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CRACKS IN SOLIDS

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Abstract—The subject of this paper is the application of basic ideas and methods of continuum mechanics to the crack propagation processes. The crack extension is governed by an additional condition at the crack-tip. As a consequence of this a problem of "fine" structure of the crack-tip is considered. The general additional condition for any model of continuum is obtained making use of the energy conservation law and of the physical concept about the fracture energy. Dynamic cracks in elastic solids and quasi-static cracks in elastic- and rigid-plastic solids are briefly considered, as well as a problem of the crack extension in dissipating viscoelastic bodies. The general approach is also applied to the case of fatigue and "fluctuation" cracks.

INTRODUCTION

CONSIDER a solid which has displacement discontinuity surfaces (cracks), deformations being small. To be certain we shall confine ourselves to the case of opening mode cracks for which the local symmetry condition holds. Single out a vicinity of an arbitrary point O on the smooth contour of the crack which is small compared to the characteristic linear size of the crack. Let xyz be a cartesian coordinate system with the point O, as a center, y-axis being perpendicular to the crack surface and z-axis being directed along the boundary. The vicinity under consideration ("fine" structure of the crack) is represented on the xy-plane as an infinite domain which has a load-free cut along the negative direction of x-axis, (Fig. 1). It is clear that the parameters describing the state of the medium in this small vicinity are independent of z.

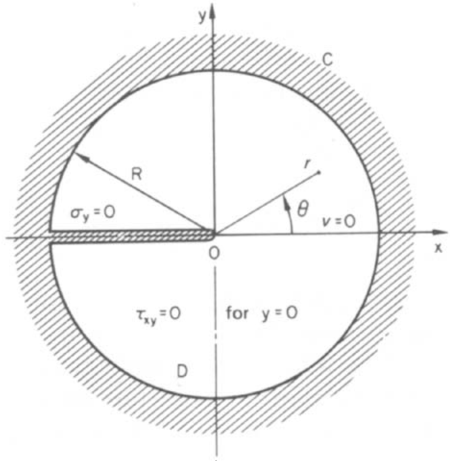

FIG. 1

Let the singular solution of the problem formulated above be determined for any given rheological model of the body with an accuracy of one free parameter N independent of x and y. The singular solution is certainly to satisfy a natural physical requirement of the energy flux finiteness at the crack-tip, i.e. at the point O. We shall make a general assumption on the existence of some physical law governing the monotonous propagation of the crack; the latter can be written in the form of some mathematical relationship between parameters of the medium and/or their functional time and spatial characteristics in the small vicinity of the crack-tip (the assumption of local fracture). This trivial assumption is generally accepted. Since all the parameters of the medium in the neighborhood of the point O depend on N, any relationship is inevitably reduced to the following one

$$\Phi N = 0 \tag{0.1}$$

where Φ is a time operator specific to the given material. A solution of the operator equation (0.1) exists and is unique due to the same assumption; it can be written in the form

$$N = N_0(t, C_1, C_2, \ldots). \tag{0.2}$$

Here t is time, $N_0(t, C_1, C_2 \ldots)$ is a function (the same for the given material), $C_1, C_2 \ldots$ are values determined by initial data. Notice that N_0 is also a function of regular parameters of state at the point O, e.g. temperature or concentration of components for a solid solution. The function N_0 includes, of course, some physical constants of the material distinguishing a fracture mechanism and depending on the crack velocity.

For the particular case of brittle and quasibrittle bodies, from a generally accepted assumption on the existence of the law which governs the crack extension and is independent of time, one obtains the following limiting condition

$$N = K_c \tag{0.3}$$

where K_c is a universal material constant (constant of Irwin†). As N in this case one usually takes the stress intensity factor. The condition equation (0.3) represents the basic concept of brittle and/or quasibrittle fracture. The latter is thus not dependent on the physical nature of fracture and is accounted for purely logical reasons.

Let us briefly recall how various authors came to this condition.

In accordance with most natural and general physical concepts by Griffith–Irwin–Orowan to form a unit of the crack surface one must spend some energy γ_*, which represents a material constant [1–3]. The mathematical formulation of these view points given by Irwin [4, 5] resolves itself into the limiting condition

$$\begin{aligned} N^2 &= \frac{E\gamma_*}{\pi} \quad \text{(plane stress‡)} \\ N^2 &= \frac{E\gamma_*}{\pi(1-\nu^2)} \quad \text{(plane strain)} \end{aligned} \tag{0.4}$$

where E is Young's modulus, ν is Poisson's ratio. Williams connected the crack extension with the limiting radius of curvature of the crack at its tip [6]. Neuber advanced the conception of the "plastic" particle, which can be reformulated as follows: the size d of the plastic region near the end of the crack is a structure material constant [7]. Leonov and Panasyuk accepted that on a length along the crack line prolongation the stress σ_y equals

† With the accuracy of a constant factor.
‡ For the plates of finite thickness γ_* depends on the plate thickness.

to the theoretical strength and the displacement v of the opposite coasts, at the point O equals to a material constant [8]. The stress σ_y on an interval of the length d along the crack line prolongation as well as the value d are believed by Barenblatt to be material characteristics [9]. McClintock assumed for the case of shear, that at an interval d ahead of the crack-tip in the plastic region the deformation attains a limiting value [10]. For all the cases the constants d are thought to be small in comparison with the crack length. All the concepts mentioned above lead to the limiting condition equation (0.3) first derived by Irwin [4, 5]. (Neuber and McClintock did not point out explicitly this condition).

1. THE LIMITING CONDITION AT THE CRACK-TIP

To find the function N_0 theoretically, we apply the most natural and general physical concepts about specific surface energy and/or fracture energy γ_*, which are analogous to the Griffith–Irwin–Orowan concepts. By γ_* we shall imply the work of irreversible deformations in the vicinity of the crack boundary which are not taken into account in the assumed model of continuum.

Let us confine ourselves to processes with contribution by mechanical and heat energy only. Let C be the circle of radius R with point O as the center (Fig. 1). The radius R is held small as compared to the crack length, but very large in the singular problem (as the region D was under the microscope). Let us fix the circle C and study the deformation and fracture process of the continuum D, located inside C. Let $r\theta$ be polar coordinates with the origin at the point O.

According to the energy conservation law the power $\dot{A}$ of the external forces plus the heat input rate $\dot{Q}$ is equal to the rate of increase of the sum of the kinetic $\dot{K}$ and intrinsic $(\dot{W}+\dot{\Pi})$ energy of the body in the domain D

$$
\begin{aligned}
&\dot{A}+\dot{Q} = \dot{K}+\dot{W}+\dot{\Pi} \\
&\dot{A} = R\int_{-\pi}^{+\pi}[(\sigma_x\cos\theta+\tau_{xy}\sin\theta)\dot{u}+(\tau_{xy}\cos\theta+\sigma_y\sin\theta)\dot{v}]\,\mathrm{d}\theta \\
&\qquad +\int_D \rho(F_x\dot{u}+F_y\dot{v})\,\mathrm{d}x\,\mathrm{d}y \\
&\dot{Q} = R\int_{-\pi}^{+\pi}(\dot{q}_x\cos\theta+\dot{q}_y\sin\theta)\,\mathrm{d}\theta \\
&\dot{K} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_D \rho(\dot{u}^2+\dot{v}^2)\,\mathrm{d}x\,\mathrm{d}y \\
&\dot{W} = \frac{\mathrm{d}}{\mathrm{d}t}\int \rho U\,\mathrm{d}x\,\mathrm{d}y \qquad \dot{\Pi} = 2\gamma_*\dot{l}.
\end{aligned}
\tag{1.1}
$$

Here u, v are displacements, $\sigma_x, \sigma_y, \tau_{xy}$ are stresses, $\dot{q}_x, \dot{q}_y$ are components of the heat flux, (F_x, F_y) is the volume force, ρ and U are the mass and intrinsic energy densities, $\dot{l}$ is the crack extension velocity.

Convert the condition (1.1) into the more convenient form [11], the singular solution at the point O being dependent on one free parameter N. Let the crack length l play the role of time. Then the energy conservation law can be written as

$$\left(\frac{\delta A}{\delta N}+\frac{\delta Q}{\delta N}-\frac{\delta K}{\delta N}-\frac{\partial W}{\partial N}\right)_{\delta l=0}\frac{\mathrm{d}N}{\mathrm{d}l}+\left(\frac{\delta A}{\delta l}+\frac{\delta Q}{\delta l}-\frac{\delta K}{\delta l}-\frac{\partial W}{\partial l}-2\gamma_*\right)_{\delta N=0}=0. \tag{1.2}$$

The factor at $\mathrm{d}N/\mathrm{d}l$ in equation (1.2) equals zero because of the energy conservation law for the fixed crack. The equation (1.2) results in the form [11]

$$\begin{aligned} &R\int_{-\pi}^{+\pi}\left[(\rho U+K_*-\rho H)\cos\theta+\frac{1}{\dot{l}}(\dot{q}_x\cos\theta+\dot{q}_y\sin\theta)-A_*\right]\mathrm{d}\theta=2\gamma_* \\ &K_*=\frac{1}{2}\rho\dot{l}^2\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2\right] \\ &A_*=(\sigma_x\cos\theta+\tau_{xy}\sin\theta)\frac{\partial u}{\partial x}+(\tau_{xy}\cos\theta+\sigma_y\sin\theta)\frac{\partial v}{\partial x} \\ &\rho H=\int\rho\left(F_x\frac{\partial u}{\partial x}+F_y\frac{\partial v}{\partial x}\right)\mathrm{d}x. \end{aligned} \tag{1.3}$$

Here all the values are calculated directly from the singular solution which corresponds to the crack-tip state at the point O, N being constant.

With the use of the local energy conservation law [12]

$$\sigma_x\dot{\varepsilon}_x+\sigma_y\dot{\varepsilon}_y+2\tau_{xy}\dot{\varepsilon}_{xy}+\frac{\partial\dot{q}_x}{\partial x}+\frac{\partial\dot{q}_y}{\partial y}=\rho\dot{U} \tag{1.4}$$

and by means of the divergence theorem the equation (1.3) can be reduced to the following most convenient form in which only stress and strain distributions in the neighborhood of the crack boundary are present

$$\begin{aligned} &R\int_{-\pi}^{+\pi}[(\mathcal{3}+K_*-\rho H)\cos\theta-A_*]\,\mathrm{d}\theta=2\gamma_* \\ &\mathcal{3}=\int\sigma_x\,\mathrm{d}\varepsilon_x+\sigma_y\,\mathrm{d}\varepsilon_y+2\tau_{xy}\,\mathrm{d}\varepsilon_{xy}. \end{aligned} \tag{1.5}$$

It is noteworthy that mechanical and heat properties of the medium were not involved; we used only its continuity. Evidently, the crack will not extend (i.e. $\dot{l}$ will be zero) if the left-hand side of the equation (1.5) will be less than $2\gamma_*$.

Each term in the integrand in equation (1.5) has to have a singularity of the type of $1/r$ at the crack-tip, so that its contribution to the total sum would be finite. Singularities of the order $r^{-\lambda}$ $(\lambda>1)$ are not allowed as they would cause the violation of the energy conservation law. Terms with weaker singularity $r^{-\lambda}$ $(\lambda<1)$ fall out, obviously.

One can demonstrate that the equation (1.5) is also valid for finite strains of the continuum, if one repeats the calculations of the work [11] and uses the local energy conservation law for the case of finite deformations [13] ($\mathcal{3}$ will stand there for the strain-energy).

2. A CONSEQUENCE OF THE SECOND LAW OF THERMODYNAMICS

Among the fundamental laws only the energy conservation law was used above. The utilization of basic laws of irreversible thermodynamics (first of all, the Gibbs equation and the second law of thermodynamics [14, 12]) allows one to obtain further results. Here we shall confine ourselves to the quasistatic cracks for which $K_* = 0$. Denote

$$\frac{\partial u}{\partial x} = \left(\frac{\partial u}{\partial x}\right)_e + \left(\frac{\partial u}{\partial x}\right)_i \qquad \frac{\partial v}{\partial x} = \left(\frac{\partial v}{\partial x}\right)_e + \left(\frac{\partial v}{\partial x}\right)_i$$
$$\rho H = \rho H_e + \rho H_i \qquad A_* = A_{*e} + A_{*i} \qquad \gamma_* = \gamma + \gamma_i. \tag{2.1}$$

Here the index e indicates the corresponding reversible (elastic) components and the index i marks the irreversible ones. The value γ represents the reversible part of the total fracture energy γ_* (the effective specific surface energy). For the case of crystalline and polycrystalline bodies γ seems to be equal to the true surface energy; for polymers it can be evidently much more than the latter. The value γ_i equals to the irreversible deformation work in the surface layer; the layer thickness and the magnitude of γ_i are prescribed by the degree of adequacy of the chosen model to the properties of the real body in question. Particularly, γ_i equals zero identically if the mathematical model describes quite exactly the mechanical and heat properties of the body. It must be emphasized that the thickness of the above-mentioned surface layer is assumed to be zero for the present formulation of the problem (i.e. it is considered to be small compared to R).

In the case under consideration one can write the Gibbs equation on the basis of equations (1.3) and (2.1) as follows

$$R\int_{-\pi}^{+\pi} [(\rho U - \rho H_e)\cos\theta - A_{*e}]\,\mathrm{d}\theta + TR\int_{-\pi}^{+\pi} \rho\eta\cos\theta\,\mathrm{d}\theta = 2\gamma. \tag{2.2}$$

Here η is the entropy density at a point of the domain D which forms the thermodynamical system under study. The absolute temperature T near the crack-tip is assumed to be finite. Because of equations (2.2) and (1.3) the entropy flux into the D per unit time equals to

$$\dot{l}R\int_{-\pi}^{+\pi} \rho\eta\cos\theta\,\mathrm{d}\theta = \frac{\dot{Q}}{T} - \frac{\dot{l}}{T}\left[2\gamma_i + R\int_{-\pi}^{+\pi} (\rho H_i\cos\theta + A_{*i})\,\mathrm{d}\theta\right] \tag{2.3}$$

The second term in the right-hand side of equation (2.3), taken with inverse sign, represents the flux of the intrinsic increment of entropy per unit time. According to the second law of thermodynamics this flux must be non-negative.

$$\frac{\dot{l}}{T}\left[2\gamma_i + R\int_{-\pi}^{+\pi} (\rho H_i\cos\theta + A_{*i})\,\mathrm{d}\theta\right] \geqslant 0. \tag{2.4}$$

Since the quantity in square brackets is the dissipation energy rate and is therefore essentially positive, equation (2.4) results in the irreversibility condition for the crack extension

$$\dot{l} \geqslant 0. \tag{2.5}$$

Within the framework of irreversible thermodynamics the crack expansion velocity $\dot{l}$ can be treated as a thermodynamical flux and the term in square brackets divided by T as a thermodynamical force.

The phenomenological linearity postulate accepted in Onsager's theory [14, 12] leads to the following expression

$$l = \frac{\alpha}{T}\left[2\gamma_i + R\int_{-\pi}^{+\pi}(\rho H_i \cos\theta + A_{*i})\,\mathrm{d}\theta\right] \tag{2.6}$$

where α is a material constant.

3. DYNAMIC CRACKS IN AN ELASTIC BODY

Consider a crack in an ideal elastic body, the contour of the crack extending at an arbitrary velocity less than that of transverse waves. For a sufficiently small time interval one can, clearly, always choose such a small vicinity of any fixed point O of the contour that the stress and strain distribution in this vicinity would correspond to the constant propagation velocity of the crack. It follows that the stress and strain distribution near any point of the dynamic crack boundary at any moment will be exactly the same as for the case of a semi-infinite straight cut moving at constant speed v ("the microscope principle").

The stresses and displacements in the stationary dynamical problem of the theory of elasticity for the cut $y = 0$, $x < vt$ (the singular solution) can be easily found with the help of the Galin's method [15, 16]

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{N}{\sqrt{r(MH-GT)}}\left[-\frac{AT\cos\frac{1}{2}\operatorname{arc\,tg}(\kappa_1\operatorname{tg}\theta)}{(\cos^2\theta+\kappa_1^2\sin^2\theta)^{\frac{1}{4}}}+\frac{BM\cos\frac{1}{2}\operatorname{arc\,tg}(\kappa_2\operatorname{tg}\theta)}{(\cos^2\theta+\kappa_2^2\sin^2\theta)^{\frac{1}{4}}}\right]\\
\frac{\partial v}{\partial x} &= \frac{N}{\sqrt{r(MH-GT)}}\left[\frac{CT\sin\frac{1}{2}\operatorname{arc\,tg}(\kappa_1\operatorname{tg}\theta)}{(\cos^2\theta+\kappa_1^2\sin^2\theta)^{\frac{1}{4}}}-\frac{DM\sin\frac{1}{2}\operatorname{arc\,tg}(\kappa_2\operatorname{tg}\theta)}{(\cos^2\theta+\kappa_2^2\sin^2\theta)^{\frac{1}{4}}}\right]\\
\sigma_x &= \frac{N}{\sqrt{r(MH-GT)}}\left[-\frac{LT\cos\frac{1}{2}\operatorname{arc\,tg}(\kappa_1\operatorname{tg}\theta)}{(\cos^2\theta+\kappa_1^2\sin^2\theta)^{\frac{1}{4}}}+\frac{FM\cos\frac{1}{2}\operatorname{arc\,tg}(\kappa_2\operatorname{tg}\theta)}{(\cos^2\theta+\kappa_2^2\sin^2\theta)^{\frac{1}{4}}}\right]\\
\sigma_y &= \frac{N}{\sqrt{r(MH-GT)}}\left[-\frac{GT\cos\frac{1}{2}\operatorname{arc\,tg}(\kappa_1\operatorname{tg}\theta)}{(\cos^2\theta+\kappa_1^2\sin^2\theta)^{\frac{1}{4}}}+\frac{MH\cos\frac{1}{2}\operatorname{arc\,tg}(\kappa_2\operatorname{tg}\theta)}{(\cos^2\theta+\kappa_2^2\sin^2\theta)^{\frac{1}{4}}}\right]\\
\tau_{xy} &= \frac{NMT}{\sqrt{r(MH-GT)}}\left[\frac{\sin\frac{1}{2}\operatorname{arc\,tg}(\kappa_1\operatorname{tg}\theta)}{(\cos^2\theta+\kappa_1^2\sin^2\theta)^{\frac{1}{4}}}-\frac{\sin\frac{1}{2}\operatorname{arc\,tg}(\kappa_2\operatorname{tg}\theta)}{(\cos^2\theta+\kappa_2^2\sin^2\theta)^{\frac{1}{4}}}\right]
\end{aligned} \tag{3.1}$$

where

$$\begin{aligned}
A &= -\frac{1}{1-2v}\sqrt{\left(1-\frac{1-2v}{2-2v}m^2\right)} \qquad B = -\frac{1}{1-2v}\sqrt{(1-m^2)}\\
C &= \frac{1}{1-2v}\left(1-\frac{1-2v}{2-2v}m^2\right) \qquad D = \frac{1}{1-2v}\\
L &= -\frac{E}{(1+v)(1-2v)}\left(1+\frac{vm^2}{2-2v}\right)\sqrt{\left(1-\frac{1-2v}{2-2v}m^2\right)}\\
H &= -F = \frac{E}{(1+v)(1-2v)}\sqrt{(1-m^2)}
\end{aligned} \tag{3.2}$$

$$G = \frac{E(2-m^2)}{2(1+\nu)(1-2\nu)}\Big/\sqrt{\left(1-\frac{1-2\nu}{2-2\nu}m^2\right)}$$

$$M = \frac{E}{(1+\nu)(1-2\nu)}\left(1-\frac{1-2\nu}{2-2\nu}m^2\right)$$

$$T = \frac{E}{(1+\nu)(1-2\nu)}(1-\tfrac{1}{2}m^2)$$

$$\kappa_1^2 = 1-\frac{v^2}{c_1^2} \qquad \kappa_2^2 = 1-\frac{v^2}{c_2^2} \qquad m = \frac{v}{c_2}$$

$$c_1^2 = \frac{(1-\nu)E}{(1+\nu)(1-2\nu)\rho} \qquad c_2^2 = \frac{E}{2(1+\nu)\rho}.$$

Following Irwin, the free parameter N will be referred to as the stress intensity factor

$$N = \lim(\sigma_y\sqrt{r}) \qquad \text{as } \theta = 0, r \to 0 \tag{3.3}$$

The parameter N is a function of time, boundary conditions, body configuration, crack speed and acceleration, coordinates of the point O in a fixed coordinate system; this function is determined from the solution of the problem as a whole. For the stationary problem N is independent of time and the crack velocity. The curve $y = \text{arc tg}(\kappa \text{ tg } \theta)$ in the interval of interest, $-\pi \lessgtr \theta \lessgtr \pi$, behaves itself roughly as shown in Fig. 2.

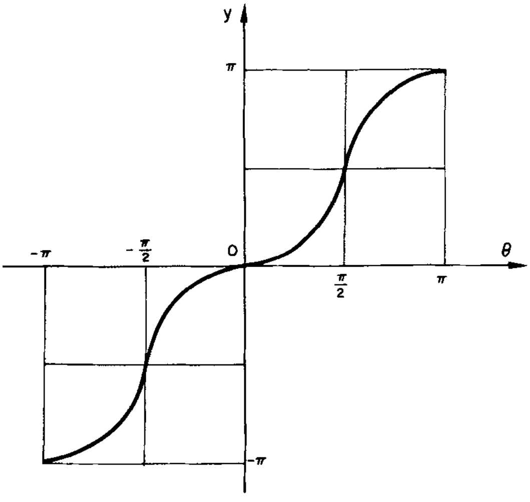

FIG. 2

Now we show the simplest form of the general condition equation (1.5). Making use of the procedure of the work [11] the following equation can be derived by varying the contour C

$$\lim \int_{-R}^{+R} \left(\sigma_y \frac{\partial v}{\partial x}\right)\Bigg|_{\text{for } y=\delta} \mathrm{d}x = \gamma_* \tag{3.4}$$

as $R \to 0$, $\delta/R \to 0$.

Next we calculate the following integral

$$\int_0^{\pi} \frac{\sin \frac{1}{2} \operatorname{arc\,tg}(\kappa_2 \operatorname{tg} \theta) \cdot \cos \frac{1}{2} \operatorname{arc\,tg}(\kappa_1 \operatorname{tg} \theta)\, d\theta}{\sin \theta (\cos^2\theta + \kappa_1^2 \sin^2\theta)^{\frac{1}{4}} (\cos^2\theta + \kappa_2^2 \sin^2\theta)^{\frac{1}{4}}} = \frac{\pi}{2}. \tag{3.5}$$

By virtue of equations (3.1), (3.2) and (3.5) the condition at the crack boundary (3.4) can be written as

$$N^2 = R(m, \nu)\Phi(m, \nu)E\gamma_* \tag{3.6}$$

where

$$R(m, \nu) = \sqrt{\left[(1-m^2)\left(1 - \frac{1-2\nu}{2-2\nu} m^2\right)\right]} - (1 - \tfrac{1}{2}m^2)^2$$

$$\Phi(m, \nu) = \frac{4}{\pi(1+\nu)m^2\sqrt{\{1 - [(1-2\nu)/(2-2\nu)]m^2\}}}.$$

The value $2N^2/(ER(m, \nu)\Phi(m, \nu))$ has the meaning of the total energy flux into the moving crack-tip. The condition equation (3.6) implies that the Rayleigh velocity m_R, which is a root of the equation $R(m_R, \nu) = 0$, is an unattainable limit for the crack-speed because it takes an infinitely great energy flux to the crack-tip to keep such a speed of the crack propagation. In homogeneous materials the maximum velocity is limited even earlier by the value m_*, at which the crack-twinning occurs. The function $m_*(\nu)$, plotted in Fig. 3, can be easily found by means of equation (3.1) (see [16]). This qualitative result was obtained first by Ioffe [17].

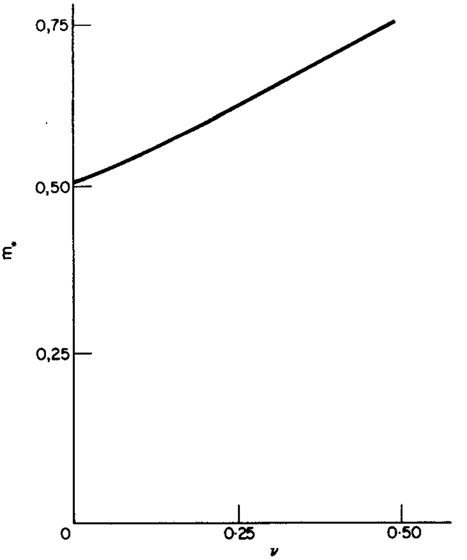

FIG. 3

The condition equation (3.6) plays a part of the boundary one for dynamic cracks; given further knowledge on the dependence of γ_* on the crack velocity (and on time for the non-steady case), this condition allows one to make the formulation of the problem closed. The present state of the dynamical crack problem is outlined in the work [18] by Irwin and Krafft.

4. QUASISTATIC CRACKS IN AN ELASTIC BODY

Let a crack in a linearly elastic body extend at a velocity much lower than that of sound, so that inertia terms can be neglected. In this case the general condition equation (1.5) and/or equation (3.4) results in the Irwin's condition equation (0.4) after substituting a proper singular solution of static elasticity (see [11, 19]). According to this condition a crack does not grow, if $N < K_c$; the equation $N = K_c$ corresponds to the crack development. Notice that Irwin obtained the limiting condition by means of the more particular procedure applicable only for linearly elastic bodies. If the crack surface is known beforehand, the Irwin's approach reduces the brittle-crack extension problem to the stress analysis near the crack boundary. At present this branch of the crack theory is developed most. There is a well selected list of works, mainly by British and American authors, in excellent reviews by Paris and Sih [20], by Weiss and Yukawa [21], and in earlier lectures by Sneddon [22].

5. CURVED BRITTLE CRACKS

If no special symmetry conditions are imposed, a brittle crack will develop on a surface, which is to be determined in the process of solution. We shall establish an additional condition determining the crack curvature radius at each point. Denote the parameter of the external load as p, and the crack length, measured from a fixed point, as l (for simplicity we confine ourselves to the plane problem). Let the equation of the crack line be $x = x_0(l)$, $y = y_0(l)$. Suppose that the values of the parameters l, p and $l+\Delta l$, $p+\Delta p$ correspond to the crack-tip state at points O and O_1, respectively (Fig. 4). The direction of

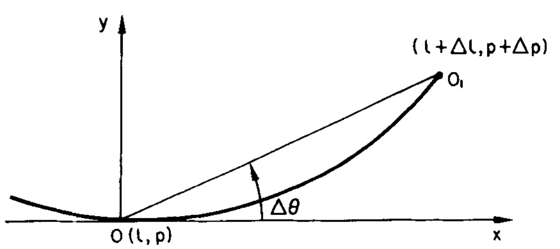

FIG. 4

the extension of a brittle crack from the point O is governed by the following rule [23–25], well confirmed experimentally: the crack is deflected in the direction of the maximum tension stress σ_θ, which is calculated in the neighborhood of the point O for the values l, $p+\Delta p$, when $\lim[\sqrt{r}(\sigma_\theta)_{\max}] = K_c$ as $r \to 0$.

The general singular solution of the theory of elasticity for the semi-infinite cut can be readily found by means of the Muskhelishvili's method [26]

$$\begin{aligned} \sigma_x+\sigma_y &= \frac{2}{\sqrt{r}}\mathscr{R}[(N+iN_1)\,e^{-\frac{1}{2}i\theta}] \qquad (r \ll R_0) \\ \sigma_y - i\tau_{xy} &= \frac{1}{\sqrt{r}}\left[(N+iN_1)\cos\frac{\theta}{2} - \tfrac{1}{4}(N-iN_1)(e^{\frac{3}{2}i\theta}-e^{\frac{1}{2}i\theta})\right]. \end{aligned} \tag{5.1}$$

Here $N_1(p, l)$ is a second stress intensity factor which is also determined only from the solution of the problem as a whole (for the case of the local symmetry $N_1 = 0$), R_0 is a curvature radius on the left from the point O (Fig. 4). According to equation (5.1) the stress σ_θ equals to

$$\sigma_\theta = \frac{1}{4\sqrt{r}}\mathscr{R}[3(N+iN_1)\,\mathrm{e}^{-\frac{1}{2}i\theta}+(N-3iN_1)\,\mathrm{e}^{\frac{3}{2}i\theta}]. \tag{5.2}$$

Hence the angle $\Delta\theta$ of the crack deflection can be found from the equation

$$N(\sin\tfrac{3}{2}\Delta\theta+\sin\tfrac{1}{2}\Delta\theta) = N_1(3\cos\tfrac{3}{2}\Delta\theta+\cos\tfrac{1}{2}\Delta\theta). \tag{5.3}$$

The angle $\Delta\theta$ can be finite, if only the parameter p jumps or the crack-tip state at the point O corresponds to an initial crack. For the case of the continuous variation of p and l the crack line will be smooth, and at the crack boundary the local symmetry condition will be satisfied. We then divide the process of the crack development into a finite number of steps, so that Δp and Δl would correspond to each step and the crack extension would be discontinuous. Performing a limiting process $\Delta p \to 0$, $\Delta l \to 0$, $\Delta\theta \to 0$ from equation (5.3) we obtain

$$2\left(\frac{\mathrm{d}N_1}{\mathrm{d}\theta}\right)_{l=\mathrm{const}} = N. \tag{5.4}$$

Here $\mathrm{d}N_1$ is an increment N_1 at the point O which corresponds to the load increase by $\mathrm{d}p$, the crack being fixed (remember that $N_1(p, l) = 0$).

Next, differentiate the identity $N[p, l(p)] = K$ which is true at any moment according to the Irwin's and local symmetry conditions.

$$\frac{\mathrm{d}l}{\mathrm{d}p} = -\frac{\partial N}{\mathrm{d}p}\bigg/\frac{\partial N}{\mathrm{d}l}. \tag{5.5}$$

The equations (5.4–5.5) allow us to find the curvature radius of the smooth crack line at any point

$$R_1 = \frac{\mathrm{d}l}{\mathrm{d}\theta} = -\frac{1}{2}N\frac{\partial N}{\partial p}\bigg/\left(\frac{\partial N}{\partial l}\frac{\partial N_1}{\partial p}\right). \tag{5.6}$$

The applied finite-differences method is convenient for a numerical solution of problems of the curved crack development; the conditions $N = K_c$ and equations (5.4–5.6) play there a part of the boundary one's at the crack border. It must be stressed that the local symmetry condition alone would be insufficient for solving the problem.

6. THE STABILITY OF THE BRITTLE-CRACK GROWTH

Let an increase of an external load and the growth of the crack correspond to an increase of the parameters p and l respectively. Then the parametric stability condition of the crack extension $\mathrm{d}l/\mathrm{d}p > 0$ takes the form (due to equation (5.5)).

$$\frac{\partial N}{\partial p}\bigg/\frac{\partial N}{\partial l} < 0. \tag{6.1}$$

If a brittle crack is growing stably, the crack velocity $\dot{l}$ at the point O must evidently be determined by the rate of the load increase $\dot{p}$

$$\dot{l} = -\dot{p}\frac{\partial N}{\partial p}\bigg/\frac{\partial N}{\partial l}. \tag{6.2}$$

For the case of an unstable crack the following condition is satisfied

$$\frac{\partial N}{\partial p}\bigg/\frac{\partial N}{\partial l} > 0 \tag{6.3}$$

so that after attaining a limiting state the crack goes over to the dynamic regime.

7. AN IDEAL ELASTIC–PLASTIC BODY. THE GRIFFITH PROBLEM

Consider a plate of an elastic–plastic material subjected to the Tresca–St. Venant yield criterion; the plate has a straight through crack of length $2l$, which is located in the homogenous field of a monotonously increasing tensile stress $\sigma_y = p$ normal to the crack line. The edges of the crack are assumed to be load-free (Fig. 5). It was shown in paper [19] that

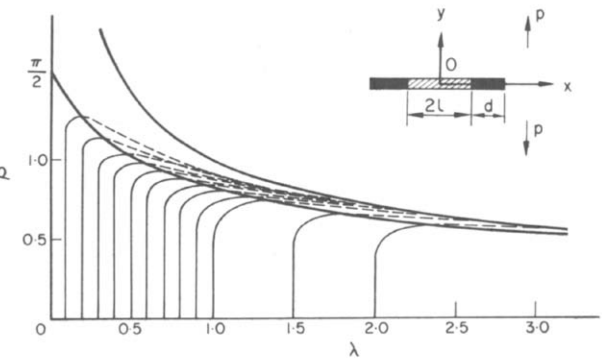

FIG. 5

for the values p, not too close to the tension yield-point σ_s, an exact solution of this problem satisfies the Dugdale's hypothesis [27], so that the plastic domains near crack-tips represent segments of length d along the crack-line extension

$$\frac{d}{l} = \sec\frac{\pi p}{2\sigma_s} - 1. \tag{7.1}$$

It was found in the same work by virtue of the energy conservation law, that the crack length is related to the load by the following first-order differential equation

$$\frac{\mathrm{d}\beta}{\mathrm{d}\lambda} = \frac{1-2\lambda(\ln\cos\beta+\beta\operatorname{tg}\beta)}{\lambda^2(\beta\sec^2\beta-\operatorname{tg}\beta)} \tag{7.2}$$

$$\left(\beta = \frac{\pi p}{2\sigma_s}, \qquad \lambda = \frac{2\sigma_s^2 l}{\pi E\gamma_*}\right).$$

Figure 5 illustrates the field of the integral curves of the equation (7.2) in the interval $0 < \lambda < \infty, 0 < \beta < \pi/2$. Continuous and dashed thin lines stand for stable and unstable parts of the curves respectively. The curve determined by the equation

$$2\lambda(\ln \cos \beta + \beta \operatorname{tg} \beta) = 1 \tag{7.3}$$

divides the whole domain of the variable variation into two parts: (1) a region where the crack is growing stably from the initial state and (2) an unstable region where $d\beta/d\lambda < 0$. At the beginning the crack extends thus monotonously with the load increase, then it attains a maximum load at the point of the intersection with the curve equation (7.3), and after that it goes over into the instability region. One sees readily that in the latter all the integral curves tend asymptotically to the Griffith–Irwin–Orowan curve as $\lambda \to \infty$

$$\lambda\beta^2 = 1 \quad \text{or} \quad p = \sqrt{\left(\frac{2E\gamma_*}{\pi l}\right)}. \tag{7.4}$$

The curves equations (7.3–7.4) are plotted with thick lines in Fig. 5.

The mentioned peculiarities of the development of the Griffith crack in elastic–plastic materials are borne out well by the experiments [2, 10].

The current state of the subject under consideration is presented in the paper [28] by McClintock and Irwin.

8. AN IDEAL ELASTIC–PLASTIC BODY. THE QUASI-BRITTLE FRACTURE CONCEPT

(i) In the work [19] it was also shown that the fracture process in plastic materials is controlled by the true surface energy γ; a dimension of the plastic zone near the crack-tip and the value of the fracture energy γ_* are fully determined by γ. Based upon the exact solution of the elastic–plastic problem for a plate with a semi-infinite crack, and upon the general approach to the crack propagation (see 1), the following relation was obtained

$$\gamma_* \approx \frac{E}{\sigma_s}\gamma. \tag{8.1}$$

Here the body is assumed to satisfy the Tresca criterion up to fracture. Thereby, in the energy conservation equation small magnitudes of the first order were taken into account.

Being well confirmed by the experiments [29, 5], the relation equation (8.1) allows us to treat the quasi-brittle fracture concept and the adsorbous Rehbinder's effect [30] from a single viewpoint. Equation (8.1) confirms the principal meaning of the true surface energy in the strength problem and this fact is in essence a return to the original idea by Griffith. It should be noted that an idea of a relation between γ_* and γ appears to be expressed first by Gilman [31]; but because of a too rough calculation, his relationship does not agree with experiments.

(ii) The Dugdale's hypothesis is valid [19] also for a crack in a plate of any shape and under any boundary conditions, if a plate material follows Tresca's criterion and the dimension of the plastic region is small as compared with a characteristic linear dimension of the body (e.g., with a crack length) (Fig. 6). The plastic domain represents thereby a

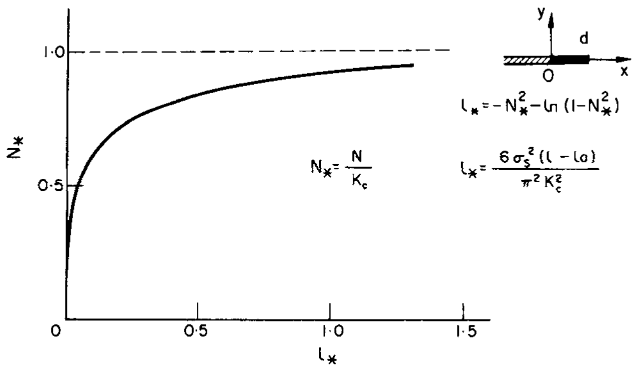

FIG. 6

segment of the length d along the crack-line prolongation

$$d = \frac{\pi^2 N^2}{4\sigma_s^2}. \tag{8.2}$$

Here the stress intensity factor N is found from a solution of the purely elastic problem as a whole; it determines the stress and strain distribution at a distance, which is large as compared to d and is small as compared to the characteristic linear dimension of the body (e.g., the curvature radius of the crack line at the point O). The displacement v of the opposite edges of the plastic line ($y = 0$, $0 < x < d$) equals to [19]

$$v = \pm\frac{2\sigma_s}{\pi E}\left[2\sqrt{[d(d-x)]}+x\ln\frac{\sqrt{d}-\sqrt{[d-x]}}{\sqrt{d}+\sqrt{[d-x]}}\right]. \tag{8.3}$$

One can calculate the fracture energy for this most typical and general case in the following manner

$$\begin{aligned}\gamma_* &= \lim_{\Delta s\to 0}\left\{\frac{\sigma_s}{\Delta s}\int_0^d [v(x-\Delta s, N+\Delta N)-v(x, N)\,\mathrm{d}x\right\}\\ &= \sigma_s\cdot\frac{4\sigma_s d}{\pi E}+\sigma_s\frac{\mathrm{d}N}{\mathrm{d}l}\int_0^d \frac{\partial v}{\partial N}\mathrm{d}x\\ &= \frac{\pi N^2}{E}+\frac{\pi^3 N^3}{3\sigma_s^2 E}\frac{\mathrm{d}N}{\mathrm{d}l}.\end{aligned} \tag{8.4}$$

The first term in equation (8.4) equals the dissipation energy rate because of the crack extension; this corresponds to the Griffith–Irwin–Orowan concept. The second term equals the dissipation energy rate in the plastic region owing to the process of loading and it is not connected with the crack growth. The following equation

$$N^2 = \frac{E\gamma_*}{\pi}-\frac{\pi^2 N^3}{3\sigma_s^2}\frac{\mathrm{d}N}{\mathrm{d}l} \tag{8.5}$$

serves as a boundary condition at the crack contour for the case when the crack has a small but finite plastic head. Having determined the function $N = N(p, l)$ from elastic stress analysis one can find by virtue of equation (8.5) a relation between the crack-length l

and load p parameters for any particular problem. Equation (8.5) can be written also in the form which resembles the formulation of the yielding law with strain-hardening of an elastic–plastic body

$$\mathrm{d}l = \frac{\pi^2 N^3\,\mathrm{d}N}{3\sigma_s^2(K_c^2-N^2)}. \tag{8.6}$$

If Irwin's constant K_c is independent of the crack velocity, we find after integrating equation (8.6)

$$l-l_0 = -\frac{\pi^2 K_c^2}{6\sigma_s^2}\left[\frac{N^2-N_0^2}{K_c^2}+\ln\frac{(1-N^2/K_c^2)}{(1-N_0^2/K_c^2)}\right] \quad (N = N_0 \quad \text{as} \quad l = l_0). \tag{8.7}$$

The family of curves equation (8.7) can be obtained by displacing the curve of Fig. 6 along the x-axis.

In the case of plane strain similar relationship can be obtained.

It is clear that in elastic–plastic bodies a crack is growing, even if the stress intensity factor is in the interval $0 < N < K_c$; the concept by Griffith–Irwin–Orowan is of asymptotic nature and holds, if the condition $6\sigma_s^2\Delta l \gg \pi^2 K_c^2$ is satisfied (practically, if $\sigma_s^2\Delta l \gtrsim 3K_c^2$, on the basis of Fig. 7).

9. A RIGID–PLASTIC BODY

Let a strip of an incompressible rigid–plastic material be stretched in y-direction with velocity v (Fig. 7). The strip is assumed to contain a crack, which is perpendicular to the

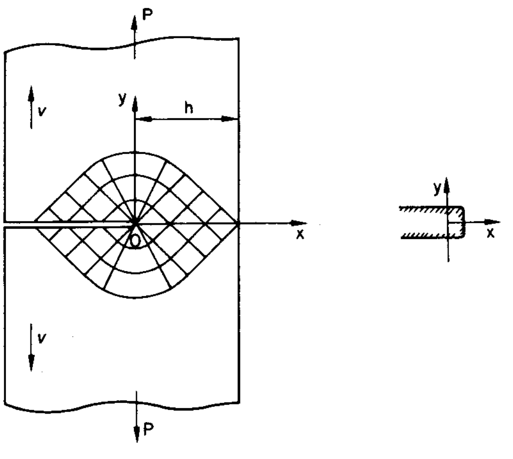

FIG. 7

load-free boundary and is at the distance h from the latter. For the case of plane strain under consideration the rigid–plastic strip yields if the force P is:

$$P = (2+\pi)\tau_s h. \tag{9.1}$$

The stress and velocity fields in the plastic region (the singular solution) are given by [32, 33]

$$
\begin{aligned}
&\text{as}\quad |\theta| < \frac{\pi}{4} && \sigma_x = \pi\tau_s \qquad \sigma_y = (2+\pi)\tau_s \\
& && \tau_{xy} = 0 \qquad v_x = 0 \qquad v_y = 0 \\
&\text{as}\quad \tfrac{3}{4}\pi > |\theta| > \frac{\pi}{4} && \sigma_x = \sigma - \tau_s \sin 2\theta \qquad \tau_{xy} = \tau_s \cos 2\theta \\
& && \sigma_y = \sigma + \tau_s \sin 2\theta \\
& && \sigma = (1+\tfrac{3}{2}\pi - 2\theta)\tau_s \\
& && v_r = v \sin\theta \qquad v_\theta = v(\cos\theta - \sqrt{2}) \\
&\text{as}\quad \pi > |\theta| > \tfrac{3}{4}\pi && \sigma_x = 2\tau_s \\
& && \sigma_y = \tau_{xy} = 0 \\
& && v_x = v \qquad v_y = 2v.
\end{aligned}
\tag{9.2}
$$

Here τ_s is a shear yield-point. The general condition equation (1.3) at the crack-tip for the solid model involved can be represented in the form [11]

$$2R\tau_s \int_{\pi/4}^{3\pi/4} \varepsilon_{r\theta} \cos\theta \,\mathrm{d}\theta - R \int_{\pi/4}^{3\pi/4} \left(\sigma_r \frac{\partial u_r}{\partial x} + \tau_{r\theta} \frac{\partial u_\theta}{\partial x} \right) \mathrm{d}\theta = \gamma_* \tag{9.3}$$

Equation (9.2) leads to

$$
\begin{aligned}
\varepsilon_{r\theta} &= \frac{1}{r\sqrt{2}} \int_0^t v \,\mathrm{d}t \qquad u_r = \sin\theta \int_0^t v \,\mathrm{d}t \\
u_\theta &= (\cos\theta - \sqrt{2}) \int_0^t v \,\mathrm{d}t \qquad (t \text{ is time})
\end{aligned}
\tag{9.4}
$$

and the edge of the crack takes a "box-like" shape (Fig. 7).

After substituting equation (9.4) into equation (9.3) the left-hand side of equation (9.3) vanishes. Therefore, there can be no crack propagation in a rigid–plastic body.†

10. THE GROWTH OF CRACKS BY CYCLIC LOADS

The growth of cracks under the application of cyclic loads whose values are much less than those of limiting ones is attributed to the qualitative peculiarities of the crack propagation in elastic–plastic materials resulting from the foregoing analysis. At present it is well established that the lifetime of materials is determined sometimes by the duration of the fatigue crack growth under the cyclic load [34, 35]. The propagation of fatigue cracks in plates can be investigated within the framework of the suggested theory by means of the basic condition at the crack contour equation (8.5) or equation (8.6) which holds

† As it will be shown below this result is valid also for purely viscous bodies. Thus, in the bodies which have no elastic properties a crack enlarges as a cavity.

for any loading conditions and for bodies of any configuration provided that the plastic region at the crack-tip is small compared to the crack length. For the case of Griffith cracks one can omit the last restriction if the equation (7.2) is used.

Since we are not striving for generality and completeness, we shall consider now as an illustrative example the case of a Griffith crack exposed to the cyclic load $p(t)$ (Fig. 8).

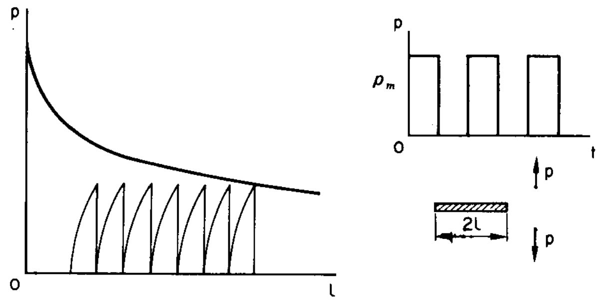

FIG. 8

For simplicity we shall confine ourselves to the case of the crack whose propagation occurs in the range of stability (β, λ) which is not too close to the boundary layer near the curve equation (7.3), so that the curves $\beta(\lambda)$ are nearly parallel to Y-axis in this region, as evidenced by Fig. 5.

Developing the right-hand side of the equation (7.2) in the neighborhood of the point $\beta = 0, \lambda = \lambda_1$ we shall find

$$\frac{d\lambda}{d\beta} = \tfrac{2}{3}\beta^3\lambda_1^2 + O(\beta^5) \tag{10.1}$$

Hence the dimensionless length of the crack increases during one loading cycle by

$$\Delta\lambda = \tfrac{1}{6}\beta_m^4\lambda_1^2 + O(\beta_m^6) \tag{10.2}$$

Here β_m is the maximum dimensionless tension load during one cycle (the residual stresses are neglected). Passing from the finite differences to differentials and denoting the number of cycles as n we get from this the rate of growth of the crack

$$\frac{dl_1}{dn} = \frac{\pi^3 p_m^4 l_1^2}{48\sigma_s^2 E\gamma_*} \tag{10.3}$$

When $d \ll l$ it is reasonable to utilise the maximum stress intensity factor N_m; for the present case it is equal to

$$N_m^2 = \tfrac{1}{2}p_m^2 l_1 \tag{10.4}$$

Then equation (10.3) can be written as

$$\frac{dl_1}{dn} = \frac{\pi^3 N_m^4}{12\sigma_s^2 E\gamma_*}. \tag{10.5}$$

Notice that it can be also derived immediately from equation (8.6).

The relationships equation (10.3) and equation (10.5) are in good agreement with experiments in the region not too close to the boundary layer [see Figs. 5 and 8, cf. e.g. data in (36–41)].

For lack of space we shall omit here such important issues, as the effects of boundaries, of the way of loading, of boundary layer phenomena, of residual stresses.

Notice only that the suggested approach does not describe nonpropagating fatigue cracks and the existence of fatigue strength. These phenomena are evidently connected with the microinhomogeneity and grain-structure of real materials omitted in the theory. To take into account these effects in the framework of our theory one must formulate the condition of the crack nonpropagation. The latter is easily obtained for the cracks which obey the condition $d \ll l$. Indeed, on the basis of general considerations of invariance (see introduction) the nonpropagation condition for these fatigue cracks is to be as follows

$$N_m \lesseqgtr K_Y \tag{10.6}$$

where K_Y is a material constant ($K_Y < K_c$).

For the case of the Griffith problem the condition equation (10.6) by means of equation (10.4) is written as

$$p_m^2 l_1 \lesseqgtr 2K_Y^2. \tag{10.7}$$

For not too small values l_1 the condition equation (10.7) agrees quite satisfactorily with the experimental data by Frost [42, 43] (he noted himself an empirical condition $p_m^3 l_1 < C$; the discrepancy with equation (10.7) is accounted for the fact, that the inequality $d \ll l$ did not hold so strictly).

For the general case the rate of crack growth by any cyclic loads can be obtained in the following form

$$\frac{\mathrm{d}l}{\mathrm{d}n} = -\beta\left(\frac{N_{\max}^2 - \delta N_{\min}^2}{K_c^2} + \ln\left[\frac{1-(N_{\max}^2/K_c^2)}{1-(\delta N_{\min}^2)/K_c^2)}\right]\right) \tag{10.8}$$

$$\delta = 1 \quad \text{if} \quad N_{\min} > 0 \quad \text{and} \quad \delta = 0 \quad \text{if} \quad N_{\min} < 0.$$

Here β is a material constant which is different in the cases of plane strain and plane stress, $N_{\max}$ and $N_{\min}$ are maximum and minimum values of N during a cycle, respectively.

11. VISCOELASTIC BODIES

Consider a linear viscoelastic body having quasistatic cracks and being in plane stress. The process is assumed to be isothermal. The stress and strain relationship can be represented for this case in the most general form [44]

$$\begin{aligned} \varepsilon_x &= E^{-1}\sigma_x - E^{-1}\nu\sigma_y & \varepsilon_y &= E^{-1}\sigma_y - E^{-1}\nu\sigma_x \\ \varepsilon_{xy} &= E^{-1}(1+\nu)\tau_{xy} & \varepsilon_z &= -E^{-1}\nu(\sigma_x+\sigma_y). \end{aligned} \tag{11.1}$$

Here E^{-1} and ν are linear commutative time-operators of the following form:

$$E^{-1}f = \int_0^t E_0(t-\tau)f(\tau)\,\mathrm{d}\tau \qquad \nu f = \int_0^t \nu_0(t-\tau)f(\tau)\,\mathrm{d}\tau. \tag{11.2}$$

The functions $E_0(x)$ and $\nu_0(x)$ belong to the class of the generalized functions. It is convenient for the practical purpose to use the Rabotnov's kernels [45]. A singular solution in the case under consideration coincides with an elastic one [11], but the stress intensity factor N must be treated as a function of time.

Calculate the fracture energy by means of the general condition equation (1.5) and the singular solution

$$2\gamma_* = R\int_{-\pi}^{+\pi}(3\cos\theta - A_*)\,\mathrm{d}\theta$$

$$\begin{aligned} R\int_{-\pi}^{+\pi} 3\cos\theta\,\mathrm{d}\theta = R\int_0^t \mathrm{d}t\Big\{\int_{-\pi}^{+\pi}(\sigma_x+\sigma_y)E^{-1}(\dot\sigma_x+\dot\sigma_y)\cos\theta\,\mathrm{d}\theta \\ - \int_{-\pi}^{+\pi}\sigma_x E^{-1}(1+\nu)\dot\sigma_y\cos\theta\,\mathrm{d}\theta - \int_{-\pi}^{+\pi}\sigma_y E^{-1}(1+\nu)\dot\sigma_x\cos\theta\,\mathrm{d}\theta \\ +2\int_{-\pi}^{+\pi}\tau_{xy}E^{-1}(1+\nu)\tau_{xy}\cos\theta\,\mathrm{d}\theta\Big\} = \pi\int_0^t NE^{-1}(1-\nu)\dot N\,\mathrm{d}t \end{aligned} \tag{11.3}$$

$$\begin{aligned} R\int_0^t A_*\,\mathrm{d}\theta = \mathrm{Im}\oint(\sigma_x - i\tau_{xy})\left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right)\mathrm{d}z \\ +R\int_{-\pi}^{+\pi}(\sigma_x+\sigma_y)\frac{\partial v}{\partial x}\sin\theta\,\mathrm{d}\theta = -\frac{\pi}{2}NE^{-1}(3+\nu)N(z = Re^{i\theta}) \end{aligned}$$

and

$$2\gamma_* = \pi\int_0^t NE^{-1}(1-\nu)\dot N\,\mathrm{d}t + \frac{\pi}{2}NE^{-1}(3+\nu)N. \tag{11.4}$$

Here t is the time from the beginning of the loading to the initial moment of the crack growth. As the montonously growing cracks are considered here, t tends to zero. Then, passing to the limit for $t \to 0$ in the formula (11.4) we obtain the following equation

$$E\gamma_* = \pi N^2 \tag{11.5}$$

This is the known Irwin's condition for the brittle cracks.

Thus, a crack in a body will behave as a brittle one, if γ_* is a material constant and the body is linear viscoelastic up to the fracture. As it is readily seen from equation (11.1) and equation (11.2), the stress and strain distribution near the end of a monotonously growing crack in a viscoelastic body will be elastic.

The equation (11.4) will be of independent interest if the dimension of the field of plastic (or high-elastic) deformations near the crack-tip is larger than that of the "elastic kernel", so that the linear dimension R of the "fine" structure of the crack-tip is large compared to the latter.

12. FLUCTUATION CRACKS

Consider briefly another possible mechanism of the crack-growth kinetics connected with the fracture of the plastic (or high-elastic) head at the crack-tip as a result of heat fluctuations. For this case the corresponding condition at the crack-tip is easily derived from the linear postulate of irreversible thermodynamics which can be written here in the form equation (2.6). Indeed, substituting equation (2.6) into equation (1.3) and assuming the material to be a linearly elastic one, one finds

$$R\int_{-\pi}^{+\pi}(\rho U\cos\theta - A_*)\,\mathrm{d}\theta = 2\gamma + \frac{T}{\alpha}l \tag{12.1}$$

and

$$\frac{2\pi N^2}{E} = 2\gamma + \frac{T}{\alpha}l. \tag{12.2}$$

Hence, ignoring a reversible part of fracture energy, we come to the following formula

$$\frac{\mathrm{d}l}{\mathrm{d}t} = \zeta N^2 \qquad \zeta = \frac{2\pi\alpha}{ET}. \tag{12.3}$$

Here a constant ζ depends only on temperature.

For the case of the Griffith problem the formula equation (12.3) becomes

$$\frac{\mathrm{d}l}{\mathrm{d}t} = \tfrac{1}{2}\zeta p^2 l \tag{12.4}$$

and for $p =$ const. during the whole process we find

$$l = l_0 \exp(\tfrac{1}{2}\zeta p^2 t). \tag{12.5}$$

The exponential expression equation (12.5) agrees well with the results of works [46–47], obtained with the help of other approaches.

However, the following equation leads to better agreement with experimental data on long-time strength:

$$\frac{\mathrm{d}l}{\mathrm{d}t} = \delta_1 \exp\frac{\delta_2 N - \delta_3}{T} \tag{12.6}$$

($\delta_1, \delta_2, \delta_3$ are material constants). It can be obtained from the fluctuation theory and the modified concept of Neuber.

CONCLUSION

On the basis of the exact mathematical approach, and the subsequent application of singular solutions, different possible mechanisms of the energy absorption at the crack-tip, due to the plastic (7)–(10) and viscous (12) dissipation in the plastic head and due to the dissipation in the bulk of the material (11), were considered in this paper. It appears that the combination of these particular mechanisms to a single one will make it possible to work out the most flexible and universal concept.

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Абстракт—Предлагаемая работа посвящена приложению основных идей и методов механики сплошных сред к процессам распространения трещин. Развитие трещины определяется дополнительным условием в её конце. В связи с этим ставится задача о "тонкой" структуре конца трещины. Используя закон сохранения энергии и физическое представление об энергии разрушения, находится общий вид дополнительного условия в произвольной сплошной среде. Кратко рассматриваются динамические трещины в упругом теле и статические трещины в упруго-пластическом и жестко-пластическом телах. Весьма кратко рассмотрен вопрос о развитии квазистатических трещин в диссипирующих вязко-упругих средах. Общий подход применяется также к усталостным и флюктуационным трещинам.