MAP 2210 – Aplicações de Álgebra Linear

1º Semestre - 2019

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Objetivos

Formação básica de álgebra linear aplicada a problemas numéricos. Resolução de problemas em microcomputadores usando linguagens e/ou software adequados fora do horário de aula.

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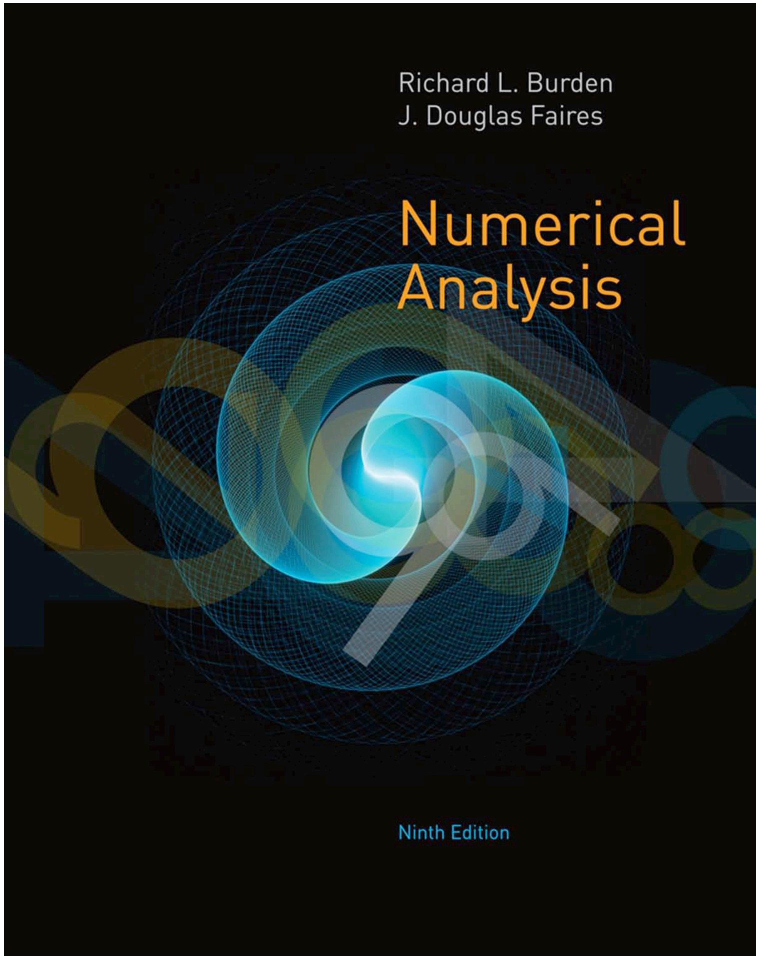

Numerical Analysis

NINTH EDITION

Richard L. Burden
Youngstown State University

J. Douglas Faires
Youngstown State University

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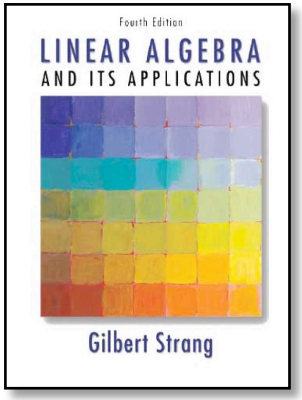
Fourth Edition
LINEAR ALGEBRA
AND ITS APPLICATIONS
Gilbert Strang

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6.6 Special Types of Matrices

We now turn attention to two classes of matrices for which Gaussian elimination can be performed effectively without row interchanges.

Diagonally Dominant Matrices

The first class is described in the following definition.

Definition 6.20 The $n \times n$ matrix A is said to be **diagonally dominant** when

$$|a_{ii}| \geq \sum_{\substack{j=1,\\ j\neq i}}^{n} |a_{ij}| \quad \text{holds for each } i = 1, 2, \cdots, n. \tag{6.10}$$

A diagonally dominant matrix is said to be **strictly diagonally dominant** when the inequality in (6.10) is strict for each n, that is, when

$$|a_{ii}| > \sum_{\substack{j=1,\\ j\neq i}}^{n} |a_{ij}| \quad \text{holds for each } i = 1, 2, \cdots, n. \qquad \blacksquare$$

Illustration

Consider the matrices

$$A = \begin{bmatrix} 7 & 2 & 0 \\ 3 & 5 & -1 \\ 0 & 5 & -6 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 6 & 4 & -3 \\ 4 & -2 & 0 \\ -3 & 0 & 1 \end{bmatrix}.$$

The nonsymmetric matrix A is strictly diagonally dominant because

$$|7| > |2| + |0|, \quad |5| > |3| + |-1|, \quad \text{and} \quad |-6| > |0| + |5|.$$

The symmetric matrix B is not strictly diagonally dominant because, for example, in the first row the absolute value of the diagonal element is $|6| < |4| + |-3| = 7$. It is interesting to note that A^t is not strictly diagonally dominant, because the middle row of A^t is $[2\ 5\ 5]$, nor, of course, is B^t because $B^t = B$. $\square$

Theorem 6.21

A strictly diagonally dominant matrix A is nonsingular. Moreover, in this case, Gaussian elimination can be performed on any linear system of the form $A\mathbf{x} = \mathbf{b}$ to obtain its unique solution without row or column interchanges, and the computations will be stable with respect to the growth of round-off errors. ■

Proof We first use proof by contradiction to show that A is nonsingular. Consider the linear system described by $A\mathbf{x} = \mathbf{0}$, and suppose that a nonzero solution $\mathbf{x} = (x_i)$ to this system exists. Let k be an index for which

$$0 < |x_k| = \max_{1 \le j \le n} |x_j|.$$

Because $\sum_{j=1}^{n} a_{ij}x_j = 0$ for each $i = 1, 2, \ldots, n$, we have, when $i = k$,

$$a_{kk}x_k = -\sum_{\substack{j=1,\\ j\neq k}}^{n} a_{kj}x_j.$$

From the triangle inequality we have

$$|a_{kk}||x_k| \le \sum_{\substack{j=1,\\ j\neq k}}^{n} |a_{kj}||x_j|, \quad \text{so} \quad |a_{kk}| \le \sum_{\substack{j=1,\\ j\neq k}}^{n} |a_{kj}|\frac{|x_j|}{|x_k|} \le \sum_{\substack{j=1,\\ j\neq k}}^{n} |a_{kj}|.$$

This inequality contradicts the strict diagonal dominance of A. Consequently, the only solution to $A\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$. This is shown in Theorem 6.17 on page 398 to be equivalent to the nonsingularity of A.

To prove that Gaussian elimination can be performed without row interchanges, we show that each of the matrices $A^{(2)}, A^{(3)}, \ldots, A^{(n)}$ generated by the Gaussian elimination process (and described in Section 6.5) is strictly diagonally dominant. This will ensure that at each stage of the Gaussian elimination process the pivot element is nonzero.

Since A is strictly diagonally dominant, $a_{11} \neq 0$ and $A^{(2)}$ can be formed. Thus for each $i = 2, 3, \ldots, n$,

$$a_{ij}^{(2)} = a_{ij}^{(1)} - \frac{a_{1j}^{(1)} a_{i1}^{(1)}}{a_{11}^{(1)}}, \quad \text{for} \quad 2 \leq j \leq n.$$

First, $a_{i1}^{(2)} = 0$. The triangle inequality implies that

$$\sum_{\substack{j=2 \\ j \neq i}}^{n} |a_{ij}^{(2)}| = \sum_{\substack{j=2 \\ j \neq i}}^{n} \left| a_{ij}^{(1)} - \frac{a_{1j}^{(1)} a_{i1}^{(1)}}{a_{11}^{(1)}} \right| \leq \sum_{\substack{j=2 \\ j \neq i}}^{n} |a_{ij}^{(1)}| + \sum_{\substack{j=2 \\ j \neq i}}^{n} \left| \frac{a_{1j}^{(1)} a_{i1}^{(1)}}{a_{11}^{(1)}} \right|.$$

But since A is strictly diagonally dominant,

$$\sum_{\substack{j=2 \\ j \neq i}}^{n} |a_{ij}^{(1)}| < |a_{ii}^{(1)}| - |a_{i1}^{(1)}| \quad \text{and} \quad \sum_{\substack{j=2 \\ j \neq i}}^{n} |a_{1j}^{(1)}| < |a_{11}^{(1)}| - |a_{1i}^{(1)}|,$$

so

$$\sum_{\substack{j=2\\ j\neq i}}^{n} |a_{ij}^{(2)}| < |a_{ii}^{(1)}| - |a_{i1}^{(1)}| + \frac{|a_{i1}^{(1)}|}{|a_{11}^{(1)}|}(|a_{11}^{(1)}| - |a_{1i}^{(1)}|) = |a_{ii}^{(1)}| - \frac{|a_{i1}^{(1)}||a_{1i}^{(1)}|}{|a_{11}^{(1)}|}.$$

The triangle inequality also implies that

$$|a_{ii}^{(1)}| - \frac{|a_{i1}^{(1)}||a_{1i}^{(1)}|}{|a_{11}^{(1)}|} \leq \left| a_{ii}^{(1)} - \frac{|a_{i1}^{(1)}||a_{1i}^{(1)}|}{|a_{11}^{(1)}|} \right| = |a_{ii}^{(2)}|.$$

which gives

$$\sum_{\substack{j=2\\ j\neq i}}^{n} |a_{ij}^{(2)}| < |a_{ii}^{(2)}|.$$

This establishes the strict diagonal dominance for rows $2, \ldots, n$. But the first row of $A^{(2)}$ and A are the same, so $A^{(2)}$ is strictly diagonally dominant.

This process is continued inductively until the upper-triangular and strictly diagonally dominant $A^{(n)}$ is obtained. This implies that all the diagonal elements are nonzero, so Gaussian elimination can be performed without row interchanges.

The demonstration of stability for this procedure can be found in [We]. ■ ■ ■

Positive Definite Matrices

The next special class of matrices is called *positive definite*.

Definition 6.22

A matrix A is **positive definite** if it is symmetric and if $\mathbf{x}^t A\mathbf{x} > 0$ for every n-dimensional vector $\mathbf{x} \neq \mathbf{0}$. ■

To be precise, Definition 6.22 should specify that the 1×1 matrix generated by the operation $\mathbf{x}^t A\mathbf{x}$ has a positive value for its only entry since the operation is performed as follows:

$$\mathbf{x}^t A\mathbf{x} = [x_1, x_2, \cdots, x_n] \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$= [x_1, x_2, \cdots, x_n] \begin{bmatrix} \sum_{j=1}^n a_{1j}x_j \\ \sum_{j=1}^n a_{2j}x_j \\ \vdots \\ \sum_{j=1}^n a_{nj}x_j \end{bmatrix} = \left[\sum_{i=1}^n \sum_{j=1}^n a_{ij}x_i x_j \right].$$

Example 1

Show that the matrix

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

is positive definite

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Solution Suppose $\mathbf{x}$ is any three-dimensional column vector. Then

$$\mathbf{x}^t A\mathbf{x} = [x_1, x_2, x_3] \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= [x_1, x_2, x_3] \begin{bmatrix} 2x_1 - x_2 \\ -x_1 + 2x_2 - x_3 \\ -x_2 + 2x_3 \end{bmatrix}$$

$$= 2x_1^2 - 2x_1x_2 + 2x_2^2 - 2x_2x_3 + 2x_3^2.$$

Rearranging the terms gives

$$\mathbf{x}^t A\mathbf{x} = x_1^2 + (x_1^2 - 2x_1x_2 + x_2^2) + (x_2^2 - 2x_2x_3 + x_3^2) + x_3^2$$

$$= x_1^2 + (x_1 - x_2)^2 + (x_2 - x_3)^2 + x_3^2,$$

which implies that

$$x_1^2 + (x_1 - x_2)^2 + (x_2 - x_3)^2 + x_3^2 > 0$$

unless $x_1 = x_2 = x_3 = 0$. ∎

Theorem 6.23

If A is an $n \times n$ positive definite matrix, then

(i) A has an inverse;

(ii) $a_{ii} > 0$, for each $i = 1, 2, \ldots, n$;

(iii) $\max_{1 \le k,j \le n} |a_{kj}| \le \max_{1 \le i \le n} |a_{ii}|$;

(iv) $(a_{ij})^2 < a_{ii}a_{jj}$, for each $i \neq j$. ■

Proof

(i) If $\mathbf{x}$ satisfies $A\mathbf{x} = \mathbf{0}$, then $\mathbf{x}^t A\mathbf{x} = 0$. Since A is positive definite, this implies $\mathbf{x} = \mathbf{0}$. Consequently, $A\mathbf{x} = \mathbf{0}$ has only the zero solution. By Theorem 6.17 on page 398, this is equivalent to A being nonsingular.

(ii) For a given i, let $\mathbf{x} = (x_j)$ be defined by $x_i = 1$ and $x_j = 0$, if $j \neq i$. Since $\mathbf{x} \neq \mathbf{0}$,

$$0 < \mathbf{x}^t A\mathbf{x} = a_{ii}.$$

(iii) For $k \neq j$, define $\mathbf{x} = (x_i)$ by

$$x_i = \begin{cases} 0, & \text{if } i \neq j \text{ and } i \neq k, \\ 1, & \text{if } i = j, \\ -1, & \text{if } i = k. \end{cases}$$

Since $\mathbf{x} \neq \mathbf{0}$,

$$0 < \mathbf{x}^t A \mathbf{x} = a_{jj} + a_{kk} - a_{jk} - a_{kj}.$$

But $A^t = A$, so $a_{jk} = a_{kj}$, which implies that

$$2a_{kj} < a_{jj} + a_{kk}. \tag{6.11}$$

Now define $\mathbf{z} = (z_i)$ by

$$z_i = \begin{cases} 0, & \text{if } i \neq j \text{ and } i \neq k, \\ 1, & \text{if } i = j \text{ or } i = k. \end{cases}$$

Then $\mathbf{z}^t A \mathbf{z} > 0$, so

$$-2a_{kj} < a_{kk} + a_{jj}. \tag{6.12}$$

Equations (6.11) and (6.12) imply that for each $k \neq j$,

$$|a_{kj}| < \frac{a_{kk} + a_{jj}}{2} \leq \max_{1 \leq i \leq n} |a_{ii}|, \quad \text{so} \quad \max_{1 \leq k, j \leq n} |a_{kj}| \leq \max_{1 \leq i \leq n} |a_{ii}|.$$

(iv) For $i \neq j$, define $\mathbf{x} = (x_k)$ by

$$x_k = \begin{cases} 0, & \text{if } k \neq j \text{ and } k \neq i, \\ \alpha, & \text{if } k = i, \\ 1, & \text{if } k = j, \end{cases}$$

where α represents an arbitrary real number. Because $\mathbf{x} \neq \mathbf{0}$,

$$0 < \mathbf{x}^t A \mathbf{x} = a_{ii}\alpha^2 + 2a_{ij}\alpha + a_{jj}.$$

As a quadratic polynomial in α with no real roots, the discriminant of $P(\alpha) = a_{ii}\alpha^2 + 2a_{ij}\alpha + a_{jj}$ must be negative. Thus

$$4a_{ij}^2 - 4a_{ii}a_{jj} < 0 \quad \text{and} \quad a_{ij}^2 < a_{ii}a_{jj}. \qquad \blacksquare \; \blacksquare \; \blacksquare$$

Definition 6.24 A **leading principal submatrix** of a matrix A is a matrix of the form

$$A_k = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \vdots & \vdots & & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kk} \end{bmatrix},$$

for some $1 \leq k \leq n$. ■

A proof of the following result can be found in [Stew2], p. 250.

Theorem 6.25 A symmetric matrix A is positive definite if and only if each of its leading principal submatrices has a positive determinant. ■

Example 2

In Example 1 we used the definition to show that the symmetric matrix

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

is positive definite. Confirm this using Theorem 6.25.

Solution Note that

$$\det A_1 = \det[2] = 2 > 0,$$

$$\det A_2 = \det \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = 4 - 1 = 3 > 0,$$

and

$$\det A_3 = \det \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} = 2 \det \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} - (-1) \det \begin{bmatrix} -1 & -1 \\ 0 & 2 \end{bmatrix}$$

$$= 2(4 - 1) + (-2 + 0) = 4 > 0.$$

in agreement with Theorem 6.25. ■

Theorem 6.26 The symmetric matrix A is positive definite if and only if Gaussian elimination without row interchanges can be performed on the linear system $A\mathbf{x} = \mathbf{b}$ with all pivot elements positive. Moreover, in this case, the computations are stable with respect to the growth of round-off errors. ■

Some interesting facts that are uncovered in constructing the proof of Theorem 6.26 are presented in the following corollaries.

Corollary 6.27 The matrix A is positive definite if and only if A can be factored in the form LDL^t, where L is lower triangular with 1s on its diagonal and D is a diagonal matrix with positive diagonal entries. ■

Corollary 6.28 The matrix A is positive definite if and only if A can be factored in the form LL^t, where L is lower triangular with nonzero diagonal entries. ■

The matrix L in this Corollary is not the same as the matrix L in Corollary 6.27. A relationship between them is presented in Exercise 32.

Corollary 6.29 Let A be a symmetric $n \times n$ matrix for which Gaussian elimination can be applied without row interchanges. Then A can be factored into LDL^t, where L is lower triangular with 1s on its diagonal and D is the diagonal matrix with $a_{11}^{(1)}, \ldots, a_{nn}^{(n)}$ on its diagonal. ■

Example 3 Determine the LDL^t factorization of the positive definite matrix

$$A = \begin{bmatrix} 4 & -1 & 1 \\ -1 & 4.25 & 2.75 \\ 1 & 2.75 & 3.5 \end{bmatrix}.$$

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Solution The LDL^t factorization has 1s on the diagonal of the lower triangular matrix L so we need to have

$$A = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{21} & a_{22} & a_{32} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} \begin{bmatrix} 1 & l_{21} & l_{31} \\ 0 & 1 & l_{32} \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} d_1 & d_1 l_{21} & d_1 l_{31} \\ d_1 l_{21} & d_2 + d_1 l_{21}^2 & d_2 l_{32} + d_1 l_{21} l_{31} \\ d_1 l_{31} & d_1 l_{21} l_{31} + d_2 l_{32} & d_1 l_{31}^2 + d_2 l_{32}^2 + d_3 \end{bmatrix}$$

Thus

$a_{11} : 4 = d_1 \Longrightarrow d_1 = 4,$ $\quad a_{21} : -1 = d_1 l_{21} \Longrightarrow l_{21} = -0.25$

$a_{31} : 1 = d_1 l_{31} \Longrightarrow l_{31} = 0.25,$ $\quad a_{22} : 4.25 = d_2 + d_1 l_{21}^2 \Longrightarrow d_2 = 4$

$a_{32} : 2.75 = d_1 l_{21} l_{31} + d_2 l_{32} \Longrightarrow l_{32} = 0.75,$ $\quad a_{33} : 3.5 = d_1 l_{31}^2 + d_2 l_{32}^2 + d_3 \Longrightarrow d_3 = 1,$

and we have

$$A = LDL^t = \begin{bmatrix} 1 & 0 & 0 \\ -0.25 & 1 & 0 \\ 0.25 & 0.75 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -0.25 & 0.25 \\ 0 & 1 & 0.75 \\ 0 & 0 & 1 \end{bmatrix}.$$

■

ALGORITHM
6.5

LDL^t Factorization

To factor the positive definite $n \times n$ matrix A into the form LDL^t, where L is a lower triangular matrix with 1s along the diagonal and D is a diagonal matrix with positive entries on the diagonal:

INPUT the dimension n; entries a_{ij}, for $1 \le i, j \le n$ of A.

OUTPUT the entries l_{ij}, for $1 \le j < i$ and $1 \le i \le n$ of L, and d_i, for $1 \le i \le n$ of D.

Step 1 For $i = 1, \ldots, n$ do Steps 2–4.

Step 2 For $j = 1, \ldots, i-1$, set $v_j = l_{ij}d_j$.

Step 3 Set $d_i = a_{ii} - \sum_{j=1}^{i-1} l_{ij}v_j$.

Step 4 For $j = i+1, \ldots, n$ set $l_{ji} = (a_{ji} - \sum_{k=1}^{i-1} l_{jk}v_k)/d_i$.

Step 5 OUTPUT (l_{ij} for $j = 1, \ldots, i-1$ and $i = 1, \ldots, n$);
OUTPUT (d_i for $i = 1, \ldots, n$);
STOP.

The LDL^t factorization described in Algorithm 6.5 requires

$$\frac{1}{6}n^3 + n^2 - \frac{7}{6}n \text{ multiplications/divisions} \quad \text{and} \quad \frac{1}{6}n^3 - \frac{1}{6}n \text{ additions/subtractions.}$$

Algorithm 6.5 provides a stable method for factoring a positive definite matrix into the form $A = LDL^t$, but it must be modified to solve the linear system $A\mathbf{x} = \mathbf{b}$. To do this, we delete the STOP statement from Step 5 in the algorithm and add the following steps to solve the lower triangular system $L\mathbf{y} = \mathbf{b}$:

Step 6 Set $y_1 = b_1$.

Step 7 For $i = 2, \ldots, n$ set $y_i = b_i - \sum_{j=1}^{i-1} l_{ij} y_j$.

The linear system $D\mathbf{z} = \mathbf{y}$ can then be solved by

Step 8 For $i = 1, \ldots, n$ set $z_i = y_i / d_i$.

Finally, the upper-triangular system $L^t\mathbf{x} = \mathbf{z}$ is solved with the steps given by

Step 9 Set $x_n = z_n$.

Step 10 For $i = n - 1, \ldots, 1$ set $x_i = z_i - \sum_{j=i+1}^{n} l_{ji} x_j$.

Step 11 OUTPUT (x_i for $i = 1, \ldots, n$);
STOP.

7. Modify the LDL^t Factorization Algorithm as suggested in the text so that it can be used to solve linear systems. Use the modified algorithm to solve the following linear systems.

b.
$$\begin{aligned} 4x_1 + 2x_2 + 2x_3 &= 0, \\ 2x_1 + 6x_2 + 2x_3 &= 1, \\ 2x_1 + 2x_2 + 5x_3 &= 0. \end{aligned}$$

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7. Modify the LDL^t Factorization Algorithm as suggested in the text so that it can be used to solve linear systems. Use the modified algorithm to solve the following linear systems.

c.
$$\begin{aligned} 4x_1 + x_2 - x_3 \phantom{{}+2x_4} &= 7, \\ x_1 + 3x_2 - x_3 \phantom{{}+2x_4} &= 8, \\ -x_1 - x_2 + 5x_3 + 2x_4 &= -4, \\ 2x_3 + 4x_4 &= 6. \end{aligned}$$

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