MAP2210 – aula 07

MAP 2210 – Aplicações de Álgebra Linear
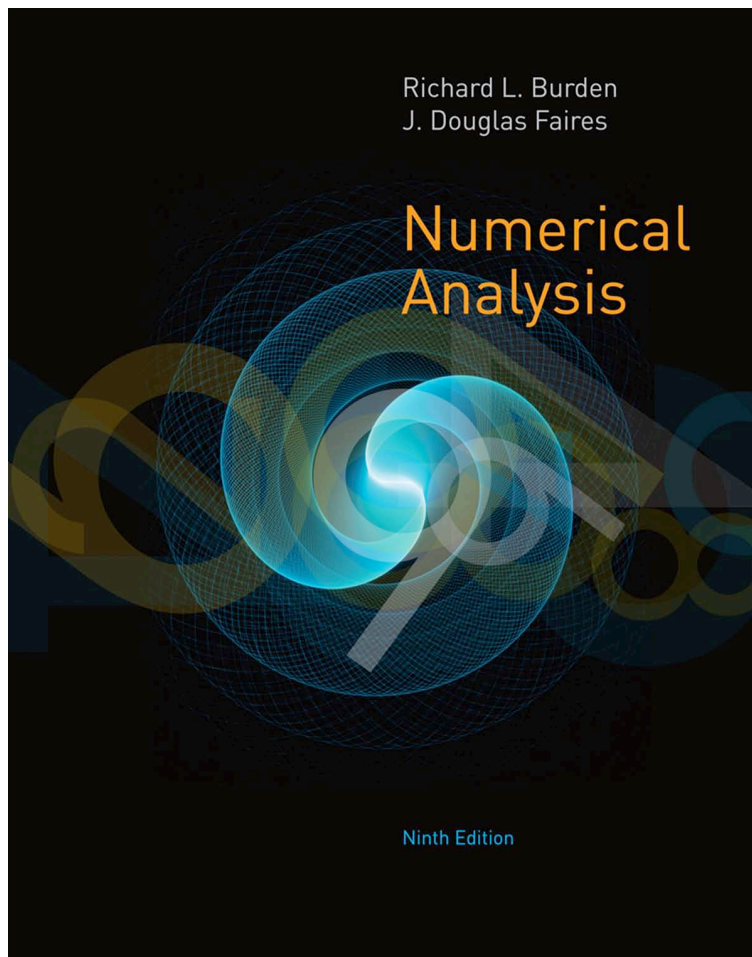# 1º Semestre - 2019

Prof. Dr. Luis Carlos de Castro Santos

lsantos@ime.usp.br

Objetivos

Formação básica de álgebra linear aplicada a problemas numéricos. Resolução de problemas em microcomputadores usando linguagens e/ou software adequados fora do horário de aula.

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Richard L. Burden
J. Douglas Faires

Numerical
Analysis

Ninth Edition

Numerical Analysis

NINTH EDITION

Richard L. Burden
Youngstown State University

J. Douglas Faires
Youngstown State University

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6 Direct Methods for Solving Linear Systems 357

7 Iterative Techniques in Matrix Algebra 431

9 Approximating Eigenvalues 561

+

Fourth Edition

LINEAR ALGEBRA
AND ITS APPLICATIONS

Gilbert Strang

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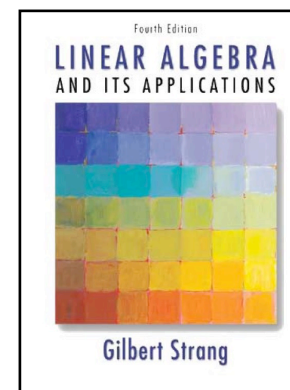
6.6 Special Types of Matrices

We now turn attention to two classes of matrices for which Gaussian elimination can be performed effectively without row interchanges.

Corollary 6.29   Let A be a symmetric $n \times n$ matrix for which Gaussian elimination can be applied without row interchanges. Then A can be factored into LDL^t, where L is lower triangular with 1s on its diagonal and D is the diagonal matrix with $a_{11}^{(1)}, \ldots, a_{nn}^{(n)}$ on its diagonal. ■

Example 3   Determine the LDL^t factorization of the positive definite matrix

$$A = \begin{bmatrix} 4 & -1 & 1 \\ -1 & 4.25 & 2.75 \\ 1 & 2.75 & 3.5 \end{bmatrix}.$$

$$A = LDL^t = \begin{bmatrix} 1 & 0 & 0 \\ -0.25 & 1 & 0 \\ 0.25 & 0.75 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -0.25 & 0.25 \\ 0 & 1 & 0.75 \\ 0 & 0 & 1 \end{bmatrix}. \quad ■$$

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Algorithm 6.5 is easily modified to factor the symmetric matrices described in Corollary 6.29. It simply requires adding a check to ensure that the diagonal elements are nonzero. The Cholesky Algorithm 6.6 produces the LL^t factorization described in Corollary 6.28.

Corollary 6.28   The matrix A is positive definite if and only if A can be factored in the form LL^t, where L is lower triangular with nonzero diagonal entries.   ■

Example 4   Determine the Cholesky LL^t factorization of the positive definite matrix

$$A = \begin{bmatrix} 4 & -1 & 1 \\ -1 & 4.25 & 2.75 \\ 1 & 2.75 & 3.5 \end{bmatrix}.$$

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Example 4

Determine the Cholesky LL^t factorization of the positive definite matrix

$$A = \begin{bmatrix} 4 & -1 & 1 \\ -1 & 4.25 & 2.75 \\ 1 & 2.75 & 3.5 \end{bmatrix}.$$

Solution The LL^t factorization does not necessarily has 1s on the diagonal of the lower triangular matrix L so we need to have

$$A = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{21} & a_{22} & a_{32} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix}$$

$$= \begin{bmatrix} l_{11}^2 & l_{11}l_{21} & l_{11}l_{31} \\ l_{11}l_{21} & l_{21}^2 + l_{22}^2 & l_{21}l_{31} + l_{22}l_{32} \\ l_{11}l_{31} & l_{21}l_{31} + l_{22}l_{32} & l_{31}^2 + l_{32}^2 + l_{33}^2 \end{bmatrix}$$

Thus

$$a_{11}: \quad 4 = l_{11}^2 \implies l_{11} = 2, \qquad\qquad a_{21}: \quad -1 = l_{11}l_{21} \implies l_{21} = -0.5$$

$$a_{31}: \quad 1 = l_{11}l_{31} \implies l_{31} = 0.5, \qquad\quad a_{22}: \quad 4.25 = l_{21}^2 + l_{22}^2 \implies l_{22} = 2$$

$$a_{32}: \quad 2.75 = l_{21}l_{31} + l_{22}l_{32} \implies l_{32} = 1.5, \quad a_{33}: \quad 3.5 = l_{31}^2 + l_{32}^2 + l_{33}^2 \implies l_{33} = 1,$$

and we have

$$A = LL^t = \begin{bmatrix} 2 & 0 & 0 \\ -0.5 & 2 & 0 \\ 0.5 & 1.5 & 1 \end{bmatrix} \begin{bmatrix} 2 & -0.5 & 0.5 \\ 0 & 2 & 1.5 \\ 0 & 0 & 1 \end{bmatrix}.$$ ∎

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ALGORITHM
6.6

Cholesky

To factor the positive definite $n \times n$ matrix A into LL^t, where L is lower triangular:

INPUT the dimension n; entries a_{ij}, for $1 \le i, j \le n$ of A.

OUTPUT the entries l_{ij}, for $1 \le j \le i$ and $1 \le i \le n$ of L. (*The entries of $U = L^t$ are* $u_{ij} = l_{ji}$, *for $i \le j \le n$ and $1 \le i \le n$.*)

Step 1 Set $l_{11} = \sqrt{a_{11}}$.

Step 2 For $j = 2, \ldots, n$, set $l_{j1} = a_{j1}/l_{11}$.

Step 3 For $i = 2, \ldots, n-1$ do Steps 4 and 5.

> *Step 4* Set $l_{ii} = \left(a_{ii} - \sum_{k=1}^{i-1} l_{ik}^2 \right)^{1/2}$.
>
> *Step 5* For $j = i+1, \ldots, n$
>
> set $l_{ji} = \left(a_{ji} - \sum_{k=1}^{i-1} l_{jk} l_{ik} \right) / l_{ii}$.

Step 6 Set $l_{nn} = \left(a_{nn} - \sum_{k=1}^{n-1} l_{nk}^2 \right)^{1/2}$.

Step 7 OUTPUT (l_{ij} for $j = 1, \ldots, i$ and $i = 1, \ldots, n$);
STOP.

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The *LDL*^{t} factorization described in Algorithm 6.5 requires

$$\frac{1}{6}n^3 + n^2 - \frac{7}{6}n \text{ multiplications/divisions} \quad \text{and} \quad \frac{1}{6}n^3 - \frac{1}{6}n \text{ additions/subtractions.}$$

The *LL*^{t} Cholesky factorization of a positive definite matrix requires only

$$\frac{1}{6}n^3 + \frac{1}{2}n^2 - \frac{2}{3}n \text{ multiplications/divisions} \quad \text{and} \quad \frac{1}{6}n^3 - \frac{1}{6}n \text{ additions/subtractions.}$$

This computational advantage of Cholesky's factorization is misleading, because it requires extracting n square roots. However, the number of operations required for computing the n square roots is a linear factor of n and will decrease in significance as n increases.

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If the Cholesky factorization given in Algorithm 6.6 is preferred, the additional steps for solving the system $A\mathbf{x} = \mathbf{b}$ are as follows. First delete the STOP statement from Step 7. Then add

Step 8   Set $y_1 = b_1/l_{11}$.

Step 9   For $i = 2, \ldots, n$ set $y_i = \left(b_i - \sum_{j=1}^{i-1} l_{ij} y_j \right) \Big/ l_{ii}$.

Step 10   Set $x_n = y_n/l_{nn}$.

Step 11   For $i = n-1, \ldots, 1$ set $x_i = \left(y_i - \sum_{j=i+1}^{n} l_{ji} x_j \right) \Big/ l_{ii}$.

Step 12   OUTPUT (x_i for $i = 1, \ldots, n$);
          STOP.

Steps 8–12 require $n^2 + n$ multiplications/divisions and $n^2 - n$ additions/ subtractions.

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9. Modify the Cholesky Algorithm as suggested in the text so that it can be used to solve linear systems, and use the modified algorithm to solve the linear systems in Exercise 7.

b. $4x_1 + x_2 + x_3 + x_4 = 0.65,$

$x_1 + 3x_2 - x_3 + x_4 = 0.05,$

$x_1 - x_2 + 2x_3 \qquad = 0,$

$x_1 + x_2 \qquad + 2x_4 = 0.5.$

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Input:

Cholesky decomposition	$\begin{pmatrix} 4 & 1 & 1 & 1 \\ 1 & 3 & -1 & 1 \\ 1 & -1 & 2 & 0 \\ 1 & 1 & 0 & 2 \end{pmatrix}$

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Result:

$$A = L^\dagger . L$$

where

$$A = \begin{pmatrix} 4 & 1 & 1 & 1 \\ 1 & 3 & -1 & 1 \\ 1 & -1 & 2 & 0 \\ 1 & 1 & 0 & 2 \end{pmatrix}$$

$$L = \begin{pmatrix} 2 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{\sqrt{11}}{2} & -\frac{5}{2\sqrt{11}} & \frac{3}{2\sqrt{11}} \\ 0 & 0 & \sqrt{\frac{13}{11}} & \frac{1}{\sqrt{143}} \\ 0 & 0 & 0 & 2\sqrt{\frac{5}{13}} \end{pmatrix}$$

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Band Matrices

The last class of matrices considered are *band matrices*. In many applications, the band matrices are also strictly diagonally dominant or positive definite.

Definition 6.30

An $n \times n$ matrix is called a **band matrix** if integers p and q, with $1 < p, q < n$, exist with the property that $a_{ij} = 0$ whenever $p \le j - i$ or $q \le i - j$. The **band width** of a band matrix is defined as $w = p + q - 1$. ∎

The number p describes the number of diagonals above, and including, the main diagonal on which nonzero entries may lie. The number q describes the number of diagonals below, and including, the main diagonal on which nonzero entries may lie. For example, the matrix

$$A = \begin{bmatrix} 7 & 2 & 0 \\ 3 & 5 & -1 \\ 0 & -5 & -6 \end{bmatrix}$$

is a band matrix with $p = q = 2$ and bandwidth $2 + 2 - 1 = 3$.

The definition of band matrix forces those matrices to concentrate all their nonzero entries about the diagonal. Two special cases of band matrices that occur frequently have $p = q = 2$ and $p = q = 4$.

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Tridiagonal Matrices

Matrices of bandwidth 3 occurring when $p = q = 2$ are called **tridiagonal** because they have the form

$$A = \begin{bmatrix} a_{11} & a_{12} & 0 & \cdots & \cdots & \cdots & 0 \\ a_{21} & a_{22} & a_{23} & & & & \vdots \\ 0 & a_{32} & a_{33} & a_{34} & & & \vdots \\ \vdots & & & & & & 0 \\ \vdots & & & & & & a_{n-1,n} \\ 0 & \cdots & \cdots & 0 & & a_{n,n-1} & a_{nn} \end{bmatrix}.$$

The factorization algorithms can be simplified considerably in the case of band matrices because a large number of zeros appear in these matrices in regular patterns. It is particularly interesting to observe the form the Crout or Doolittle method assumes in this case.

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To illustrate the situation, suppose a tridiagonal matrix A can be factored into the triangular matrices L and U. Then A has at most $(3n - 2)$ nonzero entries. Then there are only $(3n - 2)$ conditions to be applied to determine the entries of L and U, provided, of course, that the zero entries of A are also obtained.

Suppose that the matrices L and U also have tridiagonal form, that is,

$$
L = \begin{bmatrix}
l_{11} & 0 & \cdots & \cdots & \cdots & 0 \\
l_{21} & l_{22} & & & & \vdots \\
0 & & \ddots & & & \vdots \\
\vdots & & & \ddots & & 0 \\
0 & \cdots & \cdots & 0 & l_{n,n-1} & l_{nn}
\end{bmatrix}
\quad \text{and} \quad
U = \begin{bmatrix}
1 & u_{12} & 0 & \cdots & \cdots & 0 \\
0 & 1 & & & & \vdots \\
\vdots & & \ddots & & & 0 \\
\vdots & & & \ddots & & u_{n-1,n} \\
0 & \cdots & \cdots & 0 & & 1
\end{bmatrix}.
$$

There are $(2n - 1)$ undetermined entries of L and $(n - 1)$ undetermined entries of U, which totals $(3n - 2)$, the number of possible nonzero entries of A. The 0 entries of A are obtained automatically.

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The multiplication involved with $A = LU$ gives, in addition to the 0 entries,

$$a_{11} = l_{11};$$

$$a_{i,i-1} = l_{i,i-1}, \quad \text{for each } i = 2, 3, \ldots, n; \tag{6.13}$$

$$a_{ii} = l_{i,i-1}u_{i-1,i} + l_{ii}, \quad \text{for each } i = 2, 3, \ldots, n; \tag{6.14}$$

and

$$a_{i,i+1} = l_{ii}u_{i,i+1}, \quad \text{for each } i = 1, 2, \ldots, n - 1. \tag{6.15}$$

A solution to this system is found by first using Eq. (6.13) to obtain all the nonzero off-diagonal terms in L and then using Eqs. (6.14) and (6.15) to alternately obtain the remainder of the entries in U and L. Once an entry L or U is computed, the corresponding entry in A is not needed. So the entries in A can be overwritten by the entries in L and U with the result that no new storage is required.

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Example 5

Determine the Crout factorization of the symmetric tridiagonal matrix

$$
\begin{bmatrix}
2 & -1 & 0 & 0 \\
-1 & 2 & -1 & 0 \\
0 & -1 & 2 & -1 \\
0 & 0 & -1 & 2
\end{bmatrix},
$$

and use this factorization to solve the linear system

$$
\begin{aligned}
2x_1 - x_2 &= 1, \\
-x_1 + 2x_2 - x_3 &= 0, \\
- x_2 + 2x_3 - x_4 &= 0, \\
- x_3 + 2x_4 &= 1.
\end{aligned}
$$

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Solution The *LU* factorization of *A* has the form

$$
A = \begin{bmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & 0 & a_{43} & a_{44} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 & 0 \\ l_{21} & l_{22} & 0 & 0 \\ 0 & l_{32} & l_{33} & 0 \\ 0 & 0 & l_{43} & l_{44} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & 0 & 0 \\ 0 & 1 & u_{23} & 0 \\ 0 & 0 & 1 & u_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
= \begin{bmatrix} l_{11} & l_{11}u_{12} & 0 & 0 \\ l_{21} & l_{22} + l_{21}u_{12} & l_{22}u_{23} & 0 \\ 0 & l_{32} & l_{33} + l_{32}u_{23} & l_{33}u_{34} \\ 0 & 0 & l_{43} & l_{44} + l_{43}u_{34} \end{bmatrix}.
$$

Thus

$$a_{11}: \quad 2 = l_{11} \implies l_{11} = 2, \qquad\qquad a_{12}: \quad -1 = l_{11}u_{12} \implies u_{12} = -\tfrac{1}{2},$$

$$a_{21}: \quad -1 = l_{21} \implies l_{21} = -1, \qquad a_{22}: \quad 2 = l_{22} + l_{21}u_{12} \implies l_{22} = -\tfrac{3}{2},$$

$$a_{23}: \quad -1 = l_{22}u_{23} \implies u_{23} = -\tfrac{2}{3}, \qquad a_{32}: \quad -1 = l_{32} \implies l_{32} = -1,$$

$$a_{33}: \quad 2 = l_{33} + l_{32}u_{23} \implies l_{33} = \tfrac{4}{3}, \qquad a_{34}: \quad -1 = l_{33}u_{34} \implies u_{34} = -\tfrac{3}{4},$$

$$a_{43}: \quad -1 = l_{43} \implies l_{43} = -1, \qquad a_{44}: \quad 2 = l_{44} + l_{43}u_{34} \implies l_{44} = \tfrac{5}{4}.$$

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This gives the Crout factorization

$$
A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ -1 & \frac{3}{2} & 0 & 0 \\ 0 & -1 & \frac{4}{3} & 0 \\ 0 & 0 & -1 & \frac{5}{4} \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2} & 0 & 0 \\ 0 & 1 & -\frac{2}{3} & 0 \\ 0 & 0 & 1 & -\frac{3}{4} \\ 0 & 0 & 0 & 1 \end{bmatrix} = LU.
$$

Solving the system

$$
L\mathbf{z} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ -1 & \frac{3}{2} & 0 & 0 \\ 0 & -1 & \frac{4}{3} & 0 \\ 0 & 0 & -1 & \frac{5}{4} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{gives} \quad \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{3} \\ \frac{1}{4} \\ 1 \end{bmatrix},
$$

and then solving

$$
U\mathbf{x} = \begin{bmatrix} 1 & -\frac{1}{2} & 0 & 0 \\ 0 & 1 & -\frac{2}{3} & 0 \\ 0 & 0 & 1 & -\frac{3}{4} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{3} \\ \frac{1}{4} \\ 1 \end{bmatrix} \quad \text{gives} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}. \quad \blacksquare
$$

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27. Tridiagonal matrices are usually labeled by using the notation

$$A = \begin{bmatrix} a_1 & c_1 & 0 & \cdots & \cdots & 0 \\ b_2 & a_2 & c_2 & & & \vdots \\ 0 & b_3 & & & & 0 \\ \vdots & & & & & c_{n-1} \\ 0 & \cdots & \cdots & 0 & b_n & a_n \end{bmatrix}$$

to emphasize that it is not necessary to consider all the matrix entries. Rewrite the Crout Factorization Algorithm using this notation, and change the notation of the l_{ij} and u_{ij} in a similar manner.

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12. Use Crout factorization for tridiagonal systems to solve the following linear systems.

b.
$$2x_1 - x_2 \qquad = 5,$$
$$-x_1 + 3x_2 + x_3 = 4,$$
$$x_2 + 4x_3 = 0.$$

c.
$$2x_1 - x_2 \qquad = 3,$$
$$x_1 + 2x_2 - x_3 \qquad = 4,$$
$$x_2 - 2x_3 + x_4 = 0,$$
$$x_3 + 2x_4 = 6.$$

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