos

Excitação harmônica → resposta harmônica

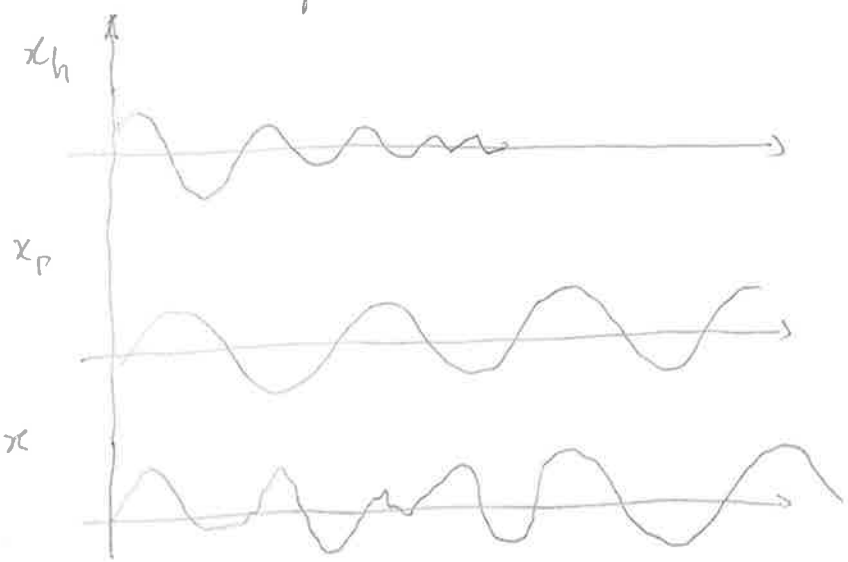
Excitação não periódica → duração curta ou longa

Excitação não periódica repentina → resposta transiente

Quando a frequência de excitação coincide com a frequência natural do sistema → ressonância

$$m\ddot{x} + c\dot{x} + kx = F(t)$$

$$x = x_h + x_p$$



Solução particular → Steady state response

Sistema sem amortecimento sob força harmônica

$$m\ddot{x} + kx = F_0 \cos \omega t$$

Considera-se que $x_p = X \cos \omega t$ $\quad$ X = amplitude máxima de x_p

$$-\omega^2 m X \cos \omega t + kX \cos \omega t = F_0 \cos \omega t$$

$$X = \frac{F_0}{k - \omega^2 m} = \text{amplitude máxima}$$

$$x_h = C_1 \cos \omega_n t + C_2 \sin \omega_n t \qquad \omega_n = \sqrt{k/m}$$

$$x = x_h + x_p$$

$$x = C_1 \cos \omega_n t + C_2 \sin \omega_n t + \frac{F_0}{k - m\omega^2} \cos \omega t$$

$$x(0) = x_0 \;, \quad \dot{x}(0) = \dot{x}_0$$

$$C_1 = x_0 - \frac{F_0}{k - m\omega^2} \qquad C_2 = \frac{\dot{x}_0}{\omega_n}$$

$$x(t) = \left(x_0 - \frac{F_0}{k - m\omega^2} \right) \cos \omega_n t + \frac{\dot{x}_0}{\omega_n} \sin \omega_n t + \frac{F_0}{k - m\omega^2} \cos \omega t$$

amplitude máxima

$$X = \frac{F_0}{k - m\omega^2}$$

$$\frac{X_0}{F_0} = \frac{1}{k - m\omega^2} = \frac{1/k}{1 - m\omega^2/k}$$

$$\frac{X_0}{F_0/k} = \frac{1}{1 - \frac{\omega^2}{\omega_n^2}}$$

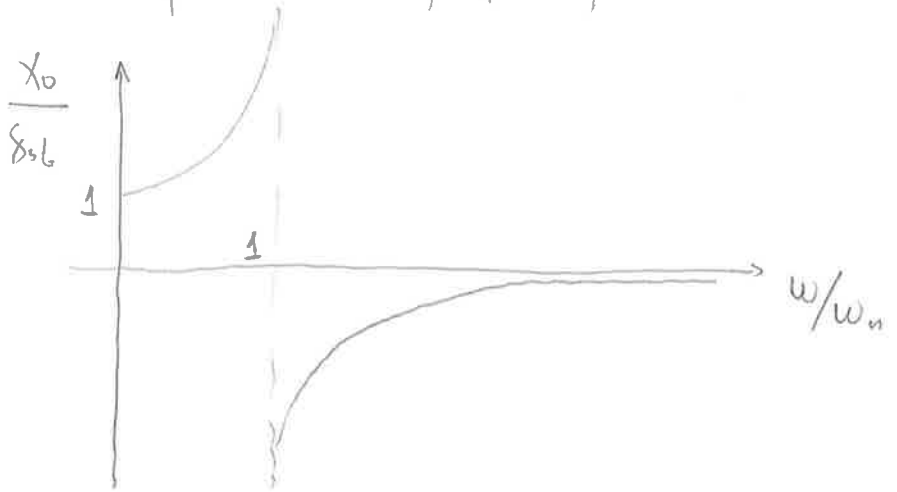
$F_0/k = \delta_{st}$

↓

static deflection

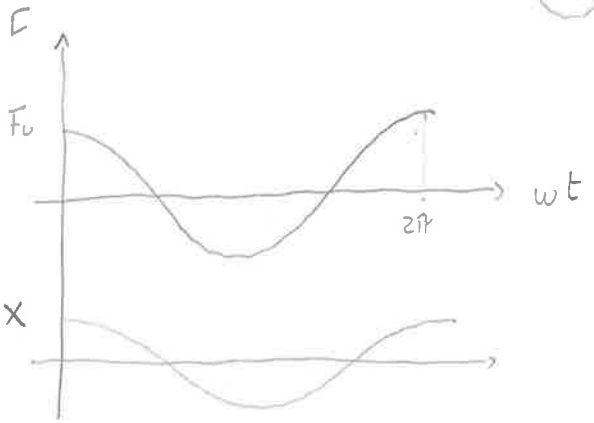
$$\frac{X_0}{\delta_{st}} = \frac{1}{1 - \left(\frac{\omega}{\omega_n} \right)^2}$$

↳ fator de amplificação



→ $0 < \omega/\omega_n < 1$

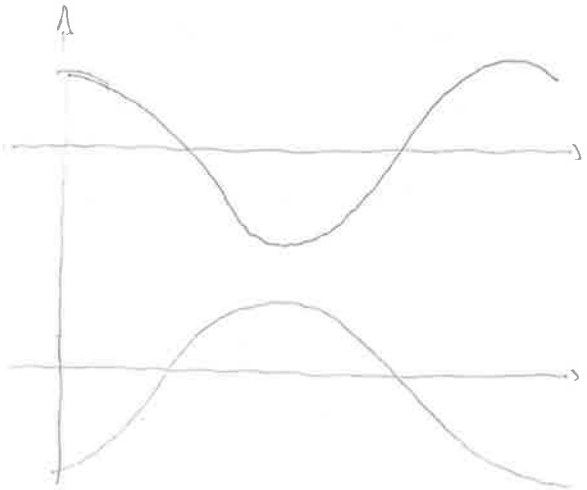
$x_p = X \cos \omega t$



→ $\frac{\omega}{\omega_n} > 1 \qquad x_p = -X \cos \omega t$

resposta fora de fase

Para altas frequências de excitação, a resposta tende a zero



$$X = \frac{\delta_s}{\left(\frac{\omega}{\omega_n}\right)^2 - 1} \simeq 0 \quad \text{para} \quad \omega >> \omega_n$$

→ $\frac{\omega}{\omega_n} = 1$ ressonância, frequência forçada igual a natural num sistema sem amortecimento

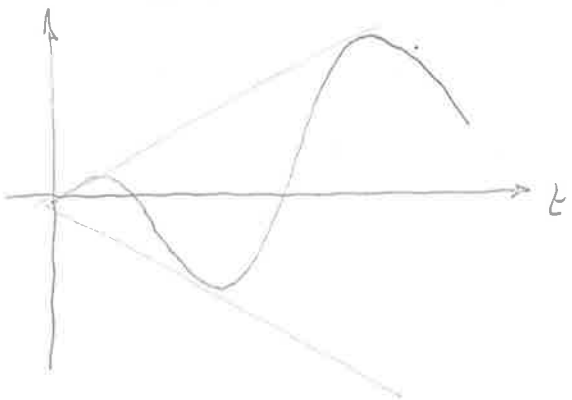
$$x(t) = \left(x_0 - \frac{F_0}{k - m\omega^2} \right) \cos \omega_n t + \frac{\dot{x}_0}{\omega_n} \sin \omega_n t + \frac{F_0}{k - m\omega^2} \cos \omega t$$

$$x(t) = x_0 \cos \omega_n t + \frac{\dot{x}_0}{\omega_n} \sin \omega_n t + \delta_{st} \frac{\cos \omega t - \cos \omega_n t}{1 - \frac{\omega^2}{\omega_n^2}}$$

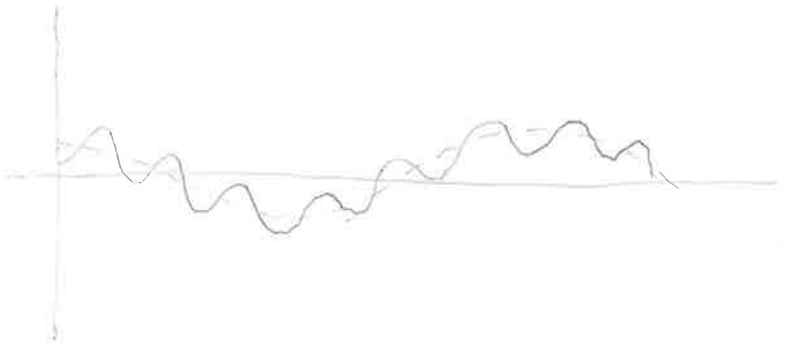
$$\lim_{\omega \to \omega_n} \frac{\cos \omega t - \cos \omega_n t}{1 - \frac{\omega^2}{\omega_n^2}} = \frac{-t \sin \omega t}{-\frac{2\omega}{\omega_n^2}} = \frac{t \omega_n \sin \omega_n t}{2}$$

$\frac{d}{d\omega}$

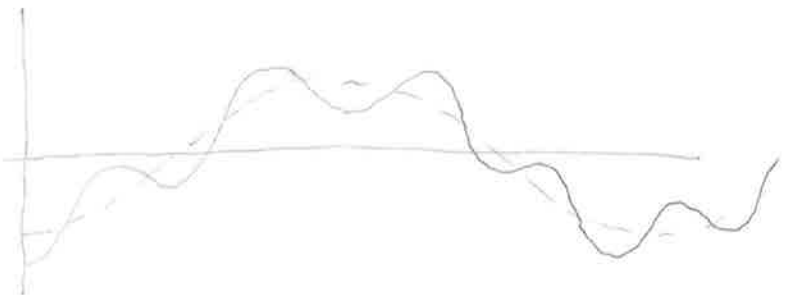
$$x(t) = x_0 \cos \omega_n t + \frac{\dot{x}_0}{\omega_n} \sin \omega_n t + \underbrace{\frac{\delta_{st} \omega_n t}{2}}_{\text{aumenta com o tempo}} \sin \omega_n t$$



$\frac{\omega}{\omega_n} < 1 \quad x(t) = A\cos(\omega_n t - \phi) + \frac{\delta_{st}}{1 - \frac{\omega^2}{\omega_n^2}} \cos\omega t$



$\frac{\omega}{\omega_n} > 1 \quad x(t) = A\cos(\omega_n t - \phi) - \frac{\delta_{st}}{1 - \frac{\omega^2}{\omega_n^2}} \cos\omega t$



$$A = \sqrt{x_0^2 + \left(\frac{\dot{x}_0}{\omega_n}\right)^2}$$

$$\phi = \tan^{-1} \frac{\dot{x}_0}{x_0 \omega_n}$$

Demo; faça de vinho!

Exemplo (3.1)

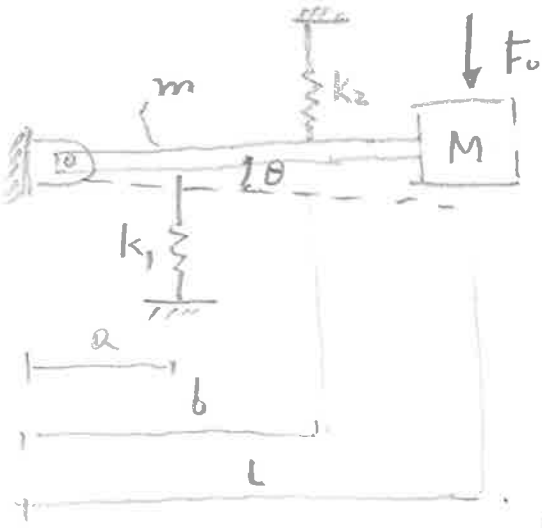


$$k = \frac{48EI}{L^3}$$

$$X = \frac{F_0}{k - m\omega^2}$$

EXEMPLOS

Exercício 3.15 → Equações do movimento e a resposta steady state



Force x dist

$k_1 a \theta \times a$

$$(J_0 + ML^2)\ddot{\theta}^2 + (k_1 a^2 + k_2 b^2)\theta = FL = F_0 L \sin \omega t$$

$$\theta = \theta_0 \sin \omega t$$

$$\dot{\theta} = \omega \theta_0 \cos \omega t \qquad \ddot{\theta} = -\omega^2 \theta \sin \omega t$$

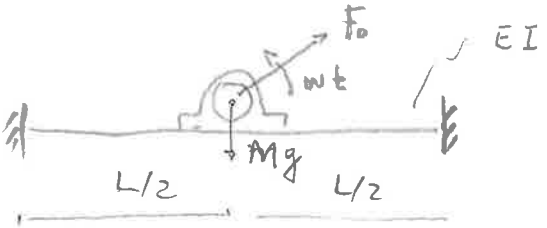
$$-(J_0 + ML^2)\omega^2 \theta_0 \sin \omega t + (k_1 a^2 + K_2 b^2)\theta_0 \sin \omega t = F_0 L \sin \omega t$$

$$\theta_0 = \frac{F_0 L}{k_1 a^2 + k_2 b^2 - (J_0 + ML^2)\omega^2}$$

$$J_0 = \frac{mL^2}{12} + m\left(\frac{L}{2}\right)^2 = \frac{1}{3} mL^2$$

Problem 3.45

Qual a amplitude da viga?



$$K = \frac{192\, EI}{L^3}$$

$$\omega_n = \sqrt{\frac{k}{M}}$$

$$r = \frac{\omega}{\omega_n}$$

$$\frac{X}{\delta_{st}} = \frac{1}{\sqrt{(1-r^2)^2 + (2\xi r)^2}}$$

$\xi \to 0$

$$\frac{X}{\delta_{st}} = \frac{1}{1-r^2}$$

$$X = \frac{\delta_{st}}{1-r^2} = \frac{F_0/k}{1-r^2}$$

Se a massa da viga for considerada: $M_{eq} = M + 0.2357\, m$

Resposta de um sistema amortecido à força harmônica

$$m\ddot{x} + c\dot{x} + kx = F_0 \cos \omega t$$

$$x_p = X \cos(\omega t - \phi)$$

$$\dot{x} = -\omega X \sin(\omega t - \phi)$$

$$\ddot{x} = -\omega^2 X \cos(\omega t - \phi)$$

$$-m\omega^2 X \cos(\omega t - \phi) - c\omega X \sin(\omega t - \phi)$$

$$+ kX \cos(\omega t - \phi) = F_0 \cos \omega t$$

$$X\left[(k - m\omega^2)\cos(\omega t - \phi) - c\omega \sin(\omega t - \phi)\right] = F_0 \cos \omega t$$

$$\cos(\omega t - \phi) = \cos \omega t \cos\phi + \sin \omega t \sin\phi$$

$$\sin(\omega t - \phi) = \sin \omega t \cos\phi - \cos \omega t \sin\phi$$

$$X\left[(k - m\omega^2)(\cos \omega t \cos\phi + \sin \omega t \sin\phi)\right.$$

$$\left. - c\omega(\sin \omega t \cos\phi - \cos \omega t \sin\phi)\right] = F_0 \cos \omega t$$

$$X\left\{\left[(k - m\omega^2)\cos\phi + c\omega \sin\phi\right]\cos \omega t + \right.$$

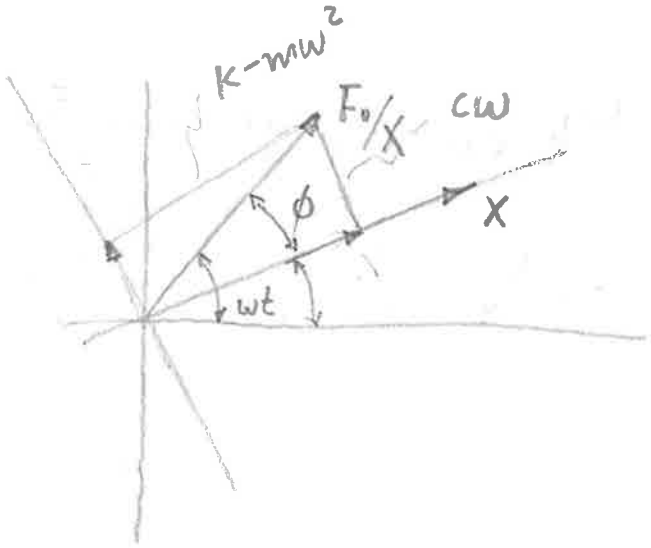
$$\left. + \left[(k - m\omega^2)\sin\phi - c\omega \cos\phi\right]\sin \omega t\right\} = F_0 \cos \omega t$$

$$X\left[(k-m\omega^2)\cos\phi + c\omega\sin\phi\right] = F_0$$

$$X\left[(k-m\omega^2)\sin\phi - c\omega\cos\phi\right] = 0$$

$$(k-m\omega^2)\tan\phi - c\omega = 0$$

$$\tan\phi = \frac{c\omega}{k-m\omega^2}$$



$$\frac{F_0}{X}^2 = (c\omega)^2 + (k-m\omega^2)^2$$

$$\frac{F_0}{X} = \sqrt{(c\omega)^2 + (k-m\omega^2)^2}$$

$$X = \frac{F_0}{\sqrt{(c\omega)^2 + (k-m\omega^2)^2}}$$

? cara lain untuk mendapatkan $\frac{X}{F_0} = \frac{1}{\sqrt{(c\omega)^2+(k-m\omega^2)^2}}$

$$X = \frac{F_0}{\sqrt{(c\omega)^2 + (k - m\omega^2)^2}}$$

$$X = \frac{F_0/k}{\sqrt{\left(\frac{c\omega}{k}\right)^2 + \left[1 - \frac{m}{k}\omega^2\right]^2}}$$

$$\omega_n^2 = \frac{k}{m} \qquad \xi = \frac{c}{c_c} = \frac{c}{2m\omega_n} = \frac{c}{2\sqrt{mk}} \Rightarrow \frac{c}{m} = 2\xi\omega_n$$

$$\delta_{st} = F_0/k \qquad r = \frac{\omega}{\omega_n}$$

$$\frac{c\omega}{k} = \frac{2m\xi\omega_n r\omega_n}{k} = 2\xi r$$

$$1 - m\frac{\omega^2}{k} = 1 - mr^2\frac{\omega_n^2}{k} = 1 - r^2\frac{mk}{mk} = 1 - r^2$$

$$X = \frac{\delta_{st}}{\sqrt{(1-r^2)^2 + (2\xi r)^2}}$$

$$\phi = \tan^{-1}\frac{2\xi r}{1 - r^2}$$

$$M = \frac{X}{\delta_{st}} = \frac{1}{\sqrt{(1-r^2)^2 + (2\xi r)^2}}$$

↳ fator de amplificação → Mostrar figura 3.11

Máxima amplificação

$$\frac{dM}{dr} = 0 \qquad -\frac{1}{2}\left[(1-r^2)^2 + (2\xi r)^2\right]^{-3/2}\left[2(1-r^2).-2r + 4\xi r\, 2\xi\right] = 0$$

$$-4r(1-r^2) + 8r\xi^2 = 0$$

$$-(1-r^2) = -2\xi^2$$

$$r^2 = 1 - 2\xi^2$$

$$r = \sqrt{1-2\xi^2}$$

$$M_{max} = \frac{1}{\sqrt{(2\xi^2)^2 + 4\xi^2(1-2\xi^2)}} = \frac{1}{\sqrt{4\xi^4 + 4\xi^2 - 8\xi^4}}$$

$$M_{max} = \frac{1}{\sqrt{4\xi^2 - 4\xi^4}} = \frac{1}{2\xi\sqrt{1-\xi^2}}$$

↓

para medir ξ de M_{max}

$$M_{\substack{\omega=\omega_n \\ r=1}} = \frac{1}{2\xi}$$

Para $\xi = 0$ excitação e resposta em fase $r < 1$

fora de fase $r > 1$

Para $\xi > 0$ $r > 1$ $90° < \phi < 180°$, resposta à frente da excitação

Para $r = 1$, $\phi = 90°$ para qualquer ξ

Resposta total

$$x = x_h + x_p$$

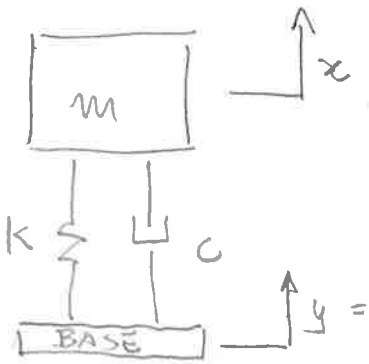
$$x(t) = X_0 e^{-\xi \omega_n t} \cos(\omega_d t - \phi_0) + X \cos(\omega t - \phi)$$

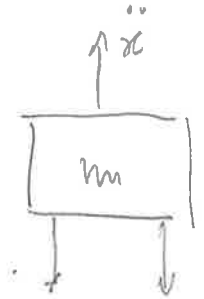
$$\omega_d = \sqrt{1 - \xi^2}\, \omega_n$$

$$r = \frac{\omega}{\omega_n} \qquad X_0, \phi_0 \rightarrow \text{from initial conditions}$$

Excitação pela base

contexto de acelerômetros



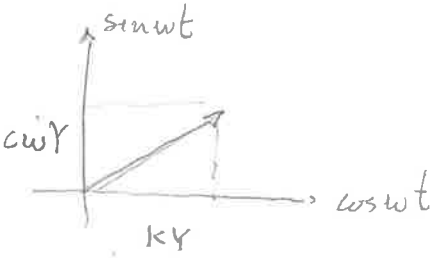


$$m\ddot{x} + c\dot{x} + kx = ky + c\dot{y} = kY\sin\omega t + c\omega Y\cos\omega t$$

$$= A\sin(\omega t - \alpha)$$

$$A = Y\sqrt{k^2 + (c\omega)^2}$$

$$\alpha = \tan^{-1}\frac{-c\omega}{k}$$



equivalente a uma força harmônica

$$A \equiv F_0$$

$$x_p(t) = \frac{F_0}{\sqrt{(k-m\omega^2)^2 + (c\omega)^2}}\sin(\omega t - \phi)$$

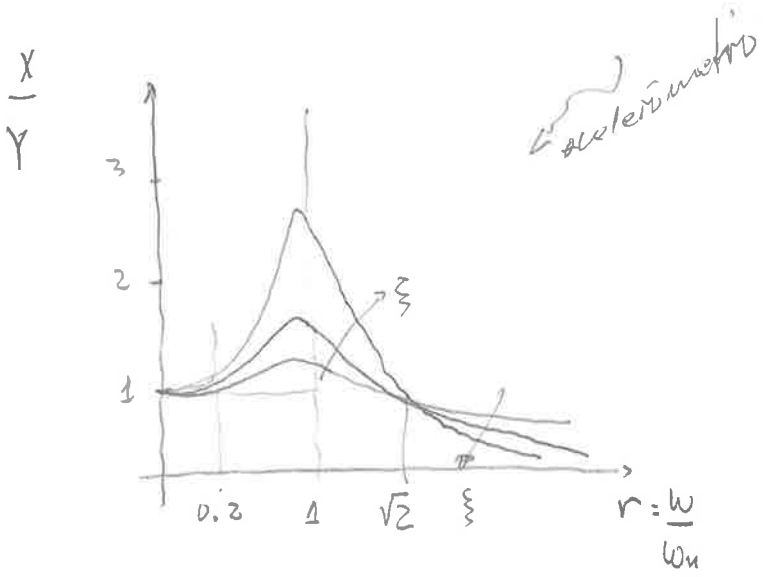
$$x_p(t) = \frac{Y\sqrt{k^2 + (c\omega)^2}}{\sqrt{(k-m\omega^2)^2 + (c\omega)^2}}\sin(\omega t - \phi_1 - \alpha)$$

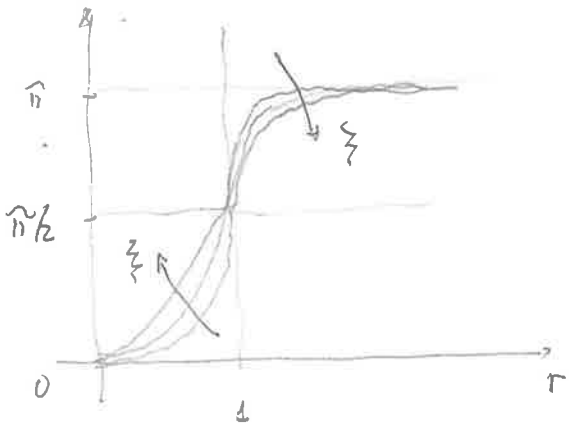
$$\phi_1 = \tan^{-1}\frac{c\omega}{k - m\omega^2}$$

$$x_p(t) = X \sin(\omega t - \phi)$$

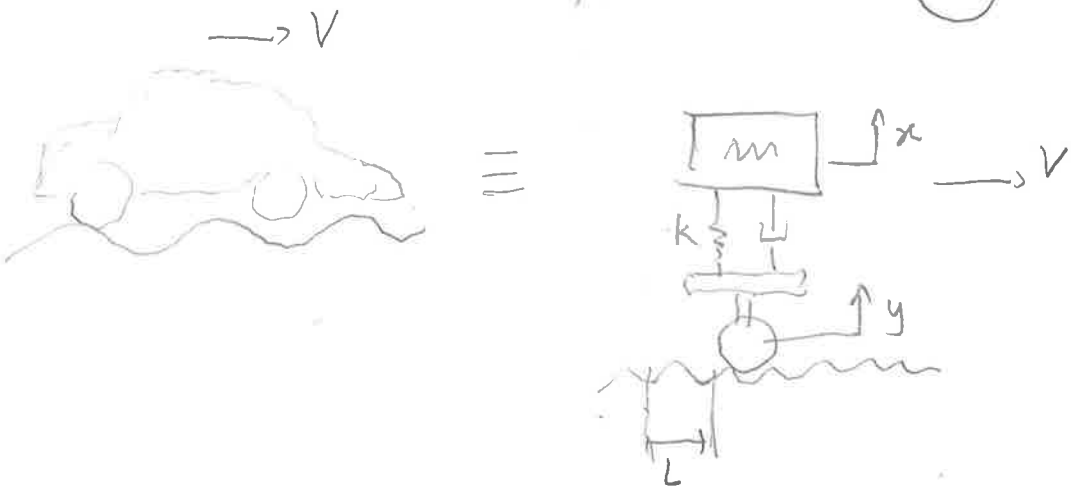
$$\frac{X}{Y} = \sqrt{\frac{k^2 + (c\omega)^2}{(k - m\omega^2)^2 + (c\omega)^2}} = \sqrt{\frac{1 + (2\xi r)^2}{(1 - r^2)^2 + (2\xi r)^2}}$$

$$\phi = \tan^{-1} \frac{mc\omega^3}{kc(k - m\omega^2) + (\omega c)^2} = \tan^{-1} \cdot \frac{2\xi r^3}{1 + (4\xi^2 - 1) r^2}$$





Exemplo



Frequência de excitação

$$\omega = 2\pi f = 2\pi \frac{V}{L}$$

$$\omega_n = \sqrt{\frac{k}{m}}$$

$$r = \frac{\omega}{\omega_n} = \frac{2\pi V}{L\sqrt{\frac{k}{m}}}$$

$$\frac{X}{Y} = \sqrt{\frac{1+(2\xi r)^2}{(1-r^2)+(2\xi r)^2}}$$

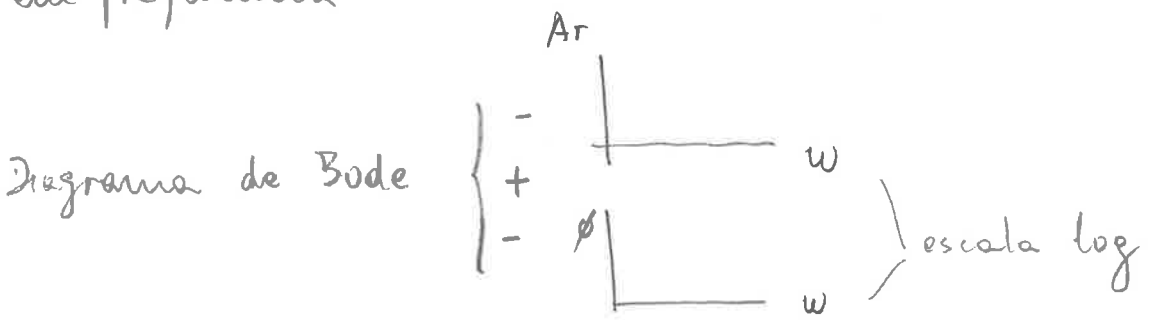
Quanto maior a velocidade do carro, V, maior r.

Quanto maior r, menor a amplitude X.

Diagrama de Bode

- Função de transferência de sistemas complexos pode ser obtida experimentalmente por testes de resposta em frequência

- A resposta em frequência pode ser bem caracterizada pelo diagrama de Bode (ou gráficos log da resposta em frequência



uma escala log $1 \times 20 \log_{10}(\text{Ganho}(w)) = 1 \text{ decibel}$

escalas log → multiplicação de magnitudes converte p/ soma
→ pode usar aproximações assintóticas
→ respostas (ganho) em baixas e altas frequências num só gráfico
→ um número maior que unidade tem um valor positivo em decibel
→ para um ganho K fixo, a curva log é paralela ao eixo da frequência

$\log i = i\frac{\pi}{2}$

$s = i\omega$

Diagrama de Bode de fatores integrais e derivativos

$$20 \log \left| \frac{1}{i\omega} \right| = 20[\log 1 - \log i\omega] = -20 \log i\omega$$

Plotando numa escala log10 para a frequência tem-se

$$(-20 \log 10\omega)\,dB = -20\,[\log 10 + \log \omega]\,dB$$

$$= (-20 - 20 \log \omega)\,dB$$

↳ slope of -20 dB/década

Magnitude de $i\omega$ (em dB)

$$20 \log |i\omega| = 20 \log \omega \ dB$$

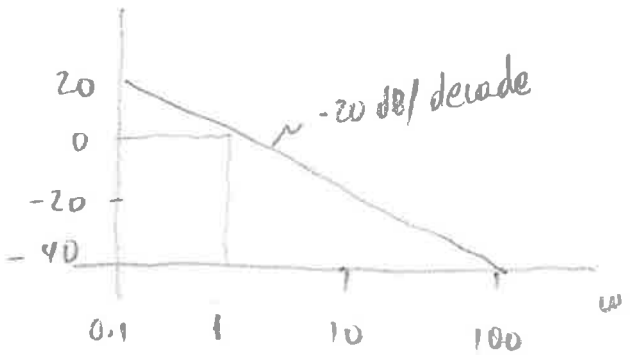
$z = a + bi$

$a = 0, \ b = 1$

$z = i$

$|z| = |i| = \sqrt{a^2 + b^2}$

$|i| = 1$



Ângulo de fase de $i\omega = \pi/2$

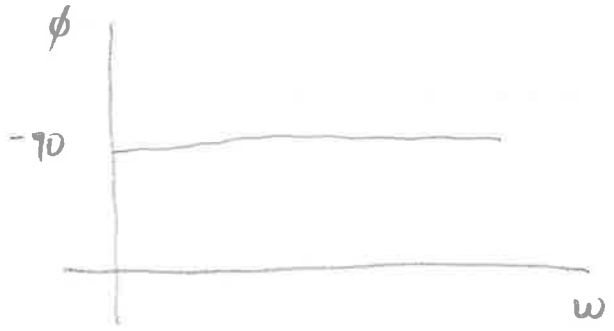


Diagrama de Bode para um sistema de 1ª ordem

$$G(i\omega) = \frac{1}{\tau\omega i + 1}$$

$$20 \log \left| \frac{1}{\tau\omega + 1} \right| = 20 \left(\log|1| - \log|\tau\omega + 1| \right)$$

$$= -20 \log|\tau\omega + 1|$$

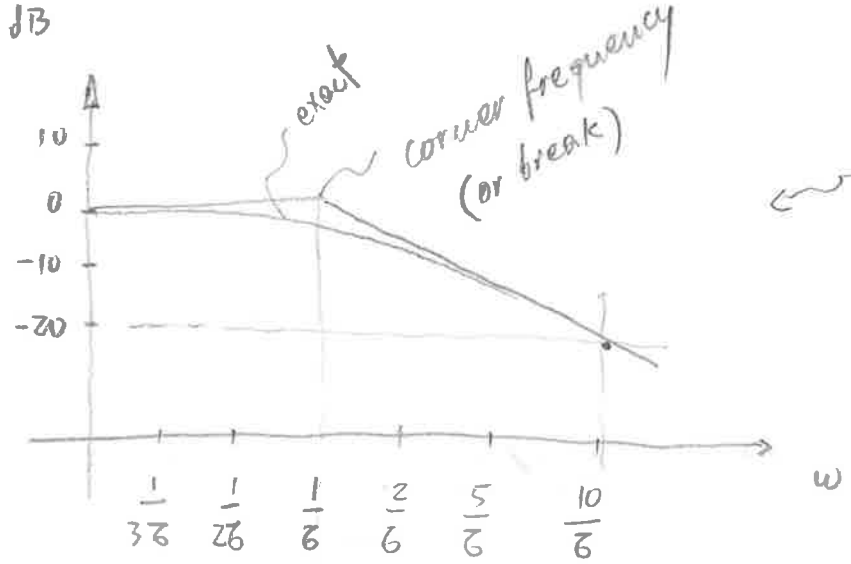
$$= -20 \log \sqrt{\tau^2\omega^2 + 1} \quad dB$$

Para $\omega \ll \frac{1}{\tau} \rightarrow -20 \log 1 = 0$ dB

Para $\omega \gg \frac{1}{\tau} \rightarrow -20 \log \omega\tau$ dB

Para $\omega = \frac{1}{\tau} \rightarrow -20 \log \sqrt{2}$ dB $\simeq 0$

Para $\omega = \frac{10}{\tau} \rightarrow -20 \log \sqrt{101}$ dB $\cong -20$ dB



erro

$-20 \log \sqrt{1+1} +$

$20 \log 1 =$

$-10 \log 2 = -3.03 dB$



Para $\omega = \frac{100}{6}$ $\rightarrow -20 \log (10^2 + 1) \approx -40$ dB

ω [10, 100] → 1 década

dB [-20, -40] → -20 dB

$-20 \log \omega T$ dB decai de 20 dB para década de ω

O ângulo de fase para o fator de ganho $\frac{1}{\tau w i + 1}$ é

$$\phi = -\tan^{-1}(w\tau)$$

$$w = 0 \longrightarrow \phi = 0$$

corner frequency ← $w = \tau \rightarrow \phi = -\tan^{-1}(1) = -45°$

$$w = \infty \rightarrow \phi = -\pi/2$$

Diagrama de Bode para sistema de 2° ordem

$$G(s) = \frac{w_n^2}{s^2 + 2\xi w_n s + w_n^2}$$

$$G(iw) = \frac{w_n^2}{(iw)^2 + 2\xi w_n iw + w_n^2}$$

$$G(iw) = \frac{1}{\left(i\frac{w}{w_n}\right)^2 + 2\xi i\frac{w}{w_n} + 1}$$

$r = \frac{w}{w_n}$

$$|G(iw)| = \left|\frac{1}{(2\xi r)i + (1 - r)}\right| = \frac{1}{\sqrt{(1-r)^2 + (2\xi r)^2}}$$

$$20 \log \frac{1}{\sqrt{(1-r)^2+(2\xi r)^2}} = 20\left[\log 1 - \log \sqrt{(1-r)^2+(2\xi r)^2}\right]$$

$$- 20 \log \sqrt{(1-r)^2+(2\xi r)^2}$$

Para $\omega \ll \omega_n$, $r \to 0$

$$- 20 \log 1 = 0 \text{ dB}$$

Para $\omega \gg \omega_n$, $r \to \infty$

$$-20 \log \sqrt{r^2+r^2} = -20 \log \sqrt{2r^2} = -20 \times 2 \log r$$

$$= (-40) \log r \text{ dB}$$

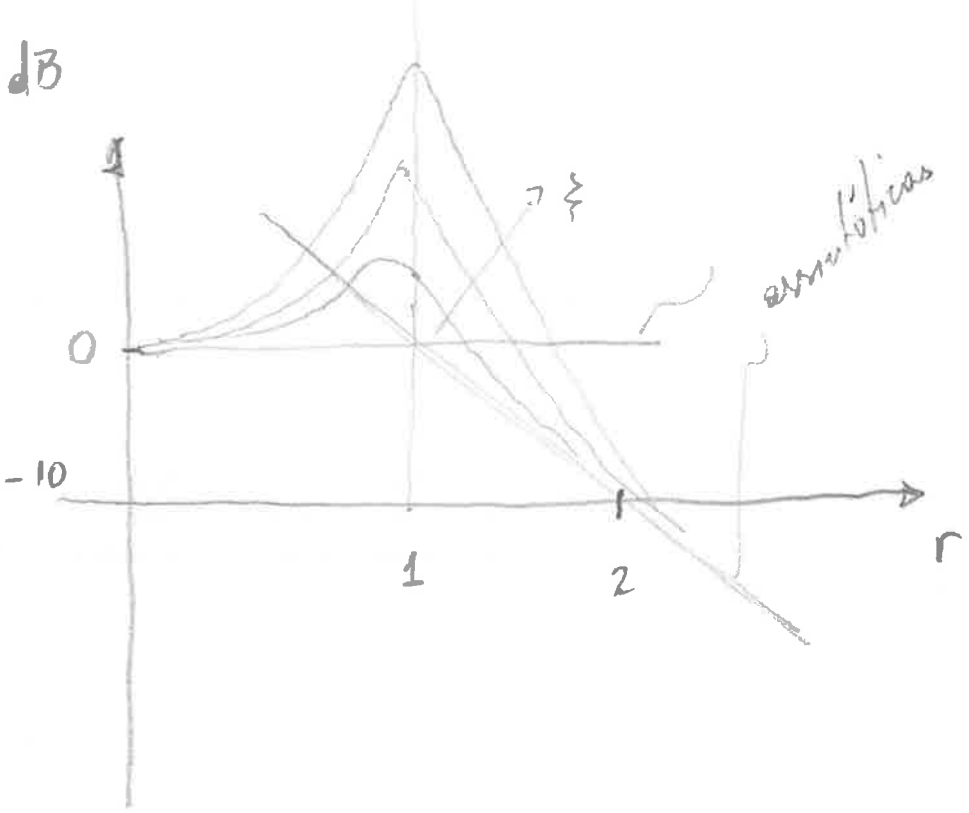
↳ slope de $\frac{-40 \text{ dB}}{\text{década}}$

$-40 \log 10r = -40 - 40 \log r$

Para $\omega = 0$, $r = 0$

$$-40 \log 1 = 0 \text{ dB}$$

mas depende de ξ



$$\phi = \tan^{-1} \frac{2\xi r}{1 - r^2}$$

$\omega = 0$, $r = 0$, $\phi = 0$

$\omega = \omega_n$, $r = 1$, $\phi = -90°$ ← não depende de ξ

$\omega = \infty$, $r = \infty$, $\phi = -180°$

Generalização para construção do diagrama de Bode

Função de transferência

$$G(s) = \frac{Y(s)}{X(s)} \qquad G(i\omega) = \frac{Y(i\omega)}{X(i\omega)}$$

Em geral, $G(s)$ pode ser reduzida a frações racionais

$$G(s) = \frac{q(s)}{p(s)} = \frac{q(i\omega)}{p(i\omega)}$$

$$q(s) = a_n s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0$$

$p(s)$: polinômio característico → suas raízes são os polos de $G(s)$

$q(s)$ → suas raízes são zeros de $G(s)$

Exemplo

$$G(s) = \frac{2(s+30)}{s(s+2)(s^2+2s+2)}$$

Zeros de $G(s) \longrightarrow s = -30$

Polos de $G(s) \longrightarrow s = 0$

$s = -2$

$s = -1 \pm i$

Ganho de Bode $\rightarrow K_B = G(s)$

Forma geral

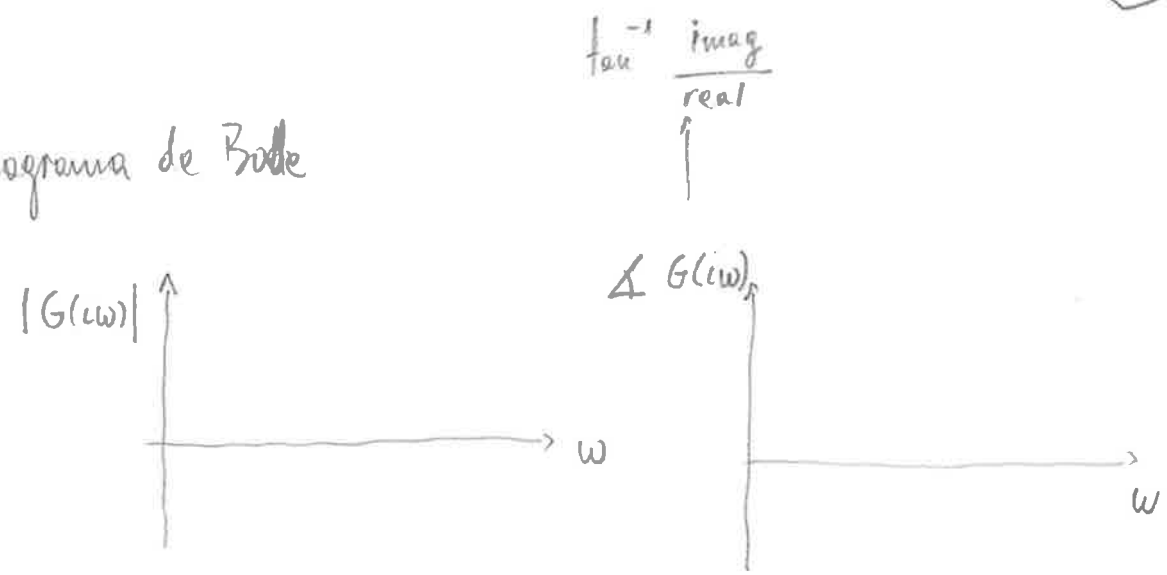
$$G(s) = \frac{K_B(\tau' s + 1)}{s(\tau s + 1)\left(\frac{s^2}{\omega_n^2} + \frac{2\xi}{\omega_n} s + 1\right)}$$

$$\rightarrow G(s) = \frac{60\left(\frac{s}{30} + 1\right)}{2s\left(\frac{s}{2} + 1\right)2\left(\frac{s^2}{2} + s + 1\right)}$$

$$K_B = \frac{60}{2 \cdot 2} = 15$$

$$\tau = \frac{1}{2} \qquad \tau' = \frac{1}{30} \quad , \quad \omega_n = \sqrt{2} \quad , \quad \xi = \frac{1}{\sqrt{2}} = 0.707$$

Diagrama de Bode



Ex. $\angle \; G(i\omega) = \frac{1}{i\omega} = a + bi$

$$\frac{i}{i\,i\omega} = \frac{i}{-\omega} \rightarrow a = 0 \quad b = -\frac{1}{\omega}$$

$$\theta = \angle \frac{1}{i\omega} = \tan^{-1} \frac{b}{a} = \tan^{-1} \frac{-1/\omega}{0} = -\infty$$

$$\theta = -\pi/2$$

$$G(i\omega) = (i\omega)^{-n}$$

$$G(i\omega)\Big|_{dB} = 20 \log \left| \frac{1}{(i\omega)^n} \right| = 20n \log \left| \frac{1}{i\omega} \right|$$

$$= -20n \log \omega \quad [dB]$$

$$\angle (i\omega)^{-n} = -n \angle i\omega$$

$$= -\frac{\pi}{2} n$$

$$G(i\omega) = (i\omega)^n$$

$$G(i\omega)\Big|_{dB} = 20 \log \left| (i\omega)^n \right|$$

$$= 20n \log |i\omega|$$

$$= 20n \log \omega \quad [dB]$$

$$\angle G(i\omega) = \frac{1}{(i\omega)^n} = -n \angle (i\omega) = -\frac{\pi}{2} n$$

$$G(i\omega) = \frac{1}{(1+i\omega\tau)^n}$$

$$|G(i\omega)|_{dB} = 20 \log \left| \frac{1}{(1+i\omega\tau)^n} \right|$$

$$= -20n \log \sqrt{1+(\omega\tau)^2} \quad [dB]$$

$$\angle G(i\omega) = \angle \frac{1}{(1+i\omega\tau)^n}$$

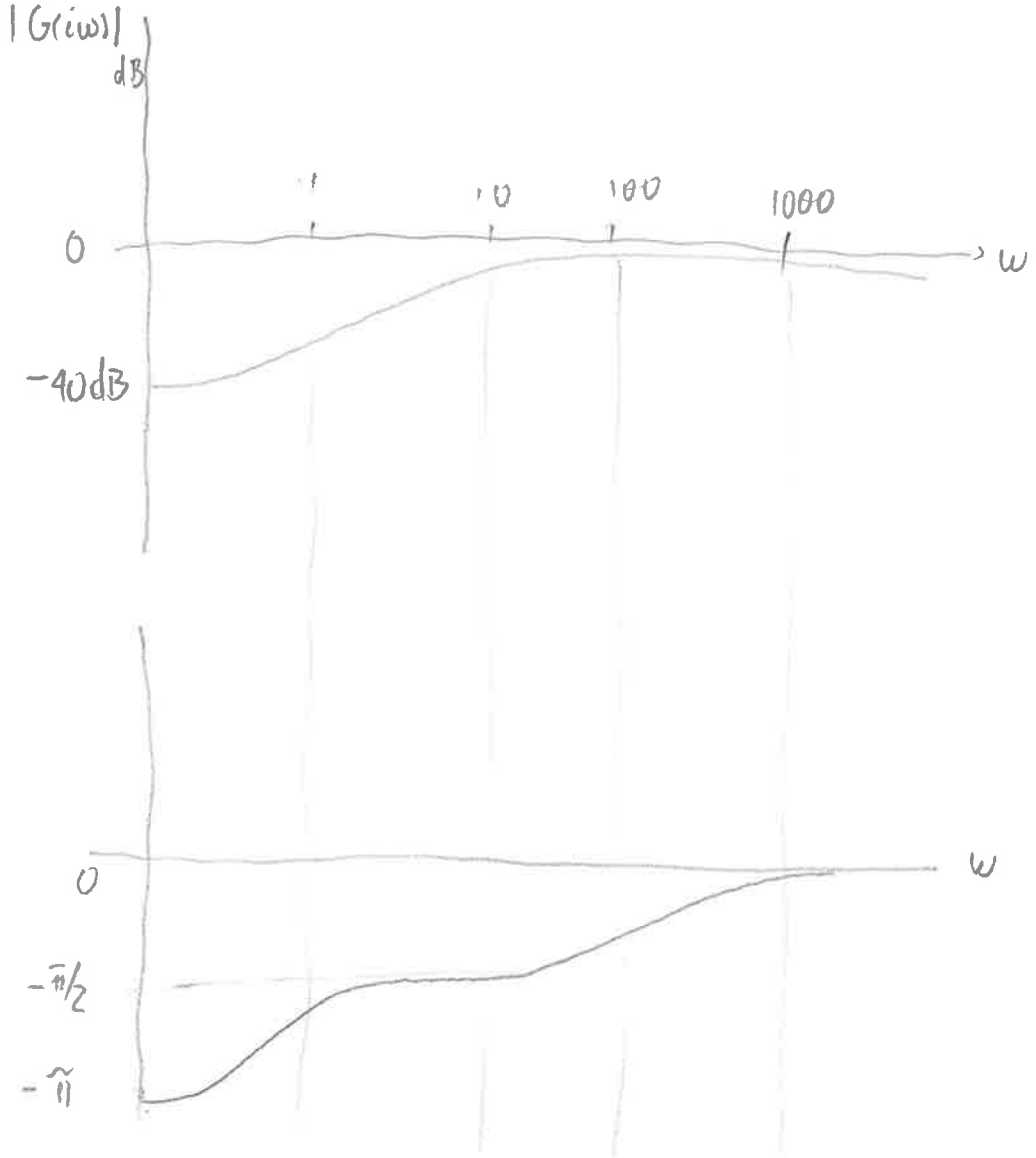
$$= -\angle (1+i\omega\tau)^n$$

$$= -n \tan^{-1}(\omega\tau)$$

$$G(i\omega) = \frac{s+1}{s-100} = \frac{\frac{1}{100}(s+1)}{\frac{s}{100}-1}$$

$$k_B = \frac{1}{100} = -40\ dB$$

$$\angle(1+i\omega) - \angle\left(-1+\frac{i\omega}{100}\right) = \angle(1+i\omega) + \pi - \angle\left(1-\frac{i\omega}{100}\right)$$



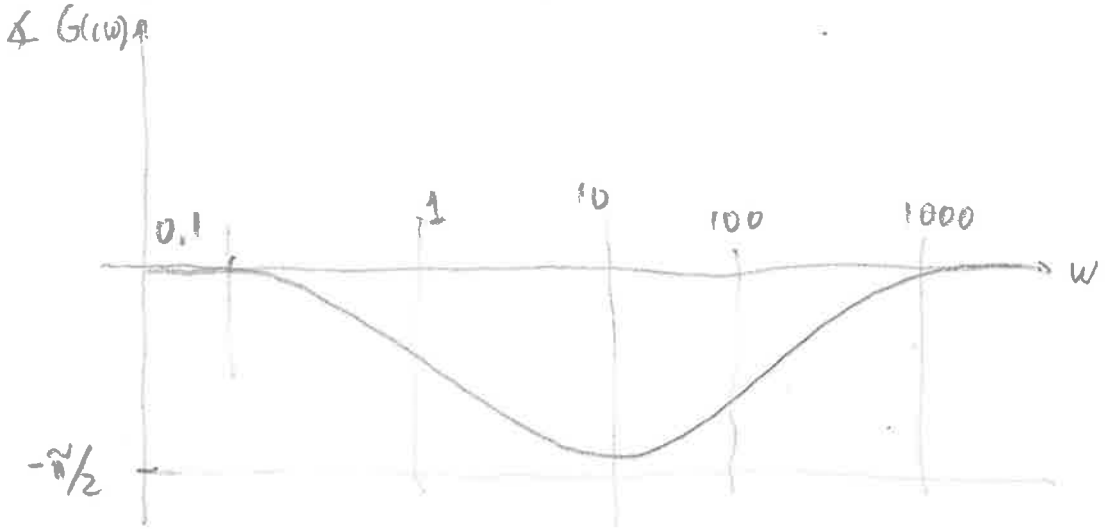
$$G(i\omega) = \frac{s-1}{s-100} = \frac{\frac{1}{100}(s-1)}{\frac{s}{100}-1}$$

$$k_B = \frac{1}{100} = -40\,dB$$

$$\angle[G(i\omega)] = \angle(-1+i\omega) - \angle\left(-\frac{1}{\cdot} + \frac{i\omega}{100}\right)$$

$$= \pi + \angle(1-i\omega) - \pi - \angle\left(1 - \frac{i\omega}{100}\right)$$

$$= \angle(1-i\omega) - \angle\left(1 - \frac{i\omega}{100}\right)$$



Comandos MatLab

[mag, phase, w] = bode (num, den, w)

magdB = 20 log (mag)

Exemplo

$$G(s) = \frac{25}{s^2+4s+25}$$

```
num = [0 0 25];
den = [1 4 25];
bode(num, den);
subplot(2,1,1);
```

Vibração sob forças quaisquer

Força periódica

$$F(t) = \frac{a_0}{2} + \sum_{i=1}^{\infty} a_i \cos i\omega t + \sum_{i=1}^{\infty} b_i \operatorname{sen} i\omega t$$

$$a_i = \frac{2}{\tau}\int_0^{\tau} F(t)\cos i\omega t\, dt \qquad i = 0, 1, 2, \ldots$$

$$b_i = \frac{2}{\tau}\int_0^{\tau} F(t)\operatorname{sen} i\omega t\, dt \qquad i = 0, 1, 2, \ldots$$

$$m\ddot{x} + c\dot{x} + kx = F(t) = \frac{a_0}{2} + \sum_{i=1}^{\infty} a_i \cos i\omega t + \sum_{i=1}^{\infty} b_i \operatorname{sen} i\omega t$$

Princípio da superposição

$$m\ddot{x} + c\dot{x} + kx = \frac{a_0}{2} \rightarrow x_p(t) = \frac{a_0}{2k}$$

$$m\ddot{x} + c\dot{x} + kx = a_i \cos\omega t \rightarrow x_p(t) = \frac{a_i/k}{\sqrt{(1 - i^2r^2)^2 + (2\xi ir)^2}} \cos(i\omega t - \phi_i)$$

$$m\ddot{x} + c\dot{x} + kx = b_i \operatorname{sen}\omega t \rightarrow x_p(t) = \frac{b_i/k}{\sqrt{(1 - i^2r^2)^2 + (2\xi ir)^2}} \operatorname{sen}(i\omega t - \phi_i)$$

$$\phi_i = \tan^{-1} \frac{2\xi r}{1 - i^2 r^2}$$

$$x_p(t) = \frac{a_0}{2k} + \sum_{i=1}^{\infty} \frac{a_i/k}{\sqrt{(1-i^2r^2)^2 + (2\xi ir)^2}} \cos(i\omega t - \phi_i) +$$

$$\sum_{i=1}^{\infty} \frac{b_i/k}{\sqrt{(1-i^2r^2)^2 + (2\xi ir)^2}} \sin(i\omega t - \phi_i)$$

steady state

For large i the amplitudes will be smaller

so the first terms are usually sufficient

solução particular

→ condições iniciais

Resposta a uma força não periódica

Força impulsiva

$$\text{Impulso} = F\Delta t = m\dot{x}_2 - m\dot{x}_1$$

$$\underset{\sim}{F} = \int_t^{t+\Delta t} F\,dt \qquad \text{magnitude do impulso}$$

Impulso unitário $\Delta t \to 0$

$$\underset{\sim}{f} = \lim_{\Delta t \to 0} \int_t^{t+\Delta t} F\,dt = F\Delta t = 1$$

$\Delta t \to 0$

$F \to \infty$

$$m\ddot{x} + c\dot{x} + kx = 0$$

$$x(t) = e^{-\zeta\omega_n t}\left[x_0 \cos\omega_d t + \frac{\dot{x}_0 + \zeta\omega_n x_0}{\omega_d}\sin\omega_d t\right]$$

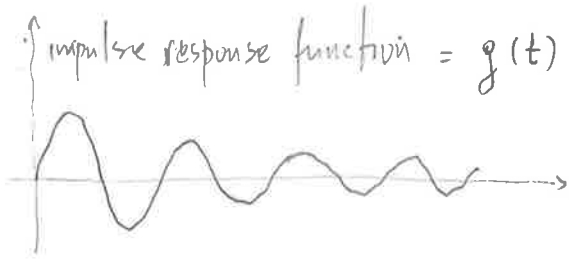
aplicar impulso unitário $\qquad x_0(0) = 0$

$$\underset{\sim}{f} = 1 = m\dot{x}(t=0) - m\dot{x}(t=0^-)$$

$$= m\dot{x}_0 - 0 = m\dot{x}_0 \longrightarrow \dot{x}_0 = \frac{1}{m}$$

$$x(t) = \frac{e^{-\xi \omega_n t}}{m \omega_d} \sin \omega_d t = g(t) \qquad \omega_d = \omega_n \sqrt{1-\xi^2}$$

$$\xi = \frac{c}{2 m \omega_n}$$



Para o caso $\underset{\sim}{F}$ $\qquad \ddot{x}_0 = \frac{F}{m}$

$$x(t) = \underset{\sim}{F} \frac{e^{-\xi \omega_n t}}{m \omega_d} \sin \omega_d t = \underset{\sim}{F}\, g(t) \qquad g(t)$$

Para impulso aplicado em $t = \tau$

$$x(t) = \underset{\sim}{F}\, g(t-\tau)$$

RESPOSTA A UMA FORÇA QUALQUER

Para uma força $F(t)$ qualquer agindo num intervalo curto

$F(t) \rightarrow$ = soma de uma série de impulsos

$$\text{Impulso} = F(\tau)\, \Delta\tau$$

$$\Delta x(t) = \underset{\sim}{F}\, g(t-\tau)$$

$$= F(\tau)\, \Delta\tau\, g(t-\tau)$$

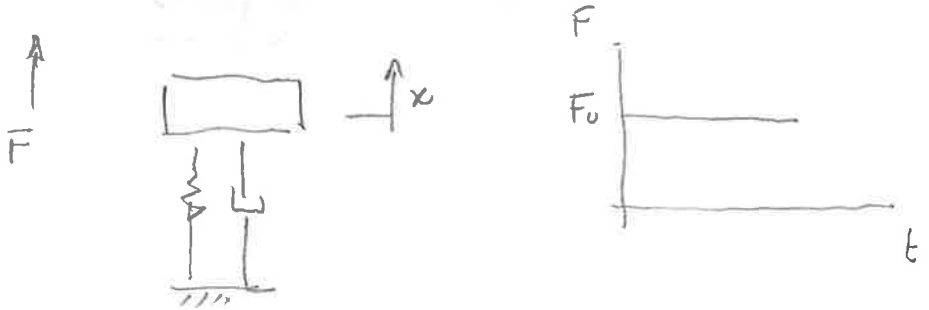
$$x(t) = \sum F(\tau)\, g(t-\tau)\, \Delta\tau$$

$$x(t) = \int F(\tau)\, g(t-\tau)\, d\tau$$

$$x(t) = \int F(\tau)\, \frac{e^{-\xi\omega_n(t-\tau)}}{m\omega_d} \sin\omega_d(t-\tau)\, d\tau$$

↳ convolução ou integral de Duhamel

Exemplo: Resposta de um sistema massa-mola-amort a função degrau

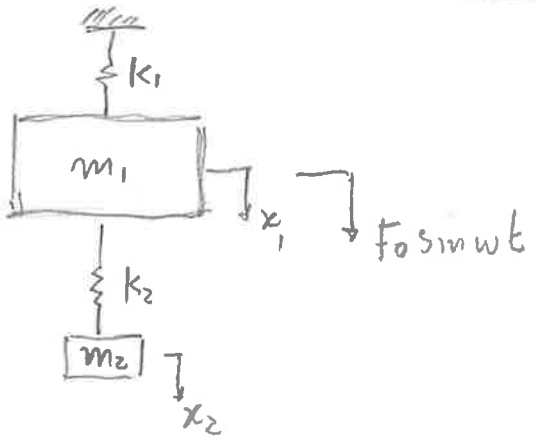


$$x(t) = \int_0^t \frac{F_0\, e^{-\xi\omega_n(t-\tau)}}{m\omega_d} \sin\omega_d(t-\tau)\, d\tau$$

$$x(t) = \frac{F_0}{k}\left[1 - \frac{1}{\sqrt{1-\xi^2}}\, e^{-\xi\omega_n t} \cos(\omega_d t - \phi)\right] \qquad \phi = \tan^{-1}\frac{\xi}{\sqrt{1-\xi^2}}$$

Solução para sistema não amortecido : vibração forçada

Caso do Absorvedor dinâmico de Vibração



$$m_1 \ddot{x}_1 + k_1 x_1 + k_2 (x_1 - x_2) = F_0 \sin \omega t$$

$$m_2 \ddot{x}_2 + k_2 (x_2 - x_1) = 0$$

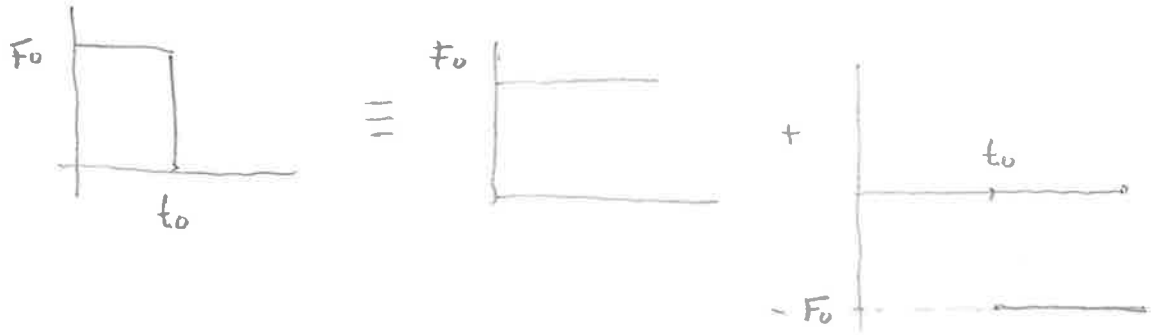
$$x_1 = X_1 \operatorname{sen} \omega t \qquad x_2 = X_2 \operatorname{sen} \omega t$$

$$X_1 = \frac{(k_2 - m_2 \omega^2) F_0}{(k_1 + k_2 - m_1 \omega^2)(k_2 - m_2 \omega^2) - k_2^2}$$

$$X_2 = \frac{k_2 F_0}{(k_1 + k_2 - m_1 \omega^2)(k_2 - m_2 \omega^2) - k_2^2}$$

Para $X_1 = 0$ $\quad k_2 - m_2 \omega^2 = 0$ $\quad \omega = \sqrt{\frac{k_2}{m_2}}$

Sistema MKC sob pulso retangular



same as a step function but starting at $t = t_0$

$$x(t) = -\frac{F_0}{k\sqrt{1-\xi^2}}\left\{\sqrt{1-\xi^2} - e^{-\xi\omega_n(t-t_0)}\cos\left[\omega_d(t-t_0) - \phi\right]\right\}$$

$$+ \frac{F_0}{k}\left\{1 - \frac{1}{\sqrt{1-\xi^2}}\, e^{-\xi\omega_n t}\cos(\omega_d t - \phi)\right\}$$

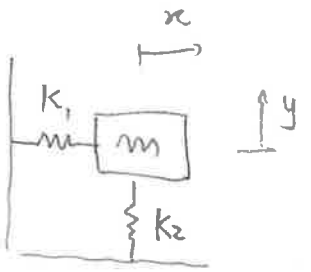
$$x(t) = \frac{F_0\, e^{-\xi\omega_n t}}{k\sqrt{1-\xi^2}}\left\{-\cos(\omega_d t - \phi) + e^{\xi\omega_n t_0}\cos\left[\omega_d(t-t_0) - \phi\right]\right\}$$

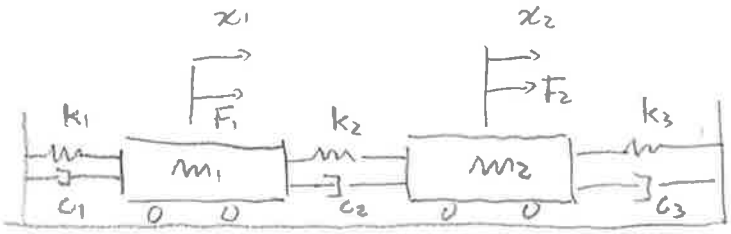
$$\phi = \tan^{-1}\frac{\xi}{\sqrt{1-\xi^2}}$$

Para $\xi = 0$

$$x(t) = \frac{F_0}{k}\left[\cos\omega_n(t-t_0) - \cos\omega_n t\right]$$

Sistemas de 2 graus de liberdade





$$m_1 \ddot{x}_1 + (c_1 + c_2)\dot{x}_1 - c_2 \dot{x}_2 + (k_1 + k_2) x_1 - k_2 x_2 = F_1$$

$$m_2 \ddot{x}_2 - c_2 \dot{x}_1 + (c_2 + c_3)\dot{x}_2 - k_2 x_1 + (k_2 + k_3) x_2 = F_2$$

sistema acoplado: movimento x_1 afeta movimento x_2

$$[m]\{\ddot{x}\} + [c]\{\dot{x}\} + [k]\{x\} = F$$

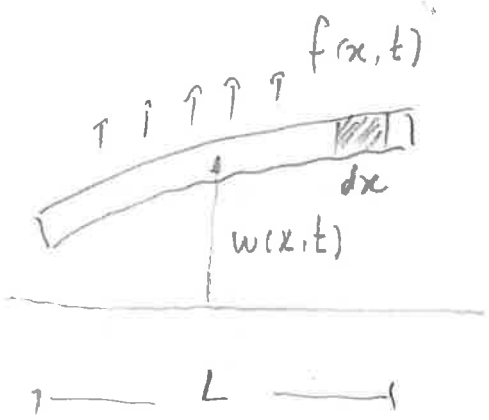
$$[m] = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \qquad \{x\} = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$$

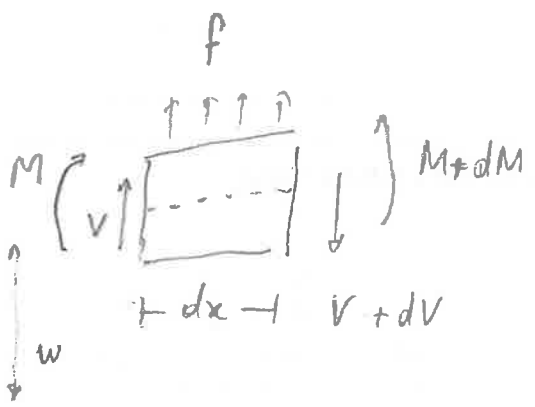
$$[c] = \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 + c_3 \end{bmatrix} \qquad \{F\} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

$$[k] = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix}$$

Sistemas contínuos

Vibração lateral de vigas





$$-(V+dV) + f\,dx + V = \rho A\,dx\,\frac{\partial^2 w}{\partial t^2}$$

$$M+dM - (V+dV)\,dx + f\,dx\,\frac{dx}{2} - M = 0$$

$$dV = \frac{\partial V}{\partial x}\,dx \qquad dM = \frac{\partial M}{\partial x}\,dx$$

$$-\frac{\partial V}{\partial x} + f = \rho A\,\frac{\partial^2 w}{\partial t^2} \qquad \frac{\partial M}{\partial x} - V = 0$$

$$-\frac{\partial^2 M}{\partial x^2} + f = \rho A\,\frac{\partial^2 w}{\partial t^2}$$



$$M = EI \frac{\partial^2 w}{\partial x^2} \longrightarrow \varepsilon = \frac{dx' - dx}{dx} = \frac{(R+y)d\theta - Rd\theta}{Rd\theta} = \frac{y}{R} = yk$$

$$k = -\frac{\partial^2 w}{\partial x^2} \qquad \sigma = E\varepsilon$$

$$EI \frac{\partial^4 w}{\partial x^4} + \rho A \frac{\partial^2 w}{\partial t^2} = f$$

$$dM = \sigma dA\, y$$

$$= E y k y dA$$

$$M = Ek \int y^2 dA$$

$$M = EIk$$

$$c^2 \frac{\partial^4 w}{\partial x^4} + \frac{\partial^2 w}{\partial t^2} = f$$

$$c = \sqrt{\frac{EI}{\rho A}}$$

Caso de vibração livre

$$w(x,t) = W(x)\,T(t)$$

$$\frac{c^2}{W} \frac{d^4 W}{dx^4} = -\frac{1}{T} \frac{d^2 T}{dt^2} = w^2$$

$$\frac{d^4 W}{dx^4} - \beta^4 W = 0$$

$$\frac{d^2 T}{dt^2} + w^2 T = 0$$

$$\beta^4 = \frac{w^2}{c^2} = \frac{\rho A w^2}{EI}$$

$$T(t) = A \cos wt + B \operatorname{sen} wt$$

$$W(x) = Ce^{sx}$$

$$s^4 - \beta^4 = 0 \qquad s_{1,2} = \pm\beta \qquad s_{3,4} = \pm i\beta$$

$$W(x) = C_1 e^{\beta x} + C_2 e^{-\beta x} + C_3 e^{i\beta x} + C_4 e^{-i\beta x}$$

$$W(x) = C_1 \cos\beta x + C_2 \operatorname{sen}\beta x + C_3 \cosh\beta x + C_4 \operatorname{senh}\beta x$$

$$w = (\beta L)^2 \sqrt{\frac{EI}{\rho A L^4}}$$

Caso de viga empotrada

$$W(0) = 0$$

$$W'(0) = 0$$

$$M(L) = \frac{\partial^2 w(L)}{\partial x^2} = 0$$

$$V(L) = \frac{\partial^3 w(L)}{\partial x^3} = 0$$

$W(0)=0$

$W'(0)=0$

$W''(L)=0$

$W'''(L)=0$

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & \beta & 0 & \beta \\ -\beta^2\cos\beta L & \beta^2\sin\beta L & \beta^2\cosh\beta L & \beta^2\sinh\beta L \\ -\sin\beta L & -\cos\beta L & \sinh\beta L & \cosh\beta L \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$C_1 + C_3 = 0 \qquad C_2 + C_4 = 0$$

$$-c\,C_1 + s\,C_2 + ch\,C_3 + sh\,C_4 = 0$$

$$-s\,C_1 - c\,C_2 + sh\,C_3 + ch\,C_4 = 0$$

$$-s\,C_1 - c\,C_2 + sh\cdot C_1 + ch\cdot -C_2 = 0$$

$$(-c-ch)\,C_2 + (-s-sh)\,C_1 = 0$$

$$C_2 = -\frac{s+sh}{c+ch}\,C_1$$

$$W = C_1\,c - \frac{s+sh}{c+ch}\,s + -C_1\,ch + \frac{s+sh}{c+ch}\,C_1\,sh$$

$$W = C_1\left[\cos\beta x - \frac{\sin\beta L+\sinh\beta L}{\cos\beta L+\cosh\beta L}\sin\beta x - \cosh\beta x + \frac{\sin\beta L+\sinh\beta L}{\cos\beta L+\cosh\beta L}\sinh\beta x\right]$$

$$\det[S] = 0$$

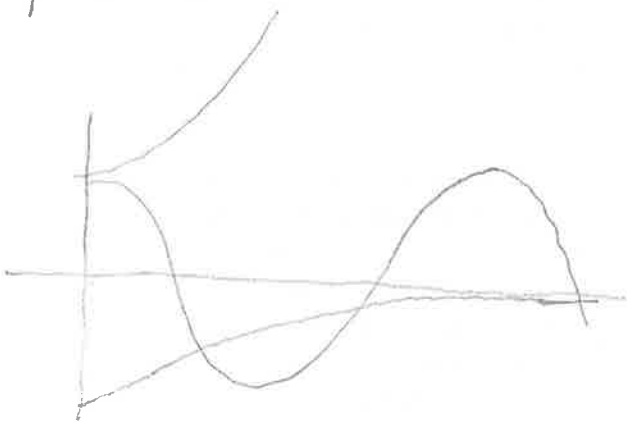
$$\begin{vmatrix} 1 & 0 & 1 \\ s & ch & sh \\ -c & sh & ch \end{vmatrix} + \begin{vmatrix} 0 & 1 & 1 \\ -c & s & sh \\ -s & -c & ch \end{vmatrix} = 0$$

$-sh^2 + ch^2 = 1$

$$ch^2 + s\,sh + c\,ch - sh^2 - s\,sh + s^2 + c\,ch + c^2 = 0$$

$$1 + 2\,c\,ch + 1 = 0$$

$$\cos\beta L \cosh\beta L = -1$$



$$\beta_1 L = 1.875 \qquad \sqrt{\frac{\omega_1}{c}}\, L = 1.875$$

$$\omega_1 = \frac{1.875^2}{L^2}\sqrt{\frac{EI}{m}}$$

$$\beta_2 L = 4.694$$

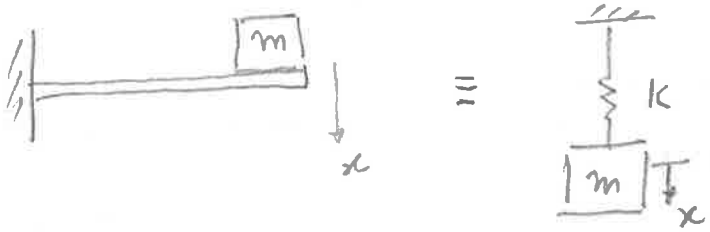
$$\beta_3 L = 7.855$$

Demo sala: régua

⋮

RESUMO DO CURSO DE VIBRAÇÕES

- Estruturas simples podem ser representados por modelos massa-mola



Rigidez equivalente de vigas

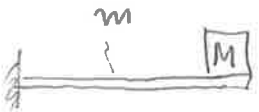
Viga em balanço $k = \frac{3EI}{L^3}$

Viga apoiada em seus extremos $k = 48\frac{EI}{L^3}$

Viga engastada em seus extremos $k = 192\frac{EI}{L^3}$

64x

Massa equivalente



$$m_{eq} = M + 0{,}23m$$

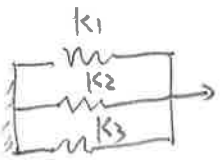


$$m_{eq} = M + \frac{m}{3}$$

Molas equivalentes



$$\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3}$$



$$k_{eq} = k_1 + k_2 + K_3$$

Rigidez axial de uma barra

$$k = \frac{EA}{L}$$

Movimento harmônico → aceleração proporcional ao deslocamento

$$x = A \sin \omega t$$

$$\ddot{x} = -\omega^2 A \sin \omega t = -\omega^2 x$$

$$x = A e^{i\omega t}$$

$$x = A \cos \omega t$$

$$\dot{x} = -\omega A \sin \omega t = \omega A \cos(\omega t + \pi/2)$$ velocidade $\frac{\pi}{2}$ à frente

$$\ddot{x} = -\omega^2 A \cos \omega t = \omega^2 A \cos(\omega t + \pi)$$ aceleração π à frente

deslocamento

Frequência natural → associada a diferença de fase entre entrada e saída de $-\pi/2$ para qualquer nível de amortecimento

Frequência de ressonância → aquela associada à máxima amplitude de resposta

Frequência natural amortecida → frequência de vibração livre de um sistema amortecido

Solução equação de equilíbrio sistema massa mola

$$m\ddot{x} + kx = 0$$

$$x(t) = Ce^{st}$$

$$ms^2Ce^{st} + kCe^{st} = 0$$

$$ms^2 + k = 0$$

$$s = \pm i\sqrt{\frac{k}{m}} \quad \rightarrow \omega_n \leftarrow \text{freq natural}$$

Condições iniciais $x(0) = x_0$, $\dot{x}(0) = \dot{x}_0$

$$x(t) = x_0 \cos \omega_n t + \frac{\dot{x}_0}{\omega_n} \operatorname{sen} \omega_n t$$

$$x(t) = A \cos(\omega_n t - \phi)$$

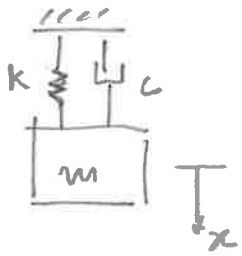
$$x(t) = A_0 \operatorname{sen}(\omega_n t + \phi_0)$$

$$A = A_0 = \sqrt{x_0^2 + \frac{\dot{x}_0^2}{\omega_n^2}}$$

$$\phi = \tan^{-1} \frac{\dot{x}_0}{\omega_n x_0}$$

$$\phi_0 = \tan^{-1} \frac{\omega_n x_0}{\dot{x}_0}$$

Vibração livre com amortecimento



$$m\ddot{x} + c\dot{x} + kx = 0$$

$$x = Ce^{st}$$

$$s_{1,2} = \frac{-c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}$$

$$x(t) = C_1 e^{s_1 t} + C_2 e^{s_2 t}$$

Amortecimento crítico $\quad c_c = 2m\omega_n$

Razão de amortecimento $\quad \xi = \frac{c}{c_c}$

$$s_{1,2} = -\xi\omega_n \pm \sqrt{\xi^2 - 1}\,\omega_n$$

$\boxed{\xi < 1}$ subamortecido

$$x(t) = e^{-\xi\omega_n t}\left\{ x_0 \cos\sqrt{1-\xi^2}\,\omega_n t + \frac{\dot{x}_0 + \xi\omega_n x_0}{\sqrt{1-\xi^2}\,\omega_n} \operatorname{sen}\sqrt{1-\xi^2}\,\omega_n t \right\}$$

$\omega_d = \sqrt{1-\xi^2}\,\omega_n \rightarrow$ freq natural amortecida

$\zeta = 1$ críticamente amortecido

$$s_1 = s_2 = -\omega_n$$

$$x(t) = e^{-\omega_n t} \left\{ x_0 + (\dot{x}_0 + \omega_n x_0)\, t \right\}$$

$\zeta > 1$ sobreamortecido

$$x(t) = C_1 e^{(-\zeta + \sqrt{\zeta^2 - 1})\omega_n t} + C_2 e^{(-\zeta - \sqrt{\zeta^2 - 1})\omega_n t}$$

$$C_1 = \frac{x_0 \omega_n (\zeta + \sqrt{\zeta^2 - 1}) + \dot{x}_0}{2\omega_n \sqrt{\zeta^2 - 1}}$$

$$C_2 = \frac{-x_0 \omega_n (\zeta - \sqrt{\zeta^2 - 1}) - \dot{x}_0}{2\omega_n \sqrt{\zeta^2 - 1}}$$

Decremento logarítmico

$$\delta = \frac{1}{n} \ln \frac{x_1}{x_{n+1}}$$

Excitação harmônica

$$m\ddot{x} + c\dot{x} + kx = F(t)$$

$$x = x_h + x_p$$

<u>Caso</u>

$F(t) = F_0 \cos \omega t$, $c = 0$

$$m\ddot{x} + kx = F_0 \cos \omega t$$

$$x_p = X \cos \omega t$$

$$X = \frac{F_0}{k - \omega^2 m}$$

$$x = C_1 \cos \omega_n t + C_2 \sin \omega_n t + \frac{F_0}{k - m\omega^2} \cos \omega t$$

$x(0) = x_0 \qquad \dot{x}(0) = \dot{x}_0$

$$C_1 = x_0 - \frac{F}{k - \omega^2 m} \qquad C_2 = \frac{\dot{x}_0}{\omega_n}$$

$$\frac{X_0}{F_0/k} = \frac{1}{1 - r^2} \qquad r = \frac{\omega}{\omega_n}$$

↳ Fator de amplificação

Sistema amortecido sob vibração harmônica forçada

$$m\ddot{x} + c\dot{x} + kx = F_0 \cos \omega t$$

$$x_p = X \cos(\omega t - \phi)$$

$$X = \frac{F_0}{\sqrt{(c\omega)^2 + (k - m\omega^2)^2}} = \frac{F_0/k}{\sqrt{(1-r^2)^2 + (2\xi r)^2}}$$

$$\phi = \tan^{-1} \frac{2\xi r}{1 - r^2}$$

Fator de amplificação

$$M = \frac{X}{\delta_{st}} = \frac{1}{\sqrt{(1-r^2)^2 + (2\xi r)^2}}$$

$$M_{max} = \frac{1}{2\xi\sqrt{1-\xi^2}} \quad \text{em} \quad r = \sqrt{1-2\xi^2}$$

$$x = x_h + x_p$$

$$x = X_0 e^{-\xi \omega_n t} \cos(\omega_d t - \phi_0) + X \cos(\omega t - \phi)$$

Vibração sob forças quaisquer periódicas

$F(t)$ → decompor em série de Fourier

$$F(t) = \frac{a_0}{2} + \sum_{i=1}^{\infty} a_i \cos i\omega t + \sum_{i=1}^{\infty} b_i \operatorname{sen} i\omega t$$

$$a_i = \frac{2}{\tau} \int_0^{\tau} F(t) \cos i\omega t \, dt$$

$$b_i = \frac{2}{\tau} \int_0^{\tau} F(t) \operatorname{sen} i\omega t \, dt$$

$$m\ddot{x} + c\dot{x} + kx = \frac{a_0}{2} + \sum_{i=1}^{\infty} a_i \cos i\omega t + \sum_{i=1}^{\infty} b_i \operatorname{sen} i\omega t$$

$$x_p(t) = \frac{a_0}{2k} + \sum_{i=1}^{\infty} \frac{a_i/k}{\sqrt{(1 - i^2 r^2)^2 + (2\xi i r)^2}} \cos(i\omega t - \phi_i)$$

$$+ \sum_{i=1}^{\infty} \frac{b_i/k}{\sqrt{(1 - i^2 r^2)^2 + (2\xi i r)^2}} \operatorname{sen}(i\omega t - \phi_i)$$

Vibrações sob forças quaisquer não periódicas

$$x(t) = \int F(\tau) \frac{e^{-\xi \omega_n (t-\tau)}}{m \omega_d} \sin \omega_d (t-\tau) \, d\tau$$

LISTA DE EXERCÍCIOS (RAO)

1. Defina grau de liberdade de um sistema vibrando.
2. Quais são os métodos principais para a solução de problemas de vibração?
3. Qual a diferença entre movimento periódico e harmônico?
4. Defina ângulo de fase, período, frequências natural, amortecida, de ressonância.
5. O que é batimento?
6. Defina decibel e oitava
7. Qual a importância de se conhecer a frequência natural de um sistema?
8. A frequência de ressonância é menor do que a natural? Pode ser igual?

9. Defina decremento logarítmico.

10. Qual a explanação física para que o fator de amplificação seja quase 1 para baixos r e quase zero para grandes r?

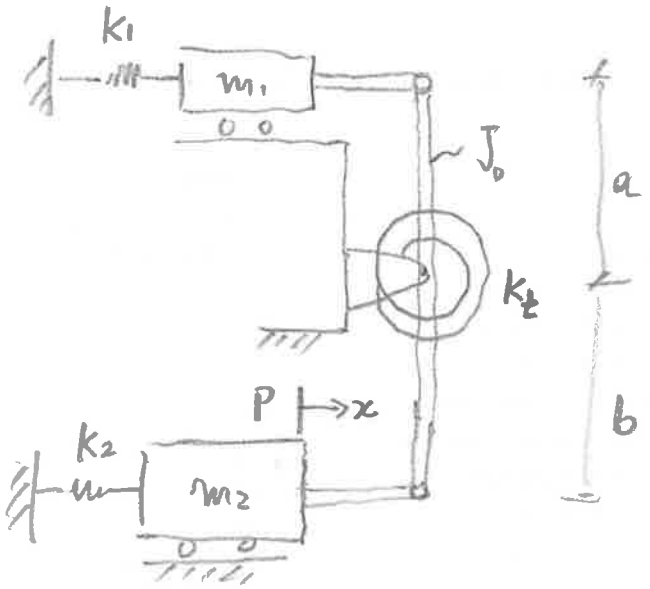
11. Se um veículo apresenta grande vibração em uma estrada, aumentar sua velocidade diminui a vibração? Qual seria a explicação física?

12. O que é um absorvedor dinâmico de vibrações?

13. Qual a condição de operação de um absorvedor dinâmico de vibrações?

LISTA DE EXERCÍCIOS (RAO)

1. Calcule a massa equivalente em P.



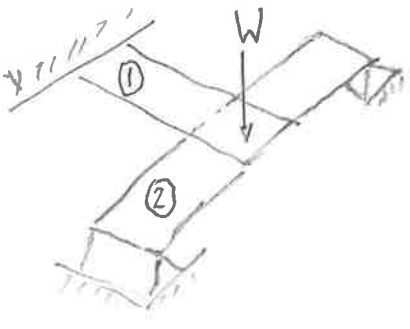
$$m_{eq} = m_1 \left(\frac{a}{b}\right)^2 + m_2 + J_0 \left(\frac{1}{b}\right)^2$$

2. Qual a máxima aceleração de um movimento harmônico de amplitude 0.05 m e frequência 10 Hz?

$a = 4.905\ m/s^2$

3. Prove que a frequência natural do sistema abaixo é

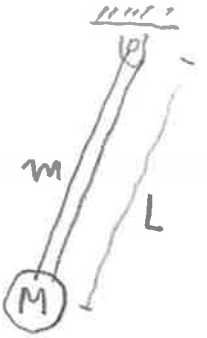
$$\omega_n = \sqrt{\left[\frac{3E_1I_1}{L_1^3} + \frac{48E_2I_2}{L_2^3}\right]\frac{g}{W}}$$



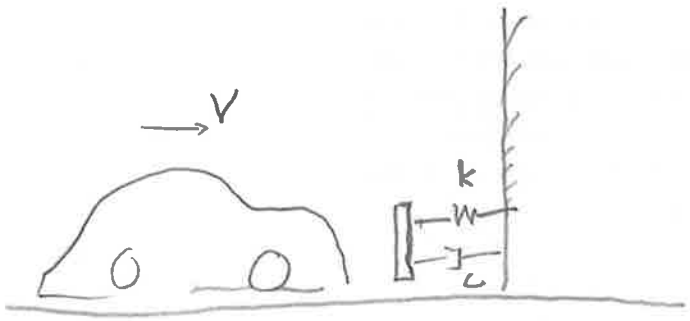
4. Prove que a frequência natural do pêndulo abaixo é

$$\omega_n = \sqrt{\frac{\left(M + \frac{m}{2}\right) g}{\left(M + \frac{m}{3}\right) L}}$$

Dica: Use $\frac{d}{dt}(T+U) = 0$



5. Determine o máximo deslocamento, e o tempo para alcançar tal configuração, do carro mostrado abaixo



$V = 10\ m/s$

$K = 40\ N/mm$

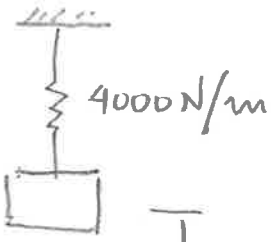
$c = 20\ N.s/mm$

$m = 2000\ kg$

Resp. $x_{max} = 1.0548 m$

$t = 0.5736\ s$

6. Obtenha o valor da massa m. A amplitude é de 20mm.

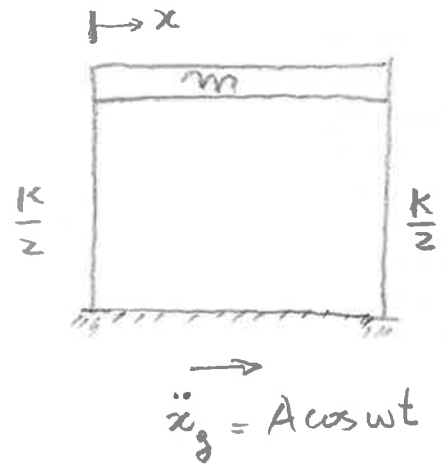


Resp.: $m = 9.1189\ kg$

$F = 100 \cos 5t$

N Hz

7. Qual a resposta em regime permanente da estrutura abaixo?



$A = 100\ mm/s^2$

$m = 2000\ kg$

$k = 0.1\ MN/m$

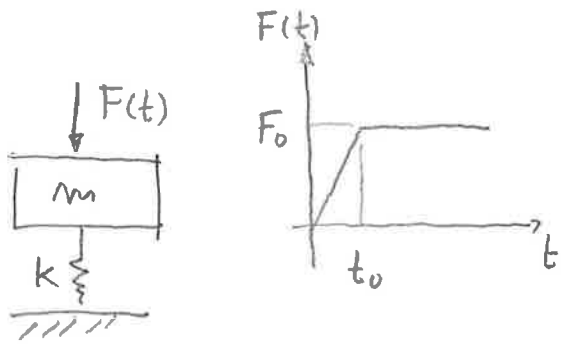
$\omega = 25\ rad/s$

$x_g(0) = \dot{x}_g(0) = 0$

$\dot{x}(0) = x(0) = 0$

Resp $x = -3.3391\ 10^{-4} \operatorname{sen} 25t$ [m]

8. Obtenha a resposta do sistema abaixo



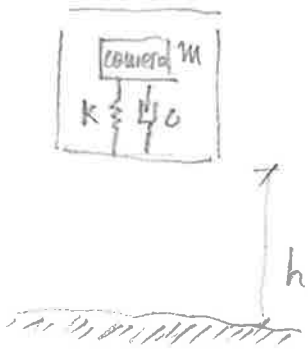
Resp. $x(t) = \frac{F_0}{k}\left\{1 + \frac{\operatorname{sen}\omega_n(t-t_0) - \operatorname{sen}\omega_n t}{\omega_n t_0}\right.$

para $t > t_0$

Dica: $F(t) = \underbrace{\frac{F_0 t}{t_0}}_{t>0} + \underbrace{-\frac{F_0(t-t_0)}{t_0}}_{t>t_0}$

9. Uma câmera de massa m está embalada em uma caixa que acidentalmente cai de uma altura h.

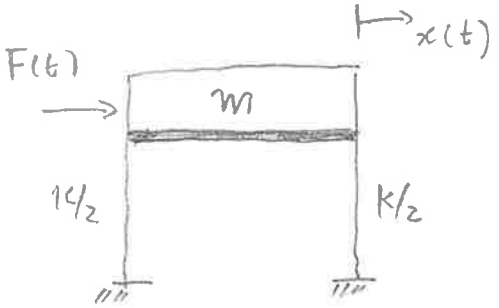
Qual a força sofrida pela câmera?

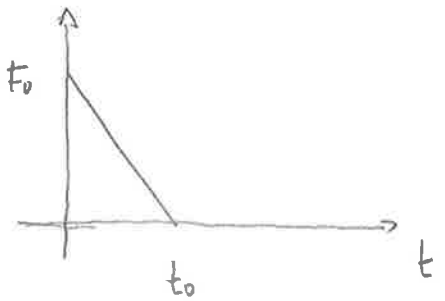


Dica: Mostre que $x(t) = e^{-\xi\omega_n t} \dfrac{\sqrt{2gh}}{\omega_n\sqrt{1-\xi^2}} \sin\sqrt{1-\xi^2}\,\omega_n t$

e daí calcule a força.

10. Obtenha a resposta da estrutura abaixo (considere $\xi = 0$)





Resp $x(t) = \dfrac{F_0}{K}\left(1 - \dfrac{t}{t_0} - \cos\omega_n t + \dfrac{1}{\omega_n t_0}\sin\omega_n t\right) \quad t \le t_0$

$$x(t) = \frac{F_0}{k\omega_n t_0}\left[(1-\cos\omega_n t_0)\sin\omega_n t - (\omega_n t_0 - \sin\omega_n t_0)\cos\omega_n t\right]$$

$t > t_0$