CHAPTER 5

Complex Waves

All of the sound waves that have been described to this point were of the **sinusoidal** form that appears in panel A of Figure 5-1. Although the sine wave is not the type of vibratory motion that we are likely to experience in our daily lives, it is important to understand it thoroughly because *the sine wave is the fundamental component of other sound waves that will be encountered.*

The other sound waves shown in Figure 5-1 are much more complex in form than the simple sine wave, and they are, indeed, called

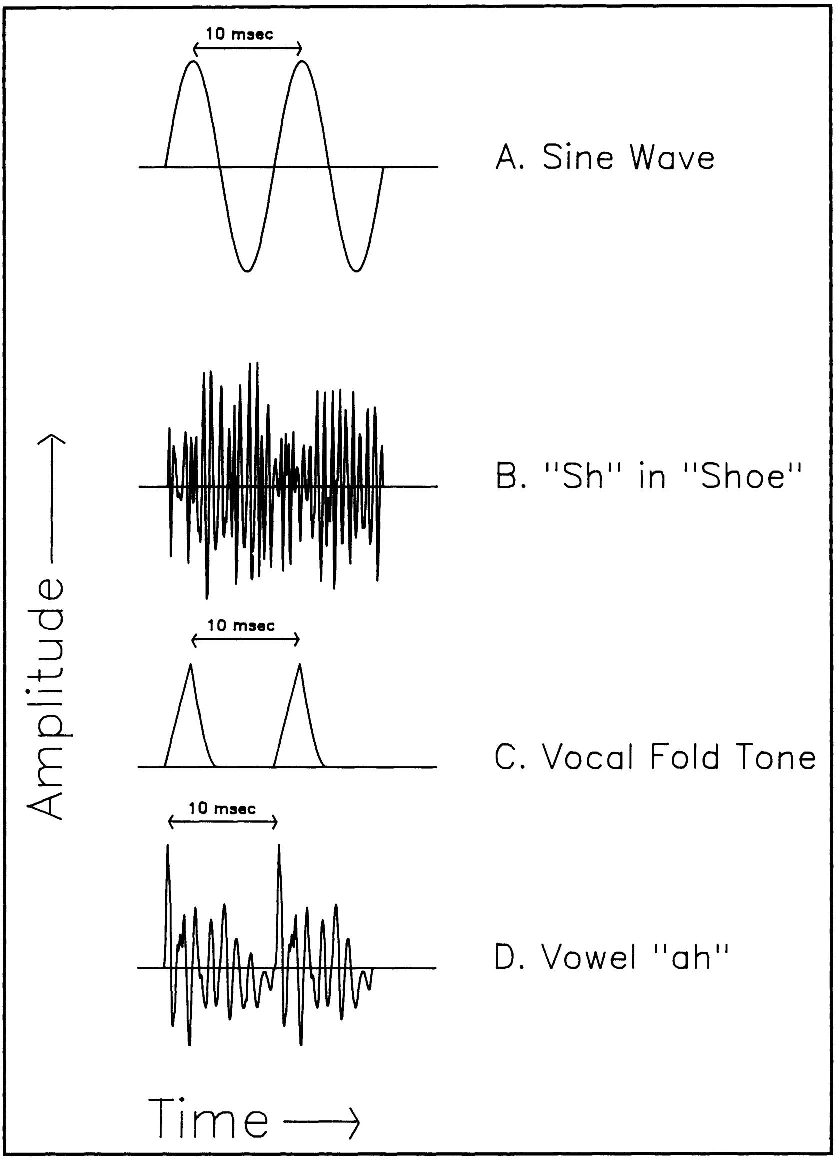

Figure 5-1. Comparison of a sine wave with three complex waves: the /sh/ sound from the word "shoe"; the sound wave created by the vibratory movement of the vocal folds; and the vowel /a/.

complex waves. The wave in panel B is similar to the "sh" sound in the word "shoe," panel C shows one kind of sound wave that results from vibratory motion of the vocal folds during production of a vowel, and panel D shows the same vowel as recorded near the mouth opening rather than deep in the throat just above the vocal folds.

FOURIER'S THEOREM

A **complex wave** can be defined as *any sound wave that is not sinusoidal*. That definition, though, does not reveal exactly what a complex wave is, but the following theorem should help. The complex waves in Figure 5-1 — and all other sound waves that are not sinusoidal — *are composed of a series of simple sinusoids that can differ in amplitude, frequency, and phase*. Thus, when two or more sine waves that differ in amplitude, frequency, or phase are added together, a complex wave is produced.

The degree of complexity of a complex sound wave depends on the number of sine waves that are combined and on the specific dimensional values (amplitude, frequency, and phase) of the sinusoidal components. This theorem was first stated by Joseph Fourier, a French mathematician who lived at the time of Napoleon I, and, hence, the series of sine waves that are combined to compose a complex wave is called a **Fourier series** in his honor.

Fourier's theorem has two important implications for the study of complex waves. First, because a complex wave consists of some number of sinusoids of different amplitudes, frequencies, and phases, the nature of any complex wave should not be difficult to comprehend if we understand the concept of simple harmonic motion that is associated with each of the sinusoidal components, and if we recall the relevant dimensions of sine waves: **amplitude**, **frequency**, and **phase**. Second, we can derive the **Fourier series** by a process called **Fourier analysis**, which means that *any complex waveform can be decomposed or analyzed to determine the amplitudes, frequencies, and phases of the sine waves that compose the complex wave*.

All sound waves can be classified by reference to (1) the presence or absence of **periodicity** in the wave and (2) the degree of complexity of the wave.

PERIODIC WAVES

A **periodic wave**, whether sinusoidal or complex, is a wave that *repeats itself at regular intervals over time*. Because the wave repeats itself periodically over time, it also can be called a **periodic time function**. The sine wave in Figure 5-1 provides an obvious example of periodicity because we can see that the characteristics of any one cycle of the wave are duplicated exactly in every other cycle — each cycle in the wave is repeated regularly over time.

Sine waves are not the only forms of wave motion that are characterized by periodicity. The vocal fold wave (panel C) and the vowel (panel D) in Figure 5–1 appear to be reasonably periodic (they are in fact called **quasiperiodic**) because we can verify that all of the features within one "cycle" of vibration are duplicated *almost exactly* during the next and every other cycle. Thus, there are two kinds of **periodic waves**: sinusoidal and complex. A **sinusoidal wave** *is a wave that results from simple harmonic motion and that comes from a relation that contains a sine function*. A **complex periodic wave** *is a periodic wave, but it is not sinusoidal.*

Components of a Complex Periodic Wave

According to Fourier's theorem, any complex periodic wave consists of some number of simple sinusoids that are summed, but the sinusoidal components cannot be selected randomly if the resultant sound wave is to be periodic. Instead, they must satisfy a basic mathematical requirement that is called a **harmonic relation**.

The term **harmonic relation** means that *the frequencies of all of the sinusoids that compose the series must be integral (whole number) multiples of the frequency of the sinusoid with the lowest frequency* in the series. For example, if the sinusoid with the lowest frequency is 100 Hz, the other sinusoidal components of the complex wave must be selected from the frequencies 200, 300, 400, 500 Hz, and so forth, because other frequency values would not satisfy the requirement of being integral multiples of the lowest frequency. Similarly, if the sinusoid with the lowest frequency is 110 Hz, the other sinusoidal components must be selected from the frequencies 220, 330, 440, 550 Hz, and so on.

A Harmonic Series

When a harmonic relation exists among frequency components, the series of frequencies is called a **harmonic series**, and all of the sinusoids in the harmonic series are called **harmonics**. The harmonics are numbered consecutively from lowest to highest frequency: 1st harmonic, which also is called the **fundamental frequency**, ($\mathbf{f_0}$), 2nd harmonic, 3rd harmonic, and so on until we reach the nth harmonic, or the last component in the series.

In the case of the first example above, the 1st harmonic (also called the $\mathbf{f_0}$), = 100 Hz, the 2nd harmonic = 200 Hz, the 3rd harmonic = 300 Hz, and so on. There also is a special circumstance in which the fundamental frequency (1st harmonic) is missing from the series. In that case, all of the higher frequencies in the harmonic series are integral multiples of what is called the repetition rate.

Figure 5–2 shows the waveform of one example of a periodic complex wave that consists of an infinite number of sinusoidal waves. Its

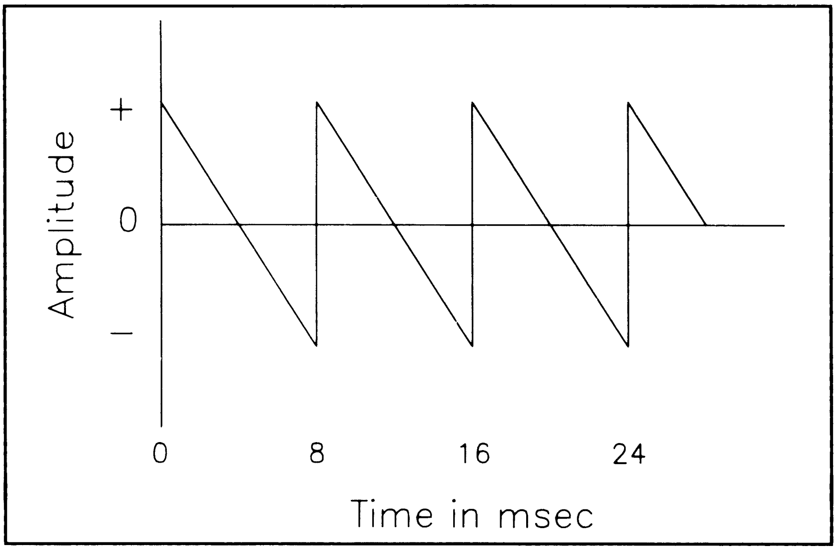

Figure 5-2. A **complex periodic** sound wave that is called a **sawtooth wave** because its shape resembles the teeth of a saw.

periodicity should be apparent, and because it obviously is not sinusoidal, it must be a complex periodic wave. This particular wave is called a **sawtooth wave** because its shape resembles the shape of the teeth of a saw. The sawtooth wave will be of interest to those who become interested in the study of the acoustics of speech because the waveform of the sawtooth wave resembles the waveform of the sound produced by the vibrating vocal folds.

The period of the fundamental frequency (the 1st harmonic) of the sawtooth wave in Figure 5-2 is 8 msec, which means that f_0, the fundamental frequency, is 125 Hz (Equation 1.12). All components — the **harmonics** — of the sawtooth wave are odd and even whole number (integral) multiples of the fundamental frequency. Thus, if the fundamental frequency is 125 Hz, the harmonic components would be: 125 (f_0 x 1), 250 (f_0 x 2), 375 (f_0 x 3), 500 (f_0 x 4), 625 (f_0 x 5), and so on for an infinity of odd and even multiples.

We see in Table 5-1 that the lowest frequency, 125 Hz, is called the fundamental fequency, f_0, and it also is called the 1st harmonic. The remaining components are labeled the 2nd harmonic, 3rd harmonic, 4th harmonic, and so on.

Partials and Overtones

You occasionally will find that the components in a complex periodic wave are called **partials** or **overtones** instead of **harmonics**. Table 5-1 also shows the relations among these different labels.

We can see from the entries in the table that the designations of **harmonic** and **partial** are synonymous *as long as all components are*

Table 5–1. Fundamental frequency, harmonics, partials, and overtones in a complex periodic sound wave.

Frequency	Harmonic	Partial	Overtone
125 (f_0)	1	1	
250	2	2	1
375	3	3	2
500	4	4	3
625	5	5	4
750	6	6	5

exact integral multiples of the fundamental frequency: the 1st harmonic (also the fundamental) is the 1st partial, the 2nd harmonic is the 2nd partial, and so on. The word, **overtone**, which might, for example, be encountered in the musical literature, derives from the fact that the complex wave can be described as consisting of a fundamental frequency, or fundamental tone, and a series of other tones whose frequencies lie *over* the fundamental. Thus, the 2nd harmonic is the 1st overtone, the 3rd harmonic is the 2nd overtone, and so on.

Summary

If the complex wave is to be periodic, the sinusoidal components *must be integral multiples of the fundamental frequency*. Thus, if f_0 = 100 Hz, the other components must be selected from 200 Hz, 300 Hz, 400 Hz, and so forth. When that occurs, the partials are indeed harmonics, and the sound wave is exactly periodic. In that circumstance, at the end of one cycle of vibration (10 msec), we will have completed one cycle of the 1st harmonic (100 Hz; T = 10 msec), two cycles of the 2nd harmonic (200 Hz; T = 5 msec), three cycles of the 3rd harmonic (300 Hz; T = 3.33 msec), four cycles of the 4th harmonic (400 Hz; T = 2.5 msec), and so on.

Summation of Sine Waves

As more and more sine waves are added (summed) in the harmonic series, the shape of the resultant complex wave changes. The left side of Figure 5–3 shows four sine waves (S_1, S_2, S_3, and S_4) that have different frequencies and amplitudes, but identical **starting phases** (180°).

The exact frequency of each wave to be summed is unimportant, but an appropriate frequency relation among the four components is maintained; the three higher frequencies must be harmonics of the lowest one — the fundamental frequency. Notice, however, that not all harmonics are present in this example. In fact, we have used only the *odd integral multiples* so that we have the 1st (f_0), 3rd, 5th, and 7th harmonics. Thus, if the frequency of S_1 were 1000 Hz, the frequencies of

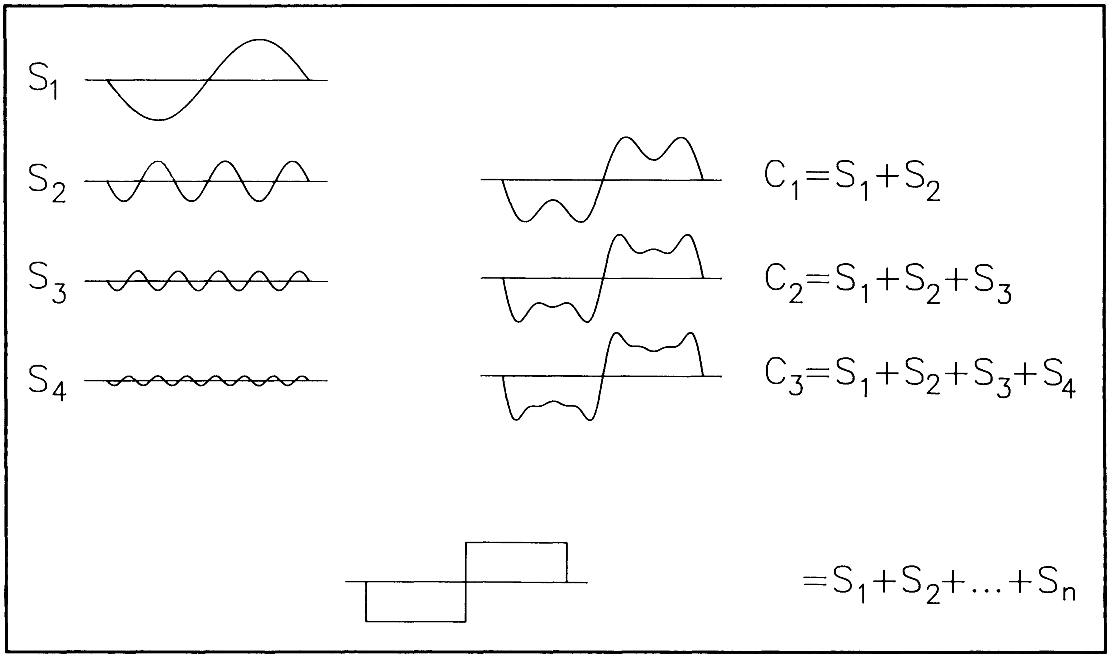

Figure 5-3. Summation of sine waves to form complex waves. The sine waves at the left (S_1 through S_4) are added progressively to form the complex waves (C_1 through C_3) at the right. At the bottom, a **square wave** is created by summation of an infinite number of sine waves of *appropriate* **amplitudes, frequencies,** and **starting phases.**

the other sinusoids would be 3000, 5000, and 7000 Hz; if the frequency of S_1 were 400 Hz, the frequencies of the other sinusoids would be 1200, 2000, and 2800.

At the right of Figure 5-3 we show what happens when the sinusoidal components are summed progressively to form three different complex waves (C_1, C_2, and C_3). The complex wave at the top of the figure (C_1) results from $S_1 + S_2$. Wave C_1 is not a sinusoid — it is complex — and it is composed of two sinusoids that differ in amplitude and frequency.

Although the **starting phases** of the two components of C_1 are identical, it should be apparent that the **instantaneous phases** of the two sinusoidal components vary from moment to moment because of their different frequencies. If the two sinusoidal components had any other values of frequency and amplitude, the complex wave that results would be different from the one shown as C_1 because the resultant wave depends on all of the specific dimensions of the sine waves that compose it.

Wave C_2 looks different from wave C_1 because its shape results from summation of three sinusoids, $S_1 + S_2 + S_3$. Wave C_3 contains all four sinusoidal components. You might notice that the complex waves at the right are becoming more and more "square" in shape as more and more sine waves are added. At the bottom of Figure 5-3 we show what happens if we combine an infinite number of odd-numbered integral multiple sinusoids (1X, 3X, 5X, 7X, 9X, + ... + nX). If, for example, the lowest frequency were 100 Hz, the other components would be 300, 500, 700 Hz, and so on. A complex wave with a perfectly square shape is created by summing an infinity of sinusoids with frequencies that are odd integral multiples of the fundamental frequency and that have appropriate relative amplitudes and identical starting phases.

Figure 5-4 provides another example of summation of sine waves to form a resultant periodic complex wave. In this case, we have added the first three of *both odd and even* integral multiples of the fundamental (f_0); all **starting phases** = 0° in this example. The two resultant waves, C_1 and C_2, look rather different from the resultant waves seen

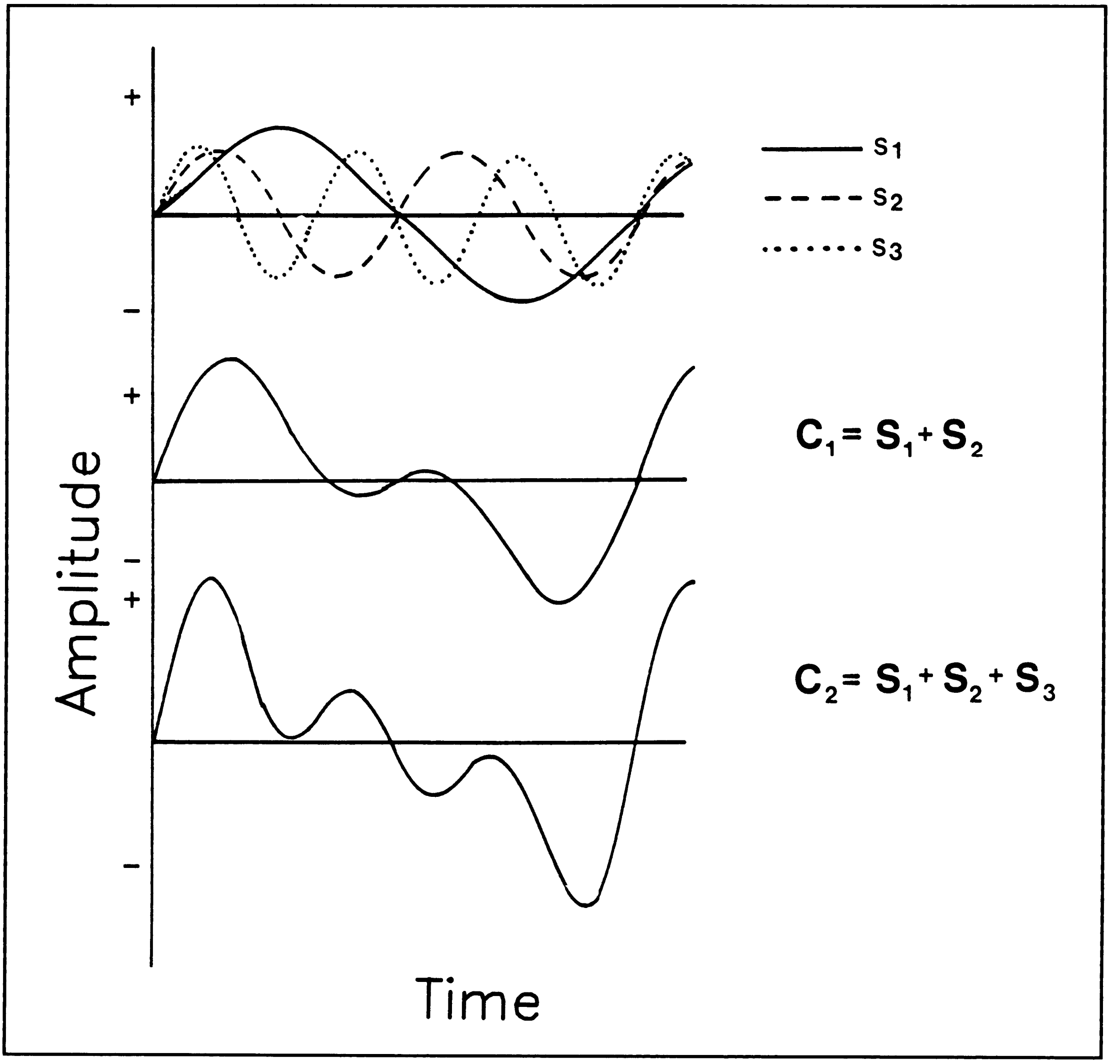

Figure 5-4. Summation of sine waves to form a different **complex periodic wave**. Because the shape of this complex wave is different from those shown previously, the parameter values for amplitude, frequency, and starting phase of the sinusoidal components must also be different.

previously in Figure 5–3 because both odd and even harmonics are included rather than just odd.

When an infinity of odd harmonics with appropriate relative amplitudes and starting phases is summed, the result is the square wave that was shown in Figure 5–3. However, when an infinity of odd and even harmonics with appropriate relative amplitudes and starting phases is summed, the result is a sawtooth wave such as that shown in Figure 5–2. In fact, with only three components, and a bit of imagination, you can see that the resultant wave in Figure 5–4 is beginning to assume a sawtooth kind of shape.

In the two examples cited thus far, the **starting phase** was identical for the individual components. However, variations in the resultant wave also will occur if we vary the starting phase of the components while holding their amplitudes and frequencies constant, as is illustrated in Figure 5–5.

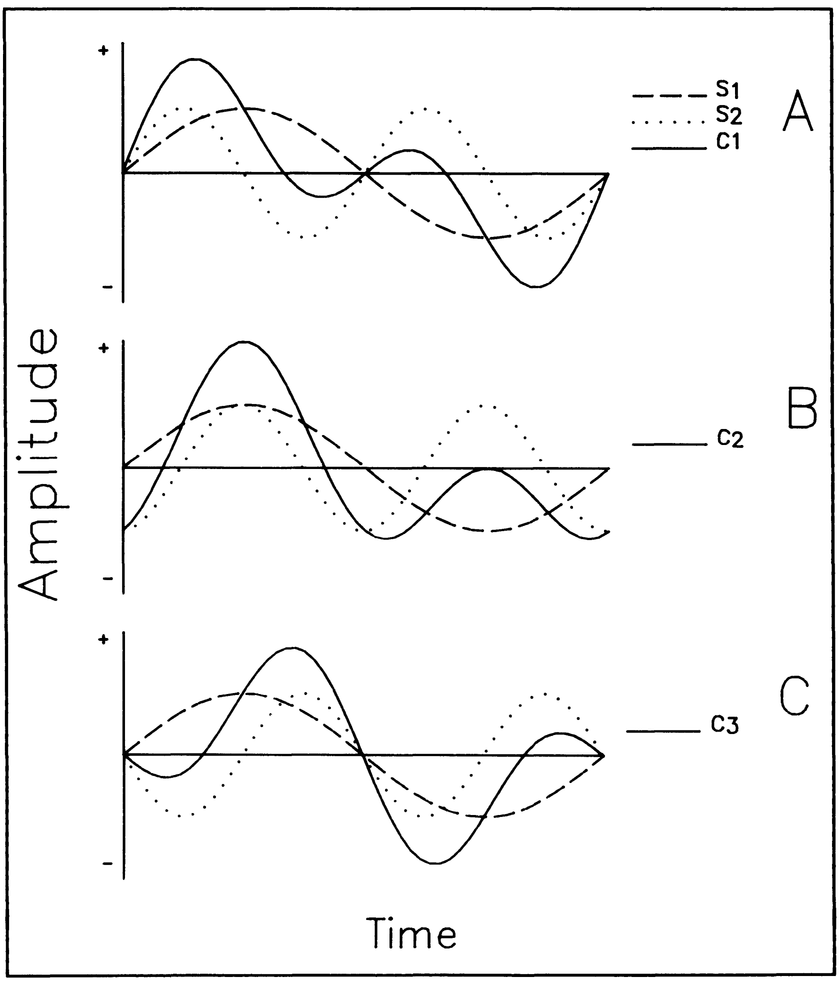

Figure 5–5. Effects of variation in **starting phase** on the shape of a complex wave that results from summation of sine waves.

In panel A of Figure 5–5, the two components (dashed and dotted lines) have identical starting phases (0°) and we can see the shape of the complex wave (solid line) that results from summation. The resultant wave in panel B is different from the one in panel A because the starting phase of S_1 has remained at 0°, but the starting phase for S_2 has been shifted from 0° to 270°. In panel C, S_1 again remains at 0°, but the starting phase of S_2 is now 180°. Thus, with these changes in starting phase of one of the components, three different complex periodic waves have been produced.

APERIODIC WAVES

The principal distinguishing characteristic of complex periodic waves is their regularity over time, or **periodicity**. They repeat themselves indefinitely. The **aperiodic wave** is a second category of waveform, and its name derives from a lack of periodicity. Thus, it is very difficult, and in the extreme case impossible, to predict what the wave will look like during one time interval from knowledge of its characteristics during another time interval of equal duration.

The vibratory motion of an aperiodic wave is **random**, and therefore unpredictable, and vibratory motions of this type are called **random time functions**. In acoustics, this is called an **aperiodic sound wave**. The sound wave shown in Figure 5–6 is an aperiodic, or random, wave, and you should see that it is virtually impossible (except by chance) to identify any two time intervals during which the characteristics of the vibratory motion are identical in all respects.

We encounter aperiodic sound waves daily. Familiar examples are the noises from aircrafts, automobiles, or speed boats. Each of those

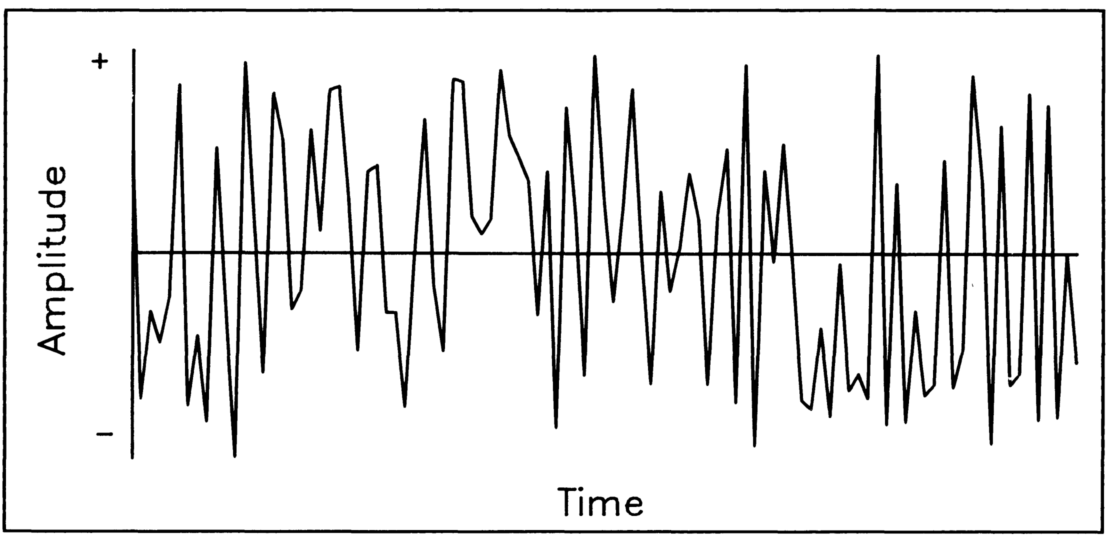

Figure 5–6. An **aperiodic** sound wave.

sounds is characterized by random vibratory motion — aperiodicity — but that does not mean that all aperiodic sounds are unpleasant. The water cascading down the side of the mountain produces an aperiodic sound wave, but under the right circumstances it might produce a very satisfying sensation. Many of the sounds of speech ("sh" in "she"; "s" in "see"; "f" in "foolish"; "th" in "three"; and so on) are characterized by random vibratory motion, but we usually don't think of such sounds as "noise."

WAVEFORM AND SPECTRUM

Waveform

Each picture of periodic (both sinusoidal and complex) and aperiodic waves that has been shown to this point has focused on the **waveform**. By that we mean, we have plotted changes in one variable (pressure, velocity, acceleration, displacement, etc.) *as a function of time*. The waveform defines, for example, the distribution of instantaneous amplitudes of a sinusoidal or complex wave over time.

Return to the waveforms for the sawtooth wave in Figure 5–2 and the square wave in Figure 5–3. We can identify the **fundamental period** of each wave, and from that we can calculate the **fundamental frequency**. However, unless we happen to remember that the square wave consists of all odd harmonics and that the sawtooth wave consists of all odd and even harmonics, we would have no way of knowing what **frequencies** other than the fundamental frequency were present by visual examination of the waveform.

We also cannot determine the **amplitudes** or the **starting phases** of the sinusoidal components by visual inspection of the waveform. We shall see subsequently that both the square wave and the sawtooth wave must satisfy very specific requirements relative to both the amplitudes and starting phases of the components, but the point we wish to emphasize now is that visual inspection of the waveform will not reveal sufficient details about these important dimensions of the sinusoidal components.

Amplitude Spectrum

A graphic alternative to the **waveform** is the **amplitude spectrum in the frequency domain**, which often is shortened to just **amplitude spectrum**. Whereas the waveform shows instantaneous magnitudes such as amplitude as a function of time, the **amplitude spectrum** *shows amplitude (in either absolute or relative values) as a function of* ***frequency***.

In Figure 5–7 the waveforms of the sawtooth and square wave are shown at the left and their respective amplitude spectra are shown

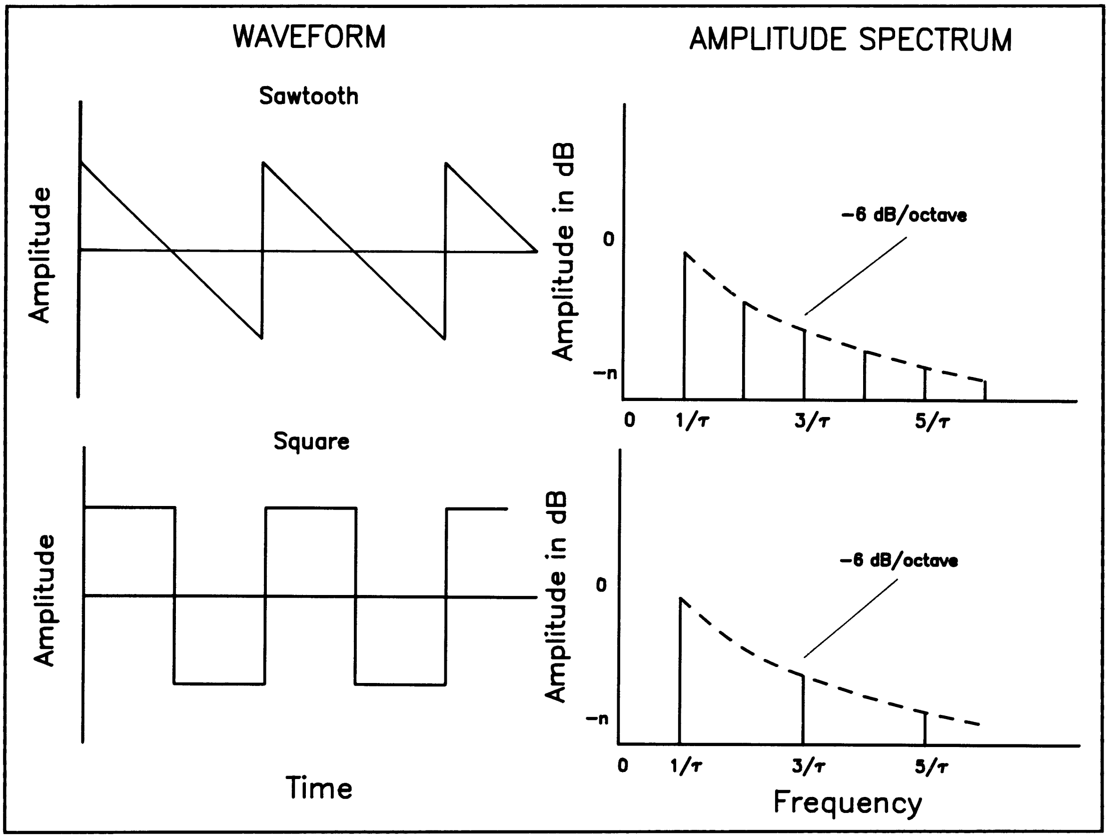

Figure 5–7. A comparison of **waveforms** and **amplitude spectra** for sawtooth and square waves, both of which are complex periodic waves. The spectrum is called a **line spectrum.**

at the right. The amplitude spectrum is shown by plotting relative amplitude in dB as a function of frequency. The location of each vertical line along the horizontal axis indicates the frequency of that component, and the height of each line is proportional to its relative amplitude; 0 dB represents the amplitude of the component with the greatest energy, and all other amplitudes therefore are shown as negative because their amplitudes are shown in dB relative to the amplitude of the fundamental.

The **envelope** of the amplitude spectrum in Figure 5–7 is shown by a dashed line that connects the peaks of each of the vertical lines. We can see that the square wave has energy at all odd harmonics and the sawtooth wave has energy at all odd and even harmonics, just as was described.

Inspection of amplitude spectra reveals information that, although present in the waveform, was not readily apparent from visual inspection of waveforms. You might have noticed that when the sinusoidal components in Figure 5–3 were summed to create a square wave, the amplitudes of the components decreased with increasing frequency. That also can be seen in the amplitude spectrum for the square wave in Figure 5–7, and now the relation among the amplitudes of the components can be seen. For the square wave, the **spectral envelope** in the frequency domain *decreases at a rate of 6 dB per octave*, which is the same as saying that the spectral envelope has a slope of −6 dB per octave.

The Octave

An **octave** refers to a *doubling in frequency* (2f). Thus, 250 Hz is one octave above 125 Hz, and 500 Hz is one octave above 250 Hz and two octaves above 125 Hz. An octave always refers to a *frequency ratio of 2:1 or 1:2, not to a frequency difference*. Thus, 2000 Hz is one octave above 1000 Hz and 200 Hz is one octave above 100 Hz because, in both cases, a ratio of 2:1 exists. The fact that the frequency difference is 1000 Hz in one case, but 100 Hz in the other, is irrelevant.

The white keys on a piano correspond to the musical notes that are designated "A,B,C,D,E,F,G,A." The lowest note is $\mathbf{A_1}$, which has a frequency of 27.5 Hz. Seven white keys to the right of $\mathbf{A_1}$ is $\mathbf{A_2}$, which has a frequency of 55 Hz. Thus, $\mathbf{A_2}$ is one octave above $\mathbf{A_1}$. At the extreme right of the keyboard is $\mathbf{A_8}$, which has a frequency of 3520 Hz. Thus, $\mathbf{A_8}$ is one octave above $\mathbf{A_7}$, which has a frequency of 1760 Hz, and seven octaves above $\mathbf{A_1}$.

Another example of octave relations can be seen by returning to Table 5–1 where the harmonic components of a complex periodic wave are listed. There we see that the 2nd harmonic is one octave above the 1st harmonic, and conversely, the 1st harmonic is one octave below the 2nd. The 4th harmonic is one octave above the 2nd and two octaves above the 1st, and so forth. In each case, the frequency ratio was either 2:1 or 1:2.

Line Spectra

The amplitude spectra in Figure 5–7 are called **line amplitude spectra**, or just **line spectra**, because the sinusoidal components of the complex periodic waves can be represented by a *set of lines*; the location of a particular line in the frequency domain (horizontal axis) identifies the frequency of that component, and the height of the line along the amplitude scale (vertical axis) identifies the amplitude.

With a line spectrum, *energy is present only at frequencies represented by the vertical lines*. Even though, for example, the spectral envelope is shown by a line that connects the harmonics of the sawtooth wave, *there is no energy at frequencies between two adjacent components*.

Continuous Spectra

The random, or aperiodic, waveform of the noise in Figure 5–6 is shown again in Figure 5–8 along with its amplitude spectrum. The result is called a **continuous amplitude spectrum**, or just **continuous spec-**

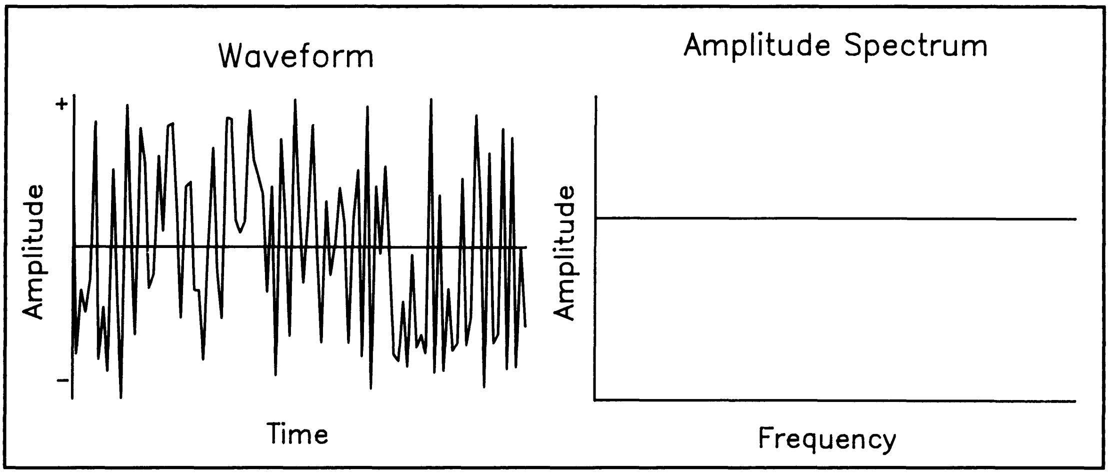

Figure 5–8. Waveform and amplitude spectrum for a complex aperiodic wave. The spectrum is called a **continuous spectrum**.

trum, in contrast to the line spectrum shown previously.

A **continuous spectrum** is one in which *energy is present at all frequencies between certain frequency limits*. Thus, the complex aperiodic wave does not result from summation of a harmonic series — odd and/or even multiples of the fundamental frequency — but rather there is energy present at all frequencies between some lower and upper limits.

In the case of the noise shown in Figure 5–8, energy is present at all frequencies and the spectral envelope has a slope of 0 dB. In other words, an identical amount of energy is present at all frequencies throughout the range. However, equal energy at all frequencies is not a requirement for all aperiodic waveforms, and subsequently we shall describe different types of aperiodic waveforms and their corresponding amplitude spectra.

Phase Spectra

In addition to the **amplitude spectrum** of a sound wave, we can also describe what is called the **phase spectrum in the frequency domain**, or just the **phase spectrum**. Whereas the amplitude spectrum describes relative amplitude as a function of frequency, the **phase spectrum** defines *the starting phase as a function of frequency*. The combination of the amplitude spectrum and the phase spectrum defines the waveform completely in the frequency domain.

EXAMPLES OF COMPLEX SOUND WAVES

Examples of several different complex signals, both periodic and aperiodic, are shown in Figure 5–9 and are compared with the familiar sine

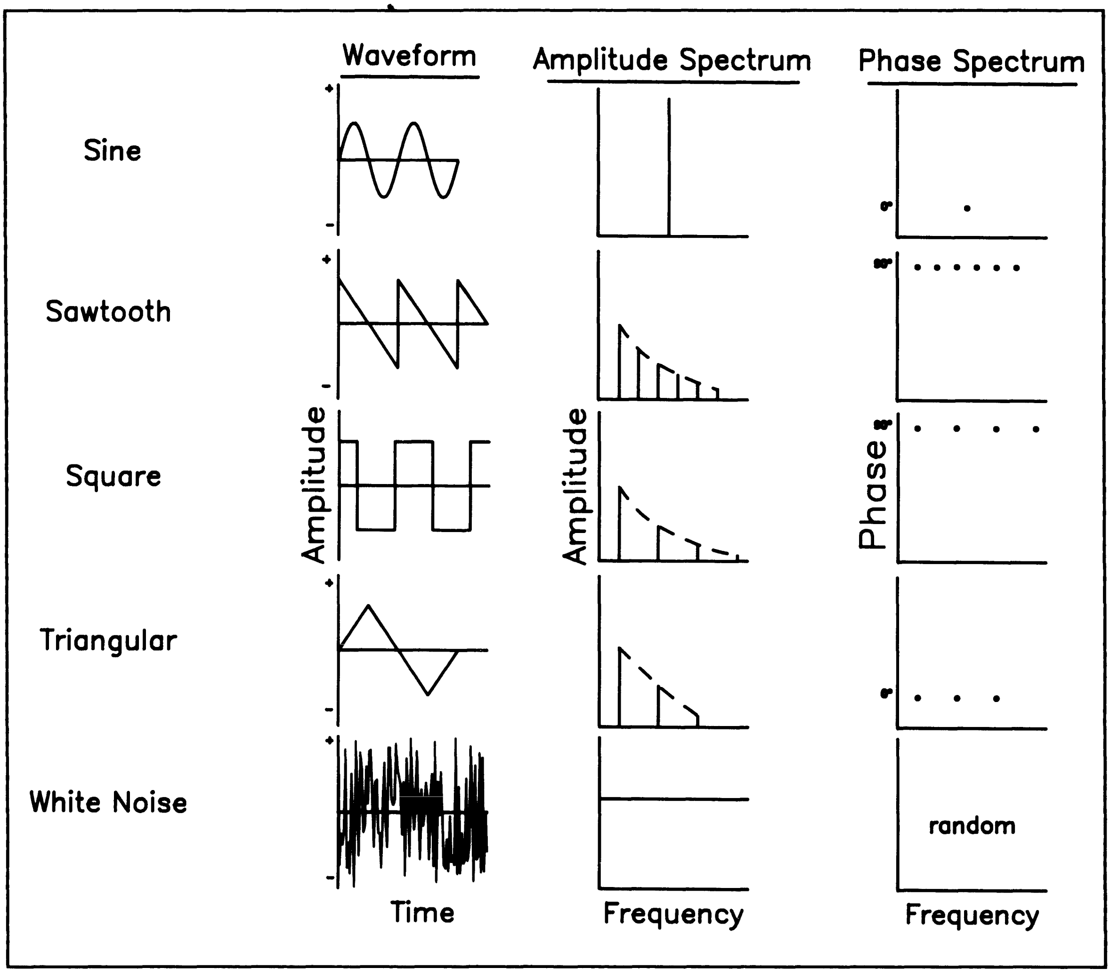

Figure 5–9. A comparison of waveforms, **amplitude spectra**, and **phase spectra** for a sine wave, sawtooth wave, square wave, triangular wave, and white noise.

wave. The panels at the left show the **waveforms**, the middle panels show the **amplitude spectra**, and the panels at the right show the **phase spectra**. The waveform of the sine wave should be thoroughly familiar by now with no further discussion.

Sawtooth Wave

A **sawtooth wave** is a complex periodic wave with energy at all, odd and even, integral multiples of the fundamental frequency. We can see from Figure 5–9 that the amplitudes of the sinusoidal components decrease with increasing frequency. Specifically, the amplitudes decrease as the inverse (the reciprocal) of the harmonic number. The relative amplitude, in decibels, for each component frequency is given by:

Equation 5.1

$$dB = 20 \log_{10} \frac{1}{h_i},$$

where h_i is the harmonic number.

By **harmonic number** (h_i) we mean 1st harmonic (h_1), 2nd harmonic (h_2), and so on to the nth harmonic. Thus, **h** always is an integer: 1, 2, 3, and so forth.

The first column of Table 5-2 lists the first nine harmonics of a sawtooth wave generated by an appropriate waveform generator. For purposes of illustration, we will arbitrarily set the rms voltage of the fundamental frequency, f_o, to be 2 V. The voltage of each of the harmonics is listed in the second column. Thus, the 2nd harmonic is 1 V (1/2 × 2 = 1 V), the 3rd harmonic is 0.67 V (1/3 × 2 = 0.67), the 4th harmonic is 0.5 V (1/4 × 2 = 0.5), and so on until the 9th harmonic where the voltage is 0.22 V (1/9 × 2 = 0.22).

Notice that the voltage is halved with each doubling of frequency. Thus, between the 1st and 2nd harmonics, the voltage decreases from 2 V to 1V. The voltage also is halved between the 2nd and 4th, the 3rd and 6th, and 4th and 8th harmonics. Recall from Chapter 4 that halving of acoustic pressure or electrical voltage corresponds to a change in amplitude of −6 dB:

$$dB = 20 \log \frac{1}{2} = -6 \text{ dB}.$$

Therefore, because each doubling of frequency corresponds to an octave, and for each octave the amplitude decreases by 6 dB, we can say that the spectral envelope has a slope of −6 dB per octave.

The third column of Table 5-2 expresses the amplitude of each harmonic in decibels re: the amplitude of the fundamental frequency. Thus, for example, with the aid of Equation 5.1 we can calculate that the level of the 5th harmonic is −14 dB:

$$dB = 20 \log \frac{1}{5} = -14\ (-13.98) \text{ dB}.$$

Table 5-2. Amplitudes (in voltage) of sinusoidal components of a **sawtooth** wave in which the amplitude of the fundamental frequency is 2 V.

Harmonic Number	rms voltage	$20 \log_{10} 1/h_i$
1 (f_0)	1/1 × 2 = 2	0
2	1/2 × 2 = 1	−6
3	1/3 × 2 = .67	−9.5
4	1/4 × 2 = .50	−12
5	1/5 × 2 = .40	−14
6	1/6 × 2 = .33	−15.6
7	1/7 × 2 = .29	−16.9
8	1/8 × 2 = .25	−18.1
9	1/9 × 2 = .22	−19.1

It is important to recognize that the *absolute amplitude* (voltage in this case because the sawtooth wave is an electrical signal) for each harmonic listed in column two of Table 5-2 *depends upon the absolute voltage of the fundamental frequency.* You might wish to try a few calculations and confirm that if the voltage of the fundamental of the sawtooth waveform were 1 V rather than 2 V, the voltages of the eight higher harmonics would be: 0.5 V; 0.33 V; 0.25 V; 0.2 V; 0.17 V; 0.14 V; 0.13 V; and 0.11 V.

We have seen, then, with those calculations that the absolute voltage of each of the harmonics in the sawtooth wave does, indeed, depend upon the absolute voltage of the fundamental. However, the relative amplitude, in decibels, for each of the harmonics in a sawtooth wave *is independent of the voltage of the fundamental frequency.* In other words, the level of the 2nd harmonic will always be −6 dB, the level of the 3rd harmonic will always be −9.5 dB, and so on.

If you continue with your computations by calculating $20 \log_{10} 1/h_i$ (Equation 5.1) for each of the calculations that you just made for the case where $\mathbf{f_0}$ = 1 V, you should obtain the same answers for a fundamental frequency of 1 V (subject to rounding error) that appear in the third column of Table 5-2 for a fundamental frequency of 2 V. For example, the relative level of the 5th harmonic still is −14 dB.

What does the amplitude spectrum of a sawtooth wave "look like?" It is a **line spectrum** because energy exists only at discrete frequencies that are integral multiples of the fundamental or lowest frequency. However, the shape of the **spectral envelope** depends on your choice of how to plot the amplitudes as a function of frequency. For example, in panel A of Figure 5-10 the voltage of each harmonic (from column two of Table 5-2) is plotted as a function of harmonic number. The scales for both the y-axis and x-axis are linear, and the resulting spectral envelope is curvilinear.

In panel B of Figure 5-10, the amplitude scale is logarithmic because we have plotted relative amplitudes in decibels re: the amplitude of the fundamental frequency ($\mathbf{f_0}$). In addition, frequency also is plotted on a logarithmic scale. The resulting spectral envelope is now linear. However, you should see that in either case, panel A or panel B, the spectral envelope has a slope of −6 dB per octave because for each doubling of frequency, the amplitude decreases by 6 dB.

In summary, a **sawtooth wave** is *a complex periodic wave with energy at odd and even integral multiples of the fundamental frequency with a spectral envelope slope of −6 dB per octave.* In Figure 5-9, each of the sinusoidal components (harmonics) of the sawtooth wave has a starting phase of 90°. The starting phases could just as well be, for example, 180°, or 0°, or 270°. However, it is essential that the starting phases of all frequency components be identical.

Square Wave

A **square wave** also is a complex periodic wave, but it has energy only at *odd integral multiples of the fundamental frequency.* We can see

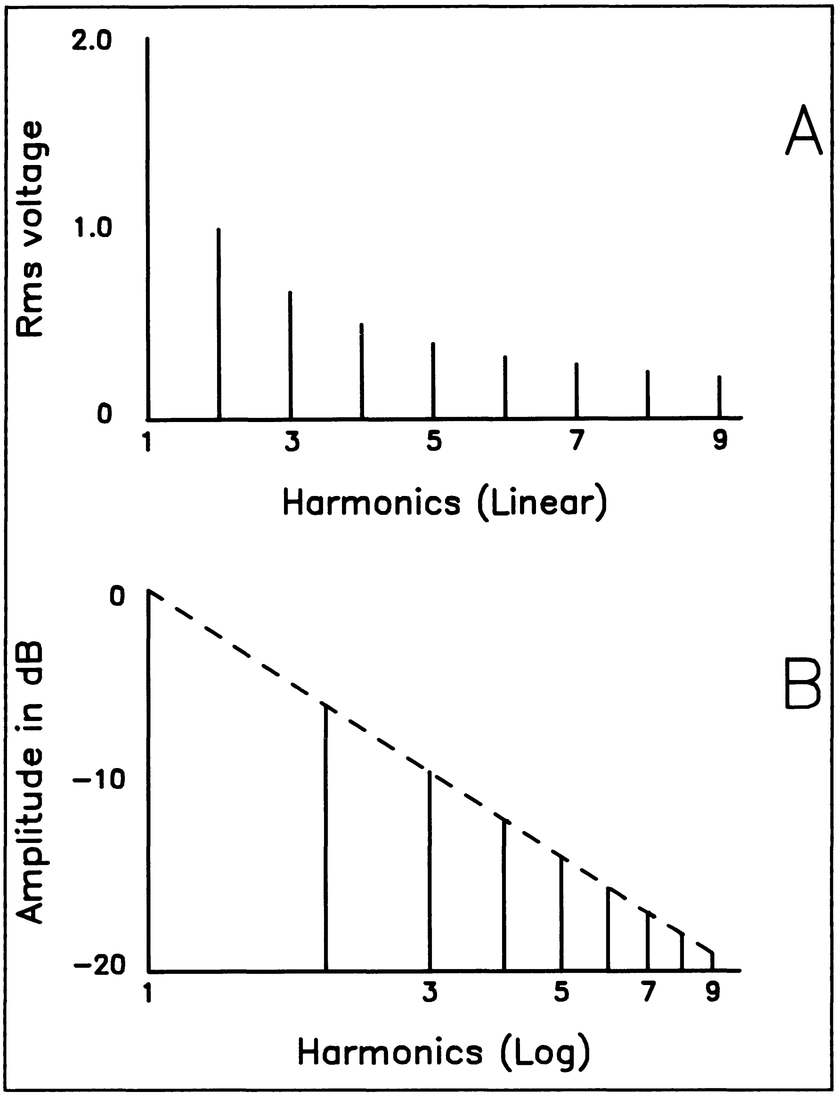

Figure 5–10. Amplitude spectra for a sawtooth wave. In panel A, the measure of amplitude is rms voltage, and the resultant **spectral envelope** is curvilinear. In panel B, the measure of amplitude is decibels, and the resultant spectral envelope is linear.

from Figure 5–9 that the amplitudes of the sinusoidal components decrease with increasing frequency, just as they did with the sawtooth wave. Moreover, we shall see that the slope of the spectral envelope of a square wave is identical to the slope of the envelope for a sawtooth wave, because the amplitudes of the frequency components also decrease as the reciprocal of the harmonic number.

The first column of Table 5–3 lists five odd harmonics (1, 3, 5, 7, and 9) of a square wave. As with the sawtooth wave, we will arbitrarily set the voltage of the fundamental frequency to be 2 V for purposes of comparison. We can see that the decrease in voltage for each of the harmonics is identical to the decrease seen for the *same harmonics* in Table

Table 5–3. Amplitudes (in voltage) of sinusoidal components of a **square** wave in which the amplitude of the fundamental frequency is 2 V.

Harmonic Number	rms voltage	20 $\log_{10} 1/h_i$
1 (f_0)	1/1 × 2 = 2	0
3	1/3 × 2 = .67	−9.5
5	1/5 × 2 = .40	−14
7	1/7 × 2 = .29	−16.9
9	1/9 × 2 = .22	−19.1

5–2 for the sawtooth wave. For example, the level of the 5th harmonic is −14 dB re: the level of the fundamental frequency for both the square wave and the sawtooth wave.

With the aid of Equation 5.1 you might wish to perform another set of computations with a voltage other than 2 V for the fundamental. Your answers for absolute voltage should differ from those in the second column in Table 5–3, but you should obtain the same answers in decibels that appear in the third column, regardless of the voltage of f_0. Thus, for example, if the voltage of the fundamental frequency is 3 V, the voltage of the 5th harmonic is 0.6 V, but its relative level in decibels is still − 14 dB. It therefore is reasonable to conceptualize a square wave as being a sawtooth wave that is devoid of even harmonics. We should reason, therefore, that the slope of the square wave also is −6 dB per octave.

In summary, a **square wave** is defined as *a complex periodic wave with energy at odd integral multiples of the fundamental and a spectral envelope slope of −6 dB per octave.* The amplitude spectrum of a square wave is a **line spectrum**. In the example shown in Figure 5–9, each of the components has a starting phase of 90°, but that is not a requirement. Those who read other introductory reference books or chapters will encounter what, at first glance, might appear to be inconsistencies. Hirsh (1952), for example, shows all components of the square wave to have 0° starting phase, whereas Yost and Nielson (1977) show all starting phases to be 90°.

Figure 5–11 should clarify any confusion. In panel A, starting phases are 0°, and the corresponding waveform begins its first excursion upward (with an infinitely steep slope) from 0°. In panel B, starting phases = 90°, and only half of the first positive-going excursion of the waveform is shown, which means in this case the waveform also begins at 90°. However, regardless of the starting phase chosen, we still are faced with the restriction that the starting phase must be identical for each frequency component.

Triangular Wave

The **triangular wave** shown in Figure 5–9 is a complex periodic wave with energy at odd integral multiples of the fundamental frequency.

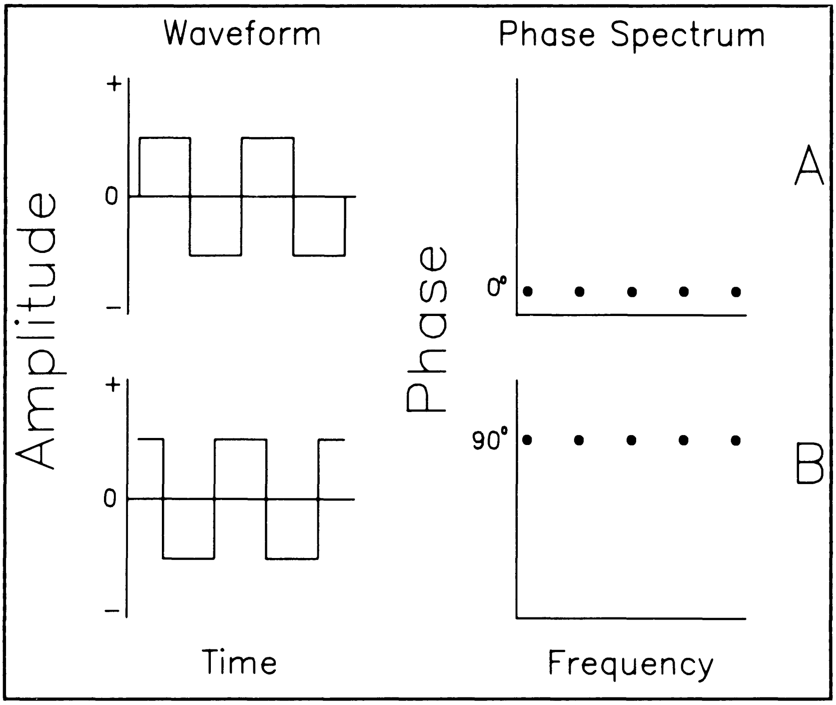

Figure 5–11. A comparison of waveforms and phase spectra for two square waves with different **starting phases**, 0° in panel A and 90° in panel B.

Because the exact same statement was made when we introduced the square wave, we must look for other explanations to account for the differences in the shapes of the two waveforms.

The first column of Table 5–4 lists the first five harmonics of a triangular wave for which the amplitude of the fundamental frequency again arbitrarily has been set at 2 V for easy comparison with our earlier calculations for sawtooth and square waves. It should be apparent that the amplitudes of the frequency components of the triangular wave decrease at a greater rate than was seen for either the sawtooth wave or the square wave, in which the amplitudes decrease as the reciprocal of the harmonic number.

The amplitudes of a triangular wave decrease as the reciprocal of the *square* of the harmonic number (rather than decrease as the reciprocal of the harmonic number itself, as with the sawtooth and square waves), and the relative amplitudes in decibels are given by:

Equation 5.2

$$dB = 20 \log_{10} \frac{1}{h_i^2},$$

where $\mathbf{h_i}$ is the harmonic number.

Consider, for example, the 3rd harmonic. For a sawtooth wave or square wave, we have seen that the amplitude of the 3rd harmonic is −9.5 dB re: the amplitude of the fundamental frequency because:

$$dB = 20 \log \frac{1}{3} = -9.5 \text{ dB}.$$

Table 5-4. Amplitudes (in voltage) of sinusoidal components of a **triangular** wave in which the amplitude of the fundamental frequency is 2 V.

Harmonic Number	rms voltage	20 $\log_{10} 1/h_i^2$
1(f_o)	$1/1^2 \times 2 = 2$	0
3	$1/3^2 \times 2 = .22$	−19.1
5	$1/5^2 \times 2 = .08$	−28
7	$1/7^2 \times 2 = .04$	−33.8
9	$1/9^2 \times 2 = .025$	−38.2

In contrast, the level of the 3rd harmonic of a triangular wave is −19.1 dB re: the level of the fundamental frequency because:

$$\text{dB} = 20 \log \frac{1}{3^2} = -19.1 \text{ dB}.$$

Thus, the slope of the spectral envelope of a triangular wave is twice as steep, −12 dB per octave, as it is for the sawtooth wave and the square wave.

Therefore, a **triangular wave** is defined as *a complex periodic wave with energy at odd integral multiples of the fundamental and a spectral envelope slope of −12 dB per octave*. The amplitude spectrum of a triangular wave also is a line spectrum because its waveform is periodic. Triangular and square waves are both characterized only by odd harmonics, but the slope of the envelope is −6 dB for the square wave and −12 dB for the triangular wave. For the example shown in Figure 5-9, all frequency components have a starting phase of 0°.

Pulse Train

Panel A of Figure 5-12 shows what is called a **pulse train**, a repetitious series of rectangularly shaped "pulses" of some width (duration, $\mathbf{P_d}$) that occur at some regular rate. For the example in the figure, the interval between the onset of one pulse and the onset of the next pulse is 10 msec. That defines the *period* (**T**) of the pulse train[1]. By taking the reciprocal of the period (1/T), we calculate the frequency of the pulse train, which for the example in the figure would be 100 Hz. This is called the **pulse repetition frequency**.

It should be apparent that the pulse train is a complex periodic waveform, and therefore, there can only be energy at *harmonics* of the pulse repetition frequency: 100 Hz, 200 Hz, 300 Hz, and so on. Panel B of Figure 5-12 shows the amplitude spectrum of the pulse train with frequency plotted on a linear scale. First, note that the component with the greatest amplitude corresponds to *0 Hz*, which refers to what is called a *dc* (direct current) component of the signal. Recall from Chapter 2 that direct current means that current is flowing only in a single

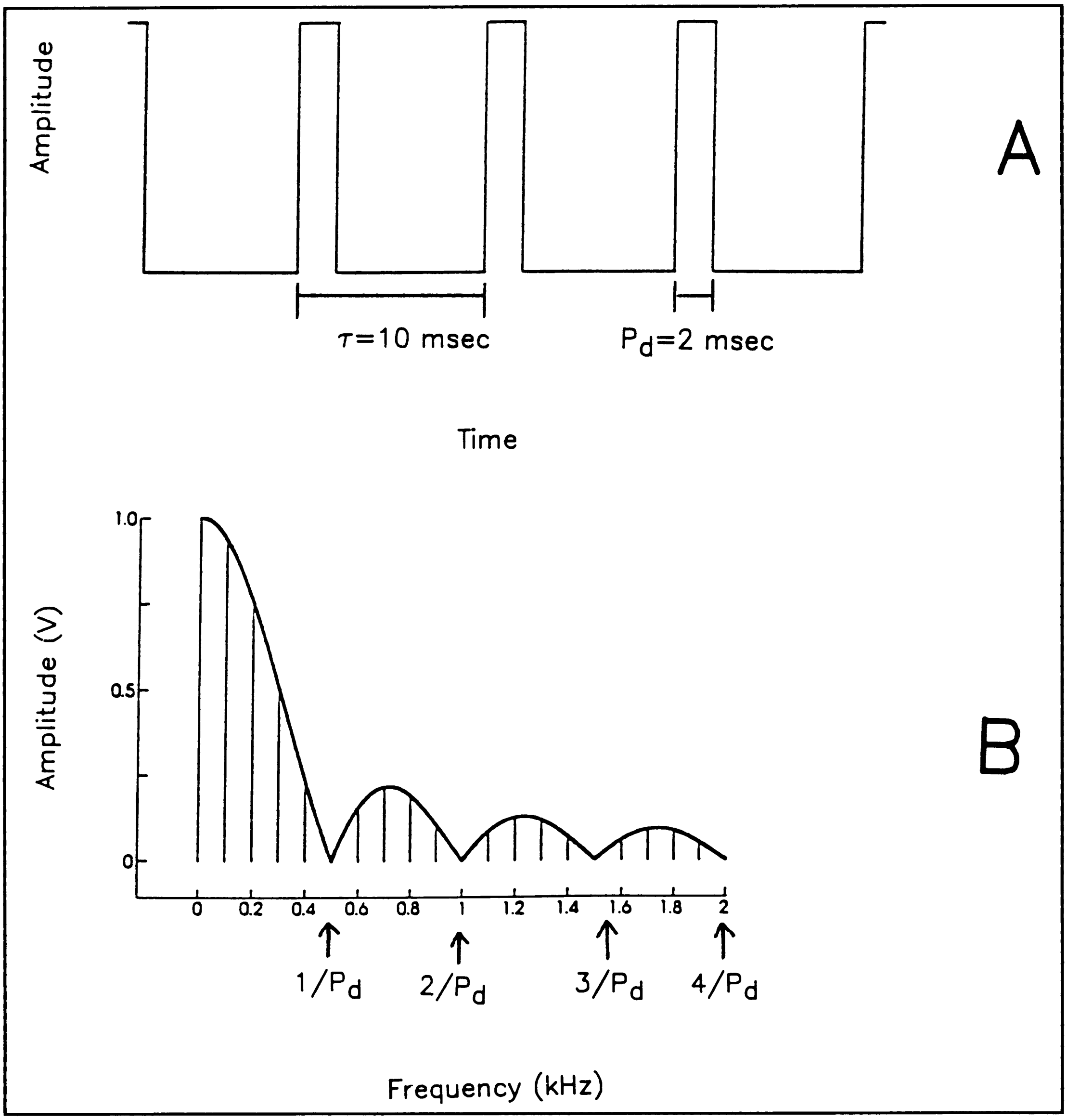

Figure 5–12. Panel A shows a **pulse train** with period = 10 msec and pulse duration (P_d) = 2 msec. Panel B shows the amplitude spectrum corresponding to the waveform in panel A. **Harmonics** are present at integral multiples of the pulse repetition frequency (100 Hz), and nulls are present at integral multiples of the reciprocal of pulse duration. Adapted from *Signals and systems for speech and hearing* (pp. 138–139) by S. Rosen and P. Howell, 1991: Academic Press, Inc., San Diego, CA. Copyright 1991 by Academic Press Limited. Printed with permission.

direction, either positive or negative, in contrast to alternating current that alternates back and forth (sinusoidally) in positive and negative directions.

Second, notice the irregularly shaped spectral envelope with lobes and valleys in panel B of Figure 5–12. Each valley or "null" occurs at integral multiples of the reciprocal of the pulse duration, $\mathbf{P_d}$. Thus, we should expect to find nulls at frequencies corresponding to $1/P_d$, $2/P_d$, $3/P_d$, and so on. The duration of each pulse in the figure is 2 msec, and therefore, the first null appears at 500 Hz (1/0.002 = 500 Hz), the next null appears at 1000 Hz (2/0.002 = 1000 Hz), and so on.

The relation among the starting phases of the frequency components is more complicated than the relation observed for the sawtooth, square, and triangular waves. The components within the first lobe below the first null at 500 Hz have a starting phase of 0°, the components within the second lobe between the first and second nulls (500 Hz and 1000 Hz) have a starting phase of 180°, and the pattern continues to alternate in this fashion from lobe to lobe as frequency increases. We shall see subsequently that it is important to emphasize that the amplitude spectrum of a pulse train is a **line spectrum**.

White, or Gaussian, Noise

White, or **Gaussian**, **noise**, which also was shown in Figure 5–9, is defined as *an aperiodic waveform with equal energy within any frequency band 1 Hz wide* (from **f** − 0.5 Hz to **f** + 0.5 Hz) *and with all phases present in a random array*. It is called **white noise** to be analogous to white light, which is characterized by equal energy at all light wavelengths.

The reason white noise is also called **Gaussian noise** is somewhat more complicated. A random time function can be described by what is called a **cumulative probability distribution**, which reveals *the percentage of the total time that any instantaneous value of the waveform's amplitude is less than some specified value*. Such a distribution for white noise is shown at the left of Figure 5–13. The slope of such a cumulative probability distribution is called a **probability density function**. For white noise, it takes the form shown at the right of Figure 5–13.

Those who have had an elementary course in descriptive statistics undoubtedly will recognize such a function as a **normal curve**, and the amplitudes (and phases) of white noise are distributed normally. A normal distribution is also called a **Gaussian** distribution in honor of Karl Friedrich Gauss, a German mathematician, astronomer, and physicist. Therefore, white noise, which is characterized by a normal probability density function, also can be called **Gaussian noise**.

The amplitude spectrum of white noise is a continuous spectrum. You can see in Figure 5–9 that the spectral envelope is a line drawn parallel to the baseline, because white noise has the same amount of energy in every frequency band that is 1 Hz wide regardless of the value of **f**. We will discuss the slope of the envelope of white noise (and introduce "pink" noise) in more detail in Chapter 6 after the concepts of **pressure spectrum level** and **octave band level** have been presented.

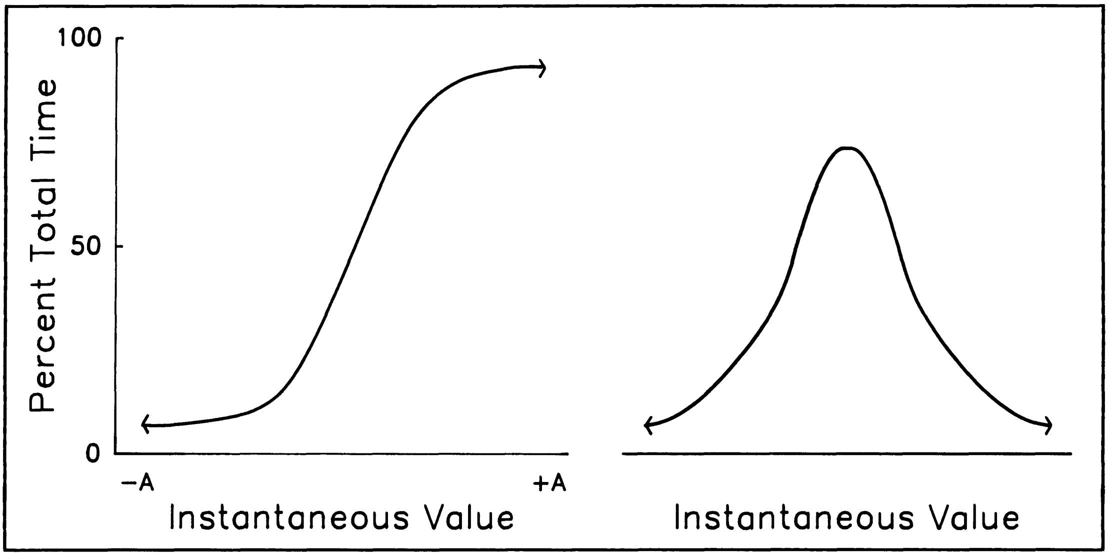

Figure 5–13. At the left is the *cumulative probability distribution* of **white noise**, which shows the percentage of total time that any instantaneous amplitude is less than some specified value. At the right is the *probability density function* for white noise, which is a plot of the slope of the function at the left. Because the probability density function assumes the shape of a normal curve, which also is called a Gaussian curve, **white noise** also is called **Gaussian noise**.

A Single Pulse

Panel A of Figure 5–14 shows the waveform of a single pulse that has the same width (duration = 2 msec) as each rectangular pulse in the pulse train that was shown in Figure 5–12. Is the waveform periodic or aperiodic? We must not be deceived because the shape of the waveform appears to be "regular" instead of random as we saw for white noise; that is irrelevant. The concept of *periodicity* means that an event occurs periodically over time. If there is only a single event (a single pulse), it cannot conceivably occur periodically.

Recall that the period of the pulse train is defined by the interval from the onset of one pulse to the onset of the next successive pulse. From that perspective, the "period" of a single pulse is infinity. If a single pulse is not periodic, we must consider it to be an aperiodic signal, and we therefore should expect that the amplitude spectrum is a **continuous spectrum** rather than a line spectrum.

Look again at the amplitude spectrum of the *pulse train* in Figure 5–12. It is a line spectrum with energy at harmonics of the pulse repetition frequency. For the example in Figure 5–12, the harmonics are spaced at 100 Hz intervals because the period of that pulse train is 10 msec (f = 1/.01 = 100 Hz).

Although it might be difficult to conceive of what would happen if the period were increased from 10 msec to infinity, we can, with a few examples, progress in that direction. If the period were increased from 10 msec to 20 msec, the pulse repetition frequency would decrease

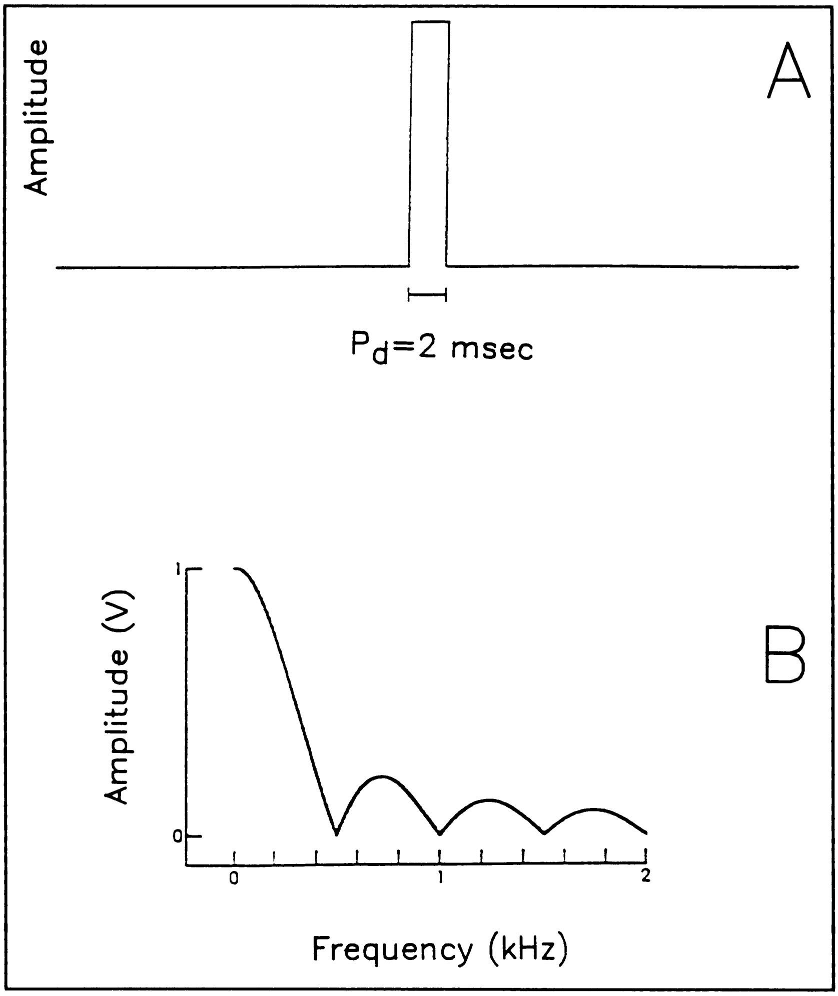

Figure 5–14. The waveform and spectrum of a **single rectangular pulse**. Nulls appear at integral multiples of the reciprocal pulse duration, just as they did for a pulse train in Figure 5–12. However, in this case the spectrum is a **continuous spectrum** in contrast to the line spectrum observed previously for a pulse train because a single pulse *cannot* be periodic. Adapted from Signals and systems for speech and hearing (pp. 143–144) by S. Rosen and P. Howell, 1991: Academic Press, Inc., San Diego, CA. Copyright 1991 by Academic Press Limited. Printed with permission.

from 100 Hz to 50 Hz (f = 1/.02 = 50 Hz), and the harmonics would be spaced twice as closely together at intervals of 50 Hz.

Recall from Chapter 1 that each time the period is doubled, frequency is halved. Therefore, for a complex wave, each time the period is doubled, the spacing between harmonics is halved. For example, if we continue to double the period to 40 msec, 80 msec, 160 msec, 320 msec, and so on, the intervals between harmonics in the amplitude spectrum progressively decrease to 25 Hz (f = 1/.04), 12.5 Hz (f = 1/.08), 6.25 Hz (f = 1/.16), 3.125 Hz (f = 1/.32), and so on.

If this process were continued to infinity, the spacing between harmonics would continue to become smaller and smaller. At infinity, the spacing between harmonics would equal 0, and the result would be a continuous spectrum of the sort shown in panel B of Figure 5-14.

The *shape of the spectral envelope* is the same as was shown previously for the pulse train because the width of the single pulse in Figure 5-14 is the same (2 msec) as the width of each of the pulses in the train of Figure 5-12. Thus, the envelope shows nulls at frequencies that correspond to integral multiples of the reciprocal of the pulse width ($1/P_d$, $2/P_d$, $3/P_d$, and so on).

MEASURES OF SOUND PRESSURE FOR COMPLEX WAVES

In Chapter 2 we described several alternative metrics by which the sound pressure of a *sine wave* could be described, and we emphasized that the various equations introduced in Chapter 2 *applied strictly only to the sine wave*. Table 5-5 contains the sine wave equations shown previously, in addition to the modifications to those equations that are required for calculating the rms, mean square, FW_{avg}, and peak sound pressure for square waves and for typical aperiodic waveforms.

It is apparent that different equations must be used for different waveforms. This introduces a problem in measurement of sound pressure. Although measurement techniques are beyond the scope of this book, one example can emphasize the importance of knowing the kind of waveform on which a measurement is being performed before the measurement is made.

Very often, an acoustical signal is converted (transduced) into an electrical signal and then various measurements are performed. Voltage is an electrical correlate of (analogous to) acoustical sound pressure. Thus, a transduced acoustical sine wave is an electrical waveform with sinusoidally fluctuating voltages over time. Measures of voltage are then performed with the aid of a voltmeter, which registers **rms** voltage.

Rms voltage is analogous to rms sound pressure. But, one type of voltmeter is called an "average-responding meter" and another type is called a "true rms meter." The average-responding meter actually "reads" the peak value of the voltage and then performs a computation

Table 5-5. Measures of sound pressure for sine, square, and random waveforms. **A** refers to the peak or **maximum amplitude** as defined in Chapter 2.

		Types of Waveforms	
Metrics	**Sine Wave**	**Square Wave**	**Random Wave**
rms	$A/\sqrt{2}$	A	~0.3 A
mean square	$A^2/2$	A^2	~0.1 A
FW_{avg}	$2A/\pi$	A	~ .25 A
peak	A	A	A

to convert the peak reading to an rms reading by dividing the peak value by $\sqrt{2}$. A true rms meter, on the other hand "reads" the rms directly, thereby avoiding any necessity for conversion. There is no problem as long as the waveform is sinusoidal.

What happens if an average-responding meter is used to measure the rms voltage of a square wave? Suppose the peak value is 1 V. We can see from Table 5–5 that the rms value also is 1 V for a square wave. However, the average-responding meter does not "know" that it is responding to a square wave. It will read the peak value of 1, divide the reading by $\sqrt{2}$, and register that the rms voltage is an erroneous 0.707.

The same measurement problem will occur with other complex waveforms. One must either know the appropriate conversions or purchase a more sophisticated and expensive measuring instrument that requires no conversion.

SIGNAL-TO-NOISE RATIO IN dB

Without exception, we listen to signals in the presence of some form of background noise. The relation between signal level and noise level is quantified by the **signal-to-noise ratio** (S/N) in dB. A positive S/N ratio means that signal level exceeds noise level, a negative S/N ratio means that noise level exceeds signal level, and an S/N ratio of 0 dB means that signal level and noise level are equal to each other. Suppose, for example, that a signal with SPL = 70 dB is presented against a background noise with SPL = 66 dB. In that case,

$$\text{dB S/N} = 70/66 = +4 \text{ dB}.$$

If the S/N ratio truly is *a ratio*, why do we solve for decibels by *subtracting* noise level from signal level rather than dividing signal level by noise level? Recall Log Law 2 from Chapter 3, which states that the log of some ratio is equal to the difference between the logs of the factors.

A decibel is (ten times) a log, and therefore we simply subtract the denominator from the numerator rather than divide the numerator by the denominator. If, on the other hand, signal intensity and noise intensity were each expressed in watt/m^2, then division would be the appropriate operation. In the example cited above, the *intensity* of the signal is 10^{-5} watt/m^2 (Equation 4.4) and the intensity of the noise is 4×10^{-6} watt/m^2. In that case,

$$\begin{aligned}\text{dB S/N} &= 10 \log 10^{-5}/(4 \times 10^{-6}),\\ &= 10 \log 0.25 \times 10^{1},\\ &= +4 \text{ dB}.\end{aligned}$$

Obviously, it is easier to simply subtract decibels.

NOTES

1. Some authors such as, for example, Yost and Nielson (1977), use the symbols **P** for **period** and **T** for **pulse duration**. For the sake of consistency, we will continue to use **T** for **period** and we have adopted the symbol $\mathbf{P_d}$ for **pulse duration**; the subscript **d** serves to distinguish $\mathbf{P_d}$ from **P**, which we have used as a symbol for **pressure**.