MAP 2210 – Aplicações de Álgebra Linear

1º Semestre - 2019

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Objetivos

Formação básica de álgebra linear aplicada a problemas numéricos. Resolução de problemas em microcomputadores usando linguagens e/ou software adequados fora do horário de aula.

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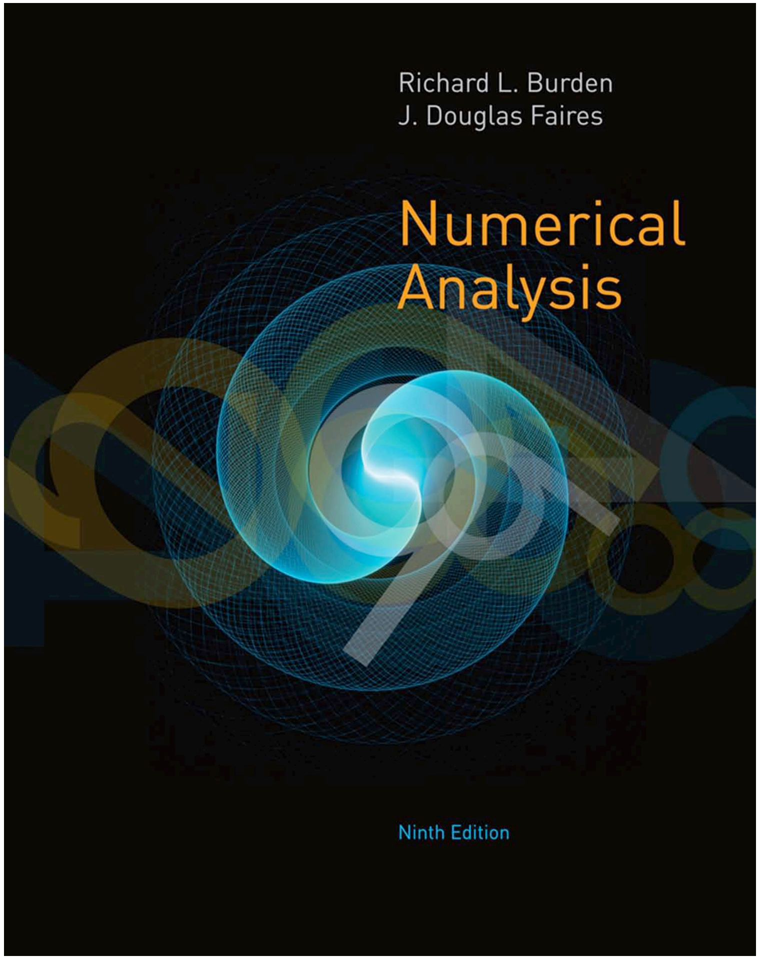

Numerical Analysis

NINTH EDITION

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Youngstown State University

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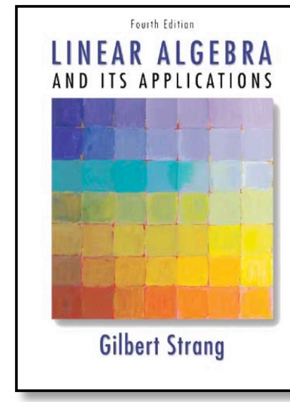

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LINEAR ALGEBRA
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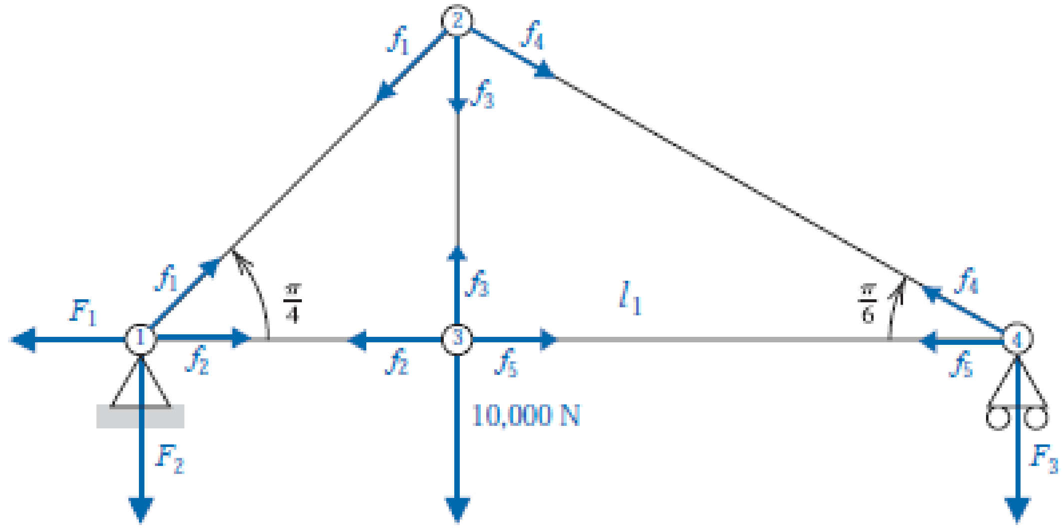
CHAPTER 7

Iterative Techniques in Matrix Algebra

Introduction



Joint	Horizontal Component	Vertical Component
①	$-F_1 + \frac{\sqrt{2}}{2} f_1 + f_2 = 0$	$\frac{\sqrt{2}}{2} f_1 - F_2 = 0$
②	$-\frac{\sqrt{2}}{2} f_1 + \frac{\sqrt{3}}{2} f_4 = 0$	$-\frac{\sqrt{2}}{2} f_1 - f_3 - \frac{1}{2} f_4 = 0$
③	$-f_2 + f_5 = 0$	$f_3 - 10{,}000 = 0$
④	$-\frac{\sqrt{3}}{2} f_4 - f_5 = 0$	$\frac{1}{2} f_4 - F_3 = 0$

8 x 8 = 64 elementos
47 zeros
17 não zeros
Esparsidade 73.4%
Densidade 26.5%

Diversos problemas aplicados dão origem a sistemas de matrizes esparsas

O princípio dos métodos iterativos é substituir o problema linear original:

A x = b

Por outro:

$$\mathbf{x}^{(i+1)} = T\mathbf{x}^{(i)} + \mathbf{c}.$$

onde na convergência as soluções são próximas dentro de uma tolerância especificada

O material desse aula provê uma revisão e preparação para esse assunto, e também para o problema de autovalores

7.1 Norms of Vectors and Matrices

Vector Norms

Let $\mathbb{R}^n$ denote the set of all n-dimensional column vectors with real-number components. To define a distance in $\mathbb{R}^n$ we use the notion of a norm, which is the generalization of the absolute value on $\mathbb{R}$, the set of real numbers.

Definition 7.1 A **vector norm** on $\mathbb{R}^n$ is a function, $\|\cdot\|$, from $\mathbb{R}^n$ into $\mathbb{R}$ with the following properties:

(i) $\|\mathbf{x}\| \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n$,

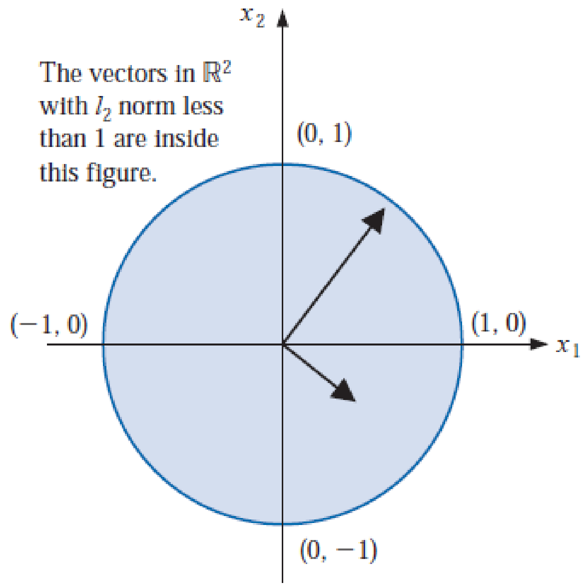
(ii) $\|\mathbf{x}\| = 0$ if and only if $\mathbf{x} = \mathbf{0}$,

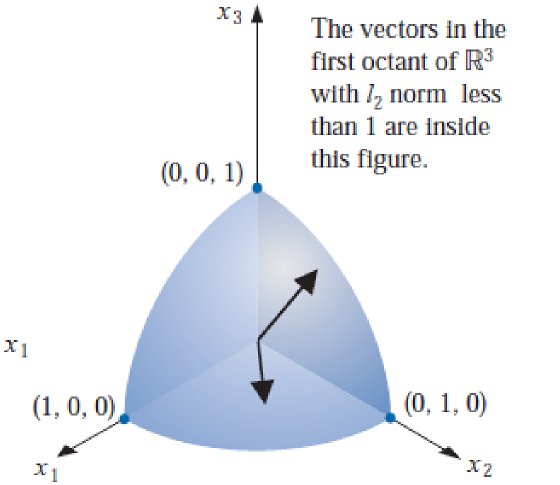
(iii) $\|\alpha\mathbf{x}\| = |\alpha|\|\mathbf{x}\|$ for all $\alpha \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^n$,

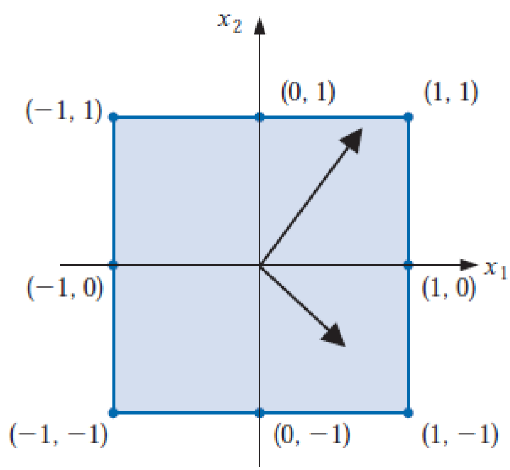
(iv) $\|\mathbf{x}+\mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. ■

Definition 7.2 The l_2 and l_∞ norms for the vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)^t$ are defined by

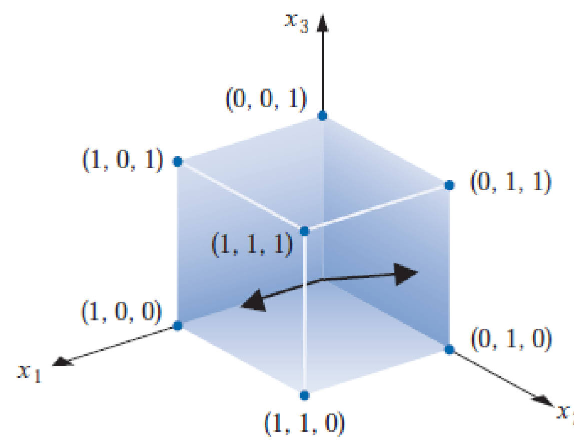
$$\|\mathbf{x}\|_2 = \left\{\sum_{i=1}^{n} x_i^2\right\}^{1/2} \quad \text{and} \quad \|\mathbf{x}\|_\infty = \max_{1\leq i\leq n} |x_i|.$$ ■







The vectors in $\mathbb{R}^2$ with l_∞ norm less than 1 are inside this figure.



The vectors in the first octant of $\mathbb{R}^3$ with l_∞ norm less than 1 are inside this figure.

Theorem 7.3 **(Cauchy-Bunyakovsky-Schwarz Inequality for Sums)**

For each $\mathbf{x} = (x_1, x_2, \ldots, x_n)^t$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)^t$ in $\mathbb{R}^n$,

$$\mathbf{x}^t\mathbf{y} = \sum_{i=1}^{n} x_i y_i \le \left\{\sum_{i=1}^{n} x_i^2\right\}^{1/2} \left\{\sum_{i=1}^{n} y_i^2\right\}^{1/2} = \|\mathbf{x}\|_2 \cdot \|\mathbf{y}\|_2. \tag{7.1}$$

■

Proof If $\mathbf{y} = \mathbf{0}$ or $\mathbf{x} = \mathbf{0}$, the result is immediate because both sides of the inequality are zero.

Suppose $\mathbf{y} \neq \mathbf{0}$ and $\mathbf{x} \neq \mathbf{0}$. Note that for each $\lambda \in \mathbb{R}$ we have

$$0 \le \|\mathbf{x} - \lambda\mathbf{y}\|_2^2 = \sum_{i=1}^{n}(x_i - \lambda y_i)^2 = \sum_{i=1}^{n} x_i^2 - 2\lambda \sum_{i=1}^{n} x_i y_i + \lambda^2 \sum_{i=1}^{n} y_i^2,$$

so that

$$2\lambda \sum_{i=1}^{n} x_i y_i \le \sum_{i=1}^{n} x_i^2 + \lambda^2 \sum_{i=1}^{n} y_i^2 = \|\mathbf{x}\|_2^2 + \lambda^2 \|\mathbf{y}\|_2^2.$$

However $\|\mathbf{x}\|_2 > 0$ and $\|\mathbf{y}\|_2 > 0$, so we can let $\lambda = \|\mathbf{x}\|_2 / \|\mathbf{y}\|_2$ to give

$$\left(2\frac{\|\mathbf{x}\|_2}{\|\mathbf{y}\|_2}\right)\left(\sum_{i=1}^{n} x_i y_i\right) \le \|\mathbf{x}\|_2^2 + \frac{\|\mathbf{x}\|_2^2}{\|\mathbf{y}\|_2^2}\|\mathbf{y}\|_2^2 = 2\|\mathbf{x}\|_2^2.$$

Hence

$$2\sum_{i=1}^{n} x_i y_i \le 2\|\mathbf{x}\|_2^2 \frac{\|\mathbf{y}\|_2}{\|\mathbf{x}\|_2} = 2\|\mathbf{x}\|_2\|\mathbf{y}\|_2,$$

and

$$\mathbf{x}^t\mathbf{y} = \sum_{i=1}^{n} x_i y_i \le \|\mathbf{x}\|_2\|\mathbf{y}\|_2 = \left\{\sum_{i=1}^{n} x_i^2\right\}^{1/2}\left\{\sum_{i=1}^{n} y_i^2\right\}^{1/2}.$$

With this result we see that for each $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$,

$$\|\mathbf{x}+\mathbf{y}\|_2^2 = \sum_{i=1}^{n}(x_i+y_i)^2 = \sum_{i=1}^{n} x_i^2 + 2\sum_{i=1}^{n} x_i y_i + \sum_{i=1}^{n} y_i^2 \le \|\mathbf{x}\|_2^2 + 2\|\mathbf{x}\|_2\|\mathbf{y}\|_2 + \|\mathbf{y}\|_2^2,$$

which gives norm property **(iv)**:

$$\|\mathbf{x}+\mathbf{y}\|_2 \le \left(\|\mathbf{x}\|_2^2 + 2\|\mathbf{x}\|_2\|\mathbf{y}\|_2 + \|\mathbf{y}\|_2^2\right)^{1/2} = \|\mathbf{x}\|_2 + \|\mathbf{y}\|_2.$$

Distance between Vectors in $\mathbb{R}^n$

The norm of a vector gives a measure for the distance between an arbitrary vector and the zero vector, just as the absolute value of a real number describes its distance from 0. Similarly, the **distance between two vectors** is defined as the norm of the difference of the vectors just as distance between two real numbers is the absolute value of their difference.

Definition 7.4 If $\mathbf{x} = (x_1, x_2, \ldots, x_n)^t$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)^t$ are vectors in $\mathbb{R}^n$, the l_2 and l_∞ distances between $\mathbf{x}$ and $\mathbf{y}$ are defined by

$$\|\mathbf{x} - \mathbf{y}\|_2 = \left\{ \sum_{i=1}^{n} (x_i - y_i)^2 \right\}^{1/2} \quad \text{and} \quad \|\mathbf{x} - \mathbf{y}\|_\infty = \max_{1 \le i \le n} |x_i - y_i|. \qquad \blacksquare$$

Definition 7.5 A sequence $\{\mathbf{x}^{(k)}\}_{k=1}^{\infty}$ of vectors in $\mathbb{R}^n$ is said to **converge** to $\mathbf{x}$ with respect to the norm $\|\cdot\|$ if, given any $\varepsilon > 0$, there exists an integer $N(\varepsilon)$ such that

$$\|\mathbf{x}^{(k)} - \mathbf{x}\| < \varepsilon, \quad \text{for all } k \ge N(\varepsilon). \qquad \blacksquare$$

Theorem 7.6 The sequence of vectors $\{\mathbf{x}^{(k)}\}$ converges to $\mathbf{x}$ in $\mathbb{R}^n$ with respect to the l_∞ norm if and only if $\lim_{k\to\infty} x_i^{(k)} = x_i$, for each $i = 1, 2, \ldots, n$. ■

Proof Suppose $\{\mathbf{x}^{(k)}\}$ converges to $\mathbf{x}$ with respect to the l_∞ norm. Given any $\varepsilon > 0$, there exists an integer $N(\varepsilon)$ such that for all $k \geq N(\varepsilon)$,

$$\max_{i=1,2,\ldots,n} |x_i^{(k)} - x_i| = \|\mathbf{x}^{(k)} - \mathbf{x}\|_\infty < \varepsilon.$$

This result implies that $|x_i^{(k)} - x_i| < \varepsilon$, for each $i = 1, 2, \ldots, n$, so $\lim_{k\to\infty} x_i^{(k)} = x_i$ for each i.

Conversely, suppose that $\lim_{k\to\infty} x_i^{(k)} = x_i$, for every $i = 1, 2, \ldots, n$. For a given $\varepsilon > 0$, let $N_i(\varepsilon)$ for each i represent an integer with the property that

$$|x_i^{(k)} - x_i| < \varepsilon,$$

whenever $k \geq N_i(\varepsilon)$.

Define $N(\varepsilon) = \max_{i=1,2,\ldots,n} N_i(\varepsilon)$. If $k \geq N(\varepsilon)$, then

$$\max_{i=1,2,\ldots,n} |x_i^{(k)} - x_i| = \|\mathbf{x}^{(k)} - \mathbf{x}\|_\infty < \varepsilon.$$

This implies that $\{\mathbf{x}^{(k)}\}$ converges to $\mathbf{x}$ with respect to the l_∞ norm. ■ ■ ■

Theorem 7.7 For each $\mathbf{x} \in \mathbb{R}^n$,

$$\|\mathbf{x}\|_\infty \leq \|\mathbf{x}\|_2 \leq \sqrt{n}\|\mathbf{x}\|_\infty.$$

Proof Let x_j be a coordinate of $\mathbf{x}$ such that $\|\mathbf{x}\|_\infty = \max_{1\leq i\leq n} |x_i| = |x_j|$. Then

$$\|\mathbf{x}\|_\infty^2 = |x_j|^2 = x_j^2 \leq \sum_{i=1}^{n} x_i^2 = \|\mathbf{x}\|_2^2,$$

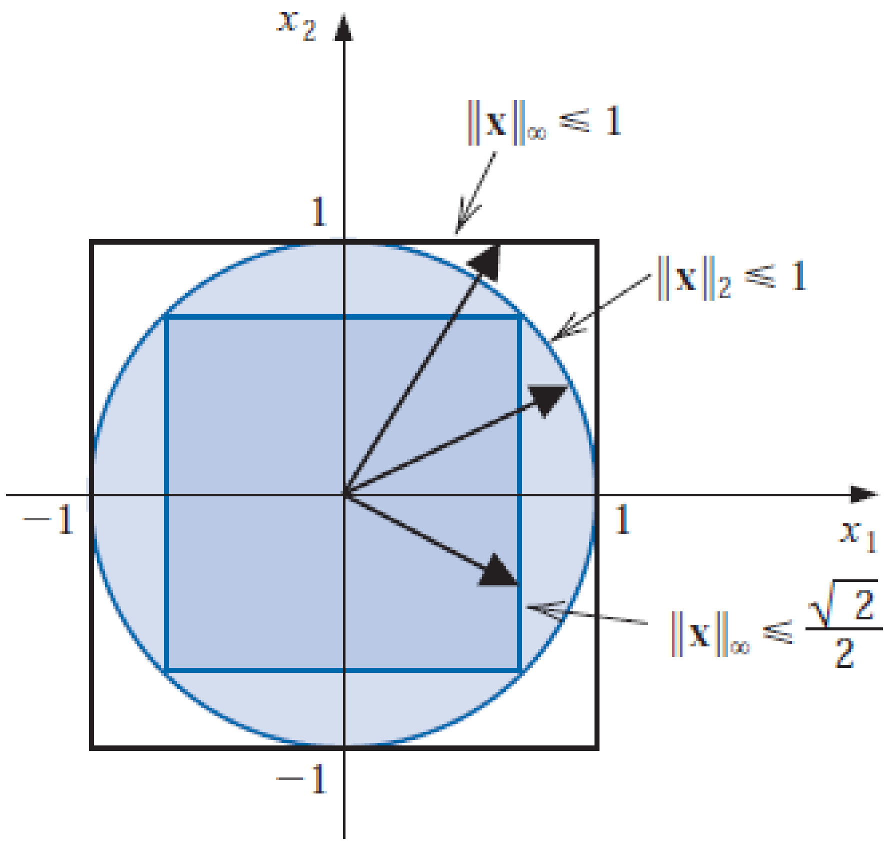
and

$$\|\mathbf{x}\|_\infty \leq \|\mathbf{x}\|_2.$$

So

$$\|\mathbf{x}\|_2^2 = \sum_{i=1}^{n} x_i^2 \leq \sum_{i=1}^{n} x_j^2 = nx_j^2 = n||\mathbf{x}||_\infty^2,$$

and $\|\mathbf{x}\|_2 \leq \sqrt{n}\|\mathbf{x}\|_\infty$.



It can be shown that all norms on $\mathbb{R}^n$ are equivalent with respect to convergence; that is, if $\|\cdot\|$ and $\|\cdot\|'$ are any two norms on $\mathbb{R}^n$ and $\{\mathbf{x}^{(k)}\}_{k=1}^{\infty}$ has the limit $\mathbf{x}$ with respect to $\|\cdot\|$, then $\{\mathbf{x}^{(k)}\}_{k=1}^{\infty}$ also has the limit $\mathbf{x}$ with respect to $\|\cdot\|'$. The proof of this fact for the general case can be found in [Or2], p. 8. The case for the l_2 and l_∞ norms follows from Theorem 7.7.

Matrix Norms and Distances

Definition 7.8 A **matrix norm** on the set of all $n \times n$ matrices is a real-valued function, $\|\cdot\|$, defined on this set, satisfying for all $n \times n$ matrices A and B and all real numbers α:

(i) $\|A\| \geq 0$;

(ii) $\|A\| = 0$, if and only if A is O, the matrix with all 0 entries;

(iii) $\|\alpha A\| = |\alpha|\|A\|$;

(iv) $\|A + B\| \leq \|A\| + \|B\|$;

(v) $\|AB\| \leq \|A\|\|B\|$. ■

The **distance between n x n matrices** A and B with respect to this matrix norm is $\|A - B\|$.

Although matrix norms can be obtained in various ways, the norms considered most frequently are those that are natural consequences of the vector norms l_2 and l_∞.

Theorem 7.9 If $||\cdot||$ is a vector norm on $\mathbb{R}^n$, then

$$\|A\| = \max_{\|\mathbf{x}\|=1} \|A\mathbf{x}\| \tag{7.2}$$

is a matrix norm. ■

Matrix norms defined by vector norms are called the **natural**, or *induced*, **matrix norm** associated with the vector norm. In this text, all matrix norms will be assumed to be natural matrix norms unless specified otherwise.

For any $\mathbf{z} \neq \mathbf{0}$, the vector $\mathbf{x} = \mathbf{z}/\|\mathbf{z}\|$ is a unit vector. Hence

$$\max_{\|\mathbf{x}\|=1} \|A\mathbf{x}\| = \max_{\mathbf{z}\neq\mathbf{0}} \left\| A\left(\frac{\mathbf{z}}{\|\mathbf{z}\|}\right) \right\| = \max_{\mathbf{z}\neq\mathbf{0}} \frac{\|A\mathbf{z}\|}{\|\mathbf{z}\|},$$

and we can alternatively write

$$\|A\| = \max_{\mathbf{z}\neq\mathbf{0}} \frac{\|A\mathbf{z}\|}{\|\mathbf{z}\|}. \tag{7.3}$$

The following corollary to Theorem 7.9 follows from this representation of $\|A\|$.

Corollary 7.10 For any vector $\mathbf{z} \neq \mathbf{0}$, matrix A, and any natural norm $\|\cdot\|$, we have

$$\|A\mathbf{z}\| \leq \|A\| \cdot \|\mathbf{z}\|.$$

■

An illustration of these norms when $n = 2$ is shown in Figures 7.4 and 7.5 for the matrix

$$A = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$$

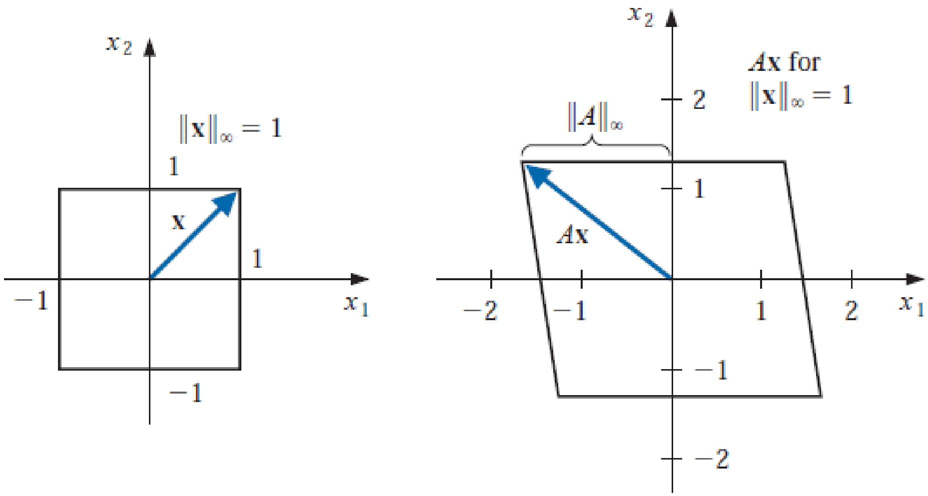


Figure 7.4

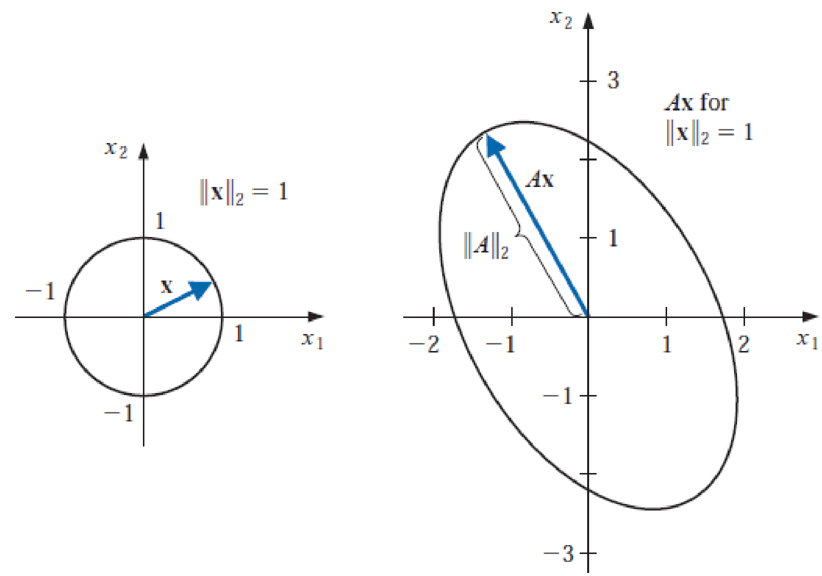


Figure 7.5

The l_∞ norm of a matrix can be easily computed from the entries of the matrix.

Theorem 7.11 If $A = (a_{ij})$ is an $n \times n$ matrix, then

$$\|A\|_\infty = \max_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}|.$$

Proof First we show that $\|A\|_\infty \le \max_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}|$.

Let $\mathbf{x}$ be an n-dimensional vector with $1 = \|\mathbf{x}\|_\infty = \max_{1 \le i \le n} |x_i|$. Since $A\mathbf{x}$ is also an n-dimensional vector,

$$\|A\mathbf{x}\|_\infty = \max_{1 \le i \le n} |(A\mathbf{x})_i| = \max_{1 \le i \le n} \left| \sum_{j=1}^{n} a_{ij} x_j \right| \le \max_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}| \max_{1 \le j \le n} |x_j|.$$

But $\max_{1 \le j \le n} |x_j| = \|\mathbf{x}\|_\infty = 1$, so

$$\|A\mathbf{x}\|_\infty \le \max_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}|,$$

and consequently,

$$\|A\|_\infty = \max_{\|\mathbf{x}\|_\infty = 1} \|A\mathbf{x}\|_\infty \le \max_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}|. \tag{7.4}$$

Now we will show the opposite inequality. Let p be an integer with

$$\sum_{j=1}^{n} |a_{pj}| = \max_{1\le i\le n} \sum_{j=1}^{n} |a_{ij}|,$$

and $\mathbf{x}$ be the vector with components

$$x_j = \begin{cases} 1, & \text{if } a_{pj} \ge 0, \\ -1, & \text{if } a_{pj} < 0. \end{cases}$$

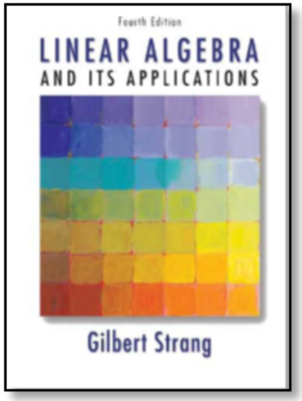
Then $\|\mathbf{x}\|_\infty = 1$ and $a_{pj}x_j = |a_{pj}|$, for all $j = 1, 2, \ldots, n$, so

$$\|A\mathbf{x}\|_\infty = \max_{1\le i\le n} \left| \sum_{j=1}^{n} a_{ij}x_j \right| \ge \left| \sum_{j=1}^{n} a_{pj}x_j \right| = \left| \sum_{j=1}^{n} |a_{pj}| \right| = \max_{1\le i\le n} \sum_{j=1}^{n} |a_{ij}|.$$

This result implies that

$$\|A\|_\infty = \max_{\|\mathbf{x}\|_\infty=1} \|A\mathbf{x}\|_\infty \ge \max_{1\le i\le n} \sum_{j=1}^{n} |a_{ij}|.$$

Putting this together with Inequality (7.4) gives $\|A\|_\infty = \max_{1\le i\le n} \sum_{j=1}^{n} |a_{ij}|$. ■ ■ ■



Chapter

5 Eigenvalues and Eigenvectors

5.1 INTRODUCTION

Eigenvalue equation $$Ax = \lambda x. \tag{8}$$

The Solutions of $Ax = \lambda x$

Notice that $Ax = \lambda x$ is a nonlinear equation; λ multiplies x. If we could discover λ, then the equation for x would be linear. In fact we could write $\lambda I x$ in place of λx, and bring this term over to the left side:

$$(A - \lambda I)x = 0. \tag{9}$$

The identity matrix keeps matrices and vectors straight; the equation $(A - \lambda)x = 0$ is shorter, but mixed up. This is the key to the problem:

The vector x is in the nullspace of $A - \lambda I$. ***The number λ is chosen so that $A - \lambda I$ has a nullspace.***

7.2 Eigenvalues and Eigenvectors

An $n \times m$ matrix can be considered as a function that uses matrix multiplication to take m-dimensional column vectors into n-dimensional column vectors. So an $n \times m$ matrix is actually a linear function from $\mathbb{R}^m$ to $\mathbb{R}^n$. A square matrix A takes the set of n-dimensional vectors into itself, which gives a linear function from $\mathbb{R}^n$ to $\mathbb{R}^n$. In this case, certain nonzero vectors $\mathbf{x}$ might be parallel to $A\mathbf{x}$, which means that a constant λ exists with $A\mathbf{x} = \lambda\mathbf{x}$. For these vectors, we have $(A - \lambda I)\mathbf{x} = \mathbf{0}$. There is a close connection between these numbers λ and the likelihood that an iterative method will converge. We will consider this connection in this section.

Definition 7.12 If A is a square matrix, the **characteristic polynomial** of A is defined by

$$p(\lambda) = \det(A - \lambda I).$$

■

Definition 7.13 If p is the characteristic polynomial of the matrix A, the zeros of p are **eigenvalues**, or characteristic values, of the matrix A. If λ is an eigenvalue of A and $\mathbf{x} \neq \mathbf{0}$ satisfies $(A - \lambda I)\mathbf{x} = \mathbf{0}$, then $\mathbf{x}$ is an **eigenvector**, or characteristic vector, of A corresponding to the eigenvalue λ. ■

To determine the eigenvalues of a matrix, we can use the fact that

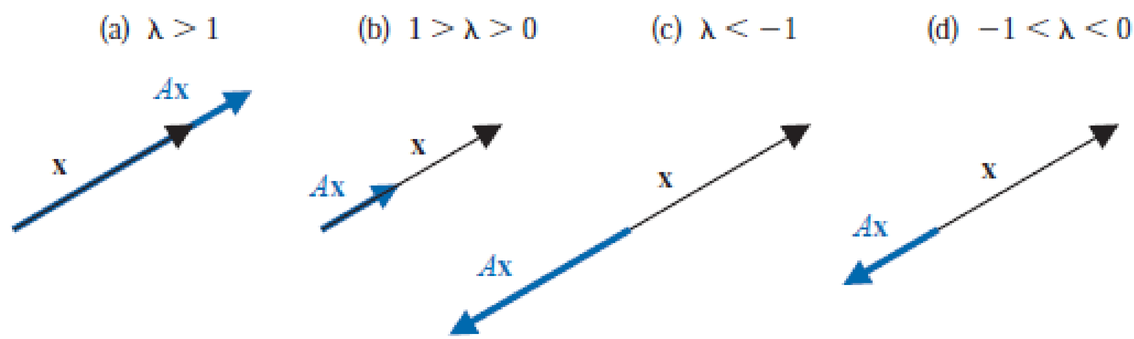
- λ is an eigenvalue of A if and only if $\det(A - \lambda I) = 0$.

Once an eigenvalue λ has been found a corresponding eigenvector $\mathbf{x} \neq \mathbf{0}$ is determined by solving the system

- $(A - \lambda I)\mathbf{x} = \mathbf{0}$.

If $\mathbf{x}$ is an eigenvector associated with the real eigenvalue λ, then $A\mathbf{x} = \lambda\mathbf{x}$, so the matrix A takes the vector $\mathbf{x}$ into a scalar multiple of itself.

- If λ is real and $\lambda > 1$, then A has the effect of stretching $\mathbf{x}$ by a factor of λ, as illustrated in Figure 7.6(a).
- If $0 < \lambda < 1$, then A shrinks $\mathbf{x}$ by a factor of λ (see Figure 7.6(b)).
- If $\lambda < 0$, the effects are similar (see Figure 7.6(c) and (d)), although the direction of $A\mathbf{x}$ is reversed.



$A\mathbf{x} = \lambda\mathbf{x}$

Notice also that if $\mathbf{x}$ is an eigenvector of A associated with the eigenvalue λ and α is any nonzero constant, then $\alpha\mathbf{x}$ is also an eigenvector since

$$A(\alpha\mathbf{x}) = \alpha(A\mathbf{x}) = \alpha(\lambda\mathbf{x}) = \lambda(\alpha\mathbf{x}).$$

An important consequence of this is that for any vector norm $||\cdot||$ we could choose the constant $\alpha = \pm||\mathbf{x}||^{-1}$, which would result in $\alpha\mathbf{x}$ being an eigenvector with norm 1. So

- For every eigenvalue and any vector norm there are eigenvectors with norm 1.

Example 2 Determine the eigenvalues and eigenvectors for the matrix

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 2 \\ 1 & -1 & 4 \end{bmatrix}.$$

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Solution The characteristic polynomial of A is

$$p(\lambda) = \det(A - \lambda I) = \det \begin{bmatrix} 2-\lambda & 0 & 0 \\ 1 & 1-\lambda & 2 \\ 1 & -1 & 4-\lambda \end{bmatrix}$$

$$= -(\lambda^3 - 7\lambda^2 + 16\lambda - 12) = -(\lambda - 3)(\lambda - 2)^2,$$

so there are two eigenvalues of A: $\lambda_1 = 3$ and $\lambda_2 = 2$.

An eigenvector $\mathbf{x}_1$ corresponding to the eigenvalue $\lambda_1 = 3$ is a solution to the vector-matrix equation $(A - 3 \cdot I)\mathbf{x}_1 = \mathbf{0}$, so

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -2 & 2 \\ 1 & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},$$

which implies that $x_1 = 0$ and $x_2 = x_3$.

Any nonzero value of x_3 produces an eigenvector for the eigenvalue $\lambda_1 = 3$. For example, when $x_3 = 1$ we have the eigenvector $\mathbf{x}_1 = (0, 1, 1)^t$, and any eigenvector of A corresponding to $\lambda = 3$ is a nonzero multiple of $\mathbf{x}_1$.

An eigenvector $\mathbf{x} \neq \mathbf{0}$ of A associated with $\lambda_2 = 2$ is a solution of the system $(A - 2 \cdot I)\mathbf{x} = \mathbf{0}$, so

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & -1 & 2 \\ 1 & -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

In this case the eigenvector has only to satisfy the equation

$$x_1 - x_2 + 2x_3 = 0,$$

which can be done in various ways. For example, when $x_1 = 0$ we have $x_2 = 2x_3$, so one choice would be $\mathbf{x}_2 = (0, 2, 1)^t$. We could also choose $x_2 = 0$, which requires that $x_1 = -2x_3$. Hence $\mathbf{x}_3 = (-2, 0, 1)^t$ gives a second eigenvector for the eigenvalue $\lambda_2 = 2$ that is not a multiple of $\mathbf{x}_2$. The eigenvectors of A corresponding to the eigenvalue $\lambda_2 = 2$ generate an entire plane. This plane is described by all vectors of the form

$$\alpha\mathbf{x}_2 + \beta\mathbf{x}_3 = (-2\beta, 2\alpha, \alpha + \beta)^t,$$

for arbitrary constants α and β, provided that at least one of the constants is nonzero. ■

Spectral Radius

Definition 7.14 The **spectral radius** $\rho(A)$ of a matrix A is defined by

$$\rho(A) = \max |\lambda|, \qquad \text{where } \lambda \text{ is an eigenvalue of } A.$$

(For complex $\lambda = \alpha + \beta i$, we define $|\lambda| = (\alpha^2 + \beta^2)^{1/2}$.) ■

Theorem 7.15 If A is an $n \times n$ matrix, then

(i) $\|A\|_2 = [\rho(A^t A)]^{1/2}$,

(ii) $\rho(A) \leq \|A\|$, for any natural norm $\|\cdot\|$. ■

Proof The proof of part (i) requires more information concerning eigenvalues than we presently have available. For the details involved in the proof, see [Or2], p. 21.

To prove part (ii), suppose λ is an eigenvalue of A with eigenvector $\mathbf{x}$ and $\|\mathbf{x}\| = 1$. Then $A\mathbf{x} = \lambda\mathbf{x}$ and

$$|\lambda| = |\lambda| \cdot \|\mathbf{x}\| = \|\lambda\mathbf{x}\| = \|A\mathbf{x}\| \leq \|A\| \|\mathbf{x}\| = \|A\|.$$

Thus

$$\rho(A) = \max |\lambda| \leq \|A\|.$$

■ ■ ■

Part (i) of Theorem 7.15 implies that if A is symmetric, then $\|A\|_2 = \rho(A)$ (see Exercise 14).

An interesting and useful result, which is similar to part (ii) of Theorem 7.15, is that for any matrix A and any $\varepsilon > 0$, there exists a natural norm $\|\cdot\|$ with the property that $\rho(A) < \|A\| < \rho(A) + \varepsilon$. Consequently, $\rho(A)$ is the greatest lower bound for the natural norms on A. The proof of this result can be found in [Or2], p. 23.

Example 3 Determine the l_2 norm of

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix}.$$

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Solution To apply Theorem 7.15 we need to calculate $\rho(A^tA)$, so we first need the eigenvalues of A^tA.

$$A^tA = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 2 & -1 \\ 2 & 6 & 4 \\ -1 & 4 & 5 \end{bmatrix}.$$

If

$$0 = \det(A^tA - \lambda I) = \det \begin{bmatrix} 3-\lambda & 2 & -1 \\ 2 & 6-\lambda & 4 \\ -1 & 4 & 5-\lambda \end{bmatrix}$$

$$= -\lambda^3 + 14\lambda^2 - 42\lambda = -\lambda(\lambda^2 - 14\lambda + 42),$$

then $\lambda = 0$ or $\lambda = 7 \pm \sqrt{7}$. By Theorem 7.15 we have

$$||A||_2 = \sqrt{\rho(A^tA)} = \sqrt{\max\{0, 7-\sqrt{7}, 7+\sqrt{7}\}} = \sqrt{7+\sqrt{7}} \approx 3.106. \quad ■$$

Convergent Matrices

In studying iterative matrix techniques, it is of particular importance to know when powers of a matrix become small (that is, when all the entries approach zero). Matrices of this type are called *convergent*.

Definition 7.16 We call an $n \times n$ matrix A **convergent** if

$$\lim_{k\to\infty} (A^k)_{ij} = 0, \quad \text{for each } i = 1, 2, \ldots, n \text{ and } j = 1, 2, \ldots, n.$$

Example 4 Show that

$$A = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{2} \end{bmatrix}$$

is a convergent matrix.

Solution Computing powers of A, we obtain:

$$A^2 = \begin{bmatrix} \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix}, \quad A^3 = \begin{bmatrix} \frac{1}{8} & 0 \\ \frac{3}{16} & \frac{1}{8} \end{bmatrix}, \quad A^4 = \begin{bmatrix} \frac{1}{16} & 0 \\ \frac{1}{8} & \frac{1}{16} \end{bmatrix},$$

and, in general,

$$A^k = \begin{bmatrix} (\frac{1}{2})^k & 0 \\ \frac{k}{2^{k+1}} & (\frac{1}{2})^k \end{bmatrix}.$$

So A is a convergent matrix because

$$\lim_{k\to\infty} \left(\frac{1}{2}\right)^k = 0 \quad \text{and} \quad \lim_{k\to\infty} \frac{k}{2^{k+1}} = 0. \qquad \blacksquare$$

Notice that the convergent matrix A in Example 4 has $\rho(A) = \frac{1}{2}$, because $\frac{1}{2}$ is the only eigenvalue of A. This illustrates an important connection that exists between the spectral radius of a matrix and the convergence of the matrix, as detailed in the following result.

Theorem 7.17 The following statements are equivalent.

(i) A is a convergent matrix.

(ii) $\lim_{n\to\infty} \|A^n\| = 0$, for some natural norm.

(iii) $\lim_{n\to\infty} \|A^n\| = 0$, for all natural norms.

(iv) $\rho(A) < 1$.

(v) $\lim_{n\to\infty} A^n\mathbf{x} = \mathbf{0}$, for every $\mathbf{x}$. ■

The proof of this theorem can be found in [IK], p. 14.

O princípio dos métodos iterativos é substituir o problema linear original:

A x = b

Por outro:

$$\mathbf{x}^{(i+1)} = T\mathbf{x}^{(i)} + \mathbf{c}.$$

onde na convergência as soluções são próximas dentro de uma tolerância especificada

Consequentemente para um método iterativo convergir a matriz T deve satisfazer a condições do teorema 7.17

EXERCISE SET 7.2

1. Compute the eigenvalues and associated eigenvectors of the following matrices.

a. $\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$

b. $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$

c. $\begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}$

d. $\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

e. $\begin{bmatrix} -1 & 2 & 0 \\ 0 & 3 & 4 \\ 0 & 0 & 7 \end{bmatrix}$

f. $\begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 1 & 1 & 2 \end{bmatrix}$

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