MAP 2210 – Aplicações de Álgebra Linear

1º Semestre - 2019

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Objetivos

Formação básica de álgebra linear aplicada a problemas numéricos. Resolução de problemas em microcomputadores usando linguagens e/ou software adequados fora do horário de aula.

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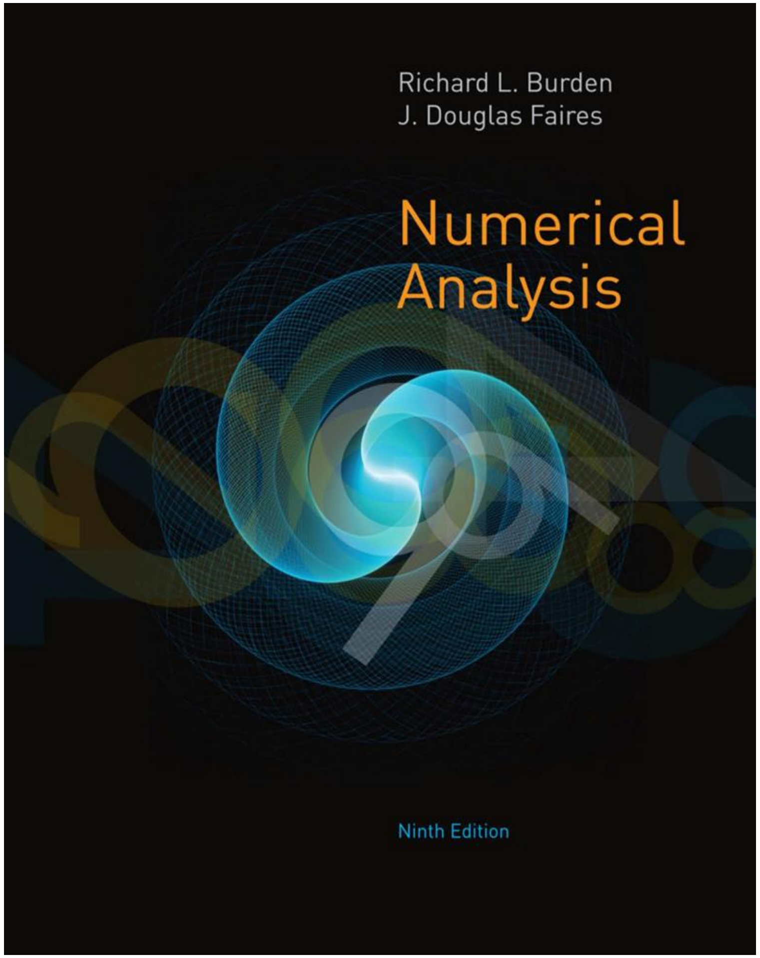

Numerical Analysis

NINTH EDITION

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9 Approximating Eigenvalues 561

EXERCISE SET 9.3

5. Find the first three iterations obtained by the Symmetric Power method applied to the following matrices.

a. $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$;
Use $\mathbf{x}^{(0)} = (1, -1, 2)^t$.

b. $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$;
Use $\mathbf{x}^{(0)} = (-1, 0, 1)^t$.

ALGORITHM 9.2

Symmetric Power Method

To approximate the dominant eigenvalue and an associated eigenvector of the $n \times n$ symmetric matrix A, given a nonzero vector $\mathbf{x}$:

INPUT dimension n; matrix A; vector $\mathbf{x}$; tolerance TOL; maximum number of iterations N.

OUTPUT approximate eigenvalue μ; approximate eigenvector $\mathbf{x}$ (with $\|\mathbf{x}\|_2 = 1$) or a message that the maximum number of iterations was exceeded.

Step 1 Set $k = 1$;
$\mathbf{x} = \mathbf{x}/\|\mathbf{x}\|_2$.

Step 2 While ($k \le N$) do Steps 3–8.

Step 3 Set $\mathbf{y} = A\mathbf{x}$.

Step 4 Set $\mu = \mathbf{x}^t\mathbf{y}$.

Step 5 If $\|\mathbf{y}\|_2 = 0$, then OUTPUT ('Eigenvector', $\mathbf{x}$);
OUTPUT ('A has eigenvalue 0, select new vector $\mathbf{x}$ and restart');
STOP.

Step 6 Set $ERR = \left\| \mathbf{x} - \dfrac{\mathbf{y}}{\|\mathbf{y}\|_2} \right\|_2$;
$\mathbf{x} = \mathbf{y}/\|\mathbf{y}\|_2$.

Step 7 If $ERR < TOL$ then OUTPUT ($\mu, \mathbf{x}$);
(*The procedure was successful.*)
STOP.

Step 8 Set $k = k + 1$.

Step 9 OUTPUT ('Maximum number of iterations exceeded');
(*The procedure was unsuccessful.*)
STOP.

Input:

eigenvalues	$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$

Results:

$\lambda_1 = 4$

$\lambda_2 = 1$

$\lambda_3 = 1$

Corresponding eigenvectors:

$v_1 = (1, 1, 1)$

$v_2 = (-1, 0, 1)$

$v_3 = (-1, 1, 0)$

EXERCISE SET 9.4

1. Use Householder's method to place the following matrices in tridiagonal form.

a. $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$;

b. $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$;

Householder Transformations

To summarize the choice of $P^{(1)}$, we have

$$\alpha = -\operatorname{sgn}(a_{21}) \left(\sum_{j=2}^{n} a_{j1}^2 \right)^{1/2},$$

$$r = \left(\frac{1}{2}\alpha^2 - \frac{1}{2}a_{21}\alpha \right)^{1/2},$$

$$w_1 = 0,$$

$$w_2 = \frac{a_{21} - \alpha}{2r},$$

and

$$w_j = \frac{a_{j1}}{2r}, \quad \text{for each } j = 3, \ldots, n.$$

$$P = I - 2\mathbf{w}\mathbf{w}^t$$

With this choice,

$$A^{(2)} = P^{(1)} A P^{(1)} = \begin{bmatrix} a_{11}^{(2)} & a_{12}^{(2)} & 0 & \cdots & 0 \\ a_{21}^{(2)} & a_{22}^{(2)} & a_{23}^{(2)} & \cdots & a_{2n}^{(2)} \\ 0 & a_{32}^{(2)} & a_{33}^{(2)} & \cdots & a_{3n}^{(2)} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & a_{n2}^{(2)} & a_{n3}^{(2)} & \cdots & a_{nn}^{(2)} \end{bmatrix}.$$

Input interpretation:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -0.707104 & -0.707106 \\ 0 & -0.707106 & 0.707108 \end{pmatrix}$$

Property:

symmetric

Inverse:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -0.70711 & -0.707108 \\ 0 & -0.707108 & 0.707106 \end{pmatrix}$$

Input:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

Properties:

symmetric

orthogonal

Inverse:

$$\frac{1}{2}\begin{pmatrix} 2 & 0 & 0 \\ 0 & -\sqrt{2} & -\sqrt{2} \\ 0 & -\sqrt{2} & \sqrt{2} \end{pmatrix}$$

Input:

eigenvalues	$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$

Results:

$\lambda_1 = 4$

$\lambda_2 = 1$

$\lambda_3 = 1$

Corresponding eigenvectors:

$v_1 = (1, 1, 1)$

$v_2 = (-1, 0, 1)$

$v_3 = (-1, 1, 0)$

Input:

eigenvalues	$\begin{pmatrix} 2 & -\sqrt{2} & 0 \\ -\sqrt{2} & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Results:

$\lambda_1 = 4$

$\lambda_2 = 1$

$\lambda_3 = 1$

Corresponding eigenvectors:

$v_1 = \left(-\frac{1}{\sqrt{2}}, 1, 0\right)$

$v_2 = (0, 0, 1)$

$v_3 = \left(\sqrt{2}, 1, 0\right)$

EXERCISE SET 9.5

1. Apply two iterations of the QR method without shifting to the following matrices.

a. $\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

b. $\begin{bmatrix} 3 & 1 & 0 \\ 1 & 4 & 2 \\ 0 & 2 & 1 \end{bmatrix}$

A rotation matrix P differs from the identity matrix in at most four elements. These four elements are of the form

$$p_{ii} = p_{jj} = \cos\theta \quad \text{and} \quad p_{ij} = -p_{ji} = \sin\theta,$$

for some θ and some $i \neq j$. ■

The factorization of $A^{(1)}$ into $A^{(1)} = Q^{(1)}R^{(1)}$ uses a product of $n-1$ rotation matrices to construct

$$R^{(1)} = P_n P_{n-1} \cdots P_2 A^{(1)}.$$

The other half of the QR factorization is the matrix

$$Q^{(1)} = P_2^t P_3^t \cdots P_n^t,$$

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k								d	2,236068										
0		**A**			**a**	**b**		**s2**	**c2**										
	2	-1	0		2	0		-0,44721	0,894427										
	-1	2	-1		2	-1													
	0	-1	2		2	-1													
												d	1,67332						
		P1				**PA**			**a**	**b**		**s3**	**c3**						
	0,894427	-0,44721	0		2,236068	-1,78885	0,447214		2,236068	0		-0,59761	0,801784						
	0,447214	0,894427	0		0	1,341641	-0,89443		1,341641	0									
	0	0	1		0	-1	2		2	-1									
		P2				**R**				**Q**				P1^T				P2^T	
	1	0	0		2,236068	-1,78885	0,447214		0,894427	0,358569	0,267261		0,894427	0,447214	0		1	0	0
	0	0,801784	-0,59761		0	1,67332	-1,91237		-0,44721	0,717137	0,534522		-0,44721	0,894427	0		0	0,801784	0,597614
	0	0,597614	0,801784		0	0	1,069045		0	-0,59761	0,801784		0	0	1		0	-0,59761	0,801784
								d	2,898275										
1		**A2=RQ**			**a**	**b**		**s2**	**c2**										
	2,8	-0,74833	0		2,8	0		-0,2582	0,966092										
	-0,74833	2,342857	-0,63888		2,342857	-0,74833													
	0	-0,63888	0,857143		0,857143	-0,63888													
												d	2,166536						
		P1				**PA**			**a**	**b**		**s3**	**c3**						
	0,966092	-0,2582	0		2,898275	-1,32788	0,164957		2,898275	0		-0,29488	0,955533						
	0,258199	0,966092	0		1,11E-16	2,070197	-0,61721		2,070197	1,11E-16									
	0	0	1		0	-0,63888	0,857143		0,857143	-0,63888									
		P2				**R**				**Q**								P2^T	
	1	0	0		2,898275	-1,32788	0,164957		0,966092	0,246718	0,076139							0	0
	0	0,955533	-0,29488		1,06E-16	2,166536	-0,84253		-0,2582	0,923133	0,284885							0,955533	0,294884
	0	0,294884	0,955533		3,27E-17	0	0,637022		0	-0,29488	0,955533							-0,29488	0,955533

2		A2=RQ	
	3,142857	-0,5594	0
	-0,5594	2,248447	-0,18785
	3,16E-17	-0,18785	0,608696

15		A2=RQ	
	3,414213	-0,00066	8,35E-17
	-0,00066	2	-2E-08
	3,06E-17	-2E-08	0,585786

Input:

eigenvalues $\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$

Results:

$\lambda_1 = 2 + \sqrt{2}$

$\lambda_2 = 2$

$\lambda_3 = 2 - \sqrt{2}$

Fim...