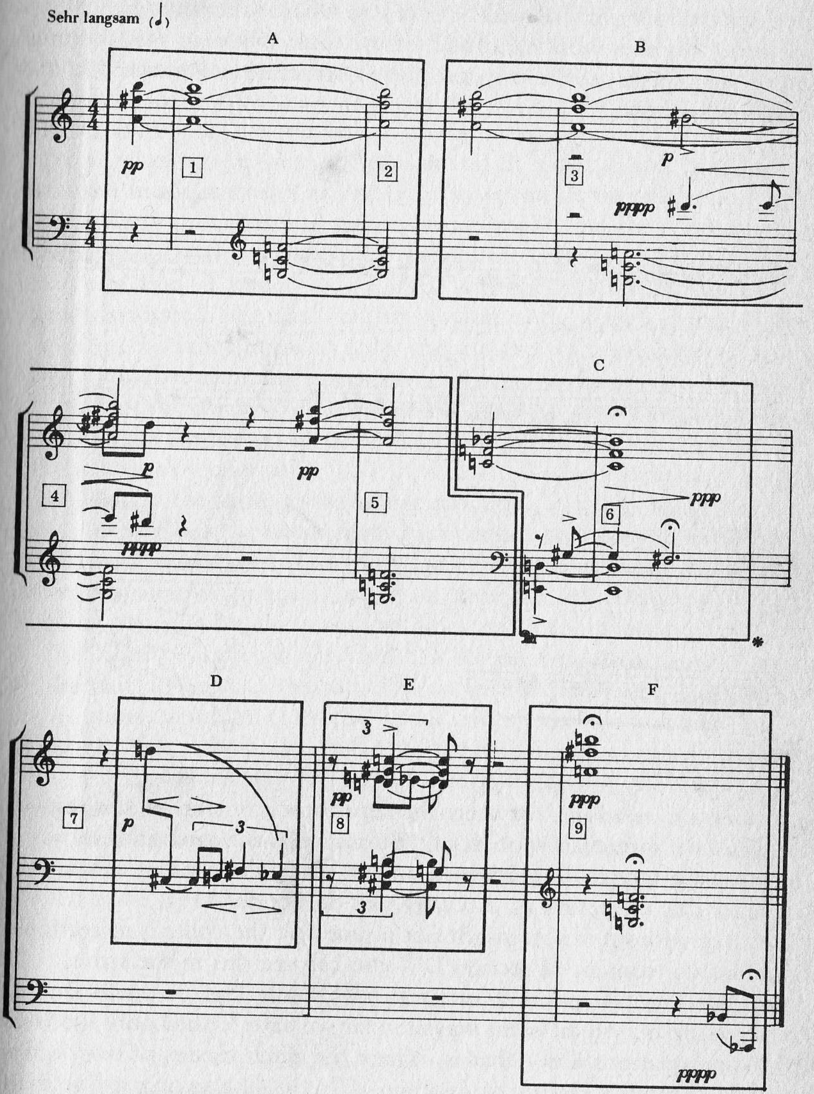
II *Set-theoretical analysis*

Fig. 53 shows the last of Schoenberg's *Six Little Piano Pieces* Op. 19. You cannot analyze this piece in terms of traditional tonal structure, in the way Kresky analyzed *Heidenröslein*; there is no tonic (at least, how could you decide what the tonic is?), and there is not the same kind of triadic elaboration you find in Schubert. What people usually do when faced with atonal music like this is to pick out certain things they regard as significant and ignore the rest. For example, you might pick out such familiar formations as the superimposed fourths in bars 1 and 5, or the whole-tones that become increasingly prominent in bars 5–6. Or you might pick out motifs that recur within this piece, for instance the way in which the prominent E – D♯ of bars 3 – 4 is echoed in the middle of the texture in bar 8. But picking out things and ignoring the rest in this way is like picking out triads in a tonal piece and ignoring the underlying structure which they prolong – which is precisely what Schenkerian analysis teaches us not to do. The aim of set-theoretical analysis, which was evolved by Allen Forte (the same Allen Forte we met in Chapter 2), is to provide the same kind of insight into the underlying structure of atonal music that Schenkerian analysis provides into tonal music: as Forte himself puts it, it 'establishes a framework for the description, interpretation and explanation of any atonal composition'.[1]

Let us begin in the same way as Kresky began with *Heidenröslein* and slice Op. 19/6 into sections. Fig. 53 shows how it falls into six sections labelled from A to F, which are distinguished from each other on the basis of surface features like texture, rhythm and dynamics. Now what we want to do is establish a network of relations between these various sections comparable to the Kresky diagram reproduced in Fig. 52, but without using the same kind of reductive techniques that are appropriate for tonal music. For example, we do not want to say that the D♯ in the left hand at bar 3 is an inessential note and the E that follows it an essential one, or the other way round, because we do not know what would make one note essential and another one inessential in an atonal piece. So rather than risk making inappropriate selections from the notes in each section, we shall try and see what structural relations exist between the entire content of each section considered as a harmonic unit. All we will assume is that register makes no difference to

[1] *The Structure of Atonal Music*, Yale University Press, 1973, p. 93. For a recent re-evaluation of set-theoretical analysis, see Forte's 'Pitch-class set analysis today', in *Musical Analysis*, 4 (1985), pp. 29–58.

Fig. 53 Schoenberg, Op. 19/6, with segmentation



the harmonic function of a note – in other words that, as in tonal harmony, a C functions the same way regardless of what octave it appears in. (In jargon, what we are interested in is *pitch classes* – Cs in general – and not pitches, such as this high C, that low C.) What this means is that our analysis will be based on what is shown in Fig. 54: we are using this as a working model of the music, hoping that the most important aspects of the original piece's structure are retained in this simplified version.[1]

Fig. 54

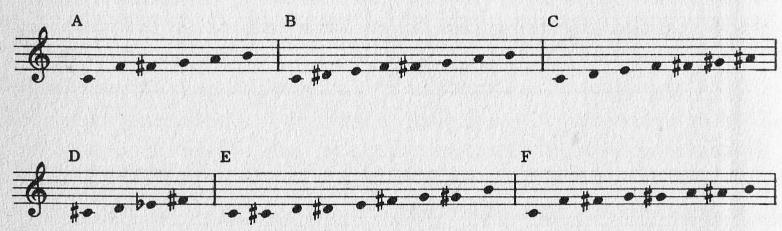
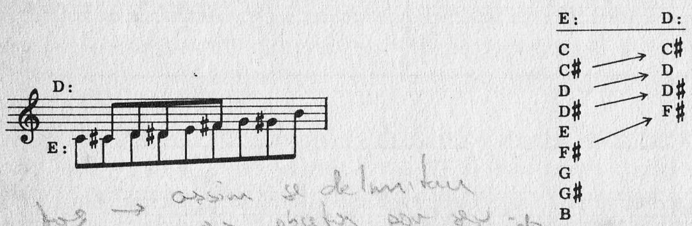


Fig. 55



Certain relations between the harmonic content of the various sections are immediately obvious. For example, the content of section B includes the content of section A, and similarly the content of section F includes that of section A. Actually you do not need Fig. 54 to tell you that! But without it you might not notice that the content of section E includes the content of section D – you can see this in the score, to be sure, but Fig. 54 makes it easier to see, while Fig. 55 spells out the relationship in two different ways. So far we have looked only for literal inclusion relationships – that is, where the pitch classes of one section include the pitch classes of another. (This is like saying a dominant seventh on G includes the G triad.) But one section might include the content of another, only at some transposition (in the way that the dominant seventh on G includes the E major triad when transposed by

[1] Is this sense? See the discussion of Op. 19/3 in Chapter 10.

a third). This is the relation between sections B and D of Schoenberg's piece, and Fig. 56 spells out how it works. However, we do not have to limit ourselves to the inclusion and transposition relationships you get in tonal harmony: we can look for other relationships too. For instance, the content of one section might include the content of another section only when it is inverted: and in fact this is the relationship between sections A and E. You can see that this is so from the music notation in Fig. 57. However, as the relationships we are dealing with become more complicated, so they become increasingly difficult to handle by means of conventional notation. So you may find it easier to see this kind of relationship if we use numerals instead. We shall call the lowest note of each group '0' and represent the other notes in it by the number of semitones by which they are higher than the lowest note. The lowest note of section E is C, so this becomes 0, C♯ becomes 1, and so on. This means that we can write the harmonic content of section E as [0, 1, 2, 3, 4, 6, 7, 8, 11] and that of section A as [0, 1, 2, 4, 6, 7]. So the numerals in Fig. 57 mean exactly the same as the music notation, and they make it a little easier to pick out the notes from each section that correspond to each other under inversion: you simply look for pairs of numbers that give the same value when added together (here the value happens to be 8, but this depends on the transpositional relationship between the two sets of notes). Some people find this kind of mathematical notation off-putting: it looks so abstract, like an arithmetic primer. But really it is no more abstract than the usual note-letter notation; it is just different. You may find it useful to practise sight-singing from these numbers. It is quite easy to pick up, and you can sing the notes as you scan the numerals, looking for patterns.

What have we done so far? We have found three ways in which the pitch content of the various sections in Op. 19/6 can relate to each

Fig. 56

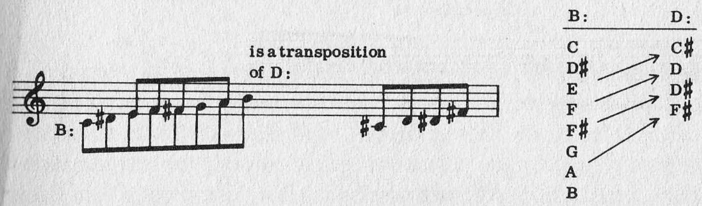


Fig. 57

other: by literal inclusion, by inclusion under transposition, and by inclusion under inversion. Now there is a further type of relationship that is important in this piece, and it is based on complementation. What is complementation? Take the pitch content of section F. It includes all the notes of the chromatic scale, except C♯, D, D♯ and E. And that means that these four notes are the *complement* of the eight notes in section F. In other words the complement of any given set of notes is simply all the other notes that together make up the chromatic scale. And we shall discover a whole lot more relations between the sections of Op. 19/6 if we take complementation into account. For example, there is not any direct relationship between the content of section F and that of section E – neither includes the notes of the other, whether literally, under transposition or under inversion. But section E does include the complement of section F, that is to say C♯, D, D♯ and E; Fig. 58 shows this, using a symbol derived from mathematics (F̄) to indicate the complement of F. So here we have the literal inclusion of a complement. Naturally, then, we can also have the inclusion of a complement under transposition. Actually there are three such relationships between the sections of Op. 19/6: E includes both the transposed complement of B and the transposed complement of C, while B includes the transposed complement of F. Fig. 59 spells out the last of these: you can work out the other two for yourself if you want to. And, again as you would expect, there is a final way in which two sets of notes can relate to each other, which is when one includes the complement of the other under inversion. There is one instance of this in Op. 19/6: the complement of A includes the inversion of D, and Fig. 60 shows this.

Fig. 58

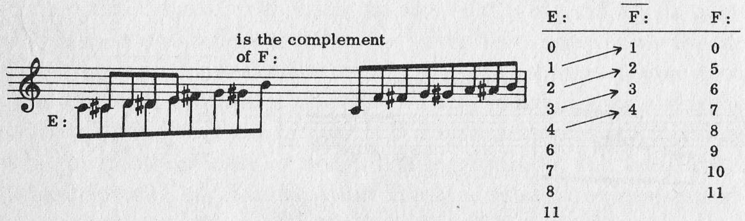


Fig. 59

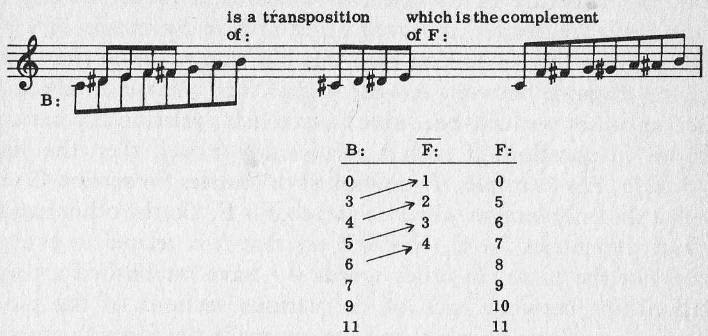
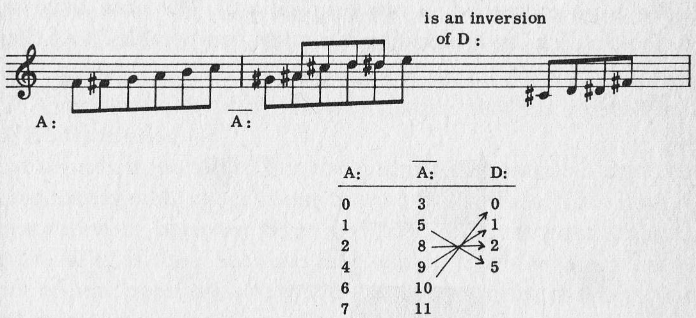


Fig. 60



Unless you have a bent for this kind of thing, all this talk of inversion and complementation may be making your head ache: but if you look back through Figs. 55–60 you'll see that the musical relationships we are talking about are really very simple and straightforward; it is merely that some of them are unfamiliar. And when you take all these relationships together, they can tell you a surprising amount about the structure of the piece as a whole. First let us express the relation of each section to every other section by means of a kind of mileage chart (Fig. 61). You read this like you read the charts that tell you the distance between towns, except that what it is telling you is whether or not we have been able to establish a relationship between the sections in question. If such a relationship exists, then the square is blacked in. For example, if you look at the entries for section C you will see that the only section which relates to it is E. On the other hand if you look at the entries for E, you will see that it is related to every other section of the piece. In other words we have established a pattern of relationships between each of the various sections of the piece that shows what relates to what, and we can make the formal consequences of this more easily visible if we draw a chart like Fig. 62. This embodies precisely the same information as Fig. 61 (the lines between sections represent relationships), and it makes it obvious how everything relates to E, whereas C is as it were out on a limb; there is no direct relationship between C and either the section before it or the section after it. And, if you think about it, this means something very like what Schenker's chart showing an 'interrupted' progression was saying (Fig. 16 above). In each case the analysis is saying that there is not a direct relationship between the two adjacent formations: they only relate to each other indirectly, in that both of them have a direct relationship to some third formation.

We have succeeded in our original aim. We now have what we were looking for, an underlying structure comparable to a Schenkerian

Fig. 61

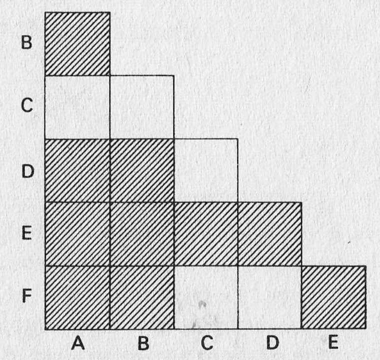
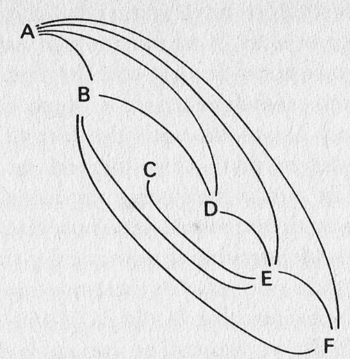


Fig. 62

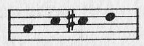


middleground; and it would be quite easy to complete the analysis in the way Kresky completed his, by looking for ways in which surface details in the music 'express' this underlying structure. And though what I have done is not really a proper set-theoretical analysis (as you will see, Allen Forte presents things rather differently), it should have given you some idea of what set-theoretical analysis is about. But the way I did it was not very convenient. I simply talked about 'the harmonic content of A'. But suppose there had been another section with the same harmonic content? Or suppose I had wanted to compare this piece with another one in which the same pitch class formation was found? What is wanted is a standardized way of referring to these pitch class formations wherever they are found. And the basis of set-theoretical analysis proper, as set out by Forte in his book *The Structure of Atonal Music*, is a complete listing of every possible pitch class formation that can appear in any piece of atonal music.

That sounds impossible! But the number of possible formations is reduced to manageable proportions by two restrictions. The first is that only formations of between three and nine different pitch classes are considered. Why is this? Suppose that section E in Op. 19/6 had consisted not of nine notes but of twelve – in other words, that the content of E had been the entire chromatic scale. In this case showing that its harmonic content included that of the other sections would have been totally meaningless: everything is contained within the content of the chromatic scale, from Beethoven's Ninth Symphony to Stockhausen's *Zeitmasse*. At the other extreme, recall what I said in the last chapter about how meaningless it would be to derive music from a single motivic cell consisting of a second (p. 109 above). At either extreme everything can be derived from anything. That is why Forte restricts himself to a central range of sizes in which the relationships you find are likely to be of some significance. So that was the first way in which the number of possible pitch class formations is kept within manageable proportions. The second has to do with the fact that in this kind of analysis we are interested in pitch class formations regardless of the particular transposition in which they occur, and regardless of whether they appear one way up or in inversion. Let us use the content of section D in Op. 19/6 as an example, writing it numerically (but you can read it as music if you like). We do not want to have one name for [0, 1, 2, 5], another name for transpositions like [1, 2, 3, 6] and another name for inversions like [0, 11, 10, 7]; we want all of these to have the same name, so that whenever we come across one of them we will immediately be able to see that it is the same as the others. And this is what Forte does. Each of these is a different version of a single pitch class set – or *pc set*, as Forte abbreviates it – which, as it happens, he calls 4–4. The first 4 means that there are four elements in the set (that is to say, there are four pitch classes in any particular version of it); the second 4 means that it comes fourth in his listing of the sets with four elements. And because there is only one pc set for this formation in all its various transpositions and inversions, the total number of possible pc sets of between three and nine elements becomes surprisingly small: there are in fact 208 of them. Forte lists them in an appendix to his book.

Fig. 63

(1) 

(2) 

(3) 

(4) [0, 1, 2, 5] [0, 1, 2, 5] [0, 1, 2, 5] [0, 1, 2, 5]

Of course you need a set of rules to tell you how to work out the correct name of any particular pitch class formation you may come across, and this is rather like identifying a butterfly from one of those books that ask you a series of questions until there is only one possibility left: it is simple in principle but a bit involved in practice. Let us take four separate versions of the pc set 4–4: the version we found in Op. 19/6; a transposition of it; an inversion; and another inversion, in which the registration is different. As shown in the top line of Fig. 63 these all look different, but we want them all to come out the same. Forte gives a formal procedure for establishing what pc sets these all belong to, and this is useful where you are dealing with very big or rather similar sets, or if you want a computer to do the work for you; but usually it is easier to do it by eye, so I am consigning Forte's procedure to a footnote.[1] First of all you have to establish whether the version you are looking at is in its most compact

[1] Rewrite whatever version you have numerically, with 0 as the lowest note (Fig. 64, line 2). Jot down the last number (for [0, 1, 8, 11] this gives 11); permutate the numbers so the first becomes the last and add 12 to it, giving [1, 8, 11, 12]; subtract the first note from the last and jot this down (12 – 1 = 11 again). Repeat the process of permutation, addition and subtraction until you are back at the first note: this gives you [8, 11, 12, 13] and [11, 12, 13, 20] and hence the new values (13 – 8 = 5) and (20 – 11 = 9). Now select the lowest of the values you've jotted down, which is 5. The normal order of the pc set is the one that gave you this value (that is, [8, 11, 12, 13]), except that you must now write the first number as 0 and subtract its value from the others, giving [0, 3, 4, 5]. Line 3 of Fig. 64 shows this; only inversely-related versions of the pc set look different now. Choose whichever version gives the lower second number, or if both yield the same second number then the lower third number, and so on. All this is essentially the same method as the one I describe informally in the main text.

Fig. 64

(1) 

(2) [0, 1, 2, 5] [0, 1, 2, 5] [0, 3, 4, 5] [0, 1, 8, 11]

(3) [0, 3, 4, 5]

(4) [0, 1, 2, 5] [0, 1, 2, 5]

form, in the sense of having the smallest possible interval between its highest and lowest notes; you can see that in this case the smallest interval into which the whole pattern can fit is a perfect fourth, which means that all except the final version are already in their most compact form (their *normal order*, as Forte calls it). So you would rewrite the final version, as in the second line of Fig. 63. Next you look at the version you are dealing with in order to see whether the interval between its first two notes is bigger or smaller than the interval between its last two notes. What you are doing here is checking it against its inversion, and you choose whichever gives the smallest interval; so the first two versions in Fig. 63 remain the same, while the last two have to be inverted. And now you turn the notes into numerals in the same way as before, calling the lowest note '0'; this gets rid of the differences in transposition between the versions, so they all come out as [0, 1, 2, 5]. This means that 0, 1, 2, 5 is the *prime form* of this pc set. And now you simply look up [0, 1, 2, 5] in the appendix to Allen Forte's book, where you find the following entry:

4–4   0, 1, 2, 5   2 1 1 1 1 0

4–4, as I said, is the name of the pc set; 0, 1, 2, 5 is its prime form; and 2 1 1 1 1 0 is its interval vector, which I shall explain shortly. And what happens if you cannot find the prime form you are looking up in Forte's table? You check your calculations, because you have made a mistake.

If you are thinking that this isn't musical analysis, then you are right, because all that it achieves is a standardized way of naming the pc set. No musical decisions have been involved; and it would not in the least matter what you called the pc set, or which version of it you took as the prime form, provided that you were always consistent. But from now on you can begin to draw genuine analytical conclusions, since the various pc sets you discover in a piece can relate to each other in a number of ways. For example, you might find that two sets were similar, in that they both contained a third, smaller set which also functioned as an independent musical element. Or you might find that the various sets used in a piece all shared the same or similar *interval vectors*. This, you remember, was the six-digit number Forte gives for each pc set in the appendix to his book; for 4–4 it was 2 1 1 1 1 0. This simply means that if you look at the intervals between all the different notes of the pc set in any given version of it, and assume octave equivalence, you will find two minor seconds; one major second; one minor third; one major third; one perfect fourth; and no augmented

fourths. Of the 208 pc sets, there are only 19 pairs that share the same interval vector; Forte calls these *Z-related* sets and puts a 'Z' in their name (for example 6–Z6), so that when you find one of these pc sets in a piece you are alerted to the possibility that interval vectors will play an important unifying role in it.

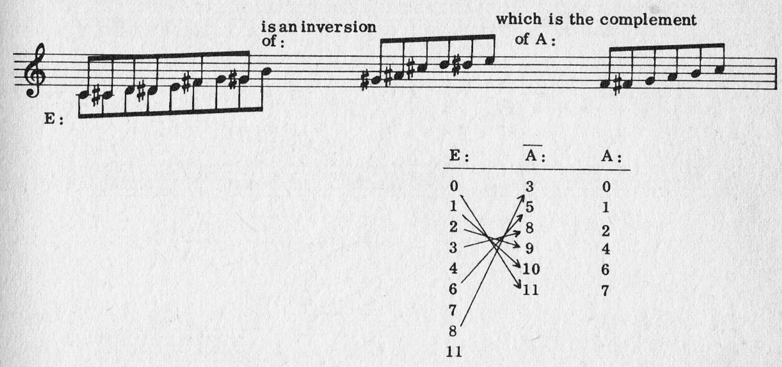
But much the most important way that different pc sets can be related, in Forte's eyes, is through their being members of the same *set complex*. Now, a set complex is a grouping of pc sets, rather in the same way that a pc set is a grouping of individual pitch class patterns; except that there is an important difference, in that a pc set is a grouping of equivalent patterns of the same size, whereas a set complex consists of a pc set plus all the pc sets of *different* sizes that can be included within it through various types of relationship. You might find it useful to think of this by analogy with a tree: the leaves belong to the set 'leaf' and the branches to the set 'branch', whereas the complex 'tree' includes the leaves and the branches, along with the trunk, the twigs and so on. Actually we have met a set complex before, though not under that name. When we looked at Op.19/6, we found that the sets of all its sections were included within the set of section E: that is, E either included the notes of the other sections, or it included them when transposed, or it included them when inverted, or else it included the complement of one of these. And this means that the sets of all of the sections of Op. 19/6 are members of the complex about the set of section E – as are also a large number of other pc sets which do not appear in this piece. When everything in a piece can be derived from a single set complex in this way, Forte calls the structure *connected*, and the main thing a set-theoretical analyst is trying to do when he analyzes a piece of music is to show how apparently unrelated pitch formations in it do in fact belong together by virtue of their common membership of a set-complex.

Forte's name for the pc set in section E of Op. 19/6 happens to be 9–4 (meaning, you remember, that it comes fourth in his list of sets with nine elements), and he would refer to the complex about this set as K(9–4). Actually it would be more correct to call it K(3–4, 9–4). This is because any set-complex involves the principle of complementation, and 3–4 is the complementary pc set to 9–4 (Forte aligns sets in his list so that complementary sets have the same order number). What this means is that K(9–4) automatically includes K(3–4), and K(3–4) automatically includes K(9–4) – in other words, there is only one set complex for 3–4 and 9–4, and therefore there really ought to be only one name for the complex: K(3–4, 9–4). However, people find it more convenient to refer to the complex either as K(3–4) or K(9–4) – depending whether it is pc set 3–4 or 9–4 that is appearing in the music – so you have to bear in mind that both names actually refer to the same thing.

Because of this principle of complementation, there are considerably fewer set complexes than there are pc sets – 114 as against 208 (the number is a bit more than half because there are a few sets that do not have complements – for example, the complement of the whole-tone scale is the whole-tone scale). However, though there is a manageable number of set complexes, there is a difficulty with them, and it is a difficulty which is rather typical of set-theoretical analysis. This is that the set complex associates so many pc sets with one another that the relationship can verge on the meaningless. As Forte says, 'examination of a particular composition . . . might yield the information that every 4-element set represented in the work belongs to K(3–2). Yet K(3–2) is but one of seven set complexes about sets of cardinal 3 which contain *all* 4-element sets . . . Reduction to a useful and significant subcomplex is evidently needed' (p. 96). So he defines a special type of relationship which holds only for certain members within a given set complex, which he calls the *subcomplex Kh* and to which he ascribes a particularly high degree of significance.

What exactly is the difference between the complex K and the subcomplex Kh? To understand this we have to look in a bit more detail at what it means for two pc sets to be members of the same set complex. Let us go back to the sets we found in Op. 19/6. You remember that we regarded one set as related to another *either* if one included the other (whether literally or under transposition or inversion) *or* if it included (or was included within) the complement of the other. For example, Fig. 56 showed how the set of B included that of D, whereas Fig. 59 showed how it included the complement of F. Now these relationships do not work the other way round: that is, the set of B neither includes nor is included in the set of F, and equally it does not include nor is it included in the complement of D. Either the one condition of set-complex membership is fulfilled or the other; but in neither of these cases are both conditions fulfilled. Sometimes, however, both conditions can be. Look at the relationship between the sets of E and A. In Fig. 57 I showed how E included A under inversion. But I could equally well have shown how E included the complement of A under inversion: Fig. 65 shows how. So here both conditions for membership of a set complex are fulfilled. And that is what defines the subcomplex Kh.

Fig. 65



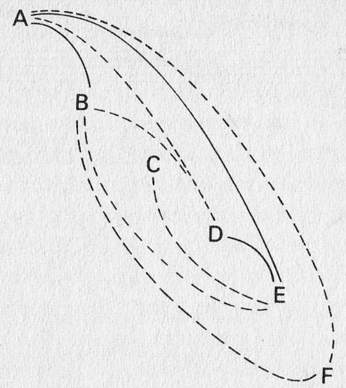
Now, in the analysis I gave of Op. 19/6 I regarded sets as related if they were in the relation K – if either condition of set membership was fulfilled, that is to say.[1] But it would have been possible to distinguish two grades of relationship, one corresponding to K and the other to Kh. Let us see how this would have affected our interpretation of the piece. Fig. 66 shows an improved version of the 'mileage chart' I gave before, while Fig. 67 refines the earlier form-chart (Fig. 62) by showing K relations between sections in a dotted line and Kh relations in a solid one. If we had considered *only* the Kh relations, then our analysis would

Fig. 66

B	Kh				
C					
D	K	K			
E	Kh	K	K	Kh	
F	K	K			K
	A	B	C	D	E

[1] Strictly this is not correct. Part of Forte's definition of a set complex is that two sets cannot be in relation K if they are of the same size (that is obvious, since otherwise they would be the same set) or if they are of complementary sizes – so that 4–n cannot be a member of K(8–m). It is true that relationships between sets of complementary sizes are not as general in their scope as true K-relations, and such sets can never be in relation Kh. But it is sometimes useful to regard them as related all the same, and I have done so in my analysis of Op. 19/6. You could always call such sets 'L-related' to avoid confusion.

Fig. 67



not have made a lot of sense: it would not have shown what sections C and F were doing in the piece at all. But the Kh relations do make sense when seen as reinforcing certain of the K relations: they underline the special role of sections C and F (C being a kind of counter-subject, and F a coda) in contrast to the continuity of the rest of the piece.

In his book Forte goes into a great deal more detail than I have been able to give. But if you understand what a pc set is and how to identify one, and if you know what a set complex is and what the subcomplex Kh is, then you have a basic working knowledge of set-theoretical analysis; so what is more to the point than elaborating the theory is to see it in action by working through one of Forte's analyses. Forte devotes ten pages of his book to *Excentrique*, the second of Stravinsky's Four Studies for Orchestra, which is a reworked version of one of his Three Pieces for String Quartet. As before, the first step in the analysis is to chop up the music into formal sections, and Fig. 68 shows how Forte does this.[1] In this piece some of the sections are very similar to others, so Forte labels them in the traditional way, with A^2 being a variant of A^1 and so on. The rest of the analysis is in effect based on this 'condensed score', as Forte calls it, which omits instrumentation, rhythms, dynamic markings and immediate repetitions – or more correctly, it includes these things but only by implication, in that they are the basis for the division into sections. So that you can see just how

[1] I have added bar numbers to Forte's chart, which is on pp. 132–3 of *The Structure of Atonal Music*. In discussing Forte's analysis I make a few minor additions to it where these are necessary for clarity.

Fig. 68 Forte, condensed score of Stravinsky's *Excentrique*