10.4 SOLUÇÕES

Revisão técnica: Ricardo Miranda Martins – IMECC – Unicamp

1. $A = \int_0^{\pi} \frac{1}{2} r^2 \, d\theta = \int_0^{\pi} \frac{1}{2} \theta^2 \, d\theta = \left[\frac{1}{6}\theta^3\right]_0^{\pi} = \frac{1}{6}\pi^3$

2. $A = \int_{-\pi/2}^{\pi/2} \frac{1}{2} e^{2\theta} \, d\theta = \left[\frac{1}{4} e^{2\theta}\right]_{-\pi/2}^{\pi/2} = \frac{1}{4}\left(e^{\pi} - e^{-\pi}\right)$

3. $A = \int_0^{\pi/6} \frac{1}{2} (2\cos\theta)^2 \, d\theta = \int_0^{\pi/6} (1 + \cos 2\theta) \, d\theta$
$= \left[\theta + \frac{1}{2} \operatorname{sen} 2\theta\right]_0^{\pi/6} = \frac{\pi}{6} + \frac{\sqrt{3}}{4}$

4. $A = \int_{\pi/6}^{5\pi/6} \frac{1}{2} (1/\theta)^2 \, d\theta = \left[-1/(2\theta)\right]_{\pi/6}^{5\pi/6} = \frac{12}{5\pi}$

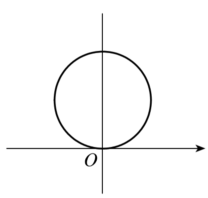
5. $A = \int_0^{\pi/6} \frac{1}{2} \operatorname{sen}^2 2\theta \, d\theta = \frac{1}{4} \int_0^{\pi/6} (1 - \cos 4\theta) \, d\theta$
$= \left[\frac{1}{4}\theta - \frac{1}{16} \operatorname{sen} 2\theta\right]_0^{\pi/6} = \frac{4\pi - 3\sqrt{3}}{96}$

6. $A = 2 \int_0^{\pi/12} \frac{1}{2} \cos^2 3\theta \, d\theta = \frac{1}{2} \int_0^{\pi/12} (1 + \cos 6\theta) \, d\theta$
$= \frac{1}{2} \left[\theta + \frac{1}{6} \operatorname{sen} 6\theta\right]_0^{\pi/12} = \frac{1}{24} (\pi + 2)$

7. $A = \int_{\pi/4}^{3\pi/4} \frac{1}{2} (3 \operatorname{sen}\theta)^2 \, d\theta = 2 \int_{\pi/4}^{\pi/2} \frac{9}{4} (1 - \cos 2\theta) \, d\theta$
$= \frac{9}{2} \left[\theta - \frac{1}{2} \operatorname{sen} 2\theta\right]_{\pi/4}^{\pi/2} = \frac{9}{8} (\pi + 2)$

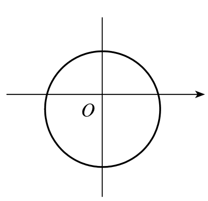
8. $A = \int_{\pi/2}^{3\pi/2} \frac{1}{2} \left(\theta^2\right)^2 \, d\theta = \left[\frac{1}{10}\theta^5\right]_{\pi/2}^{3\pi/2} = \frac{121}{160}\pi^5$

9.



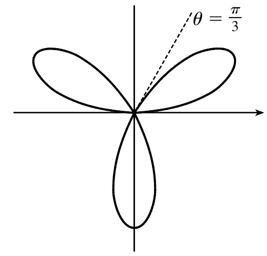
$A = \int_0^{\pi} \frac{1}{2} (5 \operatorname{sen}\theta)^2 \, d\theta = \frac{25}{4} \int_0^{\pi} (1 - \cos 2\theta) \, d\theta$
$= \frac{25}{4} \left[\theta - \frac{1}{2} \operatorname{sen} 2\theta\right]_0^{\pi} = \frac{25}{4}\pi$

10.



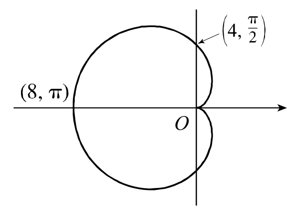
$A = 2 \int_{-\pi/2}^{\pi/2} \frac{1}{2} (4 - \operatorname{sen}\theta)^2 \, d\theta$
$= \int_{-\pi/2}^{\pi/2} \left(16 - 8 \operatorname{sen}\theta + \operatorname{sen}^2\theta\right) d\theta$
$= \int_{-\pi/2}^{\pi/2} \left(16 + \operatorname{sen}^2\theta\right) d\theta$ [pelo Teorema 5.5.7(b)]
$= 2 \int_0^{\pi/2} \left(16 + \operatorname{sen}^2\theta\right) d\theta$ [pelo Teorema 5.5.7(a)]
$= 2 \int_0^{\pi/2} \left[16 + \frac{1}{2}(1 - \cos 2\theta)\right] d\theta$
$= 2 \left[\frac{33}{2}\theta - \frac{1}{4} \operatorname{sen} 2\theta\right]_0^{\pi/2} = \frac{33\pi}{2}$

11.



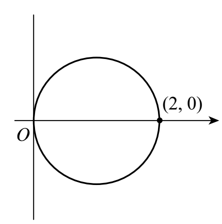
$A = 6 \int_0^{\pi/6} \frac{1}{2} \operatorname{sen}^2 3\theta \, d\theta = 3 \int_0^{\pi/6} \frac{1}{2} (1 - \cos 6\theta) \, d\theta$
$= \frac{3}{2} \left[\theta - \frac{1}{6} \operatorname{sen} 6\theta\right]_0^{\pi/6} = \frac{\pi}{4}$

12.



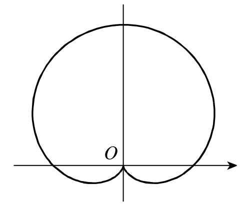
$A = 2 \int_0^{\pi} \frac{1}{2} \left[4(1 - \cos\theta)\right]^2 \, d\theta$
$= 16 \int_0^{\pi} \left(1 - 2\cos\theta + \cos^2\theta\right) d\theta$
$= 8 \int_0^{\pi} (3 - 4\cos\theta + \cos 2\theta) \, d\theta$
$= 4 \left[6\theta - 8 \operatorname{sen}\theta + \operatorname{sen} 2\theta\right]_0^{\pi} = 24\pi$

13.



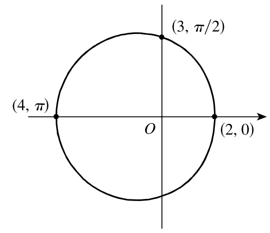
$A = 2 \int_0^{\pi/2} \frac{1}{2} (2\cos\theta)^2 \, d\theta = 2 \int_0^{\pi/2} (1 + \cos 2\theta) \, d\theta$
$= 2 \left[\theta + \frac{1}{2} \operatorname{sen} 2\theta\right]_0^{\pi/2} = \pi$

14.



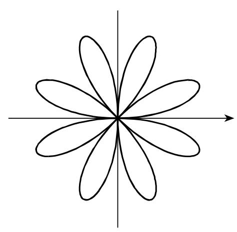
$A = 2 \int_{-\pi/2}^{\pi/2} \frac{1}{2} (1 + \operatorname{sen}\theta)^2 \, d\theta$
$= \int_{-\pi/2}^{\pi/2} \left(1 + 2 \operatorname{sen}\theta + \operatorname{sen}^2 2\theta\right) d\theta$
$= \left[\theta - 2\cos\theta\right]_{-\pi/2}^{\pi/2} + \int_{-\pi/2}^{\pi/2} \frac{1}{2} (1 - \cos 2\theta) \, d\theta$
$= \pi + \frac{1}{2} \left[\theta - \frac{1}{2} \operatorname{sen} 2\theta\right]_{-\pi/2}^{\pi/2} = \pi + \frac{\pi}{2} = \frac{3\pi}{2}$

15.



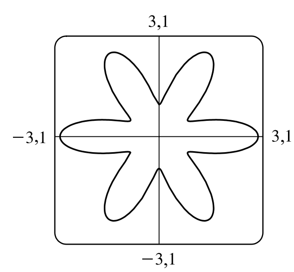
$$\begin{aligned} A &= 2\int_0^{\pi} \tfrac{1}{2}(3-\cos\theta)^2\,d\theta \\ &= \int_0^{\pi}\left(9-6\cos\theta+\cos^2\theta\right)d\theta \\ &= \left[9\theta-6\,\text{sen}\,\theta+\tfrac{1}{2}\theta+\tfrac{1}{4}\,\text{sen}\,2\theta\right]_0^{\pi} = \tfrac{19\pi}{2}\end{aligned}$$

16.



$$\begin{aligned} A &= 8\int_0^{\pi/4}\tfrac{1}{2}\,\text{sen}^2\,4\theta\,d\theta = 2\int_0^{\pi/4}(1-\cos 8\theta)\,d\theta \\ &= \left[2\theta-\tfrac{1}{4}\,\text{sen}\,8\theta\right]_0^{\pi/4} = \tfrac{\pi}{2}\end{aligned}$$

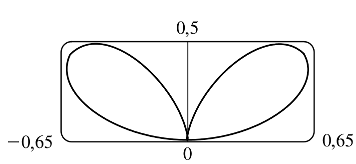
17.



Pela simetria, a área total é o dobro da área delimitada acima do eixo polar, logo

$$\begin{aligned} A &= 2\int_0^{\pi}\tfrac{1}{2}r^2\,d\theta = \int_0^{\pi}[2+\cos 6\theta]^2\,d\theta \\ &= \int_0^{\pi}\left(4+4\cos 6\theta+\cos^2 6\theta\right)d\theta \\ &= \left[4\theta+4\left(\tfrac{1}{6}\,\text{sen}\,6\theta\right)+\left(\tfrac{1}{24}\,\text{sen}\,12\theta+\tfrac{1}{2}\theta\right)\right]_0^{\pi} \\ &= 4\pi+\tfrac{\pi}{2} = \tfrac{9\pi}{2}\end{aligned}$$

18.



Observe que a curva completa $r = 2\,\text{sen}\,\theta\cos^2\theta$ é gerada $\cos\theta \in [0, \pi]$. O raio é positivo neste intervalo, então a área delimitada é

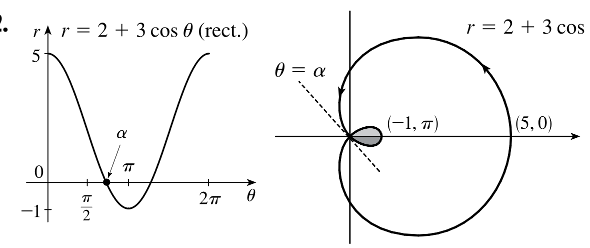
$$\begin{aligned} A &= \int_0^{\pi}\tfrac{1}{2}r^2\,d\theta = \int_0^{\pi}\tfrac{1}{2}\left(2\,\text{sen}\,\theta\cos^2\theta\right)^2 d\theta \\ &= 2\int_0^{\pi}\text{sen}^2\,\theta\cos^4\theta\,d\theta = 2\int_0^{\pi}(\text{sen}\,\theta\cos\theta)^2\cos^2\theta\,d\theta \\ &= 2\int_0^{\pi}\left(\tfrac{1}{2}\,\text{sen}\,2\theta\right)^2\cos^2\theta\,d\theta \\ &= \tfrac{1}{4}\int_0^{\pi}\text{sen}^2\,2\theta\,(\cos 2\theta+1)\,d\theta \\ &= \tfrac{1}{4}\left[\int_0^{\pi}\text{sen}^2\,2\theta\cos 2\theta\,d\theta+\int_0^{\pi}\text{sen}^2\,2\theta\,d\theta\right] \\ &= \tfrac{1}{4}\left[\tfrac{1}{2}\theta-\tfrac{1}{4}\,\text{sen}\,4\theta\right]_0^{\pi} \text{ (a primeira integral desaparece)} = \tfrac{\pi}{8}\end{aligned}$$

19.
$$\begin{aligned} A &= 2\int_0^{\pi/6}\tfrac{1}{2}\cos^2 3\theta\,d\theta = \tfrac{1}{2}\int_0^{\pi/6}(1+\cos 6\theta)\,d\theta \\ &= \tfrac{1}{2}\left[\theta+\tfrac{1}{6}\,\text{sen}\,6\theta\right]_0^{\pi/6} = \tfrac{\pi}{12}\end{aligned}$$

20.
$$\begin{aligned} A &= 2\int_0^{\pi/4}\tfrac{1}{2}(3\,\text{sen}\,2\theta)^2\,d\theta = \tfrac{9}{2}\int_0^{\pi/4}(1-\cos 4\theta)\,d\theta \\ &= \tfrac{9}{2}\left[\theta-\tfrac{1}{4}\,\text{sen}\,4\theta\right]_0^{\pi/4} = \tfrac{9\pi}{8}\end{aligned}$$

21.
$$\begin{aligned} A &= \int_0^{\pi/5}\tfrac{1}{2}\,\text{sen}^2\,5\theta\,d\theta = \tfrac{1}{4}\int_0^{\pi/5}(1-\cos 10\theta)\,d\theta \\ &= \tfrac{1}{4}\left[\theta-\tfrac{1}{10}\,\text{sen}\,10\theta\right]_0^{\pi/5} = \tfrac{\pi}{20}\end{aligned}$$

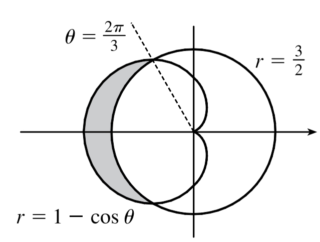
22.



$2+3\cos\theta = 0 \;\Rightarrow\; \cos\theta = -\frac{2}{3} \;\Rightarrow\; \theta = \cos^{-1}\left(-\frac{2}{3}\right)$ ou $2\pi-\cos^{-1}\left(-\frac{2}{3}\right)$. Seja $\alpha = \cos^{-1}\left(-\frac{2}{3}\right)$. Então

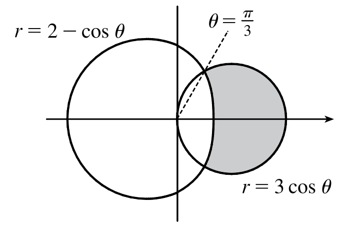
$$\begin{aligned} A &= 2\int_{\alpha}^{\pi}\tfrac{1}{2}(2+3\cos\theta)^2\,d\theta \\ &= \int_{\alpha}^{\pi}\left(4+12\cos\theta+9\cos^2\theta\right)d\theta \\ &= \int_{\alpha}^{\pi}\left(\tfrac{17}{2}+12\cos\theta+\tfrac{9}{2}\cos 2\theta\right)d\theta \\ &= \left[\tfrac{17}{2}\theta+12\,\text{sen}\,\theta+\tfrac{9}{4}\,\text{sen}\,2\theta\right]_{\alpha}^{\pi} \\ &= \tfrac{17}{2}(\pi-\alpha)-12\,\text{sen}\,\alpha-\tfrac{9}{2}\,\text{sen}\,\alpha\cos\alpha \\ &= \tfrac{17}{2}\left[\pi-\cos^{-1}\left(-\tfrac{2}{3}\right)\right]-12\left(\tfrac{\sqrt{5}}{3}\right)-\tfrac{9}{2}\left(\tfrac{\sqrt{5}}{3}\right)\left(-\tfrac{2}{3}\right) \\ &= \tfrac{17}{2}\cos^{-1}\left(\tfrac{2}{3}\right)-3\sqrt{5}\end{aligned}$$

23.



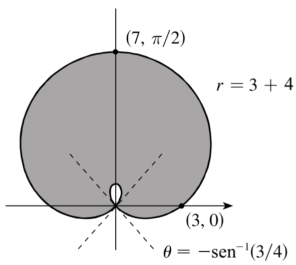
$1-\cos\theta = \frac{3}{2} \;\Rightarrow\; \cos\theta = -\frac{1}{2} \;\Rightarrow\; \theta = \frac{2\pi}{3}$ ou $\frac{4\pi}{3} \;\Rightarrow$

$$\begin{aligned} A &= 2\int_{2\pi/3}^{\pi}\tfrac{1}{2}\left[(1-\cos\theta)^2-\left(\tfrac{3}{2}\right)^2\right]d\theta \\ &= \int_{2\pi/3}^{\pi}\left(-\tfrac{5}{4}-2\cos\theta+\cos^2\theta\right)d\theta \\ &= \left[-\tfrac{5}{4}\theta-2\,\text{sen}\,\theta\right]_{2\pi/3}^{\pi}+\tfrac{1}{2}\int_{2\pi/3}^{\pi}(1+\cos 2\theta)\,d\theta \\ &= -\tfrac{5}{12}\pi+\sqrt{3}+\tfrac{1}{2}\left[\theta+\tfrac{1}{2}\,\text{sen}\,2\theta\right]_{2\pi/3}^{\pi} \\ &= -\tfrac{5}{12}\pi+\sqrt{3}+\tfrac{1}{6}\pi+\tfrac{\sqrt{3}}{8} = \tfrac{9\sqrt{3}}{8}-\tfrac{1}{4}\pi\end{aligned}$$

24. 

$3\cos\theta = 2 - \cos\theta \quad\Rightarrow\quad \cos\theta = \frac{1}{2} \quad\Rightarrow\quad \theta = \pm\frac{\pi}{3} \quad\Rightarrow$

$$\begin{aligned} A &= 2\int_0^{\pi/3} \tfrac{1}{2}\left[(3\cos\theta)^2 - (2-\cos\theta)^2\right] d\theta \\ &= \int_0^{\pi/3}\left(8\cos^2\theta + 4\cos\theta - 4\right) d\theta \\ &= \int_0^{\pi/3}(4\cos 2\theta + 4\cos\theta)\, d\theta \\ &= \left[2\,\text{sen}\,2\theta + 4\,\text{sen}\,\theta\right]_0^{\pi/3} = 3\sqrt{3} \end{aligned}$$

25. 

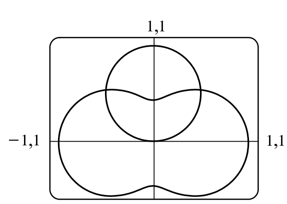
A curva cruza a si mesma quando $3 + 4\,\text{sen}\,\theta = 0 \quad\Leftrightarrow$ $\text{sen}\,\theta = -\frac{3}{4}$. Tomando $\alpha = \text{sen}^{-1}\,\frac{3}{4}$, a área desejada é

$$A = 2\left[\int_{-\alpha}^{\pi/2} \tfrac{1}{2}\,(3 + 4\,\text{sen}\,\theta)^2\, d\theta - \int_{-\pi/2}^{-\alpha} \tfrac{1}{2}\,(3 + 4\,\text{sen}\,\theta)^2\right] d\theta$$

Agora

$\int (3 + 4\,\text{sen}\,\theta)^2\, d\theta = 9\theta - 24\cos\theta + 8\theta - 4\,\text{sen}\,2\theta + C$, logo

$A = 34\alpha + 48\cos\alpha - 16\,\text{sen}\,\alpha\cos\alpha = 34\,\text{sen}^{-1}\left(\frac{3}{4}\right) + 9\sqrt{7}$.

26. 

Os pontos de intersecção ocorrem quando $\sqrt{1 - 0{,}8\,\text{sen}^2\,\theta} = \text{sen}\,\theta \quad\Leftrightarrow\quad 1{,}8\,\text{sen}^2\,\theta = 1 \quad\Leftrightarrow$ $\theta = \text{arcsen}\,\sqrt{\frac{5}{9}}$ $\left(= \alpha, \text{ logo } \cos\alpha = \frac{2}{3}\right)$. Assim a área é

$$\begin{aligned} A &= 2\int_0^{\alpha} \tfrac{1}{2}\,\text{sen}^2\,\theta d\theta + 2\int_{\alpha}^{\pi/2} \tfrac{1}{2}\left(\sqrt{1 - 0{,}8\,\text{sen}^2\,\theta}\right)^2 d\theta \\ &= \left[\tfrac{1}{2}\theta - \tfrac{1}{4}\,\text{sen}\,2\theta\right]_0^{\alpha} + \left[\theta - 0{,}8\left(\tfrac{1}{2}\theta - \tfrac{1}{4}\,\text{sen}\,2\theta\right)\right]_{\alpha}^{\pi/2} \\ &= \tfrac{1}{2}\alpha - \tfrac{1}{4}\,(2\,\text{sen}\,\alpha\cos\alpha) + 0{,}6\cdot\tfrac{\pi}{2} \\ &\qquad - \left[0{,}6\alpha + 0{,}2\,(2\,\text{sen}\,\alpha\cos\alpha)\right] \\ &= \tfrac{1}{2}\,\text{arcsen}\,\tfrac{\sqrt{5}}{3} - \tfrac{1}{2}\,\tfrac{\sqrt{5}}{3}\,\tfrac{2}{3} + 0{,}3\pi \\ &\qquad -0{,}6\,\text{arcsen}\,\tfrac{\sqrt{5}}{3} - 0{,}4\cdot\tfrac{\sqrt{5}}{3}\,\tfrac{2}{3} \\ &= \tfrac{3}{10}\pi - \tfrac{1}{10}\,\text{arcsen}\,\tfrac{\sqrt{5}}{3} - \tfrac{1}{5}\sqrt{5} \approx 0{,}411 \end{aligned}$$

27.
$$\begin{aligned} L &= \int_a^b \sqrt{r^2 + (dr/d\theta)^2}\, d\theta \\ &= \int_0^{3\pi/4} \sqrt{(5\cos\theta)^2 + (-5\,\text{sen}\,\theta)^2}\, d\theta \\ &= 5\int_0^{3\pi/4} \sqrt{\cos^2\theta + \text{sen}^2\,\theta}\, d\theta \\ &= 5\int_0^{3\pi/4} d\theta = \tfrac{15}{4}\pi \end{aligned}$$

28.
$$\begin{aligned} L &= \int_a^b \sqrt{r^2 + (dr/d\theta)^2}\, d\theta \\ &= \int_0^{2\pi} \sqrt{(2^\theta)^2 + [(\ln 2)\, 2^\theta]^2}\, d\theta = \int_0^{2\pi} 2^\theta \sqrt{1 + \ln^2 2}\, d\theta \\ &= \left[\sqrt{1 + \ln^2 2}\left(\frac{2^\theta}{\ln 2}\right)\right]_0^{2\pi} = \frac{\sqrt{1 + \ln^2 2}\,\left(2^{2\pi} - 1\right)}{\ln 2} \end{aligned}$$

29.
$$\begin{aligned} L &= 2\int_0^{\pi} \sqrt{(1 + \cos\theta)^2 + (-\,\text{sen}\,\theta)^2}\, d\theta \\ &= 2\sqrt{2}\int_0^{\pi} \sqrt{1 + \cos\theta}\, d\theta = 2\sqrt{2}\int_0^{\pi} \sqrt{2\cos^2(\theta/2)}\, d\theta \\ &= \left[8\,\text{sen}\,(\theta/2)\right]_0^{\pi} = 8 \end{aligned}$$

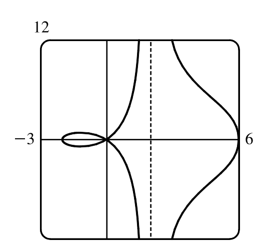
30.
$$\begin{aligned} L &= \int_0^{3\pi} \sqrt{(e^{-\theta})^2 + (-e^{-\theta})^2}\, d\theta \\ &= \sqrt{2}\int_0^{3\pi} e^{-\theta}\, d\theta = \sqrt{2}\left(1 - e^{-3\pi}\right) \end{aligned}$$

31.
$$\begin{aligned} L &= 2\int_0^{2\pi} \sqrt{\cos^8\left(\tfrac{1}{4}\theta\right) + \cos^6\left(\tfrac{1}{4}\theta\right)\text{sen}^2\left(\tfrac{1}{4}\theta\right)}\, d\theta \\ &= 2\int_0^{2\pi} \left|\cos^3\left(\tfrac{1}{4}\theta\right)\right| \sqrt{\cos^2\left(\tfrac{1}{4}\theta\right) + \text{sen}^2\left(\tfrac{1}{4}\theta\right)}\, d\theta \\ &= 2\int_0^{2\pi} \left|\cos^3\left(\tfrac{1}{4}\theta\right)\right| d\theta \\ &= 8\int_0^{\pi/2} \cos^3 u\, du \text{ (onde } u = \tfrac{1}{4}\theta) \\ &= 8\left[\text{sen}\,u - \tfrac{1}{3}\,\text{sen}^3\,u\right]_0^{\pi/2} = \tfrac{16}{3} \end{aligned}$$

Observe que a curva é retraçada após cada intervalo de comprimento 4π.

32.
$$\begin{aligned} L &= 2\int_0^{\pi} \sqrt{\left[\cos^2\left(\tfrac{1}{2}\theta\right)\right]^2 + \left[-\cos\left(\tfrac{1}{2}\theta\right)\text{sen}\left(\tfrac{1}{2}\theta\right)\right]^2}\, d\theta \\ &= 2\int_0^{\pi} \cos\left(\tfrac{1}{2}\theta\right) d\theta = 4\left[\text{sen}\left(\tfrac{1}{2}\theta\right)\right]_0^{\pi} = 4 \end{aligned}$$

33. Da Figura 4 no Exemplo 1,
$$\begin{aligned} L &= \int_{-\pi/4}^{\pi/4} \sqrt{r^2 + (r')^2}\, d\theta \\ &= 2\int_0^{\pi/4} \sqrt{\cos^2 2\theta + 4\,\text{sen}^2\,2\theta}\, d\theta \\ &\approx 2\,(1{,}211056) \approx 2{,}4221 \end{aligned}$$

34. 

$4 + 2\sec\theta = 0 \quad\Rightarrow\quad \sec\theta = -2$
$\Rightarrow\quad \cos\theta = -\frac{1}{2} \quad\Rightarrow\quad \theta = \frac{2\pi}{3}, \frac{4\pi}{3}$.

$$L = \int_{2\pi/3}^{4\pi/3} \sqrt{(4 + 2\sec\theta)^2 + (2\sec\theta\,\text{tg}\,\theta)^2}\, d\theta \approx 5{,}8128$$