Numerical methods

Roots finding

Ricardo Afonso Angélico

April 8, 2019

Newton Raphson (single variable)

Taylor polynomial for $f(x)$ expanded about p_0 and evaluated at $x = p$

$$f(p) = f(p_0) + (p - p_0)f'(p_0) + \frac{(p - p_0)^2}{2} f''(\xi) \tag{1}$$

where $\xi \in [p_0, p]$.
Taking a first order approximation:

$$0 \approx f(p_0) + (p - p_0)f'(p_0) \tag{2}$$

$$p \approx p_0 - \frac{f(p_0)}{f'(p_0)} \tag{3}$$

For an iterative process, we have:

$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}, \quad n \geq 1 \tag{4}$$

Newton Raphson (single variable)

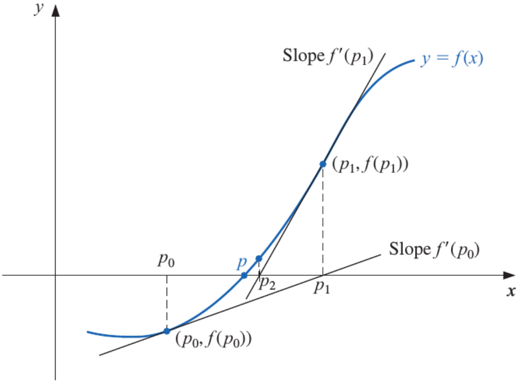

Figure: Representation of Newton Raphson method.

Newton-Raphson method (multivariate)

The method described for 1D functions can be generalized for a system of non-linear equations:

$$\begin{aligned} f_1(\mathbf{x}) &= 0 \\ f_2(\mathbf{x}) &= 0 \\ &\dots \\ f_N(\mathbf{x}) &= 0 \end{aligned}$$

where

$$\mathbf{x} = [x_1 \quad x_2 \quad \cdots x_N]^T \tag{5}$$

Defining a function vector:

$$\mathbf{f}(\mathbf{x}) = [f_1(\mathbf{x}) \quad f_2(\mathbf{x}) \quad \cdots \quad f_N(\mathbf{x})] \tag{6}$$

The system can be rewritten as:

$$\mathbf{f}(\mathbf{x}) = \mathbf{0} \tag{7}$$

Newton-Raphson method (multivariate)

Considering N = 2 (2D problem), the multidimensional equation can be geometrically interpreted as:

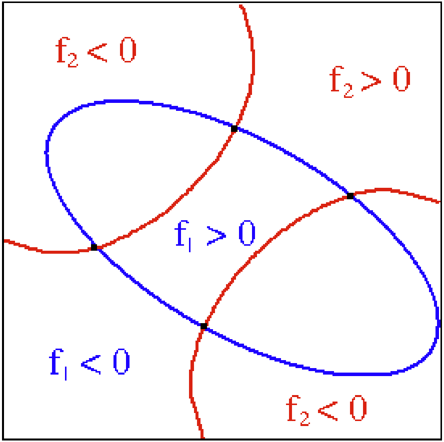

Figure: Visualization of the root finding problem in 2D.

Newton-Raphson method (multivariate)

The Taylor expansion for each function f_i can be written as:

$$f_i(\mathbf{x} + \delta\mathbf{x}) = f_i(\mathbf{x}) + \sum_{j=1}^{N} \frac{\partial f_i(x_j)}{\partial x_j} + O(\delta\mathbf{x}^2) \approx f_i(\mathbf{x}) + \sum_{j=1}^{N} \frac{\partial f_i(x_j)}{\partial x_j}\,\delta\mathbf{x} \quad (8)$$

In the vector form, the above equation can be written as:

$$\mathbf{f}(\mathbf{x} + \delta\mathbf{x}) = \mathbf{f}(\mathbf{x}) + \mathbf{J}(\mathbf{x})\,\delta\mathbf{x} \quad (9)$$

where $\mathbf{J}(\mathbf{x})$ is the *Jacobian matrix*, which is defined as:

$$\mathbf{J}(\mathbf{x}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_N} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_N}{\partial x_1} & \cdots & \frac{\partial f_N}{\partial x_N} \end{bmatrix} \quad (10)$$

Newton-Raphson method (multivariate)

Assuming $\mathbf{f}(\mathbf{x} + \delta\mathbf{x}) = \mathbf{0}$, the roots are $\mathbf{x} + \delta\mathbf{x}$, where $\delta\mathbf{x}$ can be obtained from:

$$\mathbf{f}(\mathbf{x} + \delta\mathbf{x}) = \mathbf{f}(\mathbf{x}) + \mathbf{J}(\mathbf{x})\,\delta\mathbf{x} \implies \tag{11}$$

$$\delta\mathbf{x} = \mathbf{J}(\mathbf{x})^{-1}[\mathbf{f}(\mathbf{x} + \delta\mathbf{x}) - \mathbf{f}(\mathbf{x})] = -\mathbf{J}(\mathbf{x})^{-1}\mathbf{f}(\mathbf{x}) \tag{12}$$

And, from an starting point $\mathbf{x}$:

$$\mathbf{x} + \delta\mathbf{x} = \mathbf{x} - \mathbf{J}(\mathbf{x})^{-1}\mathbf{f}(\mathbf{x}) \tag{13}$$

For nonlinear equations, the result above is only an approximation:

$$\mathbf{x}_{k+1} = \mathbf{x}_k + \delta\mathbf{x}_k = \mathbf{x}_k - \mathbf{J}(\mathbf{x}_k)^{-1}\mathbf{f}(\mathbf{x}_k) \tag{14}$$

Newton-Raphson method (multivariate)

Example 1

Computes the roots of:

$$\begin{cases} x_1^2 - 2x_1 + x_2 + 7 = 0 \\ 3x_1 - x_2 + 1 = 0 \end{cases} \tag{15}$$

Starting point: $\mathbf{x} = [1.00 \quad 1.00]^T$

Newton-Raphson method (multivariate)

Example 2

Determine the points of intersection between the circle $x^2 + y^2 = 3$ and the hyperbola $xy = 1$
Starting point: $x = 0.5; y = 1.5$

Solution: $\pm(0.618, 1.618)$ and $\pm(1.618, 0.618)$

Newton-Raphson method (multivariate)

Example 3

Computes the roots of:

$$\begin{cases} 3x_1 - \cos(x_2 x_3) - 3/2 = 0 \\ 4x_1^2 - 625x_2^2 + 2x_3 - 1 = 0 \\ 20x_3 + \exp(-x_1\, x_2) + 9 \end{cases} \tag{16}$$

Starting point: $\mathbf{x} = [1.00 \quad 1.00 \quad 1.00]^T$

Solution: $x = [0.833282 \quad 0.035335 \quad -0.498549]^T$

Exercises

Exercise 1

- The natural frequencies of a uniform cantilever beam are related to the roots β_i of the frequency equation $f(\beta) = \cosh\beta\cos\beta + 1 = 0$, where

$$\beta_i^4 = (2\pi f_i)^2 \frac{mL^3}{EI}$$

$$f_i = i\text{th natural frequency (cps)}$$

$$m = \text{mass of the beam}$$

$$L = \text{length of the beam}$$

$$E = \text{modulus of elasticity}$$

$$I = \text{moment of inertia of the cross section}$$

Determine the lowest two frequencies of a steel beam 0.9 m. long, with a rectangular cross section 25 mm wide and 2.5 mm in. high. The mass density of steel is 7850 $\mathrm{kg/m^3}$ and $E = 200$ GPa.

Exercises

Exercise 2

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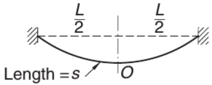


A steel cable of length s is suspended as shown in the figure. The maximum tensile stress in the cable, which occurs at the supports, is

$$\sigma_{max} = \sigma_0 \cosh \beta$$

where

$$\beta = \frac{\gamma L}{2\sigma_0}$$

$\sigma_0 =$ tensile stress in the cable at O

$\gamma =$ weight of the cable per unit volume

$L =$ horizontal span of the cable

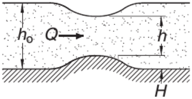
The length to span ratio of the cable is related to β by

$$\frac{s}{L} = \frac{1}{\beta} \sinh \beta$$

Find σ_{max} if $\gamma = 77 \times 10^3$ N/m^3 (steel), $L = 1000$ m and $s = 1100$ m.

Exercises

Exercise 3

- 

Bernoulli's equation for fluid flow in an open channel with a small bump is

$$\frac{Q^2}{2gb^2h_0^2} + h_0 = \frac{Q^2}{2gb^2h^2} + h + H$$

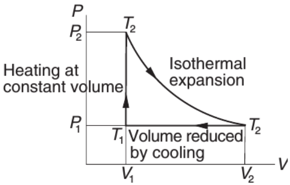
where

$$\begin{aligned} Q &= 1.2\ \mathrm{m^3/s} = \text{volume rate of flow} \\ g &= 9.81\ \mathrm{m/s^2} = \text{gravitational acceleration} \\ b &= 1.8\ \mathrm{m} = \text{width of channel} \\ h_0 &= 0.6\ \mathrm{m} = \text{upstream water level} \\ H &= 0.075\ \mathrm{m} = \text{height of bump} \\ h &= \text{water level above the bump} \end{aligned}$$

Determine h.

Exercises

Exercise 4

- 

The figure shows the thermodynamic cycle of an engine. The efficiency of this engine for monoatomic gas is

$$\eta = \frac{\ln(T_2/T_1) - (1 - T_1/T_2)}{\ln(T_2/T_1) + (1 - T_1/T_2)/(\gamma - 1)}$$

where T is the absolute temperature and $\gamma = 5/3$. Find T_2/T_1 that results in 30% efficiency ($\eta = 0.3$).

Exercises

Exercise 5

- The equations

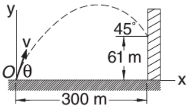
$$\sin x + 3\cos x - 2 = 0$$

$$\cos x - \sin y + 0.2 = 0$$

have a solution in the vicinity of the point $(1, 1)$. Use the Newton–Raphson method to refine the solution.

Exercises

Exercise 6

- 

A projectile is launched at O with the velocity v at the angle θ to the horizontal. The parametric equations of the trajectory are

$$x = (v\cos\theta)t$$
$$y = -\frac{1}{2}gt^2 + (v\sin\theta)t$$

where t is the time measured from the instant of launch, and $g = 9.81\ \mathrm{m/s^2}$ represents the gravitational acceleration. If the projectile is to hit the target at the 45° angle shown in the figure, determine v, θ and the time of flight.

Exercises

Exercise 7

- The equation of a circle is

$$(x-a)^2 + (y-b)^2 = R^2$$

where R is the radius and (a, b) are the coordinates of the center. If the coordinates of three points on the circle are

x	8.21	0.34	5.96
y	0.00	6.62	-1.12

determine R, a and b.