MAP 2210 – Aplicações de Álgebra Linear

1º Semestre - 2019

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Objetivos

Formação básica de álgebra linear aplicada a problemas numéricos. Resolução de problemas em microcomputadores usando linguagens e/ ou software adequados fora do horário de aula.

Fourth Edition

LINEAR ALGEBRA AND ITS APPLICATIONS

Gilbert Strang

1.5 TRIANGULAR FACTORS AND ROW EXCHANGES

We want to look again at elimination, to see what it means in terms of matrices. The starting point was the model system $Ax = b$:

$$Ax = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix} = b. \tag{1}$$

Then there were three elimination steps, with multipliers $2, -1, -1$:

Step 1. Subtract 2 times the first equation from the second;
Step 2. Subtract -1 times the first equation from the third;
Step 3. Subtract -1 times the second equation from the third.

The result was an equivalent system $Ux = c$, with a new coefficient matrix U:

$$\textbf{Upper triangular} \qquad Ux = \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 5 \\ -12 \\ 2 \end{bmatrix} = c. \tag{2}$$

This matrix U is ***upper triangular***—all entries below the diagonal are zero.

The new right side c was derived from the original vector b by the same steps that took A into U. *Forward elimination* amounted to three row operations:

Start with A and b;
Apply steps 1, 2, 3 in that order;
End with U and c.

$Ux = c$ is solved by back-substitution. Here we concentrate on connecting A to U.

Example 4 Suppose E subtracts twice the first equation from the second. Suppose F is the matrix for the next step, *to add row* 1 *to row* 3:

$$E = \begin{bmatrix} 1 & 0 & 0 \\ \mathbf{-2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \mathbf{1} & 0 & 1 \end{bmatrix}.$$

These two matrices do commute and the product does both steps at once:

$$EF = \begin{bmatrix} 1 & 0 & 0 \\ \mathbf{-2} & 1 & 0 \\ \mathbf{1} & 0 & 1 \end{bmatrix} = FE.$$

In either order, EF or FE, this changes rows 2 and 3 using row 1.

Example 5 Suppose E is the same but G *adds row 2 to row* 3. Now the order makes a difference. When we apply E and then G, the second row is altered *before* it affects the third. If E comes *after* G, then the third equation feels no effect from the first. You will see a zero in the (3, 1) entry of EG, where there is a -2 in GE:

$$GE = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \mathbf{1} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ \mathbf{-2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ \mathbf{-2} & 1 & 1 \end{bmatrix} \quad \text{but} \quad EG = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ \mathbf{0} & 1 & 1 \end{bmatrix}.$$

Thus $EG \neq GE$. A random example would show the same thing—most matrices don't commute. Here the matrices have meaning. There was a reason for $EF = FE$, and a reason for $EG \neq GE$. It is worth taking one more step, to see what happens with *all three elimination matrices at once*:

$$GFE = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ \mathbf{-1} & 1 & 1 \end{bmatrix} \quad \text{and} \quad EFG = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ \mathbf{1} & 1 & 1 \end{bmatrix}.$$

Exemplo 4A: Multiplique a matriz

$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}$$

pela matriz

$$GFE = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}$$

O que se observa do resultado ?

Perceba a equivalência entre as operações:

Step 1. Subtract 2 times the first equation from the second;
Step 2. Subtract -1 times the first equation from the third;
Step 3. Subtract -1 times the second equation from the third.

E o produto pela matriz

$$GFE = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}$$

Ambos resultam na triangularização do sistema:

$$Ax = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix} = b. \qquad Ux = \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 5 \\ -12 \\ 2 \end{bmatrix} = c.$$

The matrices E for step 1, F for step 2, and G for step 3 were introduced in the previous section. They are called ***elementary matrices***, and it is easy to see how they work. To subtract a multiple ℓ of equation j from equation i, *put the number $-\ell$ into the (i, j) position*. Otherwise keep the identity matrix, with 1s on the diagonal and 0s elsewhere. Then matrix multiplication executes the row operation.

The result of all three steps is $GFEA = U$. Note that E is the first to multiply A, then F, then G. We could multiply GFE together to find the single matrix that takes A to U (and also takes b to c). It is lower triangular (zeros are omitted):

From A to U $$GFE = \begin{bmatrix} 1 & & \\ & 1 & \\ & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ 1 & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ -2 & 1 & \\ & & 1 \end{bmatrix} = \begin{bmatrix} 1 & & \\ -2 & 1 & \\ -1 & 1 & 1 \end{bmatrix}. \quad (3)$$

This is good, but the most important question is exactly the opposite: How would we get from U back to A? ***How can we undo the steps of Gaussian elimination?***

We can invert each step of elimination, by using E^{-1} and F^{-1} and G^{-1}. I think it's not bad to see these inverses now, before the next section. The final problem is to undo the whole process at once, and see what matrix takes U back to A.

Exemplo 4B: Encontrar as inversas de E,F e G

To undo step 1 is not hard. Instead of subtracting, we *add* twice the first row to the second. (Not twice the second row to the first!) The result of doing both the subtraction and the addition is to bring back the identity matrix:

Inverse of subtraction is addition

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{4}$$

One operation cancels the other. In matrix terms, one matrix is the ***inverse*** of the other. If the elementary matrix E has the number $-\ell$ in the (i, j) position, then its inverse E^{-1} has $+\ell$ in that position. Thus $E^{-1}E = I$, which is equation (4).

Since step 3 was last in going from A to U, its matrix G must be the first to be inverted in the reverse direction. Inverses come in the opposite order! The second reverse step is F^{-1} and the last is E^{-1}:

$$\textbf{From } U \textbf{ back to } A \qquad E^{-1}F^{-1}G^{-1}U = A \quad \text{is} \quad LU = A. \tag{5}$$

You can substitute $GFEA$ for U, to see how the inverses knock out the original steps.

Exemplo 4C: Realizar a multiplicação na ordem reversa das inversas de E,F e G

Now we recognize the matrix L that takes U back to A. It is called L, because it is *lower triangular*. And it has a special property that can be seen only by multiplying the three inverse matrices in the right order:

$$E^{-1}F^{-1}G^{-1} = \begin{bmatrix} 1 & & \\ \mathbf{2} & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ \mathbf{-1} & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ & \mathbf{-1} & 1 \end{bmatrix} = \begin{bmatrix} 1 & & \\ \mathbf{2} & 1 & \\ \mathbf{-1} & \mathbf{-1} & 1 \end{bmatrix} = L. \qquad (6)$$

The special thing is that ***the entries below the diagonal are the multipliers*** $\ell = 2, -1,$ and -1. When matrices are multiplied, there is usually no direct way to read off the answer. Here the matrices come in just the right order so that their product can be written down immediately. If the computer stores each multiplier ℓ_{ij}—the number that multiplies the pivot row j when it is subtracted from row i, and produces a zero in the i, j position—then these multipliers give a complete record of elimination.

The numbers ℓ_{ij} fit right into the matrix L that takes U back to A.

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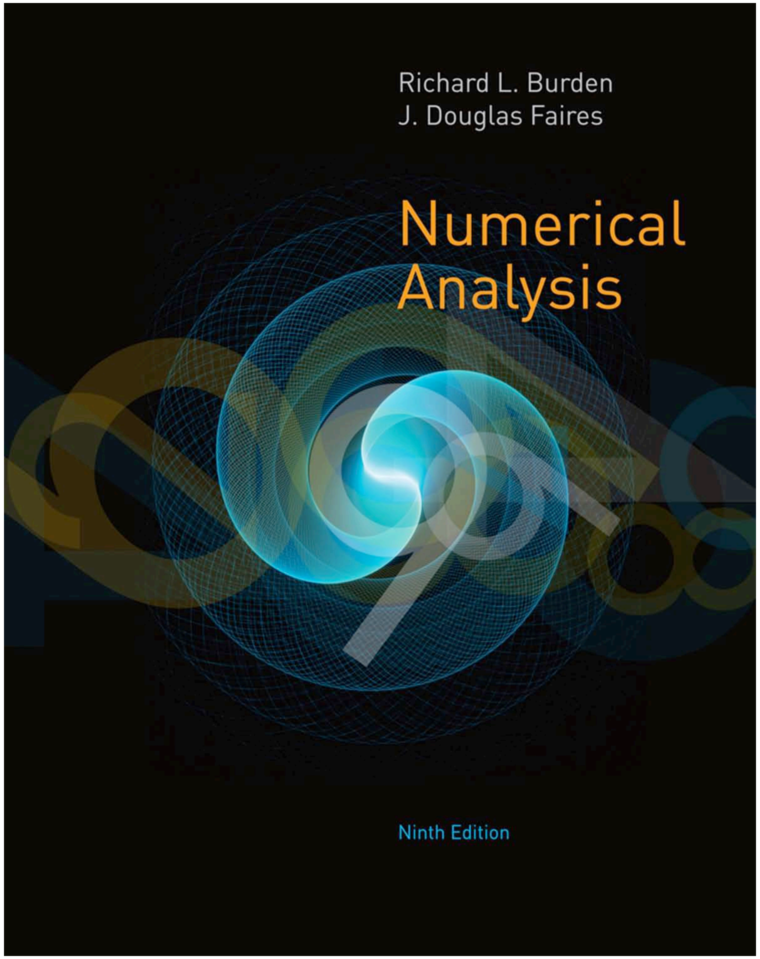

Numerical Analysis

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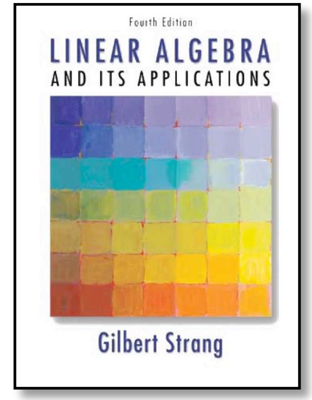

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6.5 Matrix Factorization

Gaussian elimination is the principal tool in the direct solution of linear systems of equations, so it should be no surprise that it appears in other guises. In this section we will see that the steps used to solve a system of the form $A\mathbf{x} = \mathbf{b}$ can be used to factor a matrix. The factorization is particularly useful when it has the form $A = LU$, where L is lower triangular and U is upper triangular. Although not all matrices have this type of representation, many do that occur frequently in the application of numerical techniques.

In Section 6.1 we found that Gaussian elimination applied to an arbitrary linear system $A\mathbf{x} = \mathbf{b}$ requires $O(n^3/3)$ arithmetic operations to determine $\mathbf{x}$. However, to solve a linear system that involves an upper-triangular system requires only backward substitution, which takes $O(n^2)$ operations. The number of operations required to solve a lower-triangular systems is similar.

Suppose that A has been factored into the triangular form $A = LU$, where L is lower triangular and U is upper triangular. Then we can solve for $\mathbf{x}$ more easily by using a two-step process.

- First we let $\mathbf{y} = U\mathbf{x}$ and solve the lower triangular system $L\mathbf{y} = \mathbf{b}$ for $\mathbf{y}$. Since L is triangular, determining $\mathbf{y}$ from this equation requires only $O(n^2)$ operations.
- Once $\mathbf{y}$ is known, the upper triangular system $U\mathbf{x} = \mathbf{y}$ requires only an additional $O(n^2)$ operations to determine the solution $\mathbf{x}$.

Solving a linear system $A\mathbf{x} = \mathbf{b}$ in factored form means that the number of operations needed to solve the system $A\mathbf{x} = \mathbf{b}$ is reduced from $O(n^3/3)$ to $O(2n^2)$.

To see which matrices have an LU factorization and to find how it is determined, first suppose that Gaussian elimination can be performed on the system $A\mathbf{x} = \mathbf{b}$ without row interchanges. With the notation in Section 6.1, this is equivalent to having nonzero pivot elements $a_{ii}^{(i)}$, for each $i = 1, 2, \ldots, n$.

The first step in the Gaussian elimination process consists of performing, for each $j = 2, 3, \ldots, n$, the operations

$$(E_j - m_{j,1}E_1) \to (E_j), \quad \text{where} \quad m_{j,1} = \frac{a_{j1}^{(1)}}{a_{11}^{(1)}}. \tag{6.8}$$

These operations transform the system into one in which all the entries in the first column below the diagonal are zero.

The system of operations in (6.8) can be viewed in another way. It is simultaneously accomplished by multiplying the original matrix A on the left by the matrix

$$M^{(1)} = \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ -m_{21} & 1 & \ddots & & \vdots \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ -m_{n1} & 0 & \cdots & 0 & 1 \end{bmatrix}.$$

This is called the **first Gaussian transformation matrix**. We denote the product of this matrix with $A^{(1)} \equiv A$ by $A^{(2)}$ and with $\mathbf{b}$ by $\mathbf{b}^{(2)}$, so

$$A^{(2)}\mathbf{x} = M^{(1)}A\mathbf{x} = M^{(1)}\mathbf{b} = \mathbf{b}^{(2)}.$$

In a similar manner we construct $M^{(2)}$, the identity matrix with the entries below the diagonal in the second column replaced by the negatives of the multipliers

$$m_{j,2} = \frac{a_{j2}^{(2)}}{a_{22}^{(2)}}.$$

The product of this matrix with $A^{(2)}$ has zeros below the diagonal in the first two columns, and we let

$$A^{(3)}\mathbf{x} = M^{(2)}A^{(2)}\mathbf{x} = M^{(2)}M^{(1)}A\mathbf{x} = M^{(2)}M^{(1)}\mathbf{b} = \mathbf{b}^{(3)}.$$

In general, with $A^{(k)}\mathbf{x} = \mathbf{b}^{(k)}$ already formed, multiply by the kth **Gaussian transformation matrix**

$$M^{(k)} = \begin{bmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & & & \vdots \\ \vdots & & 0 & \ddots & \ddots & & \vdots \\ \vdots & & \vdots & -m_{k+1,k} & \ddots & \ddots & \vdots \\ \vdots & & \vdots & \vdots & 0 & \ddots & 0 \\ \vdots & & \vdots & \vdots & \vdots & \ddots & \ddots \\ 0 & \cdots & 0 & -m_{n,k} & 0 & \cdots 0 & 1 \end{bmatrix},$$

to obtain

$$A^{(k+1)}\mathbf{x} = M^{(k)}A^{(k)}\mathbf{x} = M^{(k)} \cdots M^{(1)}A\mathbf{x} = M^{(k)}\mathbf{b}^{(k)} = \mathbf{b}^{(k+1)} = M^{(k)} \cdots M^{(1)}\mathbf{b}. \qquad (6.9)$$

The process ends with the formation of $A^{(n)}\mathbf{x} = \mathbf{b}^{(n)}$, where $A^{(n)}$ is the upper triangular matrix

$$A^{(n)} = \begin{bmatrix} a_{11}^{(1)} & a_{12}^{(1)} & \cdots & a_{1n}^{(1)} \\ 0 & a_{22}^{(2)} & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_{n-1,n}^{(n-1)} \\ 0 & \cdots & 0 & a_{nn}^{(n)} \end{bmatrix},$$

given by

$$A^{(n)} = M^{(n-1)}M^{(n-2)}\cdots M^{(1)}A.$$

This process forms the $U = A^{(n)}$ portion of the matrix factorization $A = LU$. To determine the complementary lower triangular matrix L, first recall the multiplication of $A^{(k)}\mathbf{x} = \mathbf{b}^{(k)}$ by the Gaussian transformation of $M^{(k)}$ used to obtain (6.9):

$$A^{(k+1)}\mathbf{x} = M^{(k)}A^{(k)}\mathbf{x} = M^{(k)}\mathbf{b}^{(k)} = \mathbf{b}^{(k+1)},$$

where $M^{(k)}$ generates the row operations

$$(E_j - m_{j,k}E_k) \to (E_j), \quad \text{for } j = k+1, \ldots, n.$$

To reverse the effects of this transformation and return to $A^{(k)}$ requires that the operations $(E_j + m_{j,k}E_k) \rightarrow (E_j)$ be performed for each $j = k+1, \ldots, n$. This is equivalent to multiplying by the inverse of the matrix $M^{(k)}$, the matrix

$$L^{(k)} = [M^{(k)}]^{-1} = \begin{bmatrix}
1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & \ddots & \ddots & & & & & \vdots \\
\vdots & \ddots & 0 & & & & & \vdots \\
\vdots & & \vdots & m_{k+1,k} & \ddots & & & \vdots \\
\vdots & & \vdots & \vdots & 0 & \ddots & & 0 \\
0 & \cdots & 0 & m_{n,k} & 0 & \cdots & 0 & 1
\end{bmatrix}.$$

The lower-triangular matrix L in the factorization of A, then, is the product of the matrices $L^{(k)}$:

$$L = L^{(1)}L^{(2)} \cdots L^{(n-1)} = \begin{bmatrix}
1 & 0 & \cdots & 0 \\
m_{21} & 1 & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
m_{n1} & \cdots & m_{n,n-1} & 1
\end{bmatrix},$$

since the product of L with the upper-triangular matrix $U = M^{(n-1)} \cdots M^{(2)} M^{(1)} A$ gives

$$
\begin{aligned}
LU &= L^{(1)} L^{(2)} \cdots L^{(n-3)} L^{(n-2)} L^{(n-1)} \cdot M^{(n-1)} M^{(n-2)} M^{(n-3)} \cdots M^{(2)} M^{(1)} A \\
&= [M^{(1)}]^{-1} [M^{(2)}]^{-1} \cdots [M^{(n-2)}]^{-1} [M^{(n-1)}]^{-1} \cdot M^{(n-1)} M^{(n-2)} \cdots M^{(2)} M^{(1)} A = A.
\end{aligned}
$$

Theorem 6.19 follows from these observations.

Theorem 6.19

If Gaussian elimination can be performed on the linear system $A\mathbf{x} = \mathbf{b}$ without row interchanges, then the matrix A can be factored into the product of a lower-triangular matrix L and an upper-triangular matrix U, that is, $A = LU$, where $m_{ji} = a_{ji}^{(i)} / a_{ii}^{(i)}$,

$$
U = \begin{bmatrix}
a_{11}^{(1)} & a_{12}^{(1)} & \cdots & a_{1n}^{(1)} \\
0 & a_{22}^{(2)} & \ddots & \vdots \\
\vdots & \ddots & \ddots & a_{n-1,n}^{(n-1)} \\
0 & \cdots & 0 & a_{nn}^{(n)}
\end{bmatrix}, \quad \text{and} \quad
L = \begin{bmatrix}
1 & 0 & \cdots & 0 \\
m_{21} & 1 & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
m_{n1} & \cdots & m_{n,n-1} & 1
\end{bmatrix}.
$$

■

Example 2

(a) Determine the LU factorization for matrix A in the linear system $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & 1 & 0 & 3 \\ 2 & 1 & -1 & 1 \\ 3 & -1 & -1 & 2 \\ -1 & 2 & 3 & -1 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ -3 \\ 4 \end{bmatrix}.$$

(b) Then use the factorization to solve the system

$$\begin{aligned} x_1 + x_2 + 3x_4 &= 8, \\ 2x_1 + x_2 - x_3 + x_4 &= 7, \\ 3x_1 - x_2 - x_3 + 2x_4 &= 14, \\ -x_1 + 2x_2 + 3x_3 - x_4 &= -7. \end{aligned}$$

(c) Com a fatorização disponível calcule o determinante e a inversa da matriz A

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Solution **(a)** The original system was considered in Section 6.1, where we saw that the sequence of operations $(E_2 - 2E_1) \to (E_2)$, $(E_3 - 3E_1) \to (E_3)$, $(E_4 - (-1)E_1) \to (E_4)$, $(E_3 - 4E_2) \to (E_3)$, $(E_4 - (-3)E_2) \to (E_4)$ converts the system to the triangular system

$$
\begin{aligned}
x_1 + x_2 \qquad\quad + \; 3x_4 &= 4, \\
-x_2 - \; x_3 - \; 5x_4 &= -7, \\
3x_3 + 13x_4 &= 13, \\
-13x_4 &= -13.
\end{aligned}
$$

The multipliers m_{ij} and the upper triangular matrix produce the factorization

$$
A = \begin{bmatrix} 1 & 1 & 0 & 3 \\ 2 & 1 & -1 & 1 \\ 3 & -1 & -1 & 2 \\ -1 & 2 & 3 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 4 & 1 & 0 \\ -1 & -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & -1 & -1 & -5 \\ 0 & 0 & 3 & 13 \\ 0 & 0 & 0 & -13 \end{bmatrix} = LU.
$$

(b) To solve

$$A\mathbf{x} = LU\mathbf{x} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 4 & 1 & 0 \\ -1 & -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & -1 & -1 & -5 \\ 0 & 0 & 3 & 13 \\ 0 & 0 & 0 & -13 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 8 \\ 7 \\ 14 \\ -7 \end{bmatrix},$$

we first introduce the substitution $\mathbf{y} = U\mathbf{x}$. Then $\mathbf{b} = L(U\mathbf{x}) = L\mathbf{y}$. That is,

$$L\mathbf{y} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 4 & 1 & 0 \\ -1 & -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 8 \\ 7 \\ 14 \\ -7 \end{bmatrix}.$$

This system is solved for $\mathbf{y}$ by a simple forward-substitution process:

$$\begin{aligned} y_1 &= 8; \\ 2y_1 + y_2 &= 7, \quad \text{so } y_2 = 7 - 2y_1 = -9; \\ 3y_1 + 4y_2 + y_3 &= 14, \quad \text{so } y_3 = 14 - 3y_1 - 4y_2 = 26; \\ -y_1 - 3y_2 + y_4 &= -7, \quad \text{so } y_4 = -7 + y_1 + 3y_2 = -26. \end{aligned}$$

We then solve $U\mathbf{x} = \mathbf{y}$ for $\mathbf{x}$, the solution of the original system; that is,

$$\begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & -1 & -1 & -5 \\ 0 & 0 & 3 & 13 \\ 0 & 0 & 0 & -13 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 8 \\ -9 \\ 26 \\ -26 \end{bmatrix}.$$

Using backward substitution we obtain $x_4 = 2$, $x_3 = 0$, $x_2 = -1$, $x_1 = 3$. ■

The factorization used in Example 2 is called *Doolittle's method* and requires that 1s be on the diagonal of L, which results in the factorization described in Theorem 6.19. In Section 6.6, we consider *Crout's method*, a factorization which requires that 1s be on the diagonal elements of U, and *Cholesky's method*, which requires that $l_{ii} = u_{ii}$, for each i.

A general procedure for factoring matrices into a product of triangular matrices is contained in Algorithm 6.4. Although new matrices L and U are constructed, the generated values can replace the corresponding entries of A that are no longer needed.

Algorithm 6.4 permits either the diagonal of L or the diagonal of U to be specified.

ALGORITHM 6.4

LU Factorization

To factor the $n \times n$ matrix $A = [a_{ij}]$ into the product of the lower-triangular matrix $L = [l_{ij}]$ and the upper-triangular matrix $U = [u_{ij}]$; that is, $A = LU$, where the main diagonal of either L or U consists of all ones:

INPUT dimension n; the entries a_{ij}, $1 \le i, j \le n$ of A; the diagonal $l_{11} = \cdots = l_{nn} = 1$ of L or the diagonal $u_{11} = \cdots = u_{nn} = 1$ of U.

OUTPUT the entries l_{ij}, $1 \le j \le i$, $1 \le i \le n$ of L and the entries, u_{ij}, $i \le j \le n$, $1 \le i \le n$ of U.

Step 1 Select l_{11} and u_{11} satisfying $l_{11}u_{11} = a_{11}$.
If $l_{11}u_{11} = 0$ then OUTPUT ('Factorization impossible');
STOP.

Step 2 For $j = 2, \ldots, n$ set $u_{1j} = a_{1j}/l_{11}$; (*First row of U.*)
$l_{j1} = a_{j1}/u_{11}$. (*First column of L.*)

Step 3 For $i = 2, \ldots, n-1$ do Steps 4 and 5.

Step 4 Select l_{ii} and u_{ii} satisfying $l_{ii}u_{ii} = a_{ii} - \sum_{k=1}^{i-1} l_{ik}u_{ki}$.

If $l_{ii}u_{ii} = 0$ then OUTPUT ('Factorization impossible');
STOP.

Step 5 For $j = i + 1, \ldots, n$

set $u_{ij} = \frac{1}{l_{ii}}\left[a_{ij} - \sum_{k=1}^{i-1} l_{ik}u_{kj}\right]$; (*ith row of U.*)

$l_{ji} = \frac{1}{u_{ii}}\left[a_{ji} - \sum_{k=1}^{i-1} l_{jk}u_{ki}\right]$. (*ith column of L.*)

Step 6 Select l_{nn} and u_{nn} satisfying $l_{nn}u_{nn} = a_{nn} - \sum_{k=1}^{n-1} l_{nk}u_{kn}$.

(*Note: If $l_{nn}u_{nn} = 0$, then $A = LU$ but A is singular.*)

Step 7 OUTPUT (l_{ij} for $j = 1, \ldots, i$ and $i = 1, \ldots, n$);
OUTPUT (u_{ij} for $j = i, \ldots, n$ and $i = 1, \ldots, n$);
STOP.

Once the matrix factorization is complete, the solution to a linear system of the form $A\mathbf{x} = LU\mathbf{x} = \mathbf{b}$ is found by first letting $\mathbf{y} = U\mathbf{x}$ and solving $L\mathbf{y} = \mathbf{b}$ for $\mathbf{y}$. Since L is lower triangular, we have

$$y_1 = \frac{b_1}{l_{11}},$$

and, for each $i = 2, 3, \ldots, n$,

$$y_i = \frac{1}{l_{ii}}\left[b_i - \sum_{j=1}^{i-1} l_{ij} y_j\right].$$

After $\mathbf{y}$ is found by this forward-substitution process, the upper-triangular system $U\mathbf{x} = \mathbf{y}$ is solved for $\mathbf{x}$ by backward substitution using the equations

$$x_n = \frac{y_n}{u_{nn}} \quad \text{and} \quad x_i = \frac{1}{u_{ii}}\left[y_i - \sum_{j=i+1}^{n} u_{ij} x_j\right].$$

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