Multivariable Controller Synthesis using SISO Design Methods

Daniel N. Mohsenizadeh, Lee H. Keel and Shankar P. Bhattacharyya

***Abstract*— This paper proposes a new method to design multivariable controllers for linear Multi-Input Multi-Output (MIMO) control systems using the Smith-McMillan form. The Smith-McMillan form of the transfer function matrix of a MIMO plant is an equivalent diagonal transfer function matrix using which the problem of multivariable controller synthesis can be reduced to multiple Single-Input Single-Output (SISO) controller designs. If the designed SISO controllers satisfy certain relative degree conditions, then the corresponding multivariable controller, to be connected to the MIMO plant, will be proper. In this paper we show how such multivariable controllers can be designed to satisfy closed loop stability and reference tracking. We also provide some illustrative examples.**

I. INTRODUCTION

Many methods have been proposed to design multivariable controllers for the class of Linear Time-Invariant (LTI) Multi-Input Multi-Output (MIMO) systems. A broad overview of linear multivariable control system design can be found in [1], [2], [3]. Matrix Fraction Decomposition (MFD) is a technique to write a rational matrix describing the dynamics of a MIMO system, known as the transfer function matrix, as two coprime polynomial matrices [4]. Using MFD allows us to develop state-space realizations of the transfer function matrix, enabling us to use a wide range of state-space design methods. Many approaches are also provided for frequency domain design. The generalization of the Nyquist criterion to MIMO system is given in [5]. A variety of techniques on this topic are also presented in [6], [7]. Another approach that is often used to deal with MIMO control problems is to decouple the different channels of a MIMO system through state feedback or output feedback [8]; however, there are systems that may not be decoupled in this fashion or even if they are, the controller may be of higher order.

In this paper, we present an alternative approach to design multivariable controllers. We use the equivalent Smith-McMillan form of the transfer function matrix of a linear MIMO plant which itself is a decoupled representation of the MIMO plant and is a diagonal matrix obtained after performing multiplications by appropriate unimodular matrices. In such a framework, the original MIMO control design problem reduces to multiple Single-Input Single-Output (SISO) controller designs. This results in the designed controller, denoted by C_d, to be of diagonal form. Therefore, many robust control methods that are well-suited for SISO systems, such as PID control [9], [10], can be effectively used toward MIMO control design. The multivariable controller C, to be connected to the original MIMO plant, can be obtained by transforming back the designed diagonal controller C_d through the same unimodular matrices used to calculate the Smith-McMillan form of the plant. The designed controller C_d must be such that the multivariable controller C becomes proper and provides stability and asymptotic tracking. This imposes a set of constraints on the relative degree of each element of C_d. We explore these conditions and also provide some illustrative design examples.

This paper is organized as follows. In Section II we provide some preliminary material on the calculation of the Smith-McMillan form of a rational matrix. We present our main result in Section III. Some illustrative examples are given in Section IV. Finally, we summarize with our concluding remarks in Section V.

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II. PRELIMINARIES

Let us denote by $P(s)$ the transfer function matrix of a linear MIMO plant. Suppose that $P(s)$ is an $n \times n$ matrix,

$$P(s) = \begin{bmatrix} p_{11}(s) & \dots & p_{1n}(s) \\ \vdots & \ddots & \vdots \\ p_{n1}(s) & \cdots & p_{nn}(s) \end{bmatrix}, \tag{1}$$

where each transfer function $p_{ij}(s),\ i,j = 1,2,,\dots,n$ is rational and proper. $P(s)$ can be written as

$$P(s) = \frac{1}{d(s)} N(s), \tag{2}$$

where $d(s)$ is the least common multiple of the denominators of all the elements in $P(s)$, and $N(s)$ is a polynomial matrix. Pre-multiplication and post-multiplication of $N(s)$ by appropriate choices of unimodular matrices results in an equivalent diagonal polynomial matrix $S(s)$, known as the Smith form,

$$S(s) = \underbrace{Y_y(s) \dots Y_2(s) Y_1(s)}_{Y(s)} N(s) \underbrace{U_1(s) U_2(s) \dots U_u(s)}_{U(s)}. \tag{3}$$

A unimodular polynomial matrix is a square polynomial matrix whose inverse is also a polynomial matrix. A necessary and sufficient condition for this is that the determinant of the polynomial matrix be a constant. Therefore, $S(s)$ can be written as

$$S = \operatorname{diag}[\epsilon_1'(s), \epsilon_2'(s), \ldots, \epsilon_n'(s)], \tag{4}$$

where

$$\epsilon_i'(s)|\epsilon_{i+1}'(s), \quad i = 1, 2, \ldots, n-1, \tag{5}$$

meaning that each polynomial $\epsilon_i'(s)$ divides $\epsilon_{i+1}'(s)$ for $i = 1, 2, \ldots, n-1$. Dividing each diagonal element of $S(s)$, $\epsilon_i'(s)$, by $d(s)$ and performing all possible cancellations yields coprime polynomials $\epsilon_i(s)$ and $\psi_i(s)$ such that

$$\frac{\epsilon_i(s)}{\psi_i(s)} = \frac{\epsilon_i'(s)}{d(s)}, \quad i = 1, 2, \ldots, n, \tag{6}$$

and

$$\begin{aligned} &\epsilon_i(s)|\epsilon_{i+1}(s), \\ &\psi_{i+1}(s)|\psi_i(s), \quad i = 1, 2, \ldots, n-1. \end{aligned} \tag{7}$$

The Smith-McMillan form of the transfer function matrix $P(s)$ will be

$$P_d(s) = \operatorname{diag}\left[\frac{\epsilon_1(s)}{\psi_1(s)}, \frac{\epsilon_2(s)}{\psi_2(s)}, \ldots, \frac{\epsilon_n(s)}{\psi_n(s)}\right]. \tag{8}$$

The poles of $P(s)$ are the roots of the following polynomial, known as the pole polynomial $p(s)$,

$$p(s) := \psi_1(s)\psi_2(s)\cdots\psi_n(s). \tag{9}$$

Similarly, the roots of the zero polynomial $z(s)$,

$$z(s) := \epsilon_1(s)\epsilon_2(s)\cdots\epsilon_n(s), \tag{10}$$

are the zeros of $P(s)$.

III. MAIN RESULT

Consider the linear MIMO plant P shown in Fig. 1 where

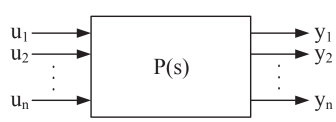

Fig. 1. A linear MIMO plant

the inputs and outputs are related by

$$y(s) = P(s)u(s). \tag{11}$$

The control problem is to design a multivariable controller $C(s)$ so that the closed loop system, shown in Fig. 2, is stable and the output signals track the reference signals.

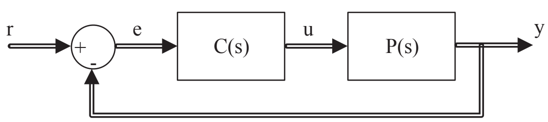

Fig. 2. A MIMO closed loop system with a multivariable controller

Let us denote the Smith-McMillan form of the transfer function matrix $P(s)$ by $P_d(s)$. Thus, we can write

$$P_d(s) = Y(s)P(s)U(s), \tag{12}$$

or

$$P(s) = Y^{-1}(s)P_d(s)U^{-1}(s), \tag{13}$$

where $Y(s)$ and $U(s)$ are appropriate unimodular matrices. Note that $Y^{-1}(s)$ and $U^{-1}(s)$ are unimodular polynomial matrices as well. Using (11) and (13) we obtain

$$y(s) = Y^{-1}(s)P_d(s)U^{-1}(s)u(s), \tag{14}$$

and by pre-multiplying (14) by $Y(s)$,

$$\underbrace{Y(s)y(s)}_{y_d(s)} = P_d(s)\underbrace{U^{-1}(s)u(s)}_{u_d(s)}, \tag{15}$$

where we define

$$u_d(s) := U^{-1}(s)u(s), \tag{16}$$
$$y_d(s) := Y(s)y(s). \tag{17}$$

A diagonal controller $C_d(s)$ is now to be designed for the equivalent diagonal system $P_d(s)$ as depicted in Fig. 3. Since the Smith-McMillan form $P_d(s)$ is of diagonal form, one can design a SISO controller for each single loop, considering each diagonal element of $P_d(s)$, independently. The transfer function matrix of the designed controller $C_d(s)$ can be written as

$$C_d(s) = \operatorname{diag}\left[C_1^d(s), C_2^d(s), \ldots, C_n^d(s)\right], \tag{18}$$

where each designed controller $C_k^d(s),\ k = 1, 2, \ldots, n$ has a relative degree r_k^d.

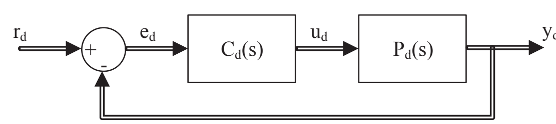

Fig. 3. Closed loop system with the designed diagonal controller

Let us define

$$r_d(s) := Y(s)r(s). \tag{19}$$

From the closed loop system in Fig. 3 we have

$$e_d(s) = r_d(s) - y_d(s), \tag{20}$$

or using (17) and (19),

$$\begin{aligned} e_d(s) &= Y(s)r(s) - Y(s)y(s) = Y(s)(r(s) - y(s)) \\ &= Y(s)e(s). \end{aligned} \tag{21}$$

The inputs and outputs of C_d are related through

$$u_d(s) = C_d(s)e_d(s), \tag{22}$$

which can be written as

$$U^{-1}(s)u(s) = C_d(s)Y(s)e(s), \tag{23}$$

using the definitions in (16) and (21). Pre-multiplying (23) by $U(s)$ yields

$$u(s) = \underbrace{U(s)C_d(s)Y(s)}_{C(s)}\, e(s), \tag{24}$$

where

$$C(s) := U(s)C_d(s)Y(s) = \begin{bmatrix} C_{11}(s) & \dots & C_{1n}(s) \\ \vdots & \ddots & \vdots \\ C_{n1}(s) & \cdots & C_{nn}(s) \end{bmatrix}, \tag{25}$$

is a non-diagonal multivariable controller to be connected to $P(s)$ (see Fig. 2). From (25) it is clear that

$$C_d(s) = U^{-1}(s)C(s)Y^{-1}(s). \tag{26}$$

We introduce the following 4 lemmas leading us toward the design of a diagonal controller C_d that guarantees stability, asymptotic tracking and properness of the corresponding multivariable controller C.

Lemma 1: The multivariable controller $C(s)$ stabilizes the MIMO plant $P(s)$ if and only if the designed diagonal controller $C_d(s)$ stabilizes the equivalent diagonal plant $P_d(s)$.

Proof: The necessary and sufficient condition to prove this is to show that both closed loop systems have the same characteristic polynomials. The characteristic polynomial of the closed loop system attained by connecting the designed diagonal controller $C_d(s)$ to the equivalent diagonal plant $P_d(s)$ (see Fig. 3) is the numerator of $\delta_d(s)$,

$$\delta_d(s) := \det[I + P_d(s)C_d(s)], \tag{27}$$

or using the forms in (8) and (18),

$$\delta_d(s) = \prod_{k=1}^{n} \det\left(1 + \frac{\epsilon_k(s)}{\psi_k(s)} C_k^d(s)\right), \tag{28}$$

because $I + P_d(s)C_d(s)$ is diagonal. Using (12) and (26), we can write

$$\begin{aligned} \delta_d(s) &= \det([I + P_d(s)C_d(s)] \\ &= \det[I + Y(s)P(s)\underbrace{U(s)U^{-1}(s)}_{I}C(s)Y^{-1}(s)] \\ &= \det[I + Y(s)P(s)C(s)Y^{-1}(s)] \\ &= \det[Y(s)Y^{-1}(s) + Y(s)P(s)C(s)Y^{-1}(s)] \\ &= \det(Y(s)[I + P(s)C(s)]Y^{-1}(s)) \\ &= \det[I + P(s)C(s)] =: \delta(s), \end{aligned} \tag{29}$$

where the numerator of $\delta(s)$ is the characteristic polynomial of the original closed loop system in Fig. 2. ■

Lemma 2: If $Y(s)$ is a unimodular polynomial matrix, then $Y(0)$ is a full rank matrix.

Proof: Let us write $Y(s)$ as

$$Y(s) = Y_0 + Y_1 s + \cdots, \tag{30}$$

and its inverse $Y^{-1}(s)$, which exists and is a unimodular polynomial matrix as well, in the following form

$$Y^{-1}(s) = L_0 + L_1 s + \cdots. \tag{31}$$

We now have

$$\begin{aligned} I &= Y(s)Y^{-1}(s) \\ &= (Y_0 + Y_1 s + \cdots)(L_0 + L_1 s + \cdots) \\ &= Y_0 L_0 + (Y_0 L_1 + Y_1 L_0)s + \cdots. \end{aligned} \tag{32}$$

Equation (32) is valid if and only if

$$\begin{aligned} Y_0 L_0 &= I, \\ Y_0 L_1 + Y_1 L_0 &= 0, \\ &\vdots \end{aligned} \tag{33}$$

Therefore, the first condition in (33), $Y_0 L_0 = I$, proves that $Y(0) = Y_0$ is a full rank matrix. ■

Lemma 3: The output signal $y(t)$ tracks the reference signal $r(t)$ in the original system (see Fig. 2) if and only if the output $y_d(t)$ tracks the reference $r_d(t)$ (see Fig. 3).

Proof: If $y_d(t)$ tracks $r_d(t)$, then

$$\lim_{t\to\infty} e_d(t) = 0, \tag{34}$$

or by the final value theorem, we have

$$\lim_{s\to 0} s e_d(s) = 0. \tag{35}$$

Substituting $e_d(s)$ by (21) gives

$$\lim_{s\to 0} sY(s)e(s) = 0. \tag{36}$$

According to Lemma 2, $Y(0)$ is a full rank matrix; thus, (36) is valid if and only if

$$\lim_{s\to 0} s e(s) = 0, \tag{37}$$

which is

$$\lim_{t\to\infty} e(t) = 0, \tag{38}$$

for the original closed loop system, implying that the output $y(t)$ tracks the reference $r(t)$. The reverse direction of the proof follows immediately. ■

Remark 1: For $y_{d_k}(t)$ to track $r_{d_k}(t)$, the controller $C_k^d(s)$ can be designed as

$$C_k^d(s) = \frac{\alpha_k(s)}{\beta_k(s)\gamma_k(s)}, \tag{39}$$

where the characteristic polynomial

$$\psi_k(s)\beta_k(s)\gamma_k(s) + \epsilon_k(s)\alpha_k(s), \tag{40}$$

is Hurwitz, and $\beta_k(s)$ has roots at the poles of $r_{d_k}(s)$, and the product $\beta_k(s)\gamma_k(s)$ has no RHP cancellations with $\epsilon_k(s)$.

Let us write the unimodular matrix $Y(s)$, in (12), as

$$Y(s) = \begin{bmatrix} y_{11}(s) & \dots & y_{1n}(s) \\ \vdots & \ddots & \vdots \\ y_{n1}(s) & \cdots & y_{nn}(s) \end{bmatrix}, \tag{41}$$

where $y_{kj}(s),\ k,j=1,2,\ldots,n$ is a polynomial of degree d^y_{kj}. Similarly, each entry of $U(s)$ in (12) is a polynomial, $u_{ik}(s),\ i,k=1,2,\ldots,n$ of degree d^u_{ik}.

The following lemma characterizes the required relative degree for each designed SISO controller $C^d_k(s),\ k=1,2,\ldots,n$, so that the corresponding multivariable controller $C(s)$ is proper.

Lemma 4: If the relative degrees r^d_k of the designed SISO controllers $C^d_k(s),\ k=1,2,\ldots,n$, satisfy

$$\min_{k=1,2,\ldots,n}\{r^d_k-d^u_{ik}-d^y_{kj}\}\geq 0,\qquad \forall i,j=1,2,\ldots,n, \tag{42}$$

where d^u_{ik} and d^y_{kj} are the degree of polynomials in the unimodular matrices $U(s)$ and $Y(s)$, respectively, then the corresponding multivariable controller $C(s)$ will be proper. For any k that $u_{ik}(s)C^d_k(s)y_{kj}(s)=0$, then the corresponding $r^d_k-d^u_{ik}-d^y_{kj}$ term in (42) should be neglected.

Proof: Performing the matrix multiplication $U(s)C_d(s)Y(s)$, given in (25), in terms of matrices' elements gives the (i,j) element of $C(s)$ as

$$C_{ij}(s)=\sum_{k=1}^{n}u_{ik}(s)C^d_k(s)y_{kj}(s), \tag{43}$$

which has at least the relative degree r^c_{ij},

$$r^c_{ij}=\min_{k=1,2,\ldots,n}\{r^d_k-d^u_{ik}-d^y_{kj}\}. \tag{44}$$

If for a specific k, $u_{ik}(s)C^d_k(s)y_{kj}(s)=0$, then the corresponding $r^d_k-d^u_{ik}-d^y_{kj}$ term in (44) needs to be neglected. $C(s)$ will be proper if $r^c_{ij}\geq 0,\ \forall i,j=1,2,\ldots,n$. ■

Now, based on Lemmas 1 to 4, we can present our main theorem.

Theorem 1: Given a rational proper transfer function matrix $P(s)$, of a linear MIMO plant, if a diagonal controller $C_d(s)$ is designed such that it stabilizes the equivalent Smith-McMillan form of $P(s)$, denoted by $P_d(s)$, and the output $y_d(t)$ tracks the reference $r_d(t)$, and $C_d(s)$ satisfies the relative degree conditions given in (42), then the corresponding multivariable controller $C(s)$ will be proper and stabilize $P(s)$, and the output $y(t)$ will track the reference $r(t)$.

Proof: The proof follows immediately from Lemmas 1 to 4. ■

IV. Illustrative Examples

Example 1: A Two-Input Two-Output (TITO) Stable Plant Consider the following transfer function matrix

$$P(s)=\begin{bmatrix}\frac{4}{(s+1)(s+2)} & \frac{-1}{s+1}\\ \frac{2}{s+1} & \frac{-1}{2(s+1)(s+2)}\end{bmatrix}, \tag{45}$$

representing a TITO stable plant and suppose that a multivariable controller is to be designed so that the closed loop system is stable and the output signals track unit step reference signals.

The least common multiple of the denominators of all the elements in $P(s)$ is

$$d(s)=(s+1)(s+2). \tag{46}$$

Thus, we can write $P(s)$ as

$$P(s)=\frac{1}{(s+1)(s+2)}\underbrace{\begin{bmatrix}4 & -s-2\\ 2s+4 & \frac{-1}{2}\end{bmatrix}}_{N(s)}. \tag{47}$$

The Smith form of $N(s)$ can be obtained by multiplying $N(s)$ by the following unimodular matrices $Y(s)$ and $U(s)$,

$$\begin{aligned}S(s)&=\underbrace{\begin{bmatrix}\frac{1}{4} & 0\\ -s-2 & 2\end{bmatrix}}_{Y(s)}\underbrace{\begin{bmatrix}4 & -s-2\\ 2s+4 & \frac{-1}{2}\end{bmatrix}}_{N(s)}\underbrace{\begin{bmatrix}1 & \frac{s+2}{4}\\ 0 & 1\end{bmatrix}}_{U(s)}\\ &=\begin{bmatrix}1 & 0\\ 0 & s^2+4s+3\end{bmatrix}.\end{aligned} \tag{48}$$

For the above unimodular matrices, we have $d^u_{11}=d^u_{21}=d^u_{22}=d^y_{11}=d^y_{12}=d^y_{22}=0$ and $d^u_{12}=d^y_{21}=1$. Dividing each element of $S(s)$, in (48), by $d(s)$, in (46), and performing all possible cancellations gives the Smith-McMillan form of $P(s)$ as

$$P_d(s)=\begin{bmatrix}\frac{1}{(s+1)(s+2)} & 0\\ 0 & \frac{s+3}{s+2}\end{bmatrix}. \tag{49}$$

Now, two SISO controllers $C^d_1(s)$ and $C^d_2(s)$ can be designed for two single loops corresponding to the diagonal elements of $P_d(s)$, to satisfy closed loop stability and reference signal tracking. Let us consider the following controller

$$C_d(s)=\begin{bmatrix}\frac{3}{s(s+4)} & 0\\ 0 & \frac{1}{2s(s+1)}\end{bmatrix}, \tag{50}$$

where both relative degrees r^d_1 and r^d_2 are 2. Based on the statement of Lemma 4, if r^d_1 and r^d_2 satisfy the following set of inequalities,

$$\begin{aligned}&\min_{k=1,2}\{r^d_k-d^u_{1k}-d^y_{k1}\}\geq 0,\\ &\min_{k=1,2}\{r^d_k-d^u_{1k}-d^y_{k2}\}\geq 0,\\ &\min_{k=1,2}\{r^d_k-d^u_{2k}-d^y_{k1}\}\geq 0,\\ &\min_{k=1,2}\{r^d_k-d^u_{2k}-d^y_{k2}\}\geq 0,\end{aligned} \tag{51}$$

then the multivariable controller $C(s)$ will be proper. Expanding the terms in (51) and substituting the values for d^u, d^y and r^d, it can be easily verified that the above set of inequalities are satisfied. The transfer function matrix $C(s)$ is

$$C(s)=U(s)C_d(s)Y(s)=\begin{bmatrix}\frac{-(s^3+8s^2+14s+10)}{8s(s+1)(s+4)} & \frac{s+2}{4s(s+1)}\\ \frac{-(s+2)}{2s(s+1)} & \frac{1}{s(s+1)}\end{bmatrix}, \tag{52}$$

which is proper. Connecting $C(s)$ to $P(s)$ and closing the loop, the closed loop system transfer function matrix $H(s)$

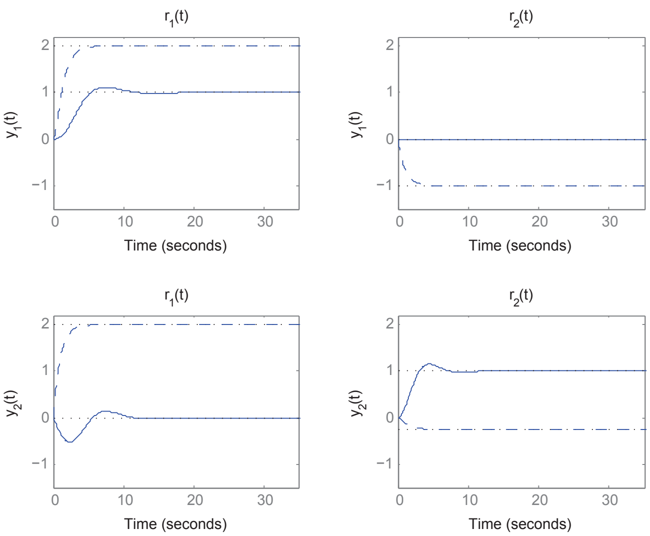

Fig. 4. Step response of the closed loop system after connecting $C(s)$ in (52) to $P(s)$ in (45) (solid line), and the step response of the open loop plant $P(s)$ in (45) (dashed line).

becomes

$$H(s) = [I + P(s)C(s)]^{-1}P(s)C(s)$$
$$= \begin{bmatrix} \frac{3}{s^4+7s^3+14s^2+8s+3} & 0 \\ \frac{-s(s+2)(s^4+10s^3+29s^2+32s+12)}{2(s^4+7s^3+14s^2+8s+3)(2s^3+6s^2+5s+3)} & \frac{s+3}{2s^3+6s^2+5s+3} \end{bmatrix}. \tag{53}$$

The response of the closed loop system to unit steps is plotted in Fig. 4 and verifies tracking of step inputs.

Example 2: A Two-Input Two-Output (TITO) Unstable Plant Consider the following TITO unstable transfer function matrix

$$P(s) = \begin{bmatrix} \frac{4}{s-1} & \frac{-1}{s+1} \\ \frac{2}{s+1} & \frac{1}{(s-1)} \end{bmatrix}. \tag{54}$$

Our objective is to design a multivariable controller $C(s)$ to stabilize the closed loop system and also make the outputs track unit steps. Here, we have $d(s) = (s+1)(s-1)$. Thus, $P(s)$ can be written as

$$P(s) = \frac{1}{(s+1)(s-1)} \underbrace{\begin{bmatrix} 4s+4 & -s+1 \\ 2s-2 & s+1 \end{bmatrix}}_{N(s)}, \tag{55}$$

and the Smith form of $N(s)$ can be calculated as follows

$$S(s) = \underbrace{\begin{bmatrix} \frac{1}{8} & \frac{-1}{4} \\ \frac{-s+1}{3} & \frac{2s+2}{3} \end{bmatrix}}_{Y(s)} \underbrace{\begin{bmatrix} 4s+4 & -s+1 \\ 2s-2 & s+1 \end{bmatrix}}_{N(s)} \underbrace{\begin{bmatrix} 1 & \frac{3s+1}{8} \\ 0 & 1 \end{bmatrix}}_{U(s)}$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & s^2 + \frac{2}{3}s + 1 \end{bmatrix}. \tag{56}$$

For this example, we have $d_{11}^u = d_{21}^u = d_{22}^u = d_{11}^y = d_{12}^y = 0$ and $d_{12}^u = d_{21}^y = d_{22}^y = 1$. The Smith-McMillan form of $P(s)$ will be

$$P_d(s) = \begin{bmatrix} \frac{1}{(s^2-1)} & 0 \\ 0 & \frac{3s^2+2s+3}{3(s^2-1)} \end{bmatrix}. \tag{57}$$

Now, two SISO stabilizing controllers $C_1^d(s)$ and $C_2^d(s)$ should be designed for the diagonal elements of $P_d(s)$ in (57) that also guarantee unit step tracking. Moreover, the relative degrees of these controllers need to satisfy the inequality conditions given in the statement of Lemma 4. Consider the following controller

$$C_d(s) = \begin{bmatrix} \frac{5s^2+5s+1}{s(.1s+1)} & 0 \\ 0 & \frac{5}{s(.1s+1)} \end{bmatrix}, \tag{58}$$

where the relative degrees of $C_1^d(s)$ and $C_2^d(s)$ meet the inequality conditions in Lemma 4. The multivariable controller $C(s)$ becomes

$$C(s) = \begin{bmatrix} \frac{1.05s+.336}{s(.1s+1)} & \frac{.415s+.166}{s(.1s+1)} \\ \frac{-1.67(s-1)}{s(.1s+1)} & \frac{3.33(s+1)}{s(.1s+1)} \end{bmatrix}, \tag{59}$$

which is strictly proper, and the closed loop system transfer function matrix $H(s)$ will be

$$H(s) = \begin{bmatrix} \frac{h_{11}(s)}{g(s)} & \frac{h_{12}(s)}{g(s)} \\ \frac{h_{21}(s)}{g(s)} & \frac{h_{22}(s)}{g(s)} \end{bmatrix}, \tag{60}$$

where

$$\begin{aligned} g(s) &= s^8 + 20s^7 + 198s^6 + 1043s^5 + 3094s^4 \\ &\quad + 3703s^3 + 3873s^2 + 2233s + 500, \\ h_{11}(s) &= 58s^6 + 605s^5 + 2688s^4 + 3862s^3 \\ &\quad + 4420s^2 + 1533s + 500, \\ h_{12}(s) &= -s(17s^5 + 143s^4 - 290s^3 \\ &\quad - 543s^2 + 273s + 400), \\ h_{21}(s) &= s(4s^5 + 28s^4 - 136s^3 \\ &\quad + 73s^2 + 132s - 100), \\ h_{22}(s) &= 42s^6 + 478s^5 + 3105s^4 + 3988s^3 \\ &\quad + 4020s^2 + 2533s + 500. \end{aligned} \tag{61}$$

The response of this closed loop system to unit steps is plotted in Fig. 5. Since $H(s)|_{s=0} = I$, the closed loop system tracks steps.

V. CONCLUDING REMARKS

In this paper we presented an alternative approach to design multivariable controllers for LTI MIMO systems using the Smith-McMillan form. We showed that the MIMO control design task can be accomplished through multiple SISO controller designs which enables the designer to use the full power of classical techniques developed for SISO systems. We also determined conditions on the relative degrees of the designed SISO controllers to guarantee the properness of the resulting multivariable controller as well as stability and reference tracking.

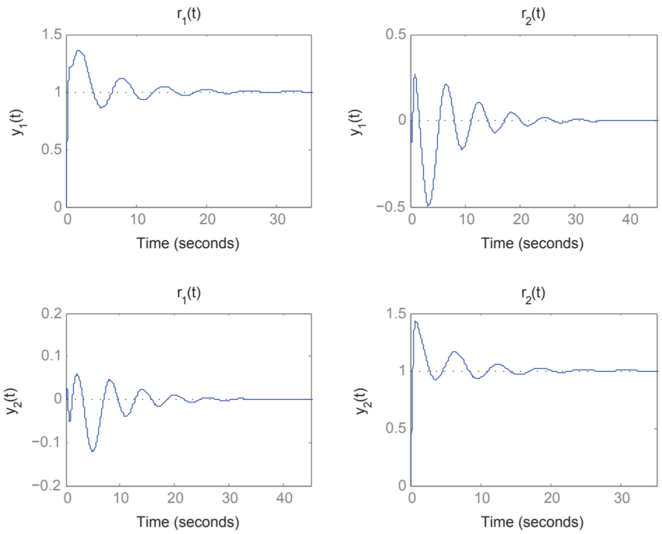

Fig. 5. Step response of the closed loop system after connecting $C(s)$ in (59) to $P(s)$ in (54).

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