14.6 SOLUÇÕES

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1. $f(x, y) = x^2y^3 + 2x^4y \quad \Rightarrow \quad fx(x, y) = 2xy^3 + 8x^3y$ e $f_y(x, y) = 3x^2y^2 + 2x^4$. Se $\mathbf{u}$ é um vetor unitário na direção de $\theta = \frac{\pi}{3}$, então, da Equação 6,

$$D_{\mathbf{u}} f(1, -2) = f_x(1, -2)\cos\tfrac{\pi}{3} + f_y(1, -2)\operatorname{sen}\tfrac{\pi}{3}$$
$$= (-32)\left(\tfrac{1}{2}\right) + (14)\left(\tfrac{\sqrt{3}}{2}\right) = 7\sqrt{3} - 16$$

2. $f(x, y) = \operatorname{sen}(x + 2y) \quad \Rightarrow \quad f_x(x, y) = \cos(x + 2y)$ e $f_y(x, y) = 2\cos(x + 2y)$. Se $\mathbf{u}$ é um vetor unitário na direção de $\theta = \frac{3\pi}{4}$, então, da Equação 6,

$$D_{\mathbf{u}} f(4, -2) = f_x(4, -2)\cos\tfrac{3\pi}{4} + f_y(4, -2)\operatorname{sen}\tfrac{3\pi}{4}$$
$$= (\cos 0)\left(-\tfrac{\sqrt{2}}{2}\right) + 2(\cos 0)\left(\tfrac{\sqrt{2}}{2}\right) = \tfrac{\sqrt{2}}{2}$$

3. $f(x, y) = xe^{-2y} \quad \Rightarrow \quad f_x(x, y) = e^{-2y}$ e $f_y(x, y) = -2xe^{-2y}$. Se $\mathbf{u}$ é um vetor unitário na direção de $\theta = \frac{\pi}{2}$, então

$$D_{\mathbf{u}} f(5, 0) = f_x(5, 0)\cos\tfrac{\pi}{2} + f_y(5, 0)\operatorname{sen}\tfrac{\pi}{2}$$
$$= 1 \cdot 0 + (-10)\,1 = -10$$

4. $f(x, y) = \left(x^2 - y\right)^3 \quad \Rightarrow \quad D_{\mathbf{u}} f(x, y) = 3\left(x^2 - y\right)^2(2x)\cos\frac{3\pi}{4} + 3\left(x^2 - y\right)^2(-1)\operatorname{sen}\frac{3\pi}{4}$. Logo, $D_{\mathbf{u}} f(3, 1) = 3(8)^2(6)\left(-\frac{\sqrt{2}}{2}\right) - 3(8)^2\left(\frac{\sqrt{2}}{2}\right) = -672\sqrt{2}$.

5. $f(x, y) = y^x \quad \Rightarrow$
$D_{\mathbf{u}} f(x, y) = (y^x \ln y)\cos\frac{\pi}{2} + \left(xy^{x-1}\right)\operatorname{sen}\frac{\pi}{2} = xy^{x-1}$.
Logo, $D_{\mathbf{u}} f(1, 2) = (1)(2)^{1-1} = 1$.

6. $f(x, y) = x^3 - 4x^2y + y^2$

(a) $\nabla f(x, y) = f_x\,\mathbf{i} + f_y\,\mathbf{j} = \left(3x^2 - 8xy\right)\mathbf{i} + \left(2y - 4x^2\right)\mathbf{j}$

(b) $\nabla f(0, -1) = -2\,\mathbf{j}$

(c) $\nabla f(0, -1) \cdot \mathbf{u} = -\frac{8}{5}$

7. $f(x, y) = e^x \operatorname{sen} y$

(a) $\nabla f(x, y) = f_x\,\mathbf{i} + f_y\,\mathbf{j} = e^x \operatorname{sen} y\,\mathbf{i} + e^x \cos y\,\mathbf{j}$

(b) $\nabla f\left(1, \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}e(\mathbf{i} + \mathbf{j})$

(c) $\nabla f\left(1, \frac{\pi}{4}\right) \cdot \mathbf{u} = \frac{\sqrt{2}}{2}e\left(\frac{1}{\sqrt{5}}\right) = \frac{1}{\sqrt{10}}e$

8. $f(x, y, z) = xy^2z^3$

(a) $\nabla f(x, y, z) = \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z)\rangle$
$= \left\langle y^2z^3, 2xyz^3, 3xy^2z^2\right\rangle$

(b) $\nabla f(1, -2, 1) = \langle 4, -4, 12\rangle$

(c) $\nabla f(1, -2, 1) \cdot \mathbf{u} = \frac{4}{\sqrt{3}} + \frac{4}{\sqrt{3}} + \frac{12}{\sqrt{3}} = \frac{20}{\sqrt{3}}$

9. $f(x, y, z) = xy + yz^2 + xz^3$

(a) $\nabla f(x, y, z) = \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z)\rangle$
$= \left\langle y + z^3, x + z^2, 2yz + 3xz^2\right\rangle$

(b) $\nabla f(2, 0, 3) = \langle 27, 11, 54\rangle$

(c) $\nabla f(2, 0, 3) \cdot \mathbf{u} = \frac{1}{3}(-54 - 11 + 108) = \frac{43}{3}$

10. $f(x, y) = x/y \quad \Rightarrow \quad \nabla f(x, y) = \left\langle 1/y, -x/y^2\right\rangle$,
$\nabla f(6, -2) = \left\langle -\frac{1}{2}, -\frac{3}{2}\right\rangle$, $\mathbf{u} = \left\langle -\frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}}\right\rangle$ e
$D_{\mathbf{u}} f(6, -2) = \frac{1}{2\sqrt{10}} - \frac{9}{2\sqrt{10}} = -\frac{4}{\sqrt{10}} = -\frac{2\sqrt{10}}{5}$.

11. $f(x, y) = \sqrt{x - y} \quad \Rightarrow$
$\nabla f(x, y) = \left\langle \frac{1}{2}(x - y)^{-1/2}, -\frac{1}{2}(x - y)^{-1/2}\right\rangle$,
$\nabla f(5, 1) = \left\langle \frac{1}{4}, -\frac{1}{4}\right\rangle$, e um vetor unitário na direção de $\mathbf{v}$ é $\mathbf{u} = \left\langle \frac{12}{13}, \frac{5}{13}\right\rangle$, então
$D_{\mathbf{u}} f(5, 1) = \nabla f(5, 1) \cdot \mathbf{u} = \frac{12}{52} - \frac{5}{52} = \frac{7}{52}$.

12. $g(x, y) = xe^{xy} \quad \Rightarrow \quad \nabla g(x, y) = \left\langle e^{xy}(1 + xy), x^2e^{xy}\right\rangle$,
$\nabla g(-3, 0) = \langle 1, 9\rangle$, $\mathbf{u} = \left\langle \frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}}\right\rangle$ e
$D_{\mathbf{u}} g(-3, 0) = \frac{2}{\sqrt{13}} + \frac{27}{\sqrt{13}} = \frac{29}{\sqrt{13}}$.

13. $g(x, y) = e^x \cos y \quad \Rightarrow$
$\nabla g(x, y) = \langle e^x \cos y, -e^x \operatorname{sen} y\rangle$,
$\nabla g\left(1, \frac{\pi}{6}\right) = \left\langle \frac{\sqrt{3}}{2}e, -\frac{1}{2}e\right\rangle$, $\mathbf{u} = \left\langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right\rangle$ e
$D_{\mathbf{u}} g\left(1, \frac{\pi}{6}\right) = \frac{\sqrt{3}}{2\sqrt{2}}e + \frac{1}{2\sqrt{2}}e = \frac{1 + \sqrt{3}}{2\sqrt{2}}e$.

14. $f(x, y, z) = \sqrt{xyz} \quad \Rightarrow$
$\nabla f(x, y, z) = \frac{1}{2}(xyz)^{-1/2}\langle yz, xz, xy\rangle$,
$\nabla f(2, 4, 2) = \left\langle 1, \frac{1}{2}, 1\right\rangle$, $\mathbf{u} = \left\langle \frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right\rangle$ e
$D_{\mathbf{u}} f(2, 4, 2) = 1 \cdot \frac{2}{3} + \frac{1}{2} \cdot \frac{1}{3} + 1\left(-\frac{2}{3}\right) = \frac{1}{6}$.

15. $g(x, y, z) = xe^{yz} + xye^z \quad \Rightarrow$
$\nabla g(x, y, z) = \langle e^{yz} + ye^z, xze^{yz} + xe^z, xy\left(e^{yz} + e^z\right)\rangle$,
$\nabla g(-2, 1, 1) = \langle 2e, -4e, -4e\rangle$, $\mathbf{u} = \frac{1}{\sqrt{14}}\langle 1, -2, 3\rangle$ e
$D_{\mathbf{u}} g(-2, 1, 1) = \frac{(2e)(1)}{\sqrt{14}} + \frac{(-4e)(-2)}{\sqrt{14}} + \frac{(-4e)(3)}{\sqrt{14}} = \frac{-2e}{\sqrt{14}}$
$= -\frac{e\sqrt{14}}{7}$

16. $g(x, y, z) = x\operatorname{tg}^{-1}(y/z) \quad \Rightarrow \quad \nabla g(x, y, z) = \left\langle \operatorname{tg}^{-1}(y/z), xz/\left(y^2 + z^2\right), -xy/\left(y^2 + z^2\right)\right\rangle$,
$\nabla g(1, 2, -2) = \left\langle -\frac{\pi}{4}, -\frac{1}{4}, -\frac{1}{4}\right\rangle$, $\mathbf{u} = \frac{1}{\sqrt{3}}\langle 1, 1, -1\rangle$ e
$D_{\mathbf{u}} g(1, 2, -2) = \frac{(-\pi)(1)}{4\sqrt{3}} + \frac{(-1)(1)}{4\sqrt{3}} + \frac{(-1)(-1)}{4\sqrt{3}} = -\frac{\pi}{4\sqrt{3}}$

17. $g(x, y, z) = z^3 - x^2y \quad \Rightarrow$
$\nabla g(x, y, z) = \left\langle -2xy, -x^2, 3z^2\right\rangle$,
$\nabla g(1, 6, 2) = \langle -12, -1, 12\rangle$, $\mathbf{u} = \left\langle \frac{3}{13}, \frac{4}{13}, \frac{12}{13}\right\rangle$, e
$D_{\mathbf{u}} g(1, 6, 2) = \frac{(-12)(3)}{13} + \frac{(-1)(4)}{13} + \frac{(12)(12)}{13} = 8$.

18. $f(x, y) = \sqrt{x^2 + 2y} \quad \Rightarrow$

$$\nabla f(x, y) = \left\langle \frac{x}{\sqrt{x^2 + 2y}}, \frac{1}{\sqrt{x^2 + 2y}}\right\rangle.$$ Então

a taxa máxima de variação é $|\nabla f(4, 10)| = \frac{\sqrt{17}}{6}$ na direção de $\left\langle \frac{2}{3}, \frac{1}{6}\right\rangle$ ou $\langle 4, 1\rangle$.

19. $f(x, y) = \cos(3x + 2y) \quad \Rightarrow$

$\nabla f(x, y) = \langle -3 \text{ sen } (3x + 2y), -2 \text{ sen } (3x + 2y) \rangle$, então

a taxa máxima de variação é $\left| \nabla f\left(\frac{\pi}{6}, -\frac{\pi}{8}\right) \right| = \sqrt{\frac{13}{2}}$

na direção de $\left\langle -\frac{3\sqrt{2}}{2}, -\sqrt{2} \right\rangle$ ou $\langle -3, -2 \rangle$.

20. $f(x, y) = xe^{-y} + 3y \Rightarrow \nabla f(x, y) = \langle e^{-y}, 3 - xe^{-y} \rangle$,

$\nabla f(1, 0) = \langle 1, 2 \rangle$, é a taxa máxima de variação e

a taxa máxima é $|\nabla f(1, 0)| = \sqrt{5}$.

21. $f(x, y) = \ln\left(x^2 + y^2\right) \quad \Rightarrow$

$\nabla f(x, y) = \left\langle \frac{2x}{x^2 + y^2}, \frac{2y}{x^2 + y^2} \right\rangle, \nabla f(1, 2) = \left\langle \frac{2}{5}, \frac{4}{5} \right\rangle$.

Então, a taxa máxima de variação é $|\nabla f(1, 2)| = \frac{2\sqrt{5}}{5}$

na direção de $\left\langle \frac{2}{5}, \frac{4}{5} \right\rangle$ ou $\langle 1, 2 \rangle$.

22. $f(x, y, z) = x + y/z \quad \Rightarrow \quad \nabla f(x, y, z) = \left\langle 1, \frac{1}{z}, -\frac{y}{z^2} \right\rangle$,

então a taxa máxima de variação é

$|\nabla f(4, 3, -1)| = \sqrt{11}$ na direção de $\langle 1, -1, -3 \rangle$.

23. $f(x, y, z) = \frac{x}{y} + \frac{y}{z} \quad \Rightarrow$

$\nabla f(x, y, z) = \left\langle \frac{1}{y}, \frac{1}{z} - \frac{x}{y^2}, -\frac{y}{z^2} \right\rangle$,

então a taxa máxima de variação é

$|\nabla f(4, 2, 1)| = \frac{\sqrt{17}}{2}$ na direção de $\left\langle \frac{1}{2}, 0, -2 \right\rangle$ ou $\langle 1, 0, -4 \rangle$.

24. $F(x, y, z) = xy + yz + zx \quad \Rightarrow$

$\nabla F(x, y, z) = \langle y + z, z + x, x + y \rangle$,

$\nabla F(1, 1, 1) = \langle 2, 2, 2 \rangle$

(a) $2x + 2y + 2z = 6$ ou $x + y + z = 3$

(b) $x - 1 = y - 1 = z - 1$ ou $x = y = z$

25. $F(x, y, z) = xyz \quad \Rightarrow \quad \nabla F(x, y, z) = \langle yz, zx, xy \rangle$,

$\nabla F(1, 2, 3) = \langle 6, 3, 2 \rangle$

(a) $6x + 3y + 2z = 18$

(b) $\frac{1}{6}(x - 1) = \frac{1}{3}(y - 2) = \frac{1}{2}(z - 3)$

26. $F(x, y, z) = x^2 + y^2 - z^2 - 2xy + 4xz \quad \Rightarrow$

$\nabla F(x, y, z) = \langle 2x - 2y + 4z, 2y - 2x, -2z + 4x \rangle$,

$\nabla F(1, 0, 1) = \langle 6, -2, 2 \rangle$

(a) $6(x - 1) - 2(y - 0) + 2(z - 1) = 0$ ou

$3x - y + z = 4$

(b) $\frac{x - 1}{3} = -y = z - 1$

27. $F(x, y, z) = x^2 - 2y^2 - 3z^2 + xyz \quad \Rightarrow$

$\nabla F(x, y, z) = \langle 2x + yz, -4y + xz, -6z + xy \rangle$,

$\nabla F(3, -2, -1) = \langle 8, 5, 0 \rangle$

(a) $8(x - 3) + 5(y + 2) + 0(z + 1) = 0$ ou $8x + 5y = 14$

(b) $\frac{x - 3}{8} = \frac{y + 2}{5}, z = -1$

28. $F(x, y, z) = xe^{yz} \quad \Rightarrow$

$\nabla F(x, y, z) = \langle e^{yz}, xze^{yz}, xye^{yz} \rangle$,

$\nabla F(1, 0, 5) = \langle 1, 5, 0 \rangle$

(a) $1(x - 1) + 5(y - 0) + 0(z - 5) = 0$ ou $x + 5y = 1$

(b) $x - 1 = \frac{y}{5}, z = 5$

29. $F(x, y, z) = 4x^2 + y^2 + z^2, \nabla F(2, 2, 2) = \langle 16, 4, 4 \rangle$

(a) $16x + 4y + 4z = 48$ ou $4x + y + z = 12$

(b) $\frac{x - 2}{16} = \frac{y - 2}{4} = \frac{z - 2}{4}$ ou $\frac{x - 2}{4} = y - 2 = z - 2$

30. $F(x, y, z) = x^2 - 2y^2 + z^2 \quad \Rightarrow$

$\nabla F(-1, 1, -2) = \langle -2, -4, -4 \rangle$

(a) $-2x - 4y - 4z = 6$ ou $x + 2y + 2z + 3 = 0$

(b) $x + 1 = \frac{y - 1}{2} = \frac{z + 2}{2}$