

# **MAP 2210 – Aplicações de Álgebra Linear**

## **1º Semestre - 2019**

**Prof. Dr. Luis Carlos de Castro Santos**

lsantos@ime.usp.br

### **Objetivos**

Formação básica de álgebra linear aplicada a problemas numéricos.  
Resolução de problemas em microcomputadores usando linguagens e/ou software adequados fora do horário de aula.

Fourth Edition

# **LINEAR ALGEBRA** AND ITS APPLICATIONS

**Gilbert Strang**

## Row Exchanges and Permutation Matrices

*A permutation matrix  $P$  has the same rows as the identity (in some order). There is a single “1” in every row and column. The most common permutation matrix is  $P = I$  (it exchanges nothing). The product of two permutation matrices is another permutation—the rows of  $I$  get reordered twice.*

After  $P = I$ , the simplest permutations exchange two rows. Other permutations exchange more rows. **There are  $n! = (n)(n - 1) \cdots (1)$  permutations of size  $n$ .** Row 1 has  $n$  choices, then row 2 has  $n - 1$  choices, and finally the last row has only one choice. We can display all 3 by 3 permutations (there are  $3! = (3)(2)(1) = 6$  matrices):

$$\begin{array}{lll}
 I = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} & P_{21} = \begin{bmatrix} & 1 & \\ 1 & & \\ & & 1 \end{bmatrix} & P_{32}P_{21} = \begin{bmatrix} & 1 & \\ & & 1 \\ 1 & & \end{bmatrix} \\
 P_{31} = \begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix} & P_{32} = \begin{bmatrix} 1 & & \\ & & 1 \\ & 1 & \end{bmatrix} & P_{21}P_{32} = \begin{bmatrix} & & 1 \\ 1 & & \\ & 1 & \end{bmatrix}.
 \end{array}$$

## Exemplo 5A: Aplicar o produtos da matrizes

$$P_{21} = \begin{bmatrix} & 1 & \\ 1 & & \\ & & 1 \end{bmatrix} \quad P_{32} = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}$$

Em sequencia a uma matriz genérica 3 x 3, e comparar com o produto pela matriz:

$$P_{32}P_{21} = \begin{bmatrix} & 1 & \\ 1 & & \\ & & 1 \end{bmatrix}$$

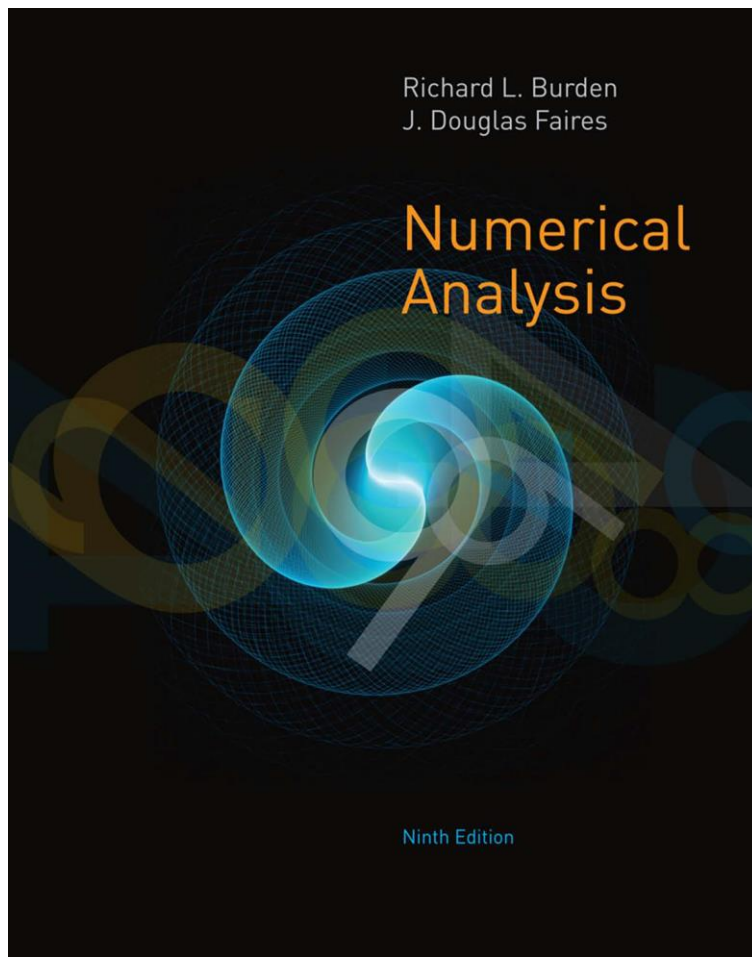
Esse exemplo ilustra a equivalência entre a troca de linhas pela multiplicação por matrizes de permutação.

Exemplo 5B: Encontre a inversa da matriz

$$\mathbf{P}_{32}\mathbf{P}_{21} = \begin{bmatrix} & 1 & \\ 1 & & \\ & & 1 \end{bmatrix}$$

Esse exemplo ilustra a propriedade.

*$\mathbf{P}^{-1}$  is always the same as  $\mathbf{P}^T$ .*



# Numerical Analysis

NINTH EDITION

**Richard L. Burden**

*Youngstown State University*

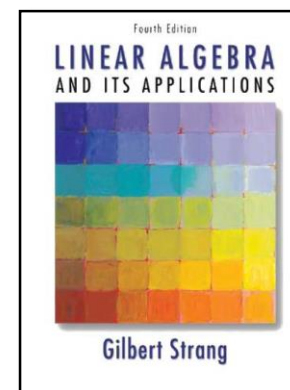
**J. Douglas Faires**

*Youngstown State University*

## 6 Direct Methods for Solving Linear Systems 357

- 6.1 Linear Systems of Equations 358
- 6.2 Pivoting Strategies 372
- 6.3 Linear Algebra and Matrix Inversion 381
- 6.4 The Determinant of a Matrix 396
- 6.5 Matrix Factorization 400
- 6.6 Special Types of Matrices 411
- 6.7 Survey of Methods and Software 428

+



+

## 7 Iterative Techniques in Matrix Algebra 431

- 7.1 Norms of Vectors and Matrices 432
- 7.2 Eigenvalues and Eigenvectors 443
- 7.3 The Jacobi and Gauss-Siedel Iterative Techniques 450
- 7.4 Relaxation Techniques for Solving Linear Systems 462
- 7.5 Error Bounds and Iterative Refinement 469
- 7.6 The Conjugate Gradient Method 479
- 7.7 Survey of Methods and Software 495

## 9 Approximating Eigenvalues 561

- 9.1 Linear Algebra and Eigenvalues 562
- 9.2 Orthogonal Matrices and Similarity Transformations 570
- 9.3 The Power Method 576
- 9.4 Householder's Method 593
- 9.5 The QR Algorithm 601
- 9.6 Singular Value Decomposition 614
- 9.7 Survey of Methods and Software 626



## Permutation Matrices

In the previous discussion we assumed that  $A\mathbf{x} = \mathbf{b}$  can be solved using Gaussian elimination without row interchanges. From a practical standpoint, this factorization is useful only when row interchanges are not required to control the round-off error resulting from the use of finite-digit arithmetic. Fortunately, many systems we encounter when using approximation methods are of this type, but we will now consider the modifications that must be made when row interchanges are required. We begin the discussion with the introduction of a class of matrices that are used to rearrange, or permute, rows of a given matrix.

An  $n \times n$  **permutation matrix**  $P = [p_{ij}]$  is a matrix obtained by rearranging the rows of  $I_n$ , the identity matrix. This gives a matrix with precisely one nonzero entry in each row and in each column, and each nonzero entry is a 1.



The matrix

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

is a  $3 \times 3$  permutation matrix. For any  $3 \times 3$  matrix  $A$ , multiplying on the left by  $P$  has the effect of interchanging the second and third rows of  $A$ :

$$PA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}.$$

Similarly, multiplying  $A$  on the right by  $P$  interchanges the second and third columns of  $A$ . □

Two useful properties of permutation matrices relate to Gaussian elimination, the first of which is illustrated in the previous example. Suppose  $k_1, \dots, k_n$  is a permutation of the integers  $1, \dots, n$  and the permutation matrix  $P = (p_{ij})$  is defined by

$$p_{ij} = \begin{cases} 1, & \text{if } j = k_i, \\ 0, & \text{otherwise.} \end{cases}$$

Then

- $PA$  permutes the rows of  $A$ ; that is,

$$PA = \begin{bmatrix} a_{k_1 1} & a_{k_1 2} & \cdots & a_{k_1 n} \\ a_{k_2 1} & a_{k_2 2} & \cdots & a_{k_2 n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k_n 1} & a_{k_n 2} & \cdots & a_{k_n n} \end{bmatrix}.$$

- $P^{-1}$  exists and  $P^{-1} = P^t$ .

At the end of Section 6.4 we saw that for any nonsingular matrix  $A$ , the linear system  $A\mathbf{x} = \mathbf{b}$  can be solved by Gaussian elimination, with the possibility of row interchanges. If we knew the row interchanges that were required to solve the system by Gaussian elimination, we could arrange the original equations in an order that would ensure that no row interchanges are needed. Hence there *is* a rearrangement of the equations in the system that permits Gaussian elimination to proceed *without* row interchanges. This implies that for any nonsingular matrix  $A$ , a permutation matrix  $P$  exists for which the system

$$PA\mathbf{x} = P\mathbf{b}$$

can be solved without row interchanges. As a consequence, this matrix  $PA$  can be factored into

$$PA = LU,$$

where  $L$  is lower triangular and  $U$  is upper triangular. Because  $P^{-1} = P^t$ , this produces the factorization

$$A = P^{-1}LU = (P^tL)U.$$

The matrix  $U$  is still upper triangular, but  $P^tL$  is not lower triangular unless  $P = I$ .

**Example 3** Determine a factorization in the form  $A = (P^t L)U$  for the matrix

$$A = \begin{bmatrix} 0 & 0 & -1 & 1 \\ 1 & 1 & -1 & 2 \\ -1 & -1 & 2 & 0 \\ 1 & 2 & 0 & 2 \end{bmatrix}.$$

**Solution** The matrix  $A$  cannot have an  $LU$  factorization because  $a_{11} = 0$ . However, using the row interchange  $(E_1) \leftrightarrow (E_2)$ , followed by  $(E_3 + E_1) \rightarrow (E_3)$  and  $(E_4 - E_1) \rightarrow (E_4)$ , produces

$$\begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 1 & 0 \end{bmatrix}.$$

Then the row interchange  $(E_2) \leftrightarrow (E_4)$ , followed by  $(E_4 + E_3) \rightarrow (E_4)$ , gives the matrix

$$U = \begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$

The permutation matrix associated with the row interchanges  $(E_1) \leftrightarrow (E_2)$  and  $(E_2) \leftrightarrow (E_4)$  is

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$

and

$$PA = \begin{bmatrix} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & 2 \\ -1 & -1 & 2 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}.$$

Gaussian elimination is performed on  $PA$  using the same operations as on  $A$ , except without the row interchanges. That is,  $(E_2 - E_1) \rightarrow (E_2)$ ,  $(E_3 + E_1) \rightarrow (E_3)$ , followed by  $(E_4 + E_3) \rightarrow (E_4)$ . The nonzero multipliers for  $PA$  are consequently,

$$m_{21} = 1, \quad m_{31} = -1, \quad \text{and} \quad m_{43} = -1,$$

and the  $LU$  factorization of  $PA$  is

$$PA = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 3 \end{bmatrix} = LU.$$

Multiplying by  $P^{-1} = P^t$  produces the factorization

$$A = P^{-1}(LU) = P^t(LU) = (P^tL)U = \begin{bmatrix} 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 3 \end{bmatrix}. \quad \blacksquare$$

Aparentemente temos que realizar a eliminação duas vezes para obter a fatorização, mas na verdade não é necessário se carregarmos os multiplicadores temporariamente na matriz L enquanto realizamos a eliminação e aplicarmos as permutações a essa matriz modificada.

O exemplo a seguir ilustra esse procedimento:

Exemplo 5C:

Encontrar a fatorização  $PA=LU$  da matriz A:

$$\begin{bmatrix} 2 & 1 & 5 \\ 4 & 4 & -4 \\ 1 & 3 & 1 \end{bmatrix}$$





Exemplo 5D:

Se a fatorização  $P^T LU = A$  estiver disponível encontre a solução do sistema  $Ax = b$  com:

$$b = \begin{bmatrix} 5 \\ 0 \\ 6 \end{bmatrix}$$



9. Obtain factorizations of the form  $A = P^tLU$  for the following matrices.

a.  $A = \begin{bmatrix} 0 & 2 & 3 \\ 1 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$

c.  $A = \begin{bmatrix} 1 & -2 & 3 & 0 \\ 3 & -6 & 9 & 3 \\ 2 & 1 & 4 & 1 \\ 1 & -2 & 2 & -2 \end{bmatrix}$

b.  $A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 2 & 3 \\ 2 & -1 & 4 \end{bmatrix}$

d.  $A = \begin{bmatrix} 1 & -2 & 3 & 0 \\ 1 & -2 & 3 & 1 \\ 1 & -2 & 2 & -2 \\ 2 & 1 & 3 & -1 \end{bmatrix}$

E verifique

Fim...

ALLA 05