MAP 2210 – Aplicações de Álgebra Linear

1º Semestre - 2019

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Objetivos

Formação básica de álgebra linear aplicada a problemas numéricos. Resolução de problemas em microcomputadores usando linguagens e/ou software adequados fora do horário de aula.

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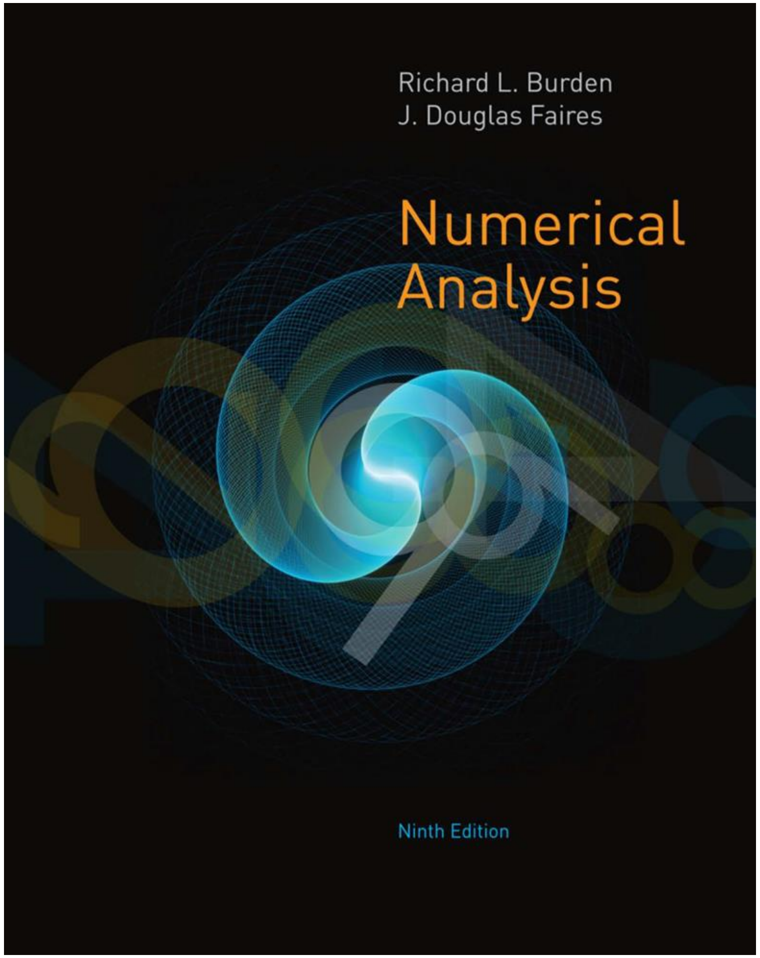

Numerical Analysis

NINTH EDITION

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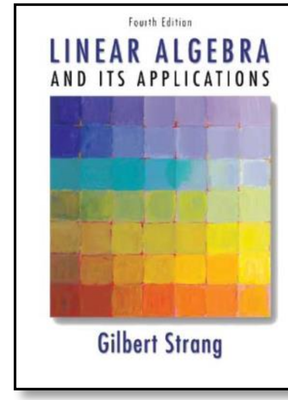

Fourth Edition
LINEAR ALGEBRA
AND ITS APPLICATIONS
Gilbert Strang

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CHAPTER 9

Approximating Eigenvalues

Introduction

The longitudinal vibrations of an elastic bar of local stiffness $p(x)$ and density $\rho(x)$ are described by the partial differential equation

$$\rho(x)\frac{\partial^2 v}{\partial t^2}(x,t) = \frac{\partial}{\partial x}\left[p(x)\frac{\partial v}{\partial x}(x,t)\right],$$

where $v(x,t)$ is the mean longitudinal displacement of a section of the bar from its equilibrium position x at time t. The vibrations can be written as a sum of simple harmonic vibrations:

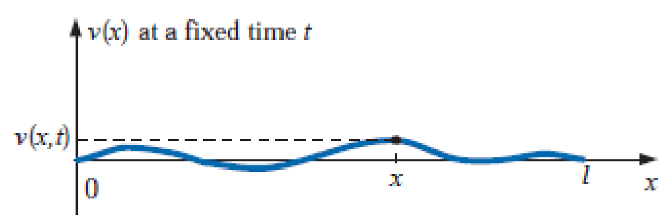
$$v(x,t) = \sum_{k=0}^{\infty} c_k u_k(x) \cos\sqrt{\lambda_k}(t-t_0),$$

where

$$\frac{d}{dx}\left[p(x)\frac{du_k}{dx}(x)\right] + \lambda_k \rho(x) u_k(x) = 0.$$

Discretização por diferenças finitas

If the bar has length l and is fixed at its ends, then this differential equation holds for $0 < x < l$ and $v(0) = v(l) = 0$.



A system of these differential equations is called a Sturm-Liouville system, and the numbers λ_k are eigenvalues with corresponding eigenfunctions $u_k(x)$.

Suppose the bar is 1 m long with uniform stiffness $p(x) = p$ and uniform density $\rho(x) = \rho$. To approximate u and λ, let $h = 0.2$. Then $x_j = 0.2j$, for $0 \le j \le 5$, and we can use the midpoint formula (4.5) in Section 4.1 to approximate the first derivatives. This gives the linear system

$$A\mathbf{w} = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix} = -0.04\frac{\rho}{p}\lambda \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix} = -0.04\frac{\rho}{p}\lambda \mathbf{w}.$$

In this system, $w_j \approx u(x_j)$, for $1 \le j \le 4$, and $w_0 = w_5 = 0$. The four eigenvalues of A approximate the eigenvalues of the *Sturm-Liouville system*. It is the approximation of eigenvalues that we will consider in this chapter. A Sturm-Liouville application is discussed in Exercise 13 of Section 9.5.

Second Derivative Midpoint Formula

- $$f''(x_0) = \frac{1}{h^2}[f(x_0 - h) - 2f(x_0) + f(x_0 + h)] - \frac{h^2}{12} f^{(4)}(\xi), \qquad (4.9)$$

for some ξ, where $x_0 - h < \xi < x_0 + h$.

9.1 Linear Algebra and Eigenvalues

Eigenvalues and eigenvectors were introduced in Chapter 7 in connection with the convergence of iterative methods for approximating the solution to a linear system. To determine the eigenvalues of an $n \times n$ matrix A, we construct the characteristic polynomial

$$p(\lambda) = \det(A - \lambda I)$$

and then determine its zeros. Finding the determinant of an $n \times n$ matrix is computationally expensive, and finding good approximations to the roots of $p(\lambda)$ is also difficult. In this chapter we will explore other means for approximating the eigenvalues of a matrix. In Section 9.6 we give an introduction to a technique for factoring a general $m \times n$ matrix into a form that has valuable applications in a number of areas.

In Chapter 7 we found that an iterative technique for solving a linear system will converge if all the eigenvalues associated with the problem have magnitude less than 1. The exact values of the eigenvalues in this case are not of primary importance—only the region of the complex plane in which they lie. An important result in this regard was first discovered by S. A. Geršgorin. It is the subject of a very interesting book by Richard Varga. [Var2]

Theorem 9.1 (Geršgorin Circle)

Let A be an $n \times n$ matrix and R_i denote the circle in the complex plane with center a_{ii} and radius $\sum_{j=1, j\neq i}^{n} |a_{ij}|$; that is,

$$R_i = \left\{ z \in \mathcal{C} \,\middle|\, |z - a_{ii}| \leq \sum_{j=1, j\neq i}^{n} |a_{ij}| \right\},$$

where $\mathcal{C}$ denotes the complex plane. The eigenvalues of A are contained within the union of these circles, $R = \cup_{i=1}^{n} R_i$. Moreover, the union of any k of the circles that do not intersect the remaining $(n-k)$ contains precisely k (counting multiplicities) of the eigenvalues. ■

Proof Suppose that λ is an eigenvalue of A with associated eigenvector $\mathbf{x}$, where $\|\mathbf{x}\|_\infty = 1$. Since $A\mathbf{x} = \lambda\mathbf{x}$, the equivalent component representation is

$$\sum_{j=1}^{n} a_{ij}x_j = \lambda x_i, \quad \text{for each } i = 1, 2, \ldots, n. \tag{9.1}$$

Let k be an integer with $|x_k| = \|\mathbf{x}\|_\infty = 1$. When $i = k$, Eq. (9.1) implies that

$$\sum_{j=1}^{n} a_{kj}x_j = \lambda x_k.$$

Thus

$$\sum_{\substack{j=1,\\ j\neq k}}^{n} a_{kj}x_j = \lambda x_k - a_{kk}x_k = (\lambda - a_{kk})x_k,$$

and

$$|\lambda - a_{kk}| \cdot |x_k| = \left| \sum_{\substack{j=1,\\ j\neq k}}^{n} a_{kj} x_j \right| \leq \sum_{\substack{j=1,\\ j\neq k}}^{n} |a_{kj}||x_j|.$$

But $|x_k| = \|\mathbf{x}\|_\infty = 1$, so $|x_j| \leq |x_k| = 1$ for all $j = 1, 2, \ldots, n$. Hence

$$|\lambda - a_{kk}| \leq \sum_{\substack{j=1,\\ j\neq k}}^{n} |a_{kj}|.$$

This proves the first assertion in the theorem, that $\lambda \in R_k$. A proof of the second statement is contained in [Var2], p. 8, or in [Or2], p. 48. ■ ■ ■

Example 1 Determine the Geršgorin circles for the matrix

$$A = \begin{bmatrix} 4 & 1 & 1 \\ 0 & 2 & 1 \\ -2 & 0 & 9 \end{bmatrix},$$

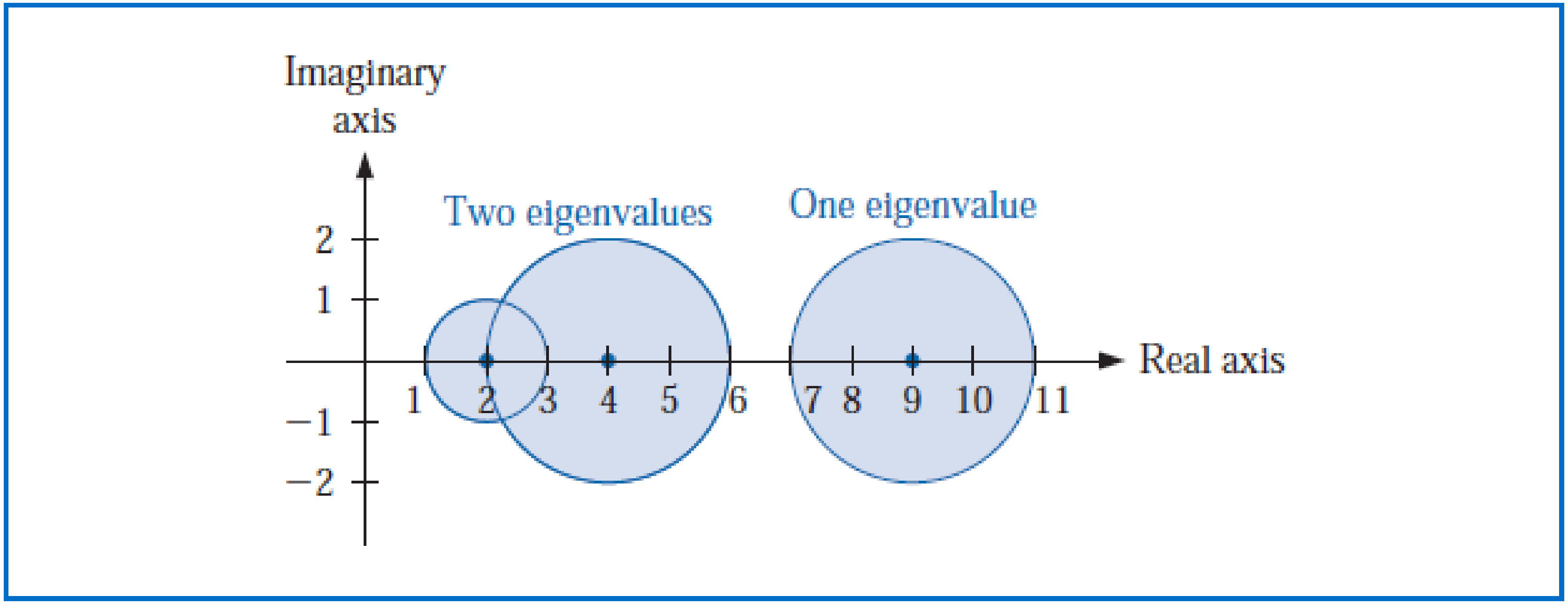
and use these to find bounds for the spectral radius of A.

Solution The circles in the Geršgorin Theorem are (see Figure 9.1)

$R_1 = \{z \in \mathcal{C} \mid |z-4| \leq 2\}$, $R_2 = \{z \in \mathcal{C} \mid |z-2| \leq 1\}$, and $R_3 = \{z \in \mathcal{C} \mid |z-9| \leq 2\}$.

Because R_1 and R_2 are disjoint from R_3, there are precisely two eigenvalues within $R_1 \cup R_2$ and one within R_3. Moreover, $\rho(A) = \max_{1\leq i\leq 3} |\lambda_i|$, so $7 \leq \rho(A) \leq 11$. ■

Figure 9.1



Even when we need to find the eigenvalues, many techniques for their approximation are iterative. Determining regions in which they lie is the first step for finding the approximation, because it provides us with an initial approximations.

Before considering further results concerning eigenvalues and eigenvectors, we need some definitions and results from linear algebra. All the general results that will be needed in the remainder of this chapter are listed here for ease of reference. The proofs of many of the results that are not given are considered in the exercises, and all can be be found in most standard texts on linear algebra (see, for example, [ND], [Poo], or [DG]).

Definition 9.2 Let $\{\mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \mathbf{v}^{(3)}, \dots, \mathbf{v}^{(k)}\}$ be a set of vectors. The set is **linearly independent** if whenever

$$\mathbf{0} = \alpha_1 \mathbf{v}^{(1)} + \alpha_2 \mathbf{v}^{(2)} + \alpha_3 \mathbf{v}^{(3)} + \cdots + \alpha_k \mathbf{v}^{(k)},$$

then $\alpha_i = 0$, for each $i = 0, 1, \dots, k$. Otherwise the set of vectors is **linearly dependent.** ■

Note that any set of vectors containing the zero vector is linearly dependent.

Theorem 9.3 Suppose that $\{\mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \mathbf{v}^{(3)}, \dots, \mathbf{v}^{(n)}\}$ is a set of n linearly independent vectors in $\mathbb{R}^n$. Then for any vector $\mathbf{x} \in \mathbb{R}^n$ a unique collection of constants $\beta_1, \beta_2, \dots, \beta_n$ exists with

$$\mathbf{x} = \beta_1 \mathbf{v}^{(1)} + \beta_2 \mathbf{v}^{(2)} + \beta_3 \mathbf{v}^{(3)} + \cdots + \beta_n \mathbf{v}^{(n)}.$$

■

Definition 9.4 Any collection of n linearly independent vectors in $\mathbb{R}^n$ is called a **basis** for $\mathbb{R}^n$. ■

The next result will be used in Section 9.3 to develop the Power method for approximating eigenvalues. A proof of this result is considered in Exercise 10.

Theorem 9.5 If A is a matrix and $\lambda_1, \ldots, \lambda_k$ are distinct eigenvalues of A with associated eigenvectors $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \ldots, \mathbf{x}^{(k)}$, then $\{\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \ldots, \mathbf{x}^{(k)}\}$ is a linearly independent set. ■

Orthogonal Vectors

In Section 8.2 we considered orthogonal and orthonormal sets of functions. Vectors with these properties are defined in a similar manner.

Definition 9.6 A set of vectors $\{\mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \ldots, \mathbf{v}^{(n)}\}$ is called **orthogonal** if $(\mathbf{v}^{(i)})^t \mathbf{v}^{(j)} = 0$, for all $i \neq j$. If, in addition, $(\mathbf{v}^{(i)})^t \mathbf{v}^{(i)} = 1$, for all $i = 1, 2, \ldots, n$, then the set is called **orthonormal**. ■

Because $\mathbf{x}^t \mathbf{x} = \|\mathbf{x}\|_2^2$ for any $\mathbf{x}$ in $\mathbb{R}^n$, a set of orthogonal vectors $\{\mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \ldots, \mathbf{v}^{(n)}\}$ is orthonormal if and only if

$$\|\mathbf{v}^{(i)}\|_2 = 1, \quad \text{for each } i = 1, 2, \ldots, n.$$

Theorem 9.7 An orthogonal set of nonzero vectors is linearly independent. ■

The **Gram-Schmidt** process for constructing a set of polynomials that are orthogonal with respect to a given weight function was described in Theorem 8.7 of Section 8.2 (see page 515). There is a parallel process, also known as Gram-Schmidt, that permits us to construct an orthogonal basis for $\mathbb{R}^n$ given a set of n linearly independent vectors in $\mathbb{R}^n$.

Theorem 9.8 Let $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k\}$ be a set of k linearly independent vectors in $\mathbb{R}^n$. Then $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ defined by

$$\begin{aligned}
\mathbf{v}_1 &= \mathbf{x}_1, \\
\mathbf{v}_2 &= \mathbf{x}_2 - \left(\frac{\mathbf{v}_1^t \mathbf{x}_2}{\mathbf{v}_1^t \mathbf{v}_1}\right) \mathbf{v}_1, \\
\mathbf{v}_3 &= \mathbf{x}_3 - \left(\frac{\mathbf{v}_1^t \mathbf{x}_3}{\mathbf{v}_1^t \mathbf{v}_1}\right) \mathbf{v}_1 - \left(\frac{\mathbf{v}_2^t \mathbf{x}_3}{\mathbf{v}_2^t \mathbf{v}_2}\right) \mathbf{v}_2, \\
&\ \vdots \\
\mathbf{v}_k &= \mathbf{x}_k - \sum_{i=1}^{k-1} \left(\frac{\mathbf{v}_i^t \mathbf{x}_k}{\mathbf{v}_i^t \mathbf{v}_i}\right) \mathbf{v}_i.
\end{aligned}$$

is set of k orthogonal vectors in $\mathbb{R}^n$. ■

Note that when the original set of vectors forms a basis for $\mathbb{R}^n$, that is, when $k = n$, then the constructed vectors form an orthogonal basis for $\mathbb{R}^n$. From this we can form an orthonormal basis $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ simply by defining for each $i = 1, 2, \ldots, n$

$$\mathbf{u}_i = \frac{\mathbf{v}_i}{||\mathbf{v}_i||_2}.$$

The following example illustrates how an orthonormal basis for $\mathbb{R}^3$ can be constructed from three linearly independent vectors in $\mathbb{R}^3$.

Example 2 (**a**) Show that $\mathbf{v}^{(1)} = (1, 0, 0)^t$, $\mathbf{v}^{(2)} = (-1, 1, 1)^t$, and $\mathbf{v}^{(3)} = (0, 4, 2)^t$ is a basis for $\mathbb{R}^3$, and (**b**) given an arbitrary vector $\mathbf{x} \in \mathbb{R}^3$ find β_1, β_2, and β_3 with

$$\mathbf{x} = \beta_1 \mathbf{v}^{(1)} + \beta_2 \mathbf{v}^{(2)} + \beta_3 \mathbf{v}^{(3)}.$$

Example 3 Show that a basis can be formed for $\mathbb{R}^3$ using the eigenvectors of the 3×3 matrix

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 2 \\ 1 & -1 & 4 \end{bmatrix}.$$

Example 4 Show that no collection of eigenvectors of the 3×3 matrix

$$B = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

can form a basis for $\mathbb{R}^3$.

Example 5 (**a**) Show that the vectors $\mathbf{v}^{(1)} = (0, 4, 2)^t$, $\mathbf{v}^{(2)} = (-5, -1, 2)^t$, and $\mathbf{v}^{(3)} = (1, -1, 2)^t$ form an orthogonal set, and (**b**) use these to determine a set of orthonormal vectors.

Example 6 Use the Gram-Schmidt process to determine a set of orthogonal vectors from the linearly independent vectors

$$\mathbf{x}^{(1)} = (1, 0, 0)^t, \quad \mathbf{x}^{(2)} = (1, 1, 0)^t, \quad \text{and} \quad \mathbf{x}^{(3)} = (1, 1, 1)^t.$$

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9.2 Orthogonal Matrices and Similarity Transformations

In this section we will consider the connection between sets of vectors and matrices formed using these vectors as their columns. We first consider some results about a class of special matrices. The terminology in the next definition follows from the fact that the columns of an orthogonal matrix will form an orthogonal set of vectors.

Definition 9.9 A matrix Q is said to be **orthogonal** if its columns $\{\mathbf{q}_1^t, \mathbf{q}_2^t, \ldots, \mathbf{q}_n^t\}$ form an orthonormal set in $\mathbb{R}^n$. ■

Theorem 9.10 Suppose that Q is an orthogonal $n \times n$ matrix. Then

(i) Q is invertible with $Q^{-1} = Q^t$;

(ii) For any $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$, $(Q\mathbf{x})^t Q\mathbf{y} = \mathbf{x}^t\mathbf{y}$;

(iii) For any $\mathbf{x}$ in $\mathbb{R}^n$, $||Q\mathbf{x}||_2 = ||\mathbf{x}||_2$. ■

In addition, the converse of part (i) holds. (See Exercise 18.) That is,

- any invertible matrix Q with $Q^{-1} = Q^t$ is orthogonal.

As an example, the permutation matrices discussed in Section 6.5 have this property, so they are orthogonal.

Property (iii) of Theorem 9.10 is often expressed by stating that orthogonal matrices are l_2-norm preserving. As an immediate consequence of this property, every orthogonal matrix Q has $||Q||_2 = 1$.

Definition 9.11 Two matrices A and B are said to be **similar** if a nonsingular matrix S exists with $A = S^{-1}BS$. ■

An important feature of similar matrices is that they have the same eigenvalues.

Theorem 9.12 Suppose A and B are similar matrices with $A = S^{-1}BS$ and λ is an eigenvalue of A with associated eigenvector $\mathbf{x}$. Then λ is an eigenvalue of B with associated eigenvector $S\mathbf{x}$. ■

A particularly important use of similarity occurs when an $n \times n$ matrix A is similar to diagonal matrix. That is, when a diagonal matrix D and an invertible matrix S exists with

$$A = S^{-1}DS \quad \text{or equivalently} \quad D = SAS^{-1}.$$

In this case the matrix A is said to be *diagonalizable*. The following result is considered in Exercise 19.

Theorem 9.13 An $n \times n$ matrix A is similar to a diagonal matrix D if and only if A has n linearly independent eigenvectors. In this case, $D = S^{-1}AS$, where the columns of S consist of the eigenvectors, and the ith diagonal element of D is the eigenvalue of A that corresponds to the ith column of S. ■

Corollary 9.14 An $n \times n$ matrix A that has n distinct eigenvalues is similar to a diagonal matrix. ■

In fact, we do not need the similarity matrix to be diagonal for this concept to be useful. Suppose that A is similar to a triangular matrix B. The determination of eigenvalues is easy for a triangular matrix B, for in this case λ is a solution to the equation

$$0 = \det(B - \lambda I) = \prod_{i=1}^{n}(b_{ii} - \lambda)$$

if and only if $\lambda = b_{ii}$ for some i. The next result describes a relationship, called a **similarity transformation**, between arbitrary matrices and triangular matrices.

Theorem 9.15 **(Schur)**

Let A be an arbitrary matrix. A nonsingular matrix U exists with the property that

$$T = U^{-1}AU,$$

where T is an upper-triangular matrix whose diagonal entries consist of the eigenvalues of A. ■

The matrix U whose existence is ensured in Theorem 9.15 satisfies the condition $\|U\mathbf{x}\|_2 = \|\mathbf{x}\|_2$ for any vector $\mathbf{x}$. Matrices with this property are called **unitary**. Although we will not make use of this norm-preserving property, it does significantly increase the application of Schur's Theorem.

Theorem 9.15 is an existence theorem that ensures that the triangular matrix T exists, but it does not provide a constructive means for finding T, since it requires a knowledge of the eigenvalues of A. In most instances, the similarity transformation U is too difficult to determine.

The following result for symmetric matrices reduces the complication, because in this case the transformation matrix is orthogonal.

Theorem 9.16 The $n \times n$ matrix A is symmetric if and only if there exists a diagonal matrix D and an orthogonal matrix Q with $A = QDQ^t$. ■

Corollary 9.17 Suppose that A is a symmetric $n \times n$ matrix. There exist n eigenvectors of A that form an orthonormal set, and the eigenvalues of A are real numbers. ■

Theorem 9.18 A symmetric matrix A is positive definite if and only if all the eigenvalues of A are positive. ■

5. For each of the following matrices determine if it diagonalizable and, if so, find P and D with $A = PDP^{-1}$.

a. $A = \begin{bmatrix} 4 & -1 \\ -4 & 1 \end{bmatrix}$

b. $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$

c. $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

d. $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

8. (i) Determine if the following matrices are positive definite, and if so, (ii) construct an orthogonal matrix Q for which $Q^tAQ = D$, where D is a diagonal matrix.

a. $A = \begin{bmatrix} 4 & 2 & 1 \\ 2 & 4 & 0 \\ 1 & 0 & 4 \end{bmatrix}$

b. $A = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

c. $A = \begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & 2 & -1 & -2 \\ -1 & -1 & 3 & 0 \\ 1 & -2 & 0 & 4 \end{bmatrix}$

d. $A = \begin{bmatrix} 8 & 4 & 2 & 1 \\ 4 & 8 & 2 & 1 \\ 2 & 2 & 8 & 1 \\ 1 & 1 & 1 & 8 \end{bmatrix}$

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