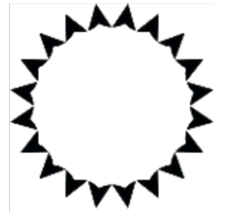




PEF2603
Estruturas na Arquitetura III -
Sistemas Reticulados e Laminares



Vigas Hiperestáticas

(25/03/2019)

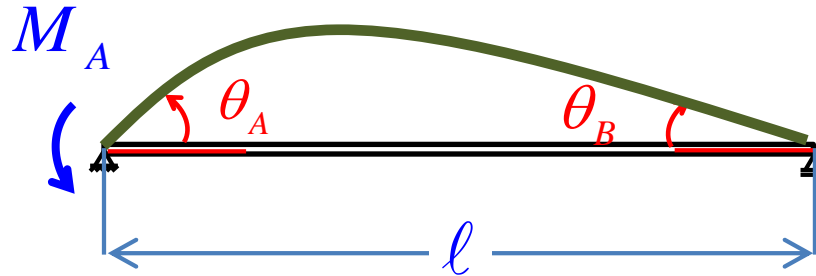
Professores

Ruy Marcelo O. Pauletti , Leila Meneghetti Valverdes, Luís Bitencourt

1º Semestre 2019

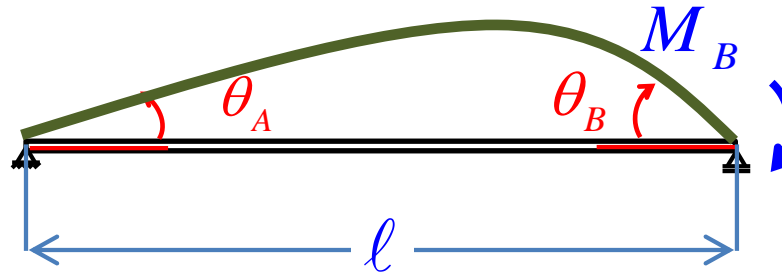
Recordando, da aula passada, as
TABELAS DE ROTAÇÕES DE APOIOS:

$$\theta_A = + \frac{M_A \ell}{3EI}$$



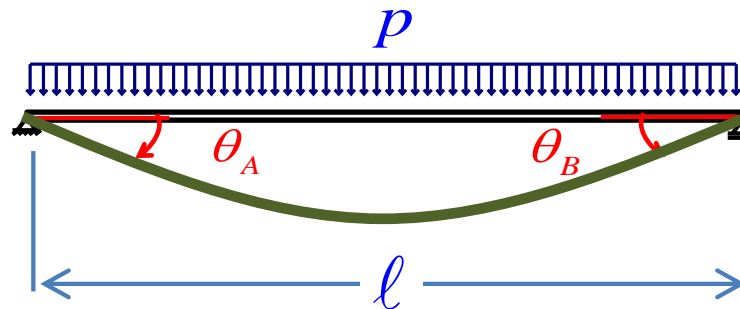
$$\theta_B = - \frac{M_A \ell}{6EI}$$

$$\theta_A = + \frac{M_B \ell}{6EI}$$



$$\theta_B = - \frac{M_B \ell}{3EI}$$

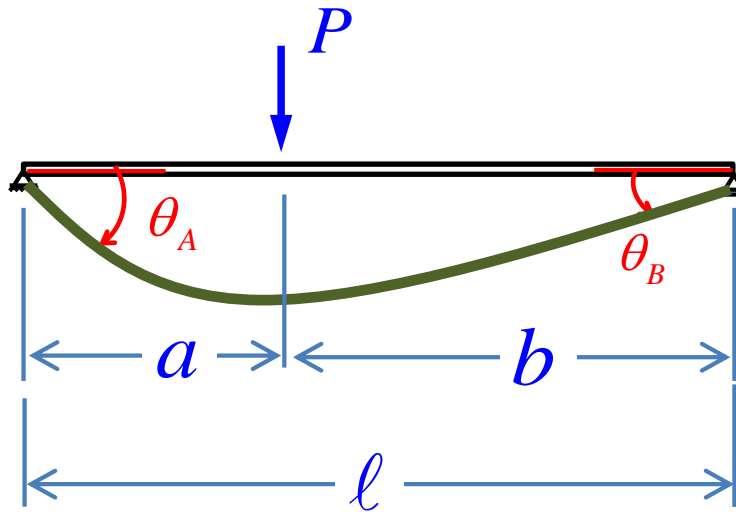
$$\theta_A = - \frac{p \ell^3}{24EI}$$



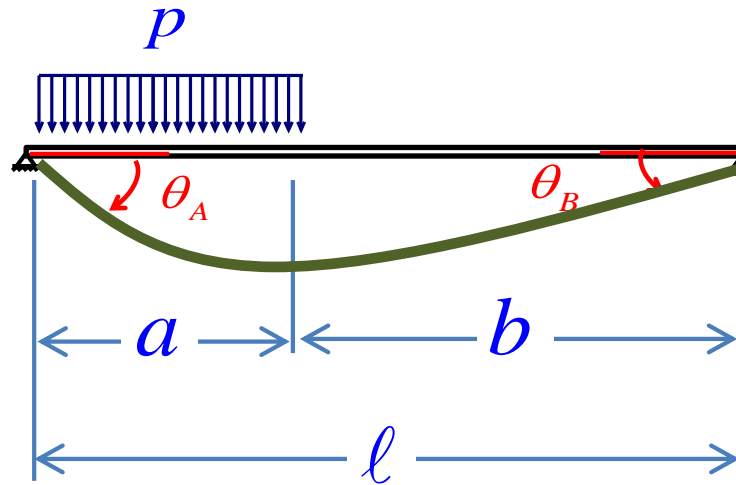
$$\theta_B = + \frac{p \ell^3}{24EI}$$



$$\theta_A = -\frac{Pb(\ell^2 - b^2)}{6\ell EI}$$



$$\theta_B = \frac{Pa(\ell^2 - a^2)}{6\ell EI}$$



$$\theta_A = -\frac{pa^2}{24\ell EI} (a^2 - 4a\ell + 4\ell^2)$$

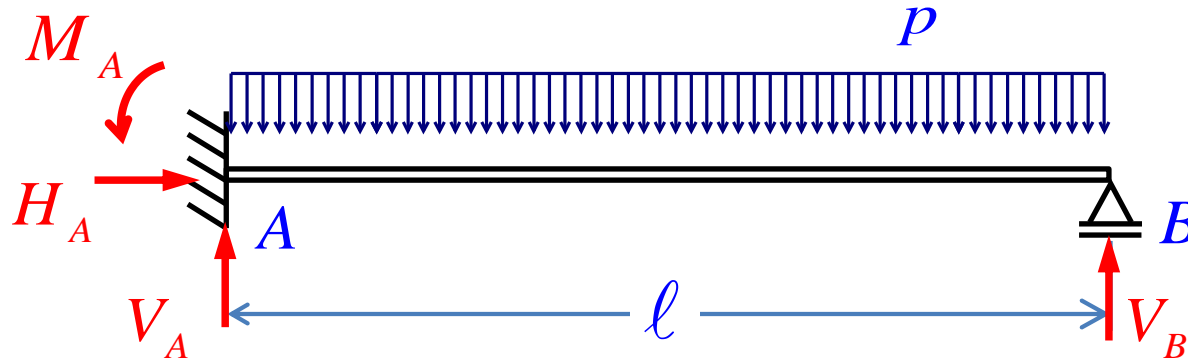
$$\theta_B = \frac{pa^2}{24\ell EI} (2\ell^2 - a^2)$$



VIGAS HIPERESTÁTICAS

Com estas expressões de rotação nos apoios, estamos em condições de resolver reações de apoio e esforços solicitantes em vigas hiperestáticas simples!

Exemplo 1: viga engastada-apoiada, sujeita a carregamento uniformemente distribuído:



- 4 reações de apoio
 - 3 equações de equilíbrio
- } \Rightarrow Viga 1x hiperestática

(1) $\sum F_X = H_A = 0$

(2) $\uparrow \sum F_Y = V_A + V_B - pl = 0 \quad \therefore \quad \boxed{V_A + V_B = pl} \quad (2')$

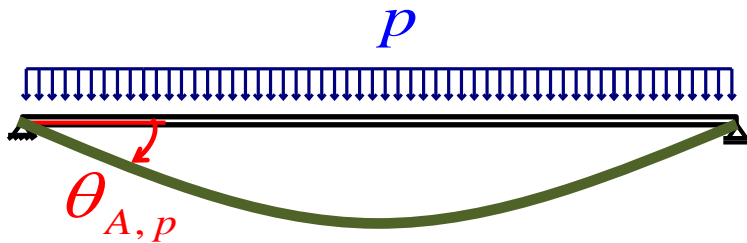
(3) $\sum M_{(A)} = M_A + V_B l - \frac{pl^2}{2} = 0 \quad \therefore \quad \boxed{M_A + V_B l = \frac{pl^2}{2}} \quad (3')$

Equações de equilíbrio:

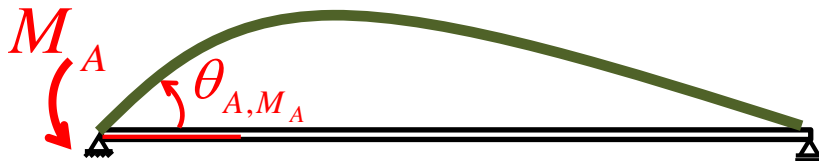


Equação de Compatibilidade

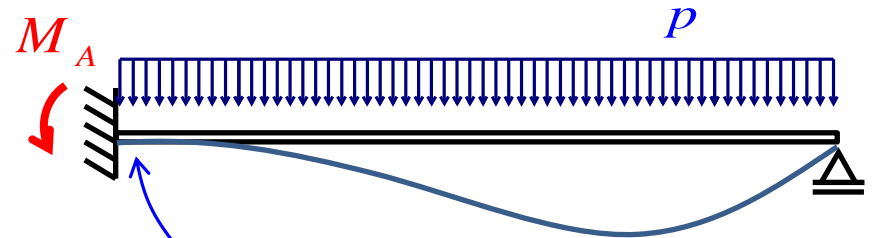
Escolhe-se uma
“Estrutura Isostática Fundamental”:



E a correspondente
“Incógnita Hiperestática”:



A combinação dos carregamentos
internos com o esforço hiperestático
deve recuperar as condições de
contorno da estrutura original:



$$\theta_A = \theta_{A,p} + \theta_{A,M_A} = 0 \quad (4)$$

(“Equação de compatibilidade”)

Das tabelas:

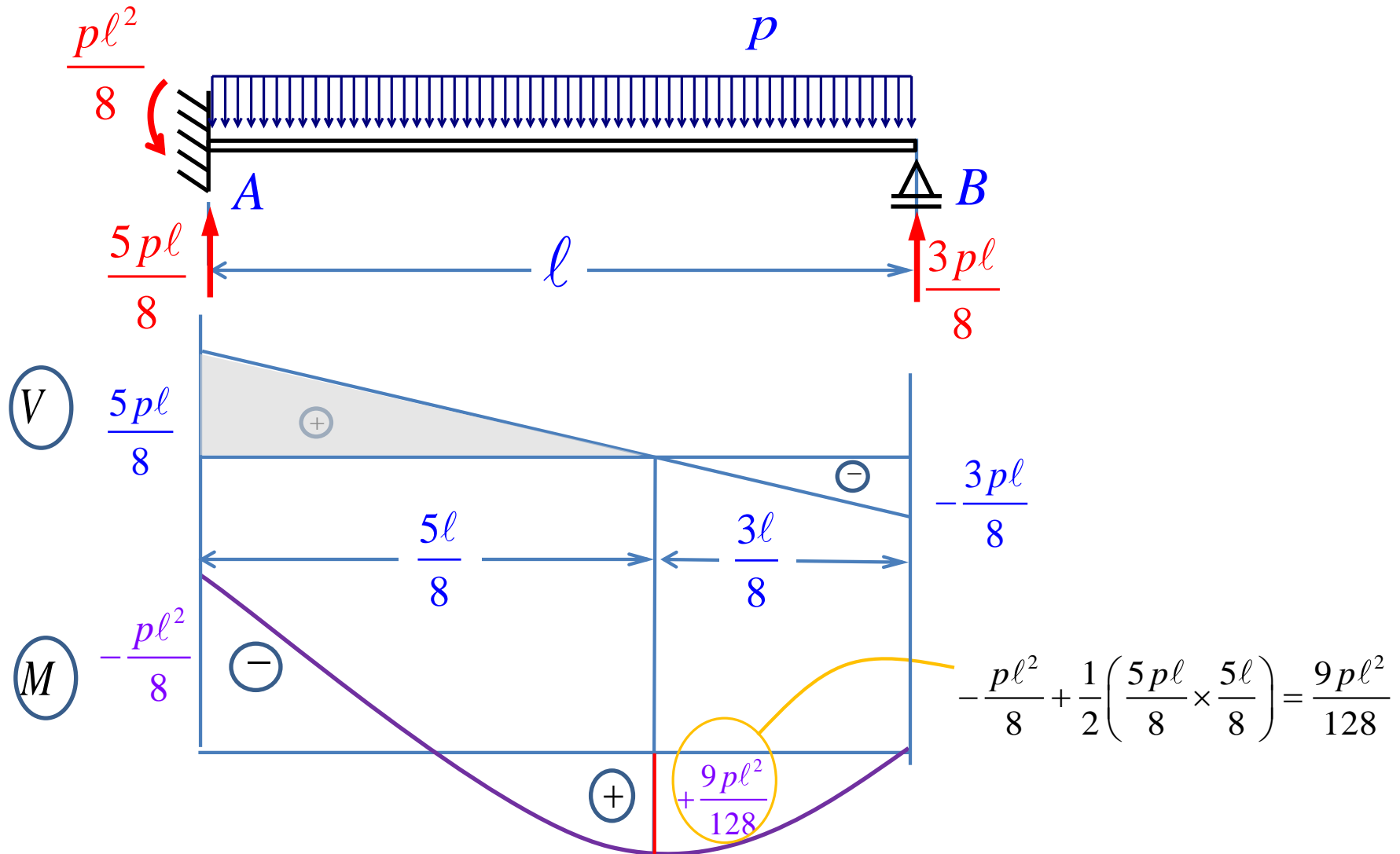
$$\theta_{A,p} = -\frac{pl^3}{24EI}$$

$$\theta_{A,M_A} = +\frac{M_A l}{3EI}$$

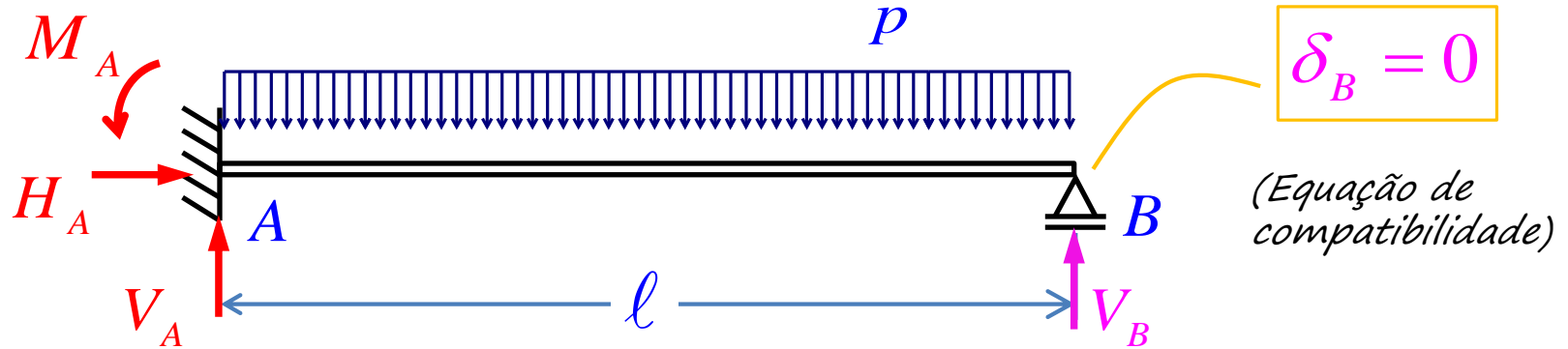
$$\theta_A = -\frac{pl^3}{24EI} + \frac{M_A l}{3EI} = 0 \quad \therefore \quad M_A = \frac{pl^2}{8} \quad (4)$$



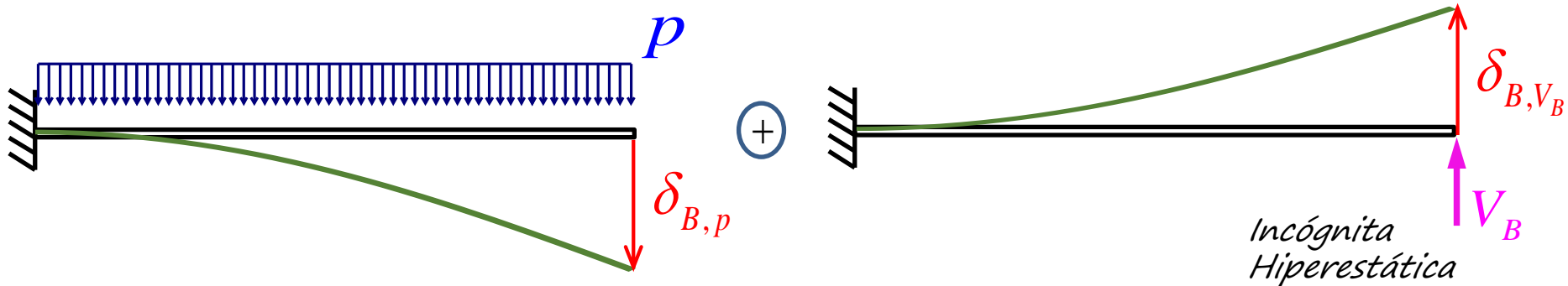
Combinando (4') \rightarrow (3'): $\frac{pl^2}{8} + V_B l = \frac{pl^2}{2} \quad \therefore \boxed{V_B = \frac{3pl}{8}} \quad \therefore \boxed{V_A = \frac{5pl}{8}}$



Exemplo 2: viga engastada-apoiada, sujeita a carregamento uniformemente distribuído
Resolução alternativa, tomando V_B como incógnita hiperestática:



Estrutura Isostática Fundamental:



Equação de compatibilidade:

$$\delta_B = \delta_{B,p} + \delta_{B,V_B} = 0$$



Das tabelas:

$$\left. \begin{aligned} \delta_{B,p} &= -\frac{pl^4}{8EI} \\ \delta_{B,V_B} &= +\frac{V_B l^3}{3EI} \end{aligned} \right\} \delta_B = -\frac{pl^4}{8EI} + \frac{V_B l^3}{3EI} = 0$$

$$\therefore V_B = \frac{3pl}{8} \quad \text{OK!}$$

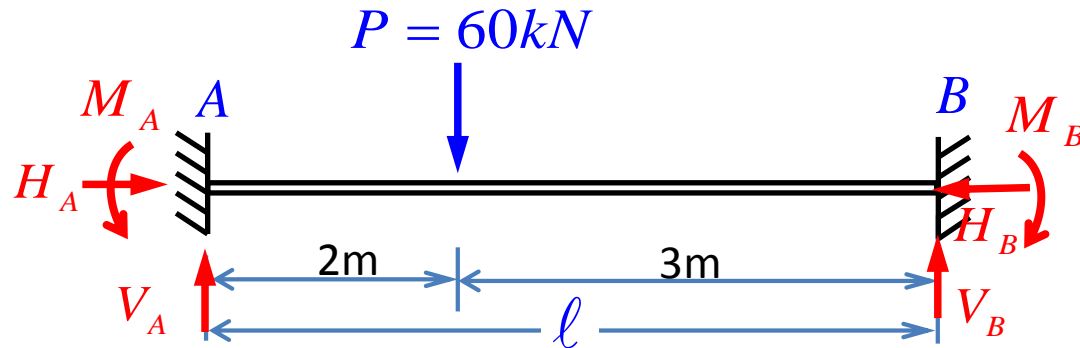
Equações de equilíbrio:

$$(1) \quad \uparrow \sum F_Y = V_A + V_B - pl = 0 \quad \therefore V_A = pl - V_B = \frac{5pl}{8} \quad \text{OK!}$$

$$(2) \quad \sum M_{(A)} = M_A + V_B l - \frac{pl^2}{2} = 0 \quad \therefore M_A = \frac{pl^2}{2} - V_B l = \frac{pl^2}{8} \quad \text{OK!}$$



Exemplo 3: Viga biengastada, sujeita a carga concentrada.



✓ Considerando todas as reações de apoio possíveis e equações de equilíbrio no plano, temos:

- 6 reações de apoio
 - 3 equações de equilíbrio
- } Viga 3x hiperestática

✓ Como não temos esforço externo na horizontal (H), $H_A = H_B = 0$. Logo temos apenas quatro reações de apoio e duas equações de equilíbrio linearmente independentes:

- 4 reações de apoio
 - 2 equações de equilíbrio
- } Viga 2x hiperestática

Equações de equilíbrio:

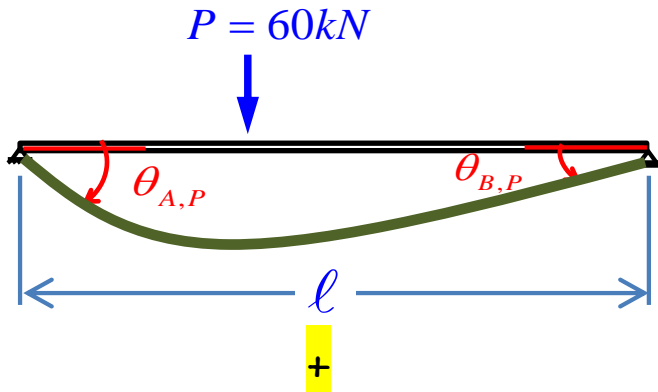
$$(1) \quad \uparrow \sum F_Y = V_A + V_B - P = 0 \quad \therefore \quad \boxed{V_A + V_B = 60} \quad (2')$$

$$(2) \quad \sum M_{(A)} = M_A - M_B + V_B \ell - Pa = 0 \quad \therefore \quad \boxed{M_A - M_B + 5V_B = 120} \quad (3')$$



Estrutura Isostática Fundamental : Viga biapoiada (eMA e MB são as incógnitas hiperestáticas:

1. carregamento externo **P**

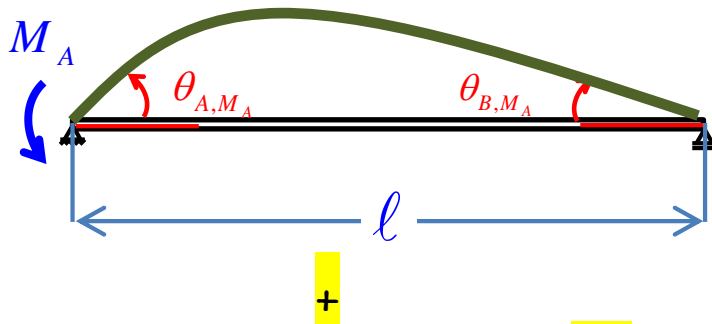


Das tabelas:

$$\theta_{A,P} = -\frac{Pb(\ell^2 - b^2)}{6\ell EI}$$

$$\theta_{B,P} = \frac{Pa(\ell^2 - a^2)}{6\ell EI}$$

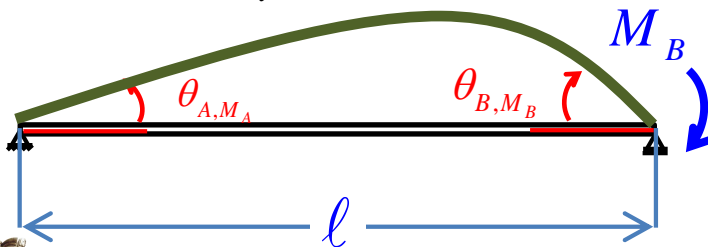
2. Incógnita hiperestática **MA**:



$$\theta_{A,M_A} = +\frac{M_A \ell}{3EI}$$

$$\theta_{B,M_A} = -\frac{M_A \ell}{6EI}$$

3. Incógnita hiperestática **MB**:



$$\theta_{A,M_B} = +\frac{M_B \ell}{6EI}$$

$$\theta_{B,M_B} = -\frac{M_B \ell}{3EI}$$



A combinação dos carregamentos internos com o esforço hiperestático deve recuperar as condições de contorno da estrutura original:

(“Equações de compatibilidade”)

$$\theta_A = \theta_{A,P} + \theta_{A,M_A} + \theta_{A,M_B} = 0$$

$$\theta_A = -\frac{Pb(\ell^2 - b^2)}{6\ell EI} + \frac{M_A \ell}{3EI} + \frac{M_B \ell}{6EI} = 0$$

$$\frac{5}{3}M_A + \frac{5}{6}M_B = 96$$

$$10M_A + 5M_B = 576 \quad (*)$$

$$\theta_B = \theta_{B,P} + \theta_{B,M_A} + \theta_{B,M_B} = 0$$

$$\theta_B = \frac{Pa(\ell^2 - a^2)}{6\ell EI} - \frac{M_A \ell}{6EI} - \frac{M_B \ell}{3EI} = 0$$

$$\frac{5}{6}M_A + \frac{5}{3}M_B = 84$$

$$5M_A + 10M_B = 504 \quad (**)$$

Resolvendo () e (**), obtemos:*

$$M_A = 43,2 \text{ kNm}$$

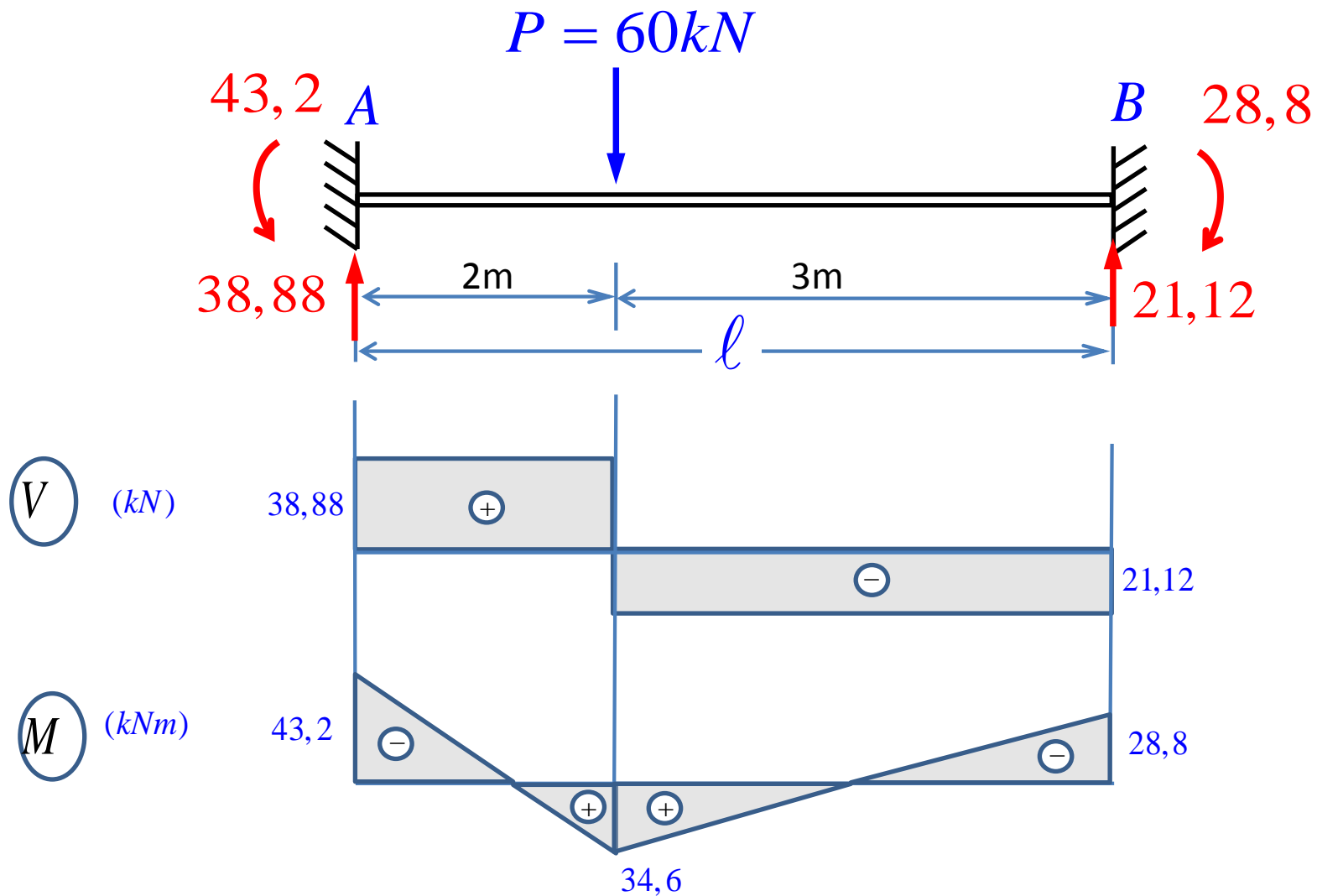
$$M_B = 28,8 \text{ kNm}$$

$$V_A = 38,88 \text{ kN}$$

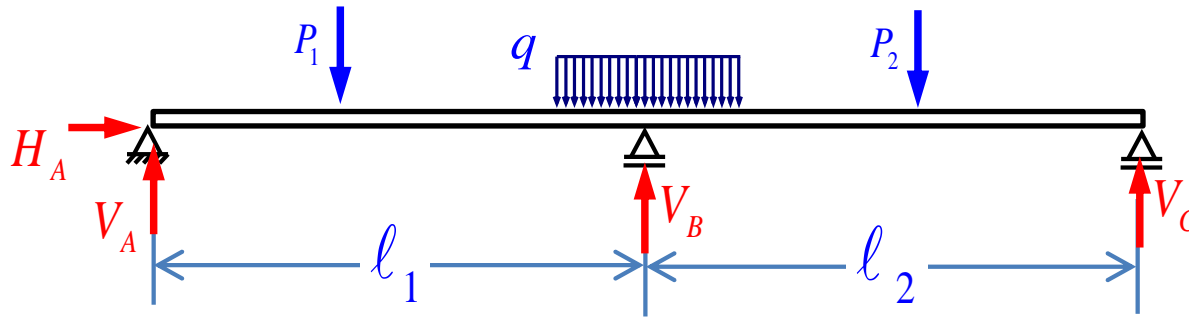
$$V_B = 21,12 \text{ kN}$$



Uma vez conhecidas as reações de apoio, o traçado dos diagramas de esforços solicitantes é direto:

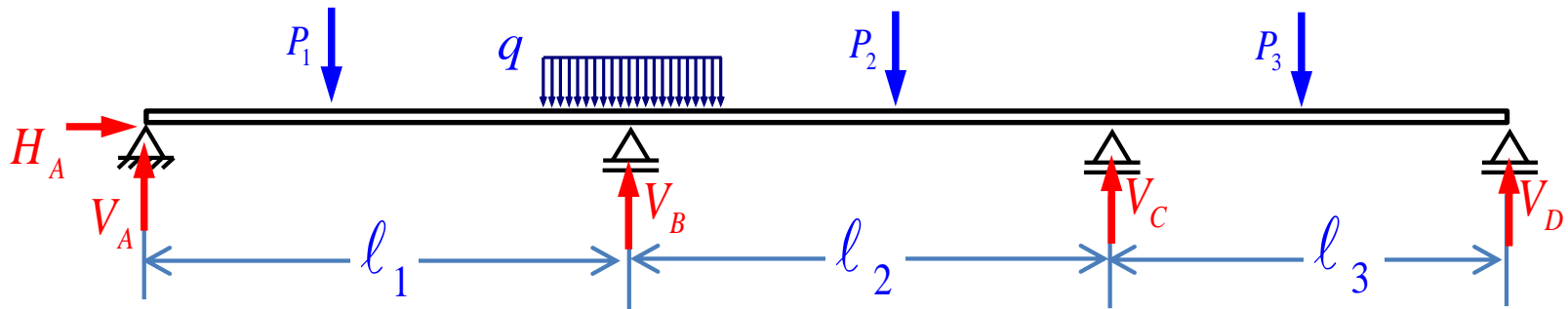


Vigas contínuas



Viga de dois tramos: 4 reações de apoio

1 x hiperestática

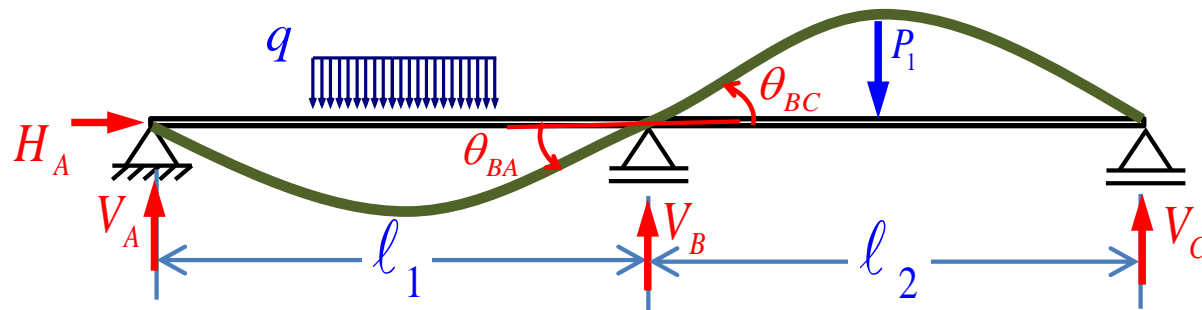


Viga de três tramos – 5 reações de apoio

2 x hiperestática



A rotação sobre cada um dos apoios intermediários oferece uma equação de compatibilidade:

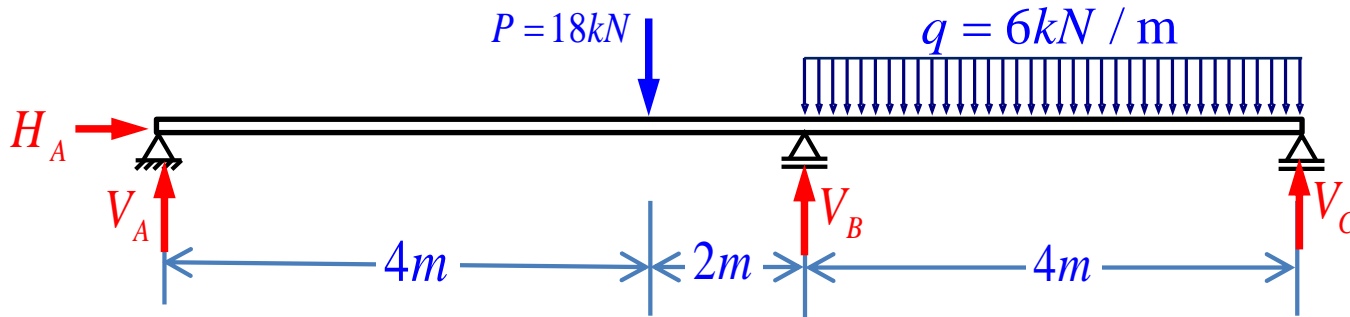


No apoio B:

$$\theta_{BA} = \theta_{BC}$$



Exemplo 4: Determinar as reações de apoio e os diagramas de esforços cortantes e momentos fletores na viga contínua da figura abaixo.



Equações de equilíbrio:

$$(1) \quad \sum_{\rightarrow} F_X = H_A = 0 \quad (1')$$

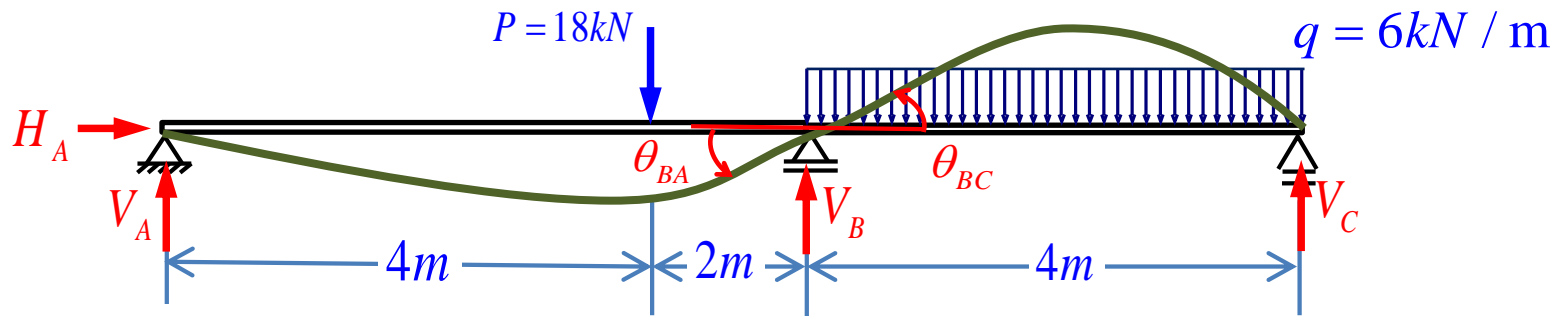
$$(2) \quad \uparrow \sum F_Y = V_A + V_B + V_C - q \ell_{BC} - P = 0$$

$$V_A + V_B + V_C - 6 \times 4 - 18 = 0 \quad \therefore \quad \boxed{V_A + V_B + V_C = 42} \quad (2')$$

$$(3) \quad \sum M_{(A)} = 10 \cdot V_C + 6 \cdot V_B - P \cdot 4 - q \cdot 4 \cdot 8 = 0$$

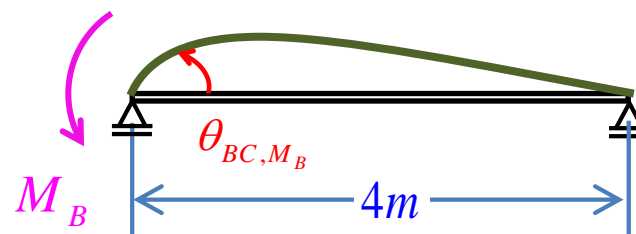
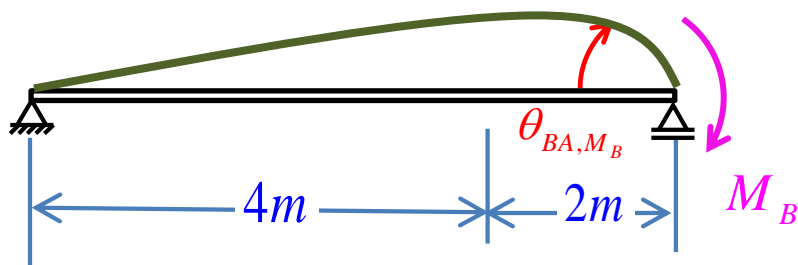
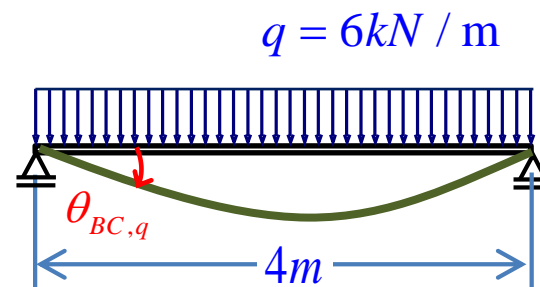
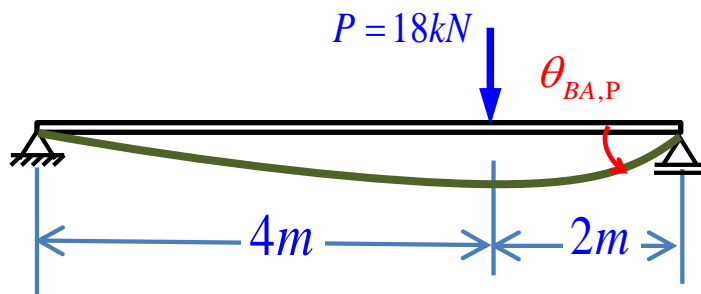
$$\therefore \quad \boxed{10V_C + 6V_B = 264} \quad (3')$$





Equação de compatibilidade:

$$\theta_{BA} = \theta_{BC}$$



$$\theta_{BA,P} + \theta_{BA,M_B} = \theta_{BC,q} + \theta_{BC,M_B}$$



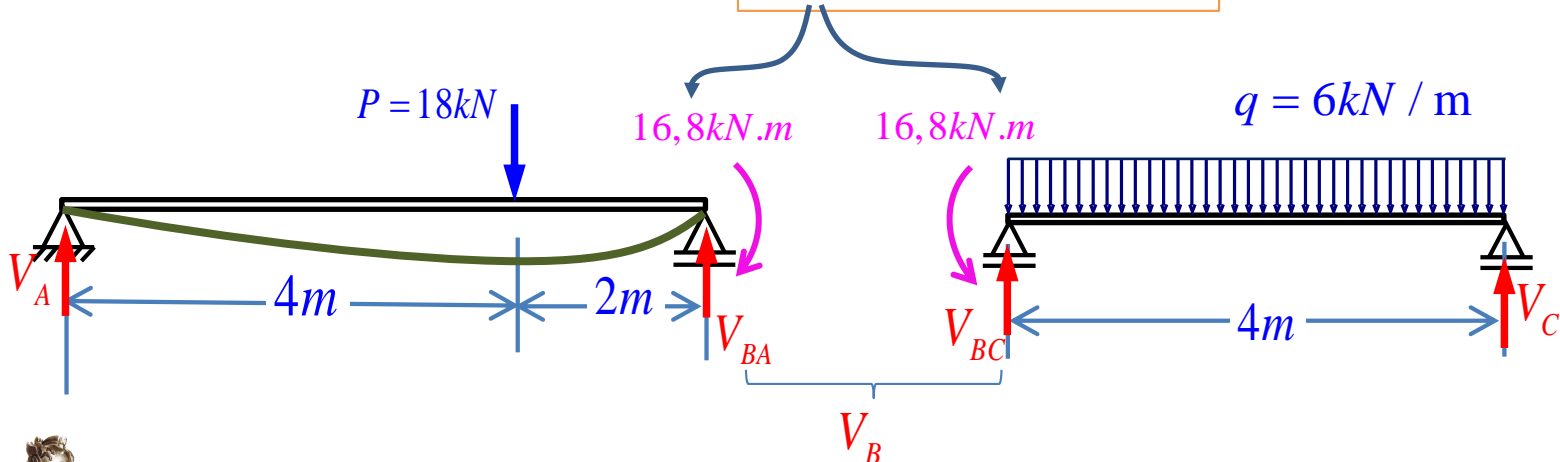
$$\theta_{BA,P} + \theta_{BA,M_B} = \theta_{BC,q} + \theta_{BC,M_B}$$

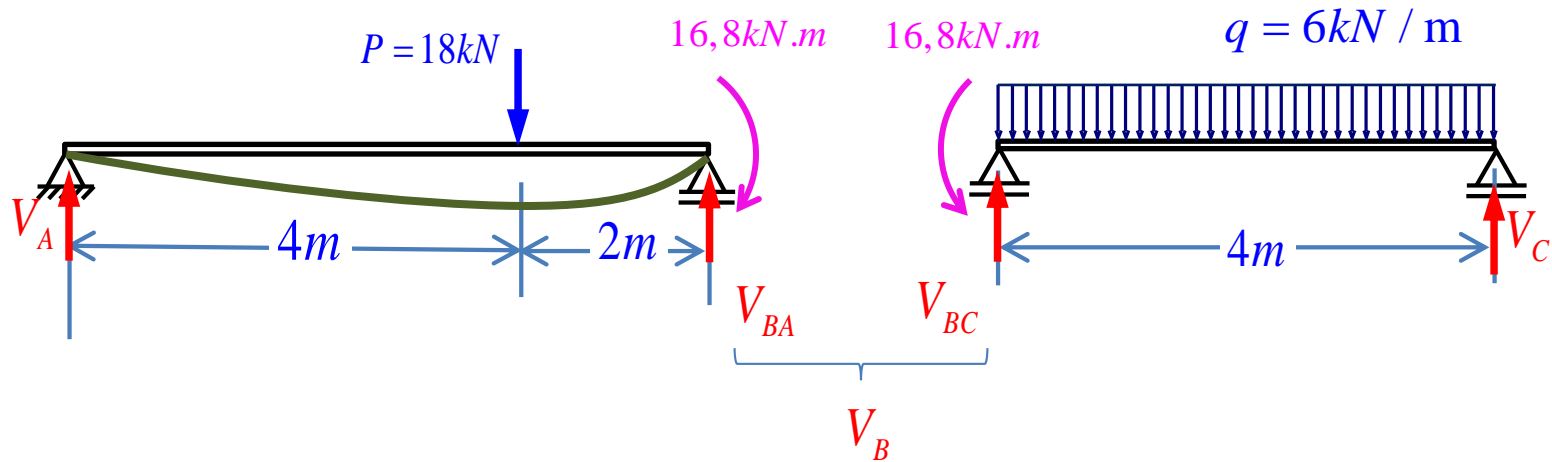
$$\frac{Pab(\ell_{BA} + a)}{6\ell_{BA}EI} - \frac{M_B \ell_{BA}}{3EI} = -\frac{q\ell_{BC}^3}{24EI} + \frac{M_B \ell_{BC}}{3EI}$$

$$\frac{18 \times 4 \times 2(6 + 4)}{6 \times 6} - \frac{M_B \times 6}{3} = -\frac{6 \times 4^3}{24} + \frac{M_B \times 4}{3}$$

$$40 - 2M_B = -16 + \frac{4}{3}M_B$$

$$M_B = 16,8 \text{ kN.m}$$





$$\sum M_{(A)} = 6 \cdot V_{BA} - 18 \cdot 4 - 16,8 = 0$$

$$\sum M_{(C)} = 4 \times V_{BC} - 6 \times 4 \times 2 - 16,8 = 0$$

$$V_{BA} = 14,8 \text{ kN}$$

$$V_{BC} = 16,2 \text{ kN}$$

$$V_B = 14,8 + 16,2 = 31 \text{ kN}$$

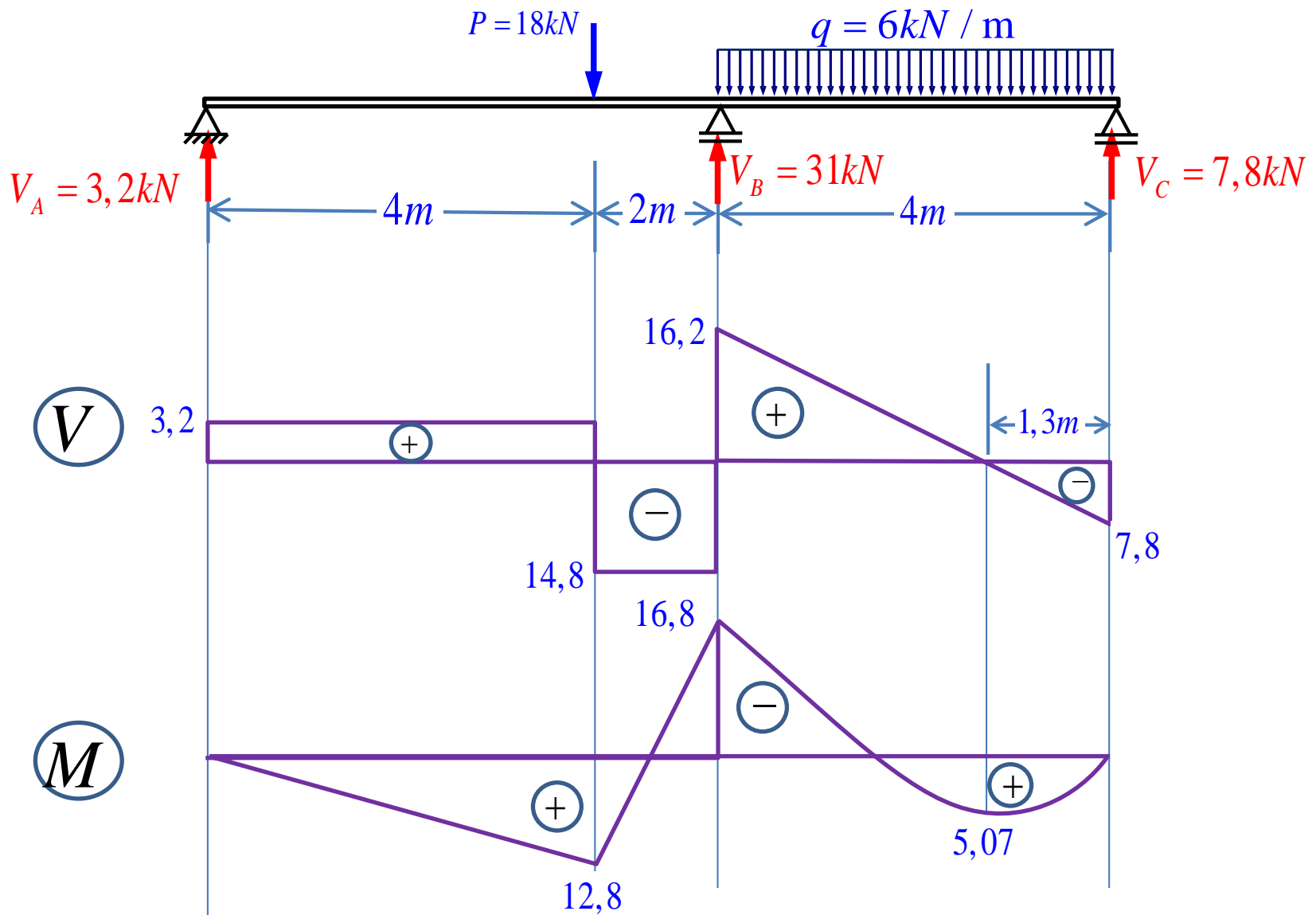
Substituindo V_B em (3') temos: $10V_C + 6 \times 31 = 264$

$$V_C = 7,8 \text{ kN}$$

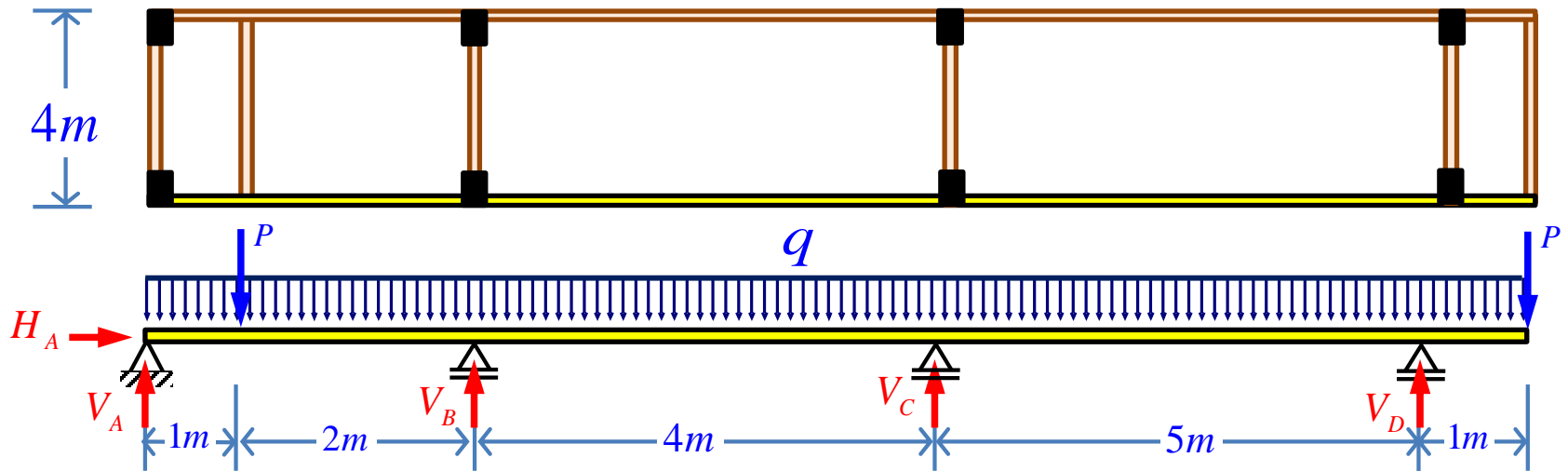
Substituindo V_B e V_C em (2') temos: $V_A + V_B + V_C = 42$

$$V_A = 3,2 \text{ kN}$$





Exemplo 5: Determinar os esforços na viga de concreto de seção transversal 20 cm x 50 cm. Sobre as vigas tem-se alvenaria de tijolos cerâmicos com pé-direito de 2,75m.



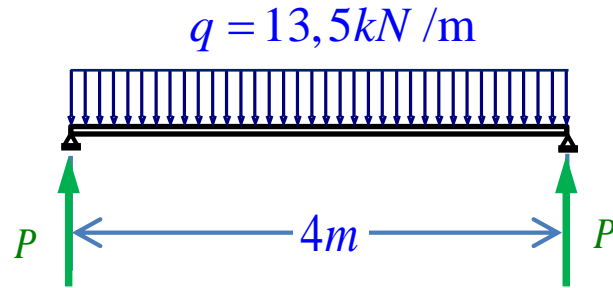
Peso próprio das vigas: $\gamma_c = 25 \text{ kN / m}^3 \quad \therefore \quad q_{\text{viga}} = 0,2 \times 0,5 \times 25 = 2,5 \text{ kN / m}$

Peso próprio da parede: $\gamma_{\text{alv}} = 16 \text{ kN / m}^3 \quad \therefore \quad q_{\text{alv}} = 0,25 \times 2,75 \times 16 = 11 \text{ kN / m}$

$$q = q_{\text{viga}} + q_{\text{alv}} = 2,5 + 11 = 13,5 \text{ kN / m}$$

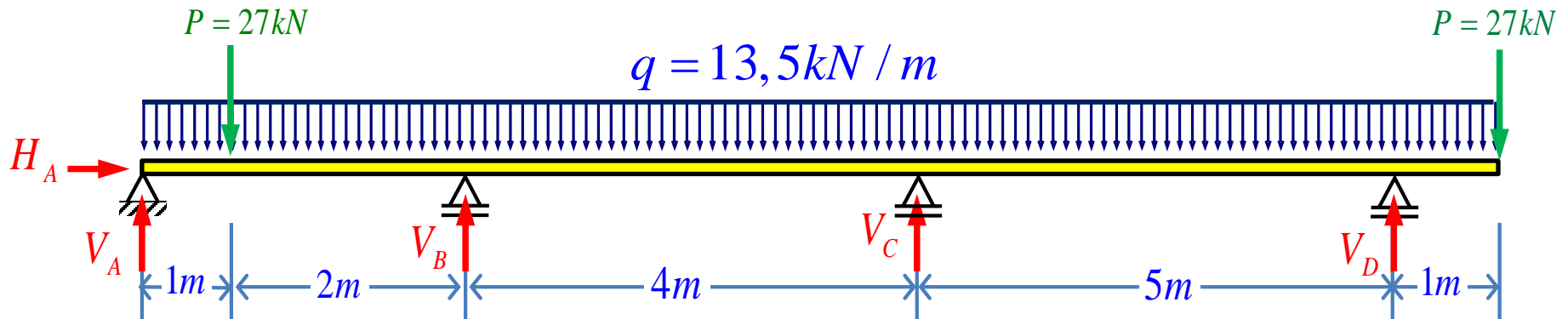


Vigas transversais:



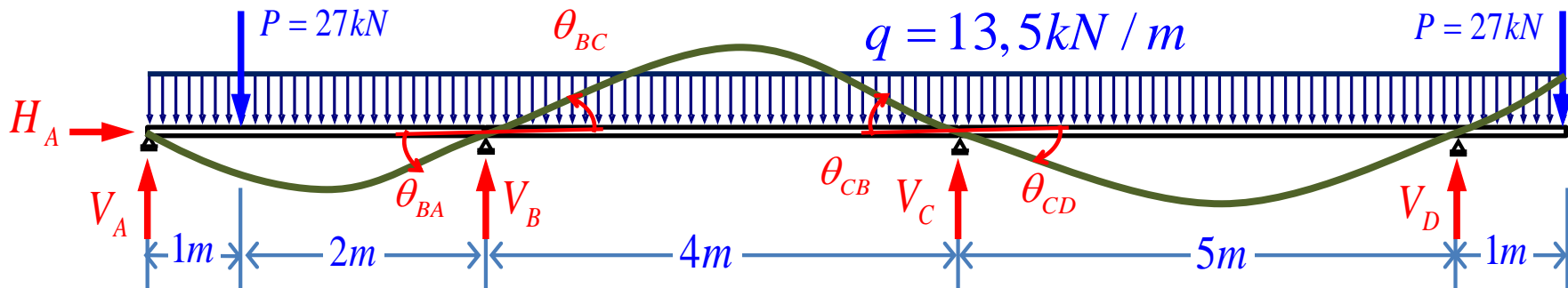
$$P = \frac{q \times \ell}{2} = \frac{13,5 \times 4}{2} = 27 \text{ kN}$$

Viga Longitudinal:



- 5 reações de apoio \Rightarrow Viga 2x hiperestática
- 3 equações de equilíbrio





Equações de equilíbrio:

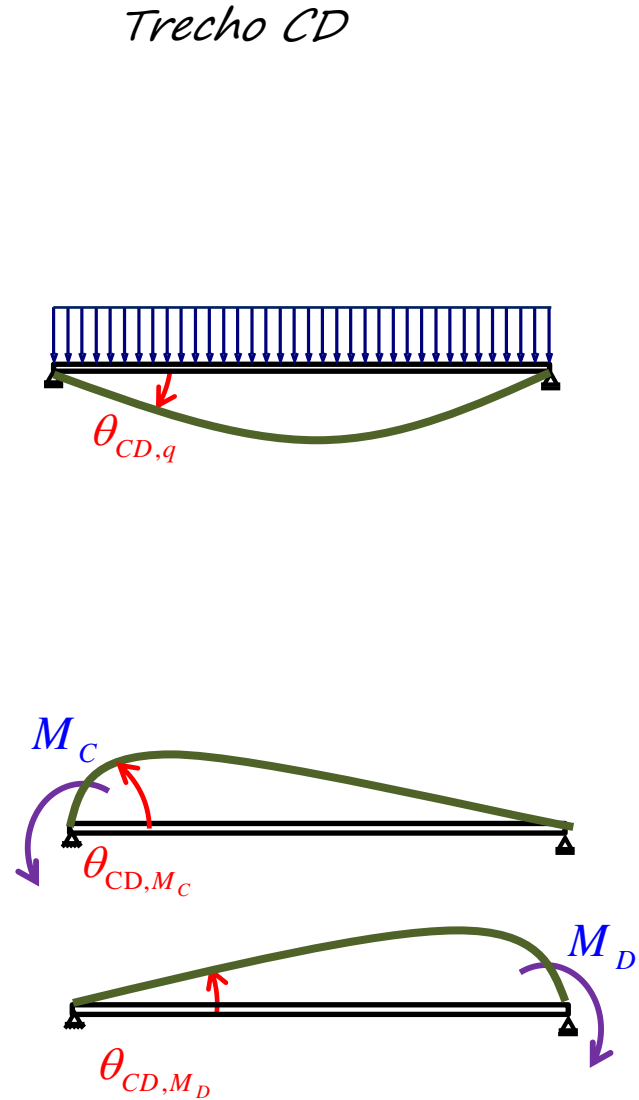
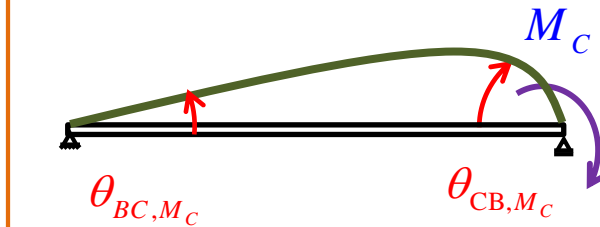
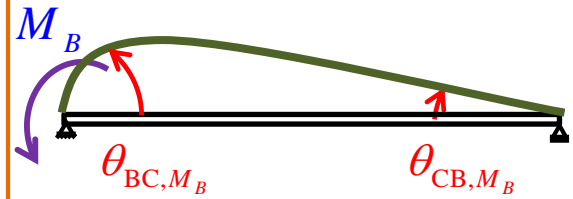
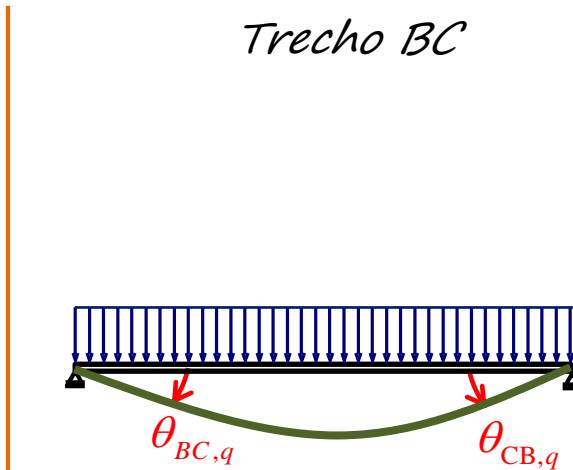
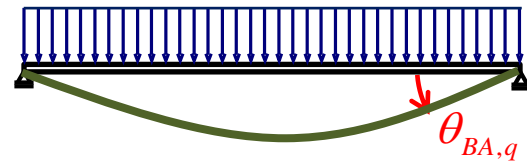
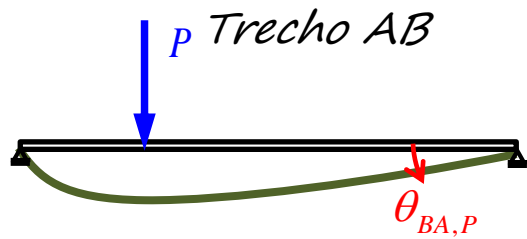
$$\left\{ \begin{array}{l} (1) \quad \sum_{\rightarrow} F_X = H_A = 0 \\ (2) \quad \uparrow \sum F_Y = V_A + V_B + V_C + V_D - q\ell - 2P = 0 \quad \therefore \boxed{V_A + V_B + V_C + V_D = 229,5} \quad (2') \\ (3) \quad \sum M_{(A)} = 3V_B + 7V_C + 12V_D - P \cdot 1 - P \cdot 13 - \frac{q \cdot 13^2}{2} = 0 \quad \therefore \\ \quad \quad \quad \boxed{3V_B + 7V_C + 12V_D = 1518,75} \quad (3') \end{array} \right.$$

Equações de compatibilidade:

$$\boxed{\theta_{BA} = \theta_{BC}}$$

$$\boxed{\theta_{CB} = \theta_{CD}}$$





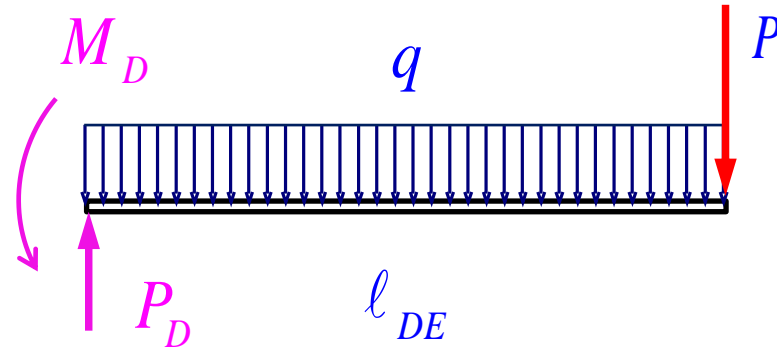
$$\theta_{BA,P} + \theta_{BA,q} + \theta_{BA,M_B} = \theta_{BC,q} + \theta_{BC,M_B} + \theta_{BC,M_C}$$

$$\theta_{CB,q} + \theta_{CB,M_B} + \theta_{CB,M_C} = \theta_{CD,q} + \theta_{CD,M_C} + \theta_{CD,M_D}$$

M_D É o momento que o balanço aplica ao ponto D



Momento fletor e carga vertical no apoio D:



$$M_D = P \times l_{DE} + \frac{q \times l_{DE}^2}{2} = 33,75 \text{ kN.m}$$

$$P_D = P + q \times l_{DE} = 40,5 \text{ kN}$$



Equação de compatibilidade para o apoio B:

$$\theta_{BA,P} + \theta_{BA,q} + \theta_{BA,M_B} = \theta_{BC,q} + \theta_{BC,M_B} + \theta_{BC,M_C}$$

$$\frac{Pab(l_{BA} + a)}{6l_{BA}EI} + \frac{ql_{BA}^3}{24EI} - \frac{M_B l_{BA}}{3EI} = -\frac{ql_{BC}^3}{24EI} + \frac{M_B l_{BC}}{3EI} + \frac{M_C l_{BC}}{6EI}$$

$$\frac{27 \times 1 \times 2(3+1)}{6 \times 3} + \frac{13,5 \times 3^3}{24} - \frac{M_B \times 3}{3} = -\frac{13,5 \times 4^3}{24} + \frac{M_B \times 4}{3} + \frac{M_C \times 4}{6}$$

$$7M_B + 2M_C = 189,5625 \quad (1)$$



Equação de compatibilidade para o apoio C:

$$\theta_{CB,q} + \theta_{CB,M_B} + \theta_{CB,M_C} = \theta_{CD,q} + \theta_{CD,M_C} + \theta_{CD,M_D}$$

$$\frac{ql_{CB}^3}{24EI} - \frac{M_B l_{CB}}{6EI} - \frac{M_C l_{CB}}{3EI} = -\frac{ql_{CD}^3}{24EI} + \frac{M_C l_{CD}}{3EI} + \frac{M_D l_{CD}}{6EI}$$

$$\frac{13,5 \times 4^3}{24} - \frac{M_B \times 4}{6} - \frac{M_C \times 4}{3} = -\frac{13,5 \times 5^3}{24} + \frac{M_C \times 5}{3} + \frac{M_D \times 5}{6}$$

$$2M_B + 9M_C = 234,5625 \quad (2)$$



$$7M_B + 2M_C = 189,5625 \quad (1)$$

$$2M_B + 9M_C = 234,5625 \quad (2)$$

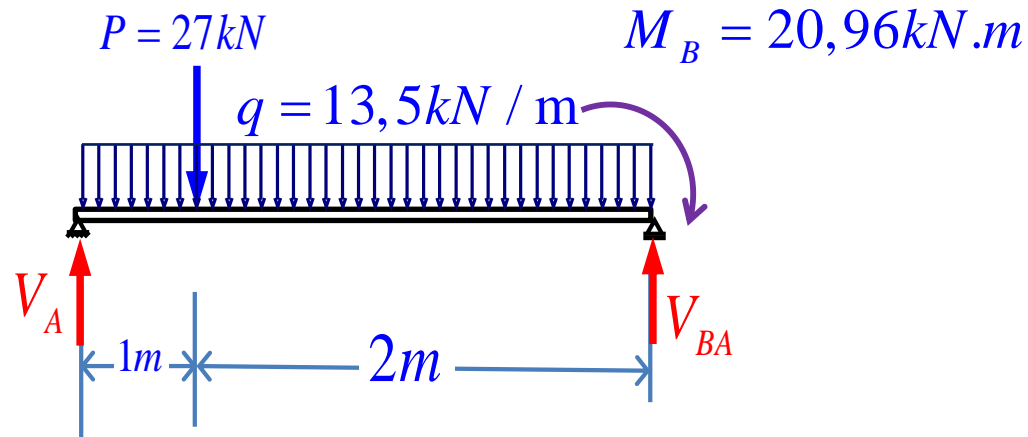
Resolvendo o sistema de equações, obtém-se:

$$M_B = 20,96kN.m$$

$$M_C = 21,4kN.m$$



Trecho AB:



$$\sum M_{(A)} = 3 \cdot V_{BA} - 20,96 - 27 \times 1 - \frac{13,5 \times 3^2}{2} = 0$$

$$V_{BA} = 36,24kN$$

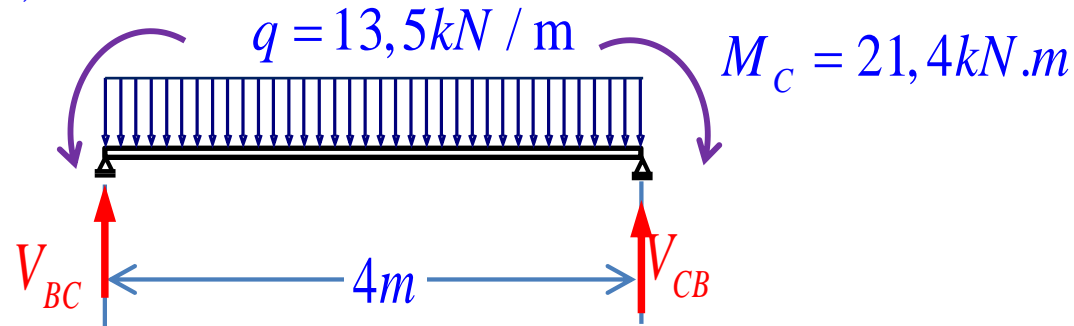
$$\uparrow \sum F_Y = V_A + V_{BA} - 13,5 \times 3 - 27 = 0$$

$$V_A = 31,26kN$$



$$M_B = 20,96 \text{ kN.m}$$

Trecho BC:



$$\sum M_{(C)} = -4 \times V_{BC} + 20,96 + \frac{13,5 \times 4^2}{2} - 21,4 = 0$$

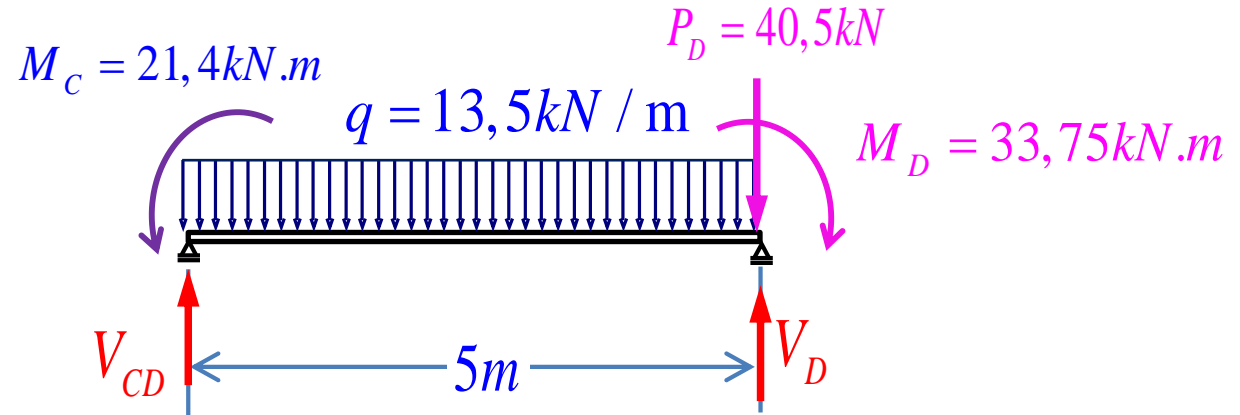
$$V_{BC} = 26,89 \text{ kN}$$

$$\sum M_{(B)} = 4 \times V_{CB} + 20,96 - \frac{13,5 \times 4^2}{2} - 21,4 = 0$$

$$V_{CB} = 27,11 \text{ kN}$$



Trecho CD:



$$\sum M_{(C)} = 5 \times V_D - 33,75 - 40,5 \times 5 - \frac{13,5 \times 5^2}{2} + 21,4 = 0$$

$$V_D = 76,72 \text{ kN}$$

$$\sum M_{(D)} = -5 \times V_{CD} - 33,75 + \frac{13,5 \times 5^2}{2} + 21,4 = 0$$

$$V_{CD} = 31,28 \text{ kN}$$



$$V_A = 31,26kN$$

$$V_D = 76,72kN$$

$$V_{BA} = 36,24kN$$

$$V_{BC} = 26,89kN$$

$$V_B = V_{BA} + V_{BC} = 36,24 + 26,89$$

$$V_B = 63,13kN$$

$$V_{CB} = 27,11kN$$

$$V_{CD} = 31,28kN$$

$$V_C = V_{CB} + V_{CD} = 27,11 + 31,28$$

$$V_C = 58,39kN$$



