MAP 2210 – Aplicações de Álgebra Linear

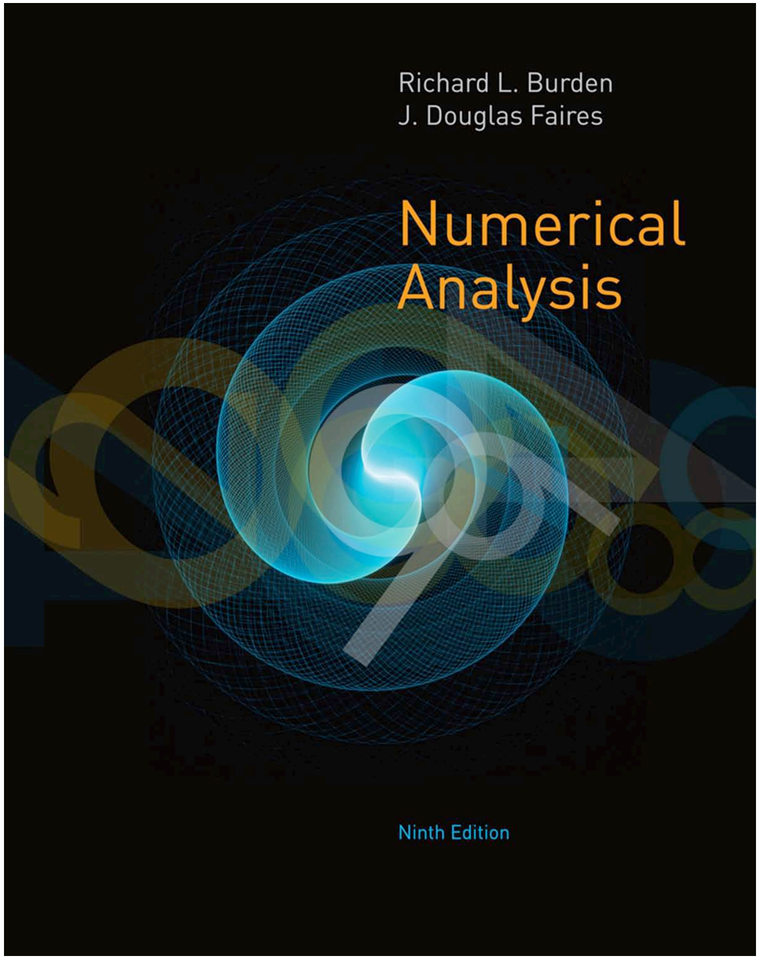
1º Semestre - 2019

Prof. Dr. Luis Carlos de Castro Santos

lsantos@ime.usp.br

Objetivos

Formação básica de álgebra linear aplicada a problemas numéricos. Resolução de problemas em microcomputadores usando linguagens e/ou software adequados fora do horário de aula.



Numerical Analysis

NINTH EDITION

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Youngstown State University

J. Douglas Faires
Youngstown State University

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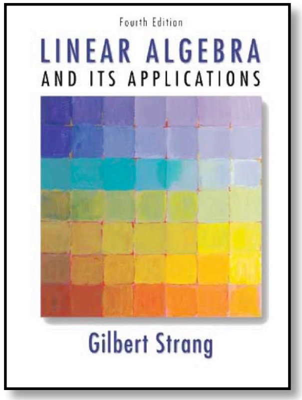

Fourth Edition
LINEAR ALGEBRA
AND ITS APPLICATIONS
Gilbert Strang

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9.3 The Power Method

The **Power method** is an iterative technique used to determine the dominant eigenvalue of a matrix—that is, the eigenvalue with the largest magnitude. By modifying the method slightly, it can also used to determine other eigenvalues. One useful feature of the Power method is that it produces not only an eigenvalue, but also an associated eigenvector. In fact, the Power method is often applied to find an eigenvector for an eigenvalue that is determined by some other means.

To apply the Power method, we assume that the $n \times n$ matrix A has n eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ with an associated collection of linearly independent eigenvectors $\{\mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \mathbf{v}^{(3)}, \ldots, \mathbf{v}^{(n)}\}$. Moreover, we assume that A has precisely one eigenvalue, λ_1, that is largest in magnitude, so that

$$|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \cdots \geq |\lambda_n| \geq 0.$$

Example 4 of Section 9.1 illustrates that an $n \times n$ matrix need not have n linearly independent eigenvectors. When it does not the Power method may still be successful, but it is not guaranteed to be.

If $\mathbf{x}$ is any vector in $\mathbb{R}^n$, the fact that $\{\mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \mathbf{v}^{(3)}, \ldots, \mathbf{v}^{(n)}\}$ is linearly independent implies that constants $\beta_1, \beta_2, \ldots, \beta_n$ exist with

$$\mathbf{x} = \sum_{j=1}^{n} \beta_j \mathbf{v}^{(j)}.$$

Multiplying both sides of this equation by $A, A^2, \ldots, A^k, \ldots$ gives

$$A\mathbf{x} = \sum_{j=1}^{n} \beta_j A\mathbf{v}^{(j)} = \sum_{j=1}^{n} \beta_j \lambda_j \mathbf{v}^{(j)}, \quad A^2\mathbf{x} = \sum_{j=1}^{n} \beta_j \lambda_j A\mathbf{v}^{(j)} = \sum_{j=1}^{n} \beta_j \lambda_j^2 \mathbf{v}^{(j)},$$

and generally, $A^k\mathbf{x} = \sum_{j=1}^{n} \beta_j \lambda_j^k \mathbf{v}^{(j)}$.

If λ_1^k is factored from each term on the right side of the last equation, then

$$A^k\mathbf{x} = \lambda_1^k \sum_{j=1}^{n} \beta_j \left(\frac{\lambda_j}{\lambda_1}\right)^k \mathbf{v}^{(j)}.$$

Since $|\lambda_1| > |\lambda_j|$, for all $j = 2, 3, \ldots, n$, we have $\lim_{k\to\infty}(\lambda_j/\lambda_1)^k = 0$, and

$$\lim_{k\to\infty} A^k\mathbf{x} = \lim_{k\to\infty} \lambda_1^k \beta_1 \mathbf{v}^{(1)}. \tag{9.2}$$

The sequence in Eq. (9.2) converges to $\mathbf{0}$ if $|\lambda_1| < 1$ and diverges if $|\lambda_1| > 1$, provided, of course, that $\beta_1 \neq 0$. As a consequence, the entries in the $A^k\mathbf{x}$ will grow with k if $|\lambda_1| > 1$ and will go to 0 if $|\lambda_1| < 1$, perhaps resulting in overflow or underflow. To take care of that possibility, we scale the powers of $A^k\mathbf{x}$ in an appropriate manner to ensure that the limit in Eq. (9.2) is finite and nonzero. The scaling begins by choosing $\mathbf{x}$ to be a unit vector $\mathbf{x}^{(0)}$ relative to $\|\cdot\|_\infty$ and choosing a component $x_{p_0}^{(0)}$ of $\mathbf{x}^{(0)}$ with

$$x_{p0}^{(0)} = 1 = \|\mathbf{x}^{(0)}\|_\infty.$$

Let $\mathbf{y}^{(1)} = A\mathbf{x}^{(0)}$, and define $\mu^{(1)} = y_{p_0}^{(1)}$. Then

$$\mu^{(1)} = y_{p_0}^{(1)} = \frac{y_{p_0}^{(1)}}{x_{p_0}^{(0)}} = \frac{\beta_1\lambda_1 v_{p_0}^{(1)} + \sum_{j=2}^{n}\beta_j\lambda_j v_{p_0}^{(j)}}{\beta_1 v_{p_0}^{(1)} + \sum_{j=2}^{n}\beta_j v_{p_0}^{(j)}} = \lambda_1\left[\frac{\beta_1 v_{p_0}^{(1)} + \sum_{j=2}^{n}\beta_j(\lambda_j/\lambda_1)v_{p_0}^{(j)}}{\beta_1 v_{p_0}^{(1)} + \sum_{j=2}^{n}\beta_j v_{p_0}^{(j)}}\right].$$

Let p_1 be the least integer such that

$$|y_{p_1}^{(1)}| = \|\mathbf{y}^{(1)}\|_\infty,$$

and define $\mathbf{x}^{(1)}$ by

$$\mathbf{x}^{(1)} = \frac{1}{y_{p_1}^{(1)}}\mathbf{y}^{(1)} = \frac{1}{y_{p_1}^{(1)}}A\mathbf{x}^{(0)}.$$

Then

$$x_{p_1}^{(1)} = 1 = \|\mathbf{x}^{(1)}\|_\infty.$$

Now define

$$\mathbf{y}^{(2)} = A\mathbf{x}^{(1)} = \frac{1}{y_{p_1}^{(1)}}A^2\mathbf{x}^{(0)}$$

and

$$\mu^{(2)} = y_{p_1}^{(2)} = \frac{y_{p_1}^{(2)}}{x_{p_1}^{(1)}} = \frac{\left[\beta_1\lambda_1^2 v_{p_1}^{(1)} + \sum_{j=2}^{n}\beta_j\lambda_j^2 v_{p_1}^{(j)}\right]\Big/ y_{p_1}^{(1)}}{\left[\beta_1\lambda_1 v_{p_1}^{(1)} + \sum_{j=2}^{n}\beta_j\lambda_j v_{p_1}^{(j)}\right]\Big/ y_{p_1}^{(1)}}$$

$$= \lambda_1\left[\frac{\beta_1 v_{p_1}^{(1)} + \sum_{j=2}^{n}\beta_j(\lambda_j/\lambda_1)^2 v_{p_1}^{(j)}}{\beta_1 v_{p_1}^{(1)} + \sum_{j=2}^{n}\beta_j(\lambda_j/\lambda_1)v_{p_1}^{(j)}}\right].$$

Let p_2 be the smallest integer with

$$|y_{p_2}^{(2)}| = \|\mathbf{y}^{(2)}\|_\infty,$$

and define

$$\mathbf{x}^{(2)} = \frac{1}{y_{p_2}^{(2)}}\mathbf{y}^{(2)} = \frac{1}{y_{p_2}^{(2)}}A\mathbf{x}^{(1)} = \frac{1}{y_{p_2}^{(2)}y_{p_1}^{(1)}}A^2\mathbf{x}^{(0)}.$$

In a similar manner, define sequences of vectors $\{\mathbf{x}^{(m)}\}_{m=0}^{\infty}$ and $\{\mathbf{y}^{(m)}\}_{m=1}^{\infty}$, and a sequence of scalars $\{\mu^{(m)}\}_{m=1}^{\infty}$ inductively by

$$\mathbf{y}^{(m)} = A\mathbf{x}^{(m-1)},$$

$$\mu^{(m)} = y_{p_{m-1}}^{(m)} = \lambda_1\left[\frac{\beta_1 v_{p_{m-1}}^{(1)} + \sum_{j=2}^{n}(\lambda_j/\lambda_1)^m\beta_j v_{p_{m-1}}^{(j)}}{\beta_1 v_{p_{m-1}}^{(1)} + \sum_{j=2}^{n}(\lambda_j/\lambda_1)^{m-1}\beta_j v_{p_{m-1}}^{(j)}}\right], \tag{9.3}$$

and

$$\mathbf{x}^{(m)} = \frac{\mathbf{y}^{(m)}}{y_{p_m}^{(m)}} = \frac{A^m\mathbf{x}^{(0)}}{\prod_{k=1}^{m} y_{p_k}^{(k)}},$$

where at each step, p_m is used to represent the smallest integer for which

$$|y_{p_m}^{(m)}| = \|\mathbf{y}^{(m)}\|_\infty.$$

By examining Eq. (9.3), we see that since $|\lambda_j/\lambda_1| < 1$, for each $j = 2, 3, \ldots, n$, $\lim_{m\to\infty}\mu^{(m)} = \lambda_1$, provided that $\mathbf{x}^{(0)}$ is chosen so that $\beta_1 \neq 0$. Moreover, the sequence of vectors $\{\mathbf{x}^{(m)}\}_{m=0}^{\infty}$ converges to an eigenvector associated with λ_1 that has l_∞ norm equal to one.

Illustration

The matrix

$$A = \begin{bmatrix} -2 & -3 \\ 6 & 7 \end{bmatrix}$$

Has eigenvalues $\lambda_1 = 4$ and $\lambda_2 = 1$ with corresponding eigenvectors $\mathbf{v}_1 = (1, -2)^t$ and $\mathbf{v}_2 = (1, -1)^t$. If we start with the arbitrary vector $\mathbf{x}_0 = (1, 1)^t$ and multiply by the matrix A we obtain

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$$\mathbf{x}_1 = A\mathbf{x}_0 = \begin{bmatrix} -5 \\ 13 \end{bmatrix}, \qquad \mathbf{x}_2 = A\mathbf{x}_1 = \begin{bmatrix} -29 \\ 61 \end{bmatrix}, \qquad \mathbf{x}_3 = A\mathbf{x}_2 = \begin{bmatrix} -125 \\ 253 \end{bmatrix},$$

$$\mathbf{x}_4 = A\mathbf{x}_3 = \begin{bmatrix} -509 \\ 1021 \end{bmatrix}, \qquad \mathbf{x}_5 = A\mathbf{x}_4 = \begin{bmatrix} -2045 \\ 4093 \end{bmatrix}, \qquad \mathbf{x}_6 = A\mathbf{x}_5 = \begin{bmatrix} -8189 \\ 16381 \end{bmatrix}.$$

As a consequence, approximations to the dominant eigenvalue $\lambda_1 = 4$ are

$$\lambda_1^{(1)} = \frac{61}{13} = 4.6923, \qquad \lambda_1^{(2)} = \frac{253}{61} = 4.14754, \qquad \lambda_1^{(3)} = \frac{1021}{253} = 4.03557,$$

$$\lambda_1^{(4)} = \frac{4093}{1021} = 4.00881, \qquad \lambda_1^{(5)} = \frac{16381}{4093} = 4.00200.$$

An approximate eigenvector corresponding to $\lambda_1^{(5)} = \dfrac{16381}{4093} = 4.00200$ is

$$\mathbf{x}_6 = \begin{bmatrix} -8189 \\ 16381 \end{bmatrix}, \quad \text{which, divided by 16381, normalizes to} \quad \begin{bmatrix} -0.49908 \\ 1 \end{bmatrix} \approx \mathbf{v}_1.$$

□

ALGORITHM 9.1

Power Method

To approximate the dominant eigenvalue and an associated eigenvector of the $n \times n$ matrix A given a nonzero vector $\mathbf{x}$:

INPUT dimension n; matrix A; vector $\mathbf{x}$; tolerance TOL; maximum number of iterations N.

OUTPUT approximate eigenvalue μ; approximate eigenvector $\mathbf{x}$ (with $||\mathbf{x}||_\infty = 1$) or a message that the maximum number of iterations was exceeded.

Step 1 Set $k = 1$.

Step 2 Find the smallest integer p with $1 \leq p \leq n$ and $|x_p| = ||\mathbf{x}||_\infty$.

Step 3 Set $\mathbf{x} = \mathbf{x}/x_p$.

Step 4 While ($k \leq N$) do Steps 5–11.

Autovalor

Step 5 Set $\mathbf{y} = A\mathbf{x}$.

Step 6 Set $\mu = y_p$.

Step 7 Find the smallest integer p with $1 \leq p \leq n$ and $|y_p| = \|\mathbf{y}\|_\infty$.

Step 8 If $y_p = 0$ then OUTPUT ('Eigenvector', $\mathbf{x}$);
OUTPUT ('A has the eigenvalue 0, select a new vector $\mathbf{x}$ and restart');
STOP.

Step 9 Set $ERR = ||\mathbf{x} - (\mathbf{y}/y_p)||_\infty$;
$\mathbf{x} = \mathbf{y}/y_p$.

Autovetor

Step 10 If $ERR < TOL$ then OUTPUT ($\mu, \mathbf{x}$);
(*The procedure was successful.*)
STOP.

Step 11 Set $k = k + 1$.

Step 12 OUTPUT ('The maximum number of iterations exceeded');
(*The procedure was unsuccessful.*)
STOP.

The Power method has the disadvantage that it is unknown at the outset whether or not the matrix has a single dominant eigenvalue. Nor is it known how $\mathbf{x}^{(0)}$ should be chosen so as to ensure that its representation in terms of the eigenvectors of the matrix will contain a nonzero contribution from the eigenvector associated with the dominant eigenvalue, should it exist.

Accelerating Convergence

Choosing, in Step 7, the smallest integer p_m for which $|y_{p_m}^{(m)}| = \|\mathbf{y}^{(m)}\|_\infty$ will generally ensure that this index eventually becomes invariant. The rate at which $\{\mu^{(m)}\}_{m=1}^{\infty}$ converges to λ_1 is determined by the ratios $|\lambda_j/\lambda_1|^m$, for $j = 2, 3, \ldots, n$, and in particular by $|\lambda_2/\lambda_1|^m$. The rate of convergence is $O(|\lambda_2/\lambda_1|^m)$ (see [IK, p. 148]), so there is a constant k such that for large m,

$$|\mu^{(m)} - \lambda_1| \approx k \left|\frac{\lambda_2}{\lambda_1}\right|^m,$$

which implies that

$$\lim_{m\to\infty} \frac{|\mu^{(m+1)} - \lambda_1|}{|\mu^{(m)} - \lambda_1|} \approx \left|\frac{\lambda_2}{\lambda_1}\right| < 1.$$

The sequence $\{\mu^{(m)}\}$ converges linearly to λ_1, so Aitken's Δ^2 procedure discussed in Section 2.5 can be used to speed the convergence. Implementing the Δ^2 procedure in Algorithm 9.1 is accomplished by modifying the algorithm as follows:

2.5 Accelerating Convergence

Aitken's Δ^2 Method

Suppose $\{p_n\}_{n=0}^{\infty}$ is a linearly convergent sequence with limit p. To motivate the construction of a sequence $\{\hat{p}_n\}_{n=0}^{\infty}$ that converges more rapidly to p than does $\{p_n\}_{n=0}^{\infty}$, let us first assume that the signs of $p_n - p$, $p_{n+1} - p$, and $p_{n+2} - p$ agree and that n is sufficiently large that

$$\frac{p_{n+1} - p}{p_n - p} \approx \frac{p_{n+2} - p}{p_{n+1} - p}.$$

Then

$$(p_{n+1} - p)^2 \approx (p_{n+2} - p)(p_n - p),$$

so

$$p_{n+1}^2 - 2p_{n+1}p + p^2 \approx p_{n+2}p_n - (p_n + p_{n+2})p + p^2$$

and

$$(p_{n+2} + p_n - 2p_{n+1})p \approx p_{n+2}p_n - p_{n+1}^2.$$

Solving for p gives

$$p \approx \frac{p_{n+2}p_n - p_{n+1}^2}{p_{n+2} - 2p_{n+1} + p_n}.$$

Adding and subtracting the terms p_n^2 and $2p_np_{n+1}$ in the numerator and grouping terms appropriately gives

$$p \approx \frac{p_np_{n+2} - 2p_np_{n+1} + p_n^2 - p_{n+1}^2 + 2p_np_{n+1} - p_n^2}{p_{n+2} - 2p_{n+1} + p_n}$$

$$= \frac{p_n(p_{n+2} - 2p_{n+1} + p_n) - (p_{n+1}^2 - 2p_np_{n+1} + p_n^2)}{p_{n+2} - 2p_{n+1} + p_n}$$

$$= p_n - \frac{(p_{n+1} - p_n)^2}{p_{n+2} - 2p_{n+1} + p_n}.$$

Aitken's $\mathbf{\Delta^2}$ method is based on the assumption that the sequence $\{\hat{p}_n\}_{n=0}^{\infty}$, defined by

$$\hat{p}_n = p_n - \frac{(p_{n+1} - p_n)^2}{p_{n+2} - 2p_{n+1} + p_n}, \tag{2.14}$$

converges more rapidly to p than does the original sequence $\{p_n\}_{n=0}^{\infty}$.

Step 1 Set $k = 1$;

$\mu_0 = 0$;

$\mu_1 = 0$.

.

.

.

Step 6 Set $\mu = y_p$;

$$\hat{\mu} = \mu_0 - \frac{(\mu_1 - \mu_0)^2}{\mu - 2\mu_1 + \mu_0}.$$

Step 10 If $ERR < TOL$ and $k \geq 4$ then OUTPUT $(\hat{\mu}, \mathbf{x})$;
STOP.

Step 11 Set $k = k + 1$;

$\mu_0 = \mu_1$;

$\mu_1 = \mu$.

In actuality, it is not necessary for the matrix to have distinct eigenvalues for the Power method to converge. If the matrix has a unique dominant eigenvalue, λ_1, with multiplicity r greater than 1 and $\mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \ldots, \mathbf{v}^{(r)}$ are linearly independent eigenvectors associated with λ_1, the procedure will still converge to λ_1. The sequence of vectors $\{\mathbf{x}^{(m)}\}_{m=0}^{\infty}$ will, in this case, converge to an eigenvector of λ_1 of l_∞ norm equal to one that depends on the choice of the initial vector $\mathbf{x}^{(0)}$ and is a linear combination of $\mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \ldots, \mathbf{v}^{(r)}$. (See [Wil2], page 570.)

Example 1

Use the Power method to approximate the dominant eigenvalue of the matrix

$$A = \begin{bmatrix} -4 & 14 & 0 \\ -5 & 13 & 0 \\ -1 & 0 & 2 \end{bmatrix},$$

and then apply Aitken's Δ^2 method to the approximations to the eigenvalue of the matrix to accelerate the convergence.

Solution This matrix has eigenvalues $\lambda_1 = 6, \lambda_2 = 3$, and $\lambda_3 = 2$, so the Power method described in Algorithm 9.1 will converge. Let $\mathbf{x}^{(0)} = (1, 1, 1)^t$, then

$$\mathbf{y}^{(1)} = A\mathbf{x}^{(0)} = (10, 8, 1)^t,$$

so

$$||\mathbf{y}^{(1)}||_\infty = 10, \quad \mu^{(1)} = y_1^{(1)} = 10, \quad \text{and} \quad \mathbf{x}^{(1)} = \frac{\mathbf{y}^{(1)}}{10} = (1, 0.8, 0.1)^t.$$

Continuing in this manner leads to the values in Table 9.1, where $\hat{\mu}^{(m)}$ represents the sequence generated by the Aitken's Δ^2 procedure. An approximation to the dominant eigenvalue, 6, at this stage is $\hat{\mu}^{(10)} = 6.000000$. The approximate l_∞-unit eigenvector for the eigenvalue 6 is $(\mathbf{x}^{(12)})^t = (1, 0.714316, -0.249895)^t$.

Although the approximation to the eigenvalue is correct to the places listed, the eigenvector approximation is considerably less accurate to the true eigenvector, $(1, 5/7, -1/4)^t \approx (1, 0.714286, -0.25)^t$. ■

Table 9.1

m	$(\mathbf{x}^{(m)})^t$	$\mu^{(m)}$	$\hat{\mu}^{(m)}$
0	(1, 1, 1)		
1	(1, 0.8, 0.1)	10	6.266667
2	(1, 0.75, −0.111)	7.2	6.062473
3	(1, 0.730769, −0.188803)	6.5	6.015054
4	(1, 0.722200, −0.220850)	6.230769	6.004202
5	(1, 0.718182, −0.235915)	6.111000	6.000855
6	(1, 0.716216, −0.243095)	6.054546	6.000240
7	(1, 0.715247, −0.246588)	6.027027	6.000058
8	(1, 0.714765, −0.248306)	6.013453	6.000017
9	(1, 0.714525, −0.249157)	6.006711	6.000003
10	(1, 0.714405, −0.249579)	6.003352	6.000000
11	(1, 0.714346, −0.249790)	6.001675	
12	(1, 0.714316, −0.249895)	6.000837	

Symmetric Matrices

When A is symmetric, a variation in the choice of the vectors $\mathbf{x}^{(m)}$ and $\mathbf{y}^{(m)}$ and the scalars $\mu^{(m)}$ can be made to significantly improve the rate of convergence of the sequence $\{\mu^{(m)}\}_{m=1}^{\infty}$ to the dominant eigenvalue λ_1. In fact, although the rate of convergence of the general Power method is $O(|\lambda_2/\lambda_1|^m)$, the rate of convergence of the modified procedure given in Algorithm 9.2 for symmetric matrices is $O(|\lambda_2/\lambda_1|^{2m})$. (See [IK, pp. 149 ff].) Because the sequence $\{\mu^{(m)}\}$ is still linearly convergent, Aitken's Δ^2 procedure can also be applied.

ALGORITHM 9.2

Symmetric Power Method

To approximate the dominant eigenvalue and an associated eigenvector of the $n \times n$ symmetric matrix A, given a nonzero vector $\mathbf{x}$:

INPUT dimension n; matrix A; vector $\mathbf{x}$; tolerance TOL; maximum number of iterations N.

OUTPUT approximate eigenvalue μ; approximate eigenvector $\mathbf{x}$ (with $\|\mathbf{x}\|_2 = 1$) or a message that the maximum number of iterations was exceeded.

Step 1 Set $k = 1$;
$\mathbf{x} = \mathbf{x}/\|\mathbf{x}\|_2$.

Step 2 While ($k \le N$) do Steps 3–8.

Step 3 Set $\mathbf{y} = A\mathbf{x}$.

Step 4 Set $\mu = \mathbf{x}^t\mathbf{y}$.

Autovalor

Step 5 If $\|\mathbf{y}\|_2 = 0$, then OUTPUT ('Eigenvector', $\mathbf{x}$);
OUTPUT ('A has eigenvalue 0, select new vector $\mathbf{x}$ and restart');
STOP.

Step 6 Set $ERR = \left\| \mathbf{x} - \dfrac{\mathbf{y}}{\|\mathbf{y}\|_2} \right\|_2$;
$\mathbf{x} = \mathbf{y}/\|\mathbf{y}\|_2$.

Autovetor

Step 7 If $ERR < TOL$ then OUTPUT ($\mu, \mathbf{x}$);
(The procedure was successful.)
STOP.

Step 8 Set $k = k + 1$.

Step 9 OUTPUT ('Maximum number of iterations exceeded');
(The procedure was unsuccessful.)
STOP.

Example 2

Apply both the Power method and the Symmetric Power method to the matrix

$$A = \begin{bmatrix} 4 & -1 & 1 \\ -1 & 3 & -2 \\ 1 & -2 & 3 \end{bmatrix},$$

using Aitken's Δ^2 method to accelerate the convergence.

Solution This matrix has eigenvalues $\lambda_1 = 6, \lambda_2 = 3$, and $\lambda_3 = 1$. An eigenvector for the eigenvalue 6 is $(1, -1, 1)^t$. Applying the Power method to this matrix with initial vector $(1, 0, 0)^t$ gives the values in Table 9.2. ■

Table 9.2

m	$(\mathbf{y}^{(m)})^t$	$\mu^{(m)}$	$\hat{\mu}^{(m)}$	$(\mathbf{x}^{(m)})^t$ with $\lVert\mathbf{x}^{(m)}\rVert_\infty = 1$
0				$(1, 0, 0)$
1	$(4, -1, 1)$	4		$(1, -0.25, 0.25)$
2	$(4.5, -2.25, 2.25)$	4.5	7	$(1, -0.5, 0.5)$
3	$(5, -3.5, 3.5)$	5	6.2	$(1, -0.7, 0.7)$
4	$(5.4, -4.5, 4.5)$	5.4	6.047617	$(1, -0.833\bar{3}, 0.833\bar{3})$
5	$(5.66\bar{6}, -5.166\bar{6}, 5.166\bar{6})$	$5.66\bar{6}$	6.011767	$(1, -0.911765, 0.911765)$
6	$(5.823529, -5.558824, 5.558824)$	5.823529	6.002931	$(1, -0.954545, 0.954545)$
7	$(5.909091, -5.772727, 5.772727)$	5.909091	6.000733	$(1, -0.976923, 0.976923)$
8	$(5.953846, -5.884615, 5.884615)$	5.953846	6.000184	$(1, -0.988372, 0.988372)$
9	$(5.976744, -5.941861, 5.941861)$	5.976744		$(1, -0.994163, 0.994163)$
10	$(5.988327, -5.970817, 5.970817)$	5.988327		$(1, -0.997076, 0.997076)$

We will now apply the Symmetric Power method to this matrix with the same initial vector $(1, 0, 0)^t$. The first steps are

$$\mathbf{x}^{(0)} = (1, 0, 0)^t, \quad A\mathbf{x}^{(0)} = (4, -1, 1)^t, \mu^{(1)} = 4,$$

and

$$\mathbf{x}^{(1)} = \frac{1}{||A\mathbf{x}^{(0)}||_2} \cdot A\mathbf{x}^{(0)} = (0.942809, -0.235702, 0.235702)^t.$$

The remaining entries are shown in Table 9.3.

Table 9.3

m	$(\mathbf{y}^{(m)})^t$	$\mu^{(m)}$	$\hat{\mu}^{(m)}$	$(\mathbf{x}^{(m)})^t$ with $\lVert\mathbf{x}^{(m)}\rVert_2 = 1$
0	(1, 0, 0)			(1, 0, 0)
1	(4, −1, 1)	4	7	(0.942809, −0.235702, 0.235702)
2	(4.242641, −2.121320, 2.121320	5	6.047619	(0.816497, −0.408248, 0.408248)
3	(4.082483, −2.857738, 2.857738)	5.666667	6.002932	(0.710669, −0.497468, 0.497468)
4	(3.837613, −3.198011, 3.198011)	5.909091	6.000183	(0.646997, −0.539164, 0.539164)
5	(3.666314, −3.342816, 3.342816)	5.976744	6.000012	(0.612836, −0.558763, 0.558763)
6	(3.568871, −3.406650, 3.406650)	5.994152	6.000000	(0.595247, −0.568190, 0.568190)
7	(3.517370, −3.436200, 3.436200)	5.998536	6.000000	(0.586336, −0.572805, 0.572805)
8	(3.490952, −3.450359, 3.450359)	5.999634		(0.581852, −0.575086, 0.575086)
9	(3.477580, −3.457283, 3.457283)	5.999908		(0.579603, −0.576220, 0.576220)
10	(3.470854, −3.460706, 3.460706)	5.999977		(0.578477, −0.576786, 0.576786)

The Symmetric Power method gives considerably faster convergence for this matrix than the Power method. The eigenvector approximations in the Power method converge to $(1, -1, 1)^t$, a vector with unit l_∞-norm. In the Symmetric Power method, the convergence is to the parallel vector $(\sqrt{3}/3, -\sqrt{3}/3, \sqrt{3}/3)^t$, which has unit l_2-norm.

If λ is a real number that approximates an eigenvalue of a symmetric matrix A and $\mathbf{x}$ is an associated approximate eigenvector, then $A\mathbf{x} - \lambda\mathbf{x}$ is approximately the zero vector. The following theorem relates the norm of this vector to the accuracy of λ to the eigenvalue.

Theorem 9.19

Suppose that A is an $n \times n$ symmetric matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. If we have $\|A\mathbf{x} - \lambda\mathbf{x}\|_2 < \varepsilon$ for some real number λ and vector $\mathbf{x}$ with $\|\mathbf{x}\|_2 = 1$, then

$$\min_{1\leq j\leq n} |\lambda_j - \lambda| < \varepsilon. \qquad \blacksquare$$

Proof Suppose that $\mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \ldots, \mathbf{v}^{(n)}$ form an orthonormal set of eigenvectors of A associated, respectively, with the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. By Theorems 9.5 and 9.3, $\mathbf{x}$ can be expressed, for some unique set of constants $\beta_1, \beta_2, \ldots, \beta_n$, as

$$\mathbf{x} = \sum_{j=1}^{n} \beta_j \mathbf{v}^{(j)}.$$

Thus

$$\|A\mathbf{x} - \lambda\mathbf{x}\|_2^2 = \left\| \sum_{j=1}^{n} \beta_j(\lambda_j - \lambda)\mathbf{v}^{(j)} \right\|_2^2 = \sum_{j=1}^{n} |\beta_j|^2 |\lambda_j - \lambda|^2 \geq \min_{1\leq j\leq n} |\lambda_j - \lambda|^2 \sum_{j=1}^{n} |\beta_j|^2.$$

But

$$\sum_{j=1}^{n} |\beta_j|^2 = \|\mathbf{x}\|_2^2 = 1, \quad \text{so} \quad \varepsilon \geq \|A\mathbf{x} - \lambda\mathbf{x}\|_2 > \min_{1\leq j\leq n} |\lambda_j - \lambda|. \qquad \blacksquare\;\blacksquare\;\blacksquare$$

Inverse Power Method

The **Inverse Power method** is a modification of the Power method that gives faster convergence. It is used to determine the eigenvalue of A that is closest to a specified number q.

Suppose the matrix A has eigenvalues $\lambda_1, \ldots, \lambda_n$ with linearly independent eigenvectors $\mathbf{v}^{(1)}, \ldots, \mathbf{v}^{(n)}$. The eigenvalues of $(A - qI)^{-1}$, where $q \neq \lambda_i$, for $i = 1, 2, \ldots, n$, are

$$\frac{1}{\lambda_1 - q}, \quad \frac{1}{\lambda_2 - q}, \quad \ldots, \quad \frac{1}{\lambda_n - q},$$

with these same eigenvectors $\mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \ldots, \mathbf{v}^{(n)}$. (See Exercise 15 of Section 7.2.)

Applying the Power method to $(A - qI)^{-1}$ gives

$$\mathbf{y}^{(m)} = (A - qI)^{-1}\mathbf{x}^{(m-1)},$$

$$\mu^{(m)} = y^{(m)}_{p_{m-1}} = \frac{y^{(m)}_{p_{m-1}}}{x^{(m-1)}_{p_{m-1}}} = \frac{\sum_{j=1}^{n} \beta_j \dfrac{1}{(\lambda_j - q)^m} v^{(j)}_{p_{m-1}}}{\sum_{j=1}^{n} \beta_j \dfrac{1}{(\lambda_j - q)^{m-1}} v^{(j)}_{p_{m-1}}}, \tag{9.4}$$

and

$$\mathbf{x}^{(m)} = \frac{\mathbf{y}^{(m)}}{y^{(m)}_{p_m}},$$

where, at each step, p_m represents the smallest integer for which $|y^{(m)}_{p_m}| = ||\mathbf{y}^{(m)}||_\infty$. The sequence $\{\mu^{(m)}\}$ in Eq. (9.4) converges to $1/(\lambda_k - q)$, where

$$\frac{1}{|\lambda_k - q|} = \max_{1 \leq i \leq n} \frac{1}{|\lambda_i - q|},$$

and $\lambda_k \approx q + 1/\mu^{(m)}$ is the eigenvalue of A closest to q.

With k known, Eq. (9.4) can be written as

$$\mu^{(m)} = \frac{1}{\lambda_k - q}\left[\frac{\beta_k v_{p_{m-1}}^{(k)} + \sum_{\substack{j=1\\ j\neq k}}^{n} \beta_j \left[\frac{\lambda_k - q}{\lambda_j - q}\right]^m v_{p_{m-1}}^{(j)}}{\beta_k v_{p_{m-1}}^{(k)} + \sum_{\substack{j=1\\ j\neq k}}^{n} \beta_j \left[\frac{\lambda_k - q}{\lambda_j - q}\right]^{m-1} v_{p_{m-1}}^{(j)}}\right]. \tag{9.5}$$

Thus, the choice of q determines the convergence, provided that $1/(\lambda_k - q)$ is a unique dominant eigenvalue of $(A - qI)^{-1}$ (although it may be a multiple eigenvalue). The closer q is to an eigenvalue λ_k, the faster the convergence since the convergence is of order

$$O\left(\left|\frac{(\lambda - q)^{-1}}{(\lambda_k - q)^{-1}}\right|^m\right) = O\left(\left|\frac{(\lambda_k - q)}{(\lambda - q)}\right|^m\right),$$

where λ represents the eigenvalue of A that is second closest to q.

The vector $\mathbf{y}^{(m)}$ is obtained by solving the linear system

$$(A - qI)\mathbf{y}^{(m)} = \mathbf{x}^{(m-1)}.$$

In general, Gaussian elimination with pivoting is used, but as in the case of the LU factorization, the multipliers can be saved to reduce the computation. The selection of q can be based on the Geršgorin Circle Theorem or on another means of localizing an eigenvalue.

Algorithm 9.3 computes q from an initial approximation to the eigenvector $\mathbf{x}^{(0)}$ by

$$q = \frac{\mathbf{x}^{(0)t} A \mathbf{x}^{(0)}}{\mathbf{x}^{(0)t} \mathbf{x}^{(0)}}.$$

This choice of q results from the observation that if $\mathbf{x}$ is an eigenvector of A with respect to the eigenvalue λ, then $A\mathbf{x} = \lambda\mathbf{x}$. So $\mathbf{x}^t A\mathbf{x} = \lambda\mathbf{x}^t\mathbf{x}$ and

$$\lambda = \frac{\mathbf{x}^t A\mathbf{x}}{\mathbf{x}^t\mathbf{x}} = \frac{\mathbf{x}^t A\mathbf{x}}{\|\mathbf{x}\|_2^2}.$$

If q is close to an eigenvalue, the convergence will be quite rapid, but a pivoting technique should be used in Step 6 to avoid contamination by round-off error.

Algorithm 9.3 is often used to approximate an eigenvector when an approximate eigenvalue q is known.

ALGORITHM 9.3

Inverse Power Method

To approximate an eigenvalue and an associated eigenvector of the $n \times n$ matrix A given a nonzero vector $\mathbf{x}$:

INPUT dimension n; matrix A; vector $\mathbf{x}$; tolerance TOL; maximum number of iterations N.

OUTPUT approximate eigenvalue μ; approximate eigenvector $\mathbf{x}$ (with $\|\mathbf{x}\|_\infty = 1$) or a message that the maximum number of iterations was exceeded.

Step 1 Set $q = \dfrac{\mathbf{x}^t A \mathbf{x}}{\mathbf{x}^t \mathbf{x}}$.

Step 2 Set $k = 1$.

Step 3 Find the smallest integer p with $1 \le p \le n$ and $|x_p| = \|\mathbf{x}\|_\infty$.

Step 4 Set $\mathbf{x} = \mathbf{x}/x_p$.

Step 5 While ($k \le N$) do Steps 6–12.

> *Step 6* Solve the linear system $(A - qI)\mathbf{y} = \mathbf{x}$.
>
> *Step 7* If the system does not have a unique solution, then
> OUTPUT (‘q is an eigenvalue’, q);
> STOP.
>
> *Step 8* Set $\mu = y_p$.
>
> *Step 9* Find the smallest integer p with $1 \le p \le n$ and $|y_p| = \|\mathbf{y}\|_\infty$.
>
> *Step 10* Set $ERR = \left\|\mathbf{x} - (\mathbf{y}/y_p)\right\|_\infty$;
> $\mathbf{x} = \mathbf{y}/y_p$.
>
> *Step 11* If $ERR < TOL$ then set $\mu = (1/\mu) + q$;
> OUTPUT $(\mu, \mathbf{x})$;
> (*The procedure was successful.*)
> STOP.
>
> *Step 12* Set $k = k + 1$.

Step 13 OUTPUT (‘Maximum number of iterations exceeded’);
(*The procedure was unsuccessful.*)
STOP.

The convergence of the Inverse Power method is linear, so Aitken Δ^2 method can again be used to speed convergence. The following example illustrates the fast convergence of the Inverse Power method if q is close to an eigenvalue.

Example 3 Apply the Inverse Power method with $\mathbf{x}^{(0)} = (1, 1, 1)^t$ to the matrix

$$A = \begin{bmatrix} -4 & 14 & 0 \\ -5 & 13 & 0 \\ -1 & 0 & 2 \end{bmatrix} \quad \text{with} \quad q = \frac{\mathbf{x}^{(0)t} A \mathbf{x}^{(0)}}{\mathbf{x}^{(0)t}\mathbf{x}^{(0)}} = \frac{19}{3},$$

and use Aitken's Δ^2 method to accelerate the convergence.

Solution The Power method was applied to this matrix in Example 1 using the initial vector $\mathbf{x}^{(0)} = (1, 1, 1)^t$. It gave the approximate eigenvalue $\mu^{(12)} = 6.000837$ and eigenvector $(\mathbf{x}^{(12)})^t = (1, 0.714316, -0.249895)^t$.

For the Inverse Power method we consider

$$A - qI = \begin{bmatrix} -\frac{31}{3} & 14 & 0 \\ -5 & \frac{20}{3} & 0 \\ -1 & 0 & -\frac{13}{3} \end{bmatrix}$$

With $\mathbf{x}^{(0)} = (1, 1, 1)^t$, the method first finds $\mathbf{y}^{(1)}$ by solving $(A - qI)\mathbf{y}^{(1)} = \mathbf{x}^{(0)}$. This gives

$$\mathbf{y}^{(1)} = \left(-\frac{33}{5}, -\frac{24}{5}, \frac{84}{65}\right)^t = (-6.6, -4.8, 1.292\overline{307692})^t.$$

So

$$||\mathbf{y}^{(1)}||_\infty = 6.6, \quad \mathbf{x}^{(1)} = \frac{1}{-6.6}\mathbf{y}^{(1)} = (1, 0.7272727, -0.1958042)^t,$$

and

$$\mu^{(1)} = -\frac{1}{6.6} + \frac{19}{3} = 6.1818182.$$

Subsequent results are listed in Table 9.4, and the right column lists the results of Aitken's Δ^2 method applied to the $\mu^{(m)}$. These are clearly superior results to those obtained with the Power method. ■

Table 9.4

m	$\mathbf{x}^{(m)t}$	$\mu^{(m)}$	$\hat{\mu}^{(m)}$
0	(1, 1, 1)		
1	(1, 0.7272727, −0.1958042)	6.1818182	6.000098
2	(1, 0.7155172, −0.2450520)	6.0172414	6.000001
3	(1, 0.7144082, −0.2495224)	6.0017153	6.000000
4	(1, 0.7142980, −0.2499534)	6.0001714	6.000000
5	(1, 0.7142869, −0.2499954)	6.0000171	
6	(1, 0.7142858, −0.2499996)	6.0000017	

If A is symmetric, then for any real number q, the matrix $(A - qI)^{-1}$ is also symmetric, so the Symmetric Power method, Algorithm 9.2, can be applied to $(A - qI)^{-1}$ to speed the convergence to

$$O\left(\left|\frac{\lambda_k - q}{\lambda - q}\right|^{2m}\right).$$

Deflation Methods

Numerous techniques are available for obtaining approximations to the other eigenvalues of a matrix once an approximation to the dominant eigenvalue has been computed. We will restrict our presentation to **deflation techniques**.

Deflation techniques involve forming a new matrix B whose eigenvalues are the same as those of A, except that the dominant eigenvalue of A is replaced by the eigenvalue 0 in B. The following result justifies the procedure. The proof of this theorem can be found in [Wil2], p. 596.

Theorem 9.20

Suppose $\lambda_1, \lambda_2, \ldots, \lambda_n$ are eigenvalues of A with associated eigenvectors $\mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \ldots, \mathbf{v}^{(n)}$ and that λ_1 has multiplicity 1. Let $\mathbf{x}$ be a vector with $\mathbf{x}^t\mathbf{v}^{(1)} = 1$. Then the matrix

$$B = A - \lambda_1 \mathbf{v}^{(1)}\mathbf{x}^t$$

has eigenvalues $0, \lambda_2, \lambda_3, \ldots, \lambda_n$ with associated eigenvectors $\mathbf{v}^{(1)}$, $\mathbf{w}^{(2)}$, $\mathbf{w}^{(3)}, \ldots, \mathbf{w}^{(n)}$, where $\mathbf{v}^{(i)}$ and $\mathbf{w}^{(i)}$ are related by the equation

$$\mathbf{v}^{(i)} = (\lambda_i - \lambda_1)\mathbf{w}^{(i)} + \lambda_1(\mathbf{x}^t\mathbf{w}^{(i)})\mathbf{v}^{(1)}, \tag{9.6}$$

for each $i = 2, 3, \ldots, n$. ■

There are many choices of the vector $\mathbf{x}$ that could be used in Theorem 9.20. **Wielandt deflation** proceeds from defining

$$\mathbf{x} = \frac{1}{\lambda_1 v_i^{(1)}}(a_{i1}, a_{i2}, \ldots, a_{in})^t, \tag{9.7}$$

where $v_i^{(1)}$ is a nonzero coordinate of the eigenvector $\mathbf{v}^{(1)}$, and the values $a_{i1}, a_{i2}, \ldots, a_{in}$ are the entries in the ith row of A.

With this definition,

$$\mathbf{x}^t\mathbf{v}^{(1)} = \frac{1}{\lambda_1 v_i^{(1)}}[a_{i1}, a_{i2}, \ldots, a_{in}](v_1^{(1)}, v_2^{(1)}, \ldots, v_n^{(1)})^t = \frac{1}{\lambda_1 v_i^{(1)}}\sum_{j=1}^{n} a_{ij}v_j^{(1)},$$

where the sum is the ith coordinate of the product $A\mathbf{v}^{(1)}$. Since $A\mathbf{v}^{(1)} = \lambda_1\mathbf{v}^{(1)}$, we have

$$\sum_{j=1}^{n} a_{ij}v_j^{(1)} = \lambda_1 v_i^{(1)},$$

which implies that

$$\mathbf{x}^t\mathbf{v}^{(1)} = \frac{1}{\lambda_1 v_i^{(1)}}(\lambda_1 v_i^{(1)}) = 1.$$

So $\mathbf{x}$ satisfies the hypotheses of Theorem 9.20. Moreover (see Exercise 20), the ith row of $B = A - \lambda_1\mathbf{v}^{(1)}\mathbf{x}^t$ consists entirely of zero entries.

If $\lambda \neq 0$ is an eigenvalue with associated eigenvector $\mathbf{w}$, the relation $B\mathbf{w} = \lambda\mathbf{w}$ implies that the ith coordinate of $\mathbf{w}$ must also be zero. Consequently the ith column of the matrix B makes no contribution to the product $B\mathbf{w} = \lambda\mathbf{w}$. Thus, the matrix B can be replaced by an $(n-1) \times (n-1)$ matrix B' obtained by deleting the ith row and column from B. The matrix B' has eigenvalues $\lambda_2, \lambda_3, \ldots, \lambda_n$.

If $|\lambda_2| > |\lambda_3|$, the Power method is reapplied to the matrix B' to determine this new dominant eigenvalue and an eigenvector, $\mathbf{w}^{(2)'}$, associated with λ_2, with respect to the matrix B'. To find the associated eigenvector $\mathbf{w}^{(2)}$ for the matrix B, insert a zero coordinate between the coordinates $w_{i-1}^{(2)'}$ and $w_i^{(2)'}$ of the $(n-1)$-dimensional vector $\mathbf{w}^{(2)'}$ and then calculate $\mathbf{v}^{(2)}$ by the use of Eq. (9.6).

Example 4 The matrix

$$A = \begin{bmatrix} 4 & -1 & 1 \\ -1 & 3 & -2 \\ 1 & -2 & 3 \end{bmatrix}$$

has the dominant eigenvalue $\lambda_1 = 6$ with associated unit eigenvector $\mathbf{v}^{(1)} = (1, -1, 1)^t$. Assume that this dominant eigenvalue is known and apply deflation to approximate the other eigenvalues and eigenvectors.

Solution The procedure for obtaining a second eigenvalue λ_2 proceeds as follows:

$$\mathbf{x} = \frac{1}{6}\begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix} = \left(\frac{2}{3}, -\frac{1}{6}, \frac{1}{6}\right)^t,$$

$$\mathbf{v}^{(1)}\mathbf{x}^t = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\begin{bmatrix} \frac{2}{3}, & -\frac{1}{6}, & \frac{1}{6} \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{6} & \frac{1}{6} \\ -\frac{2}{3} & \frac{1}{6} & -\frac{1}{6} \\ \frac{2}{3} & -\frac{1}{6} & \frac{1}{6} \end{bmatrix},$$

and

$$B = A - \lambda_1 \mathbf{v}^{(1)}\mathbf{x}^t = \begin{bmatrix} 4 & -1 & 1 \\ -1 & 3 & -2 \\ 1 & -2 & 3 \end{bmatrix} - 6\begin{bmatrix} \frac{2}{3} & -\frac{1}{6} & \frac{1}{6} \\ -\frac{2}{3} & \frac{1}{6} & -\frac{1}{6} \\ \frac{2}{3} & -\frac{1}{6} & \frac{1}{6} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 2 & -1 \\ -3 & -1 & 2 \end{bmatrix}.$$

Deleting the first row and column gives

$$B' = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix},$$

which has eigenvalues $\lambda_2 = 3$ and $\lambda_3 = 1$. For $\lambda_2 = 3$, the eigenvector $\mathbf{w}^{(2)'}$ can be obtained by solving the linear system

$$(B' - 3I)\mathbf{w}^{(2)'} = \mathbf{0}, \quad \text{resulting in} \quad \mathbf{w}^{(2)'} = (1, -1)^t.$$

Adding a zero for the first component gives $\mathbf{w}^{(2)} = (0, 1, -1)^t$ and, from Eq. (9.6), we have the eigenvector $\mathbf{v}^{(2)}$ of A corresponding to $x_2 = 3$:

$$\begin{aligned} \mathbf{v}^{(2)} &= (\lambda_2 - \lambda_1)\mathbf{w}^{(2)} + \lambda_1(\mathbf{x}^t\mathbf{w}^{(2)})\mathbf{v}^{(1)} \\ &= (3-6)(0, 1, -1)^t + 6\left[\left(\frac{2}{3}, -\frac{1}{6}, \frac{1}{6}\right)(0, 1, -1)^t\right](1, -1, 1)^t = (-2, -1, 1)^t. \end{aligned}$$ ■

Although this deflation process can be used to find approximations to all of the eigenvalues and eigenvectors of a matrix, the process is susceptible to round-off error. After deflation is used to approximate an eigenvalue of a matrix, the approximation should be used as a starting value for the Inverse Power method applied to the original matrix. This will ensure convergence to an eigenvalue of the original matrix, not to one of the reduced matrix, which likely contains errors. When all the eigenvalues of a matrix are required, techniques considered in Section 9.5, based on similarity transformations, should be used.

We close this section with Algorithm 9.4, which calculates the second most dominant eigenvalue and associated eigenvector for a matrix, once the dominant eigenvalue and associated eigenvector have been determined.

ALGORITHM 9.4

Wielandt Deflation

To approximate the second most dominant eigenvalue and an associated eigenvector of the $n \times n$ matrix A given an approximation λ to the dominant eigenvalue, an approximation $\mathbf{v}$ to a corresponding eigenvector, and a vector $\mathbf{x} \in \mathbb{R}^{n-1}$:

INPUT dimension n; matrix A; approximate eigenvalue λ with eigenvector $\mathbf{v} \in \mathbb{R}^n$; vector $\mathbf{x} \in \mathbb{R}^{n-1}$, tolerance TOL, maximum number of iterations N.

OUTPUT approximate eigenvalue μ; approximate eigenvector $\mathbf{u}$ or a message that the method fails.

Step 1 Let i be the smallest integer with $1 \le i \le n$ and $|v_i| = \max_{1 \le j \le n} |v_j|$.

Step 2 If $i \ne 1$ then
for $k = 1, \ldots, i-1$
for $j = 1, \ldots, i-1$

$$\text{set } b_{kj} = a_{kj} - \frac{v_k}{v_i} a_{ij}.$$

Step 3 If $i \ne 1$ and $i \ne n$ then
for $k = i, \ldots, n-1$
for $j = 1, \ldots, i-1$

$$\text{set } b_{kj} = a_{k+1,j} - \frac{v_{k+1}}{v_i} a_{ij};$$

$$b_{jk} = a_{j,k+1} - \frac{v_j}{v_i} a_{i,k+1}.$$

Step 4 If $i \ne n$ then
for $k = i, \ldots, n-1$
for $j = i, \ldots, n-1$

$$\text{set } b_{kj} = a_{k+1,j+1} - \frac{v_{k+1}}{v_i} a_{i,j+1}.$$

Step 5 Perform the power method on the $(n-1) \times (n-1)$ matrix $B' = (b_{kj})$ with $\mathbf{x}$ as initial approximation.

Step 6 If the method fails, then OUTPUT ('Method fails');
STOP
else let μ be the approximate eigenvalue and
$\mathbf{w}' = (w'_1, \ldots, w'_{n-1})^t$ the approximate eigenvector.

Step 7 If $i \neq 1$ then for $k = 1, \ldots, i-1$ set $w_k = w'_k$.

Step 8 Set $w_i = 0$.

Step 9 If $i \neq n$ then for $k = i+1, \ldots, n$ set $w_k = w'_{k-1}$.

Step 10 For $k = 1, \ldots, n$

$$\text{set } u_k = (\mu - \lambda) w_k + \left(\sum_{j=1}^{n} a_{ij} w_j \right) \frac{v_k}{v_i}.$$

(*Compute the eigenvector using Eq.* (9.6).)

Step 11 OUTPUT $(\mu, \mathbf{u})$; (*The procedure was successful.*)
STOP.

EXERCISE SET 9.3

1. Find the first three iterations obtained by the Power method applied to the following matrices.

a. $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$; Use $\mathbf{x}^{(0)} = (1, -1, 2)^t$.

b. $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$; Use $\mathbf{x}^{(0)} = (-1, 0, 1)^t$.

c. $\begin{bmatrix} 1 & -1 & 0 \\ -2 & 4 & -2 \\ 0 & -1 & 2 \end{bmatrix}$; Use $\mathbf{x}^{(0)} = (-1, 2, 1)^t$.

d. $\begin{bmatrix} 4 & 1 & 1 & 1 \\ 1 & 3 & -1 & 1 \\ 1 & -1 & 2 & 0 \\ 1 & 1 & 0 & 2 \end{bmatrix}$; Use $\mathbf{x}^{(0)} = (1, -2, 0, 3)^t$.

3. Repeat Exercise 1 using the Inverse Power method.

5. Find the first three iterations obtained by the Symmetric Power method

13. Use Wielandt deflation and the results of Exercise 7 to approximate the second most dominant eigenvalue of the matrices in Exercise 1. Iterate until a tolerance of 10^{-4} is achieved or until the number of iterations exceeds 25.

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