(a) barrels rolled in opposite direction (b) barrels rolled in same direction

Fig. 4.23 - Double-circuit transposition scheme

circuits as well, with the nonzero pattern of the matrix in Eq. (4.71) changing to

$$\left[\begin{array}{ccc|ccc} X & 0 & 0 & X & 0 & 0 \\ 0 & X & 0 & 0 & X & 0 \\ 0 & 0 & X & 0 & 0 & X \\ \hline X & 0 & 0 & X & 0 & 0 \\ 0 & X & 0 & 0 & X & 0 \\ 0 & 0 & X & 0 & 0 & X \end{array}\right]$$

where "X" indicates nonzero terms. Re-assigning the phases in Fig. 4.23(b) to CI, BI, AI, AII, BII, CII from top to bottom would change the matrix further to cross-couplings between positive sequence of one circuit and negative sequence of the other circuit, and vice versa,

$$\left[\begin{array}{ccc|ccc} X & 0 & 0 & X & 0 & 0 \\ 0 & X & 0 & 0 & 0 & X \\ 0 & 0 & X & 0 & X & 0 \\ \hline X & 0 & 0 & X & 0 & 0 \\ 0 & 0 & X & 0 & X & 0 \\ 0 & X & 0 & 0 & 0 & X \end{array}\right].$$

4.1.5 Modal Parameters

From the discussions of Section 4.1.3 it should have become obvious that the solution of M-phase transmission line equations becomes simpler if the M coupled equations can be transformed to M decoupled equations. These decoupled equations can then be solved as if they were single-phase equations. For balanced lines, this transformation is achieved with Eq. (4.58).

Many lines are untransposed, however, or each section of a transposition barrel may no longer be short compared with the wave length of the highest frequencies occurring in a particular study, in which case each section must be represented as an untransposed line. Fortunately, the matrices of untransposed lines can be diagonalized as well, with transformations to "modal" parameters derived from eigenvalue/eigenvector theory. The transformation matrices for untransposed lines are no longer known a priori, however, and must be calculated for each particular pair of parameter matrices $[Z'_{phase}]$ and $[Y'_{phase}]$.

To explain the theory, let us start again from the two systems of equations (4.31) and (4.32),

$$-\left[\frac{dV_{phase}}{dx}\right] = [Z'_{phase}]\,[I_{phase}] \tag{4.72a}$$

and

$$-\left[\frac{dI_{phase}}{dx}\right] = [Y'_{phase}][V_{phase}] \tag{4.72b}$$

with $[Y'_{phase}] = j\omega[C'_{phase}]$ if shunt conductances are ignored, as is customarily done. By differentiating the first equation with respect to x, and replacing the current derivative with the second equation, a second-order differential equation for voltages only is obtained,

$$\left[\frac{d^2V_{phase}}{dx^2}\right] = [Z'_{phase}]\,[Y'_{phase}]\,[V_{phase}] \tag{4.73a}$$

Similarly, a second-order differential equation for currents only can be obtained,

$$\left[\frac{d^2I_{phase}}{dx^2}\right] = [Y'_{phase}]\,[Z'_{phase}]\,[I_{phase}] \tag{4.73b}$$

where the matrix products are now in reverse order from that in Eq. (4.73a), and therefore different. Only for balanced matrices, and for the lossless high-frequency approximations discussed in Section 4.1.5.2, would the matrix products in Eq. (4.73a) and (4.73b) be identical.

With eigenvalue theory, it becomes possible to transform the two coupled equations (4.73) from phase quantities to "modal" quantities in such a way that the equations become decoupled, or in terms of matrix algebra, that the associated matrices become diagonal, e.g., for the voltages,

$$\left[\frac{d^2V_{mode}}{dx^2}\right] = [\Lambda]\,[V_{mode}] \tag{4.74}$$

with $[\Lambda]$ being a diagonal matrix. To get from Eq. (4.73a) to (4.74), the phase voltages must be transformed to mode voltages, with

$$[V_{phase}] = [T_v]\,[V_{mode}] \tag{4.75a}$$

and

$$[V_{mode}] = [T_v]^{-1}\,[V_{phase}] \tag{4.75b}$$

Then Eq. (4.73a) becomes

$$\left[\frac{d^2V_{mode}}{dx^2}\right] = [T_v]^{-1}\,[Z'_{phase}]\,[Y'_{phase}]\,[T_v]\,[V_{mode}] \tag{4.76a}$$

which, when compared with Eq. (4.74), shows us that

$$[\Lambda] = [T_v]^{-1} [Z'_{phase}] [Y'_{phase}] [T_v] \tag{4.76b}$$

To find the matrix $[T_v]$ which diagonalizes $[Z'_{phase}][Y'_{phase}]$ is the eigenvalue/eigenvector problem. The diagonal elements of $[\Lambda]$ are the eigenvalues of the matrix product $[Z'_{phase}][Y'_{phase}]$, and $[T_v]$ is the matrix of eigenvectors or modal matrix of that matrix product. There are many methods for finding eigenvalues and eigenvectors. The most reliable method for finding the eigenvalues seems to be the QR-transformation due to Francis [3], while the most efficient method for the eigenvector calculation seems to be the inverse iteration scheme due to Wilkinson [4, 5]. In the supporting routines LINE CONSTANTS and CABLE CONSTANTS, the "EISPACK"-subroutines [67] are used, in which the eigenvalues and eigenvectors of a complex upper Hessenberg matrix are found by the modified LR-method due to Rutishauser. This method is a predecessor of the QR-method, and where applicable, as in the case of positive definite matrices, is more efficient than the QR-method [68]. To transform the original complex matrix to upper Hessenberg form, stabilized elementary similarity transformations are used. For a given eigenvalue λ_k, the corresponding eigenvector $[t_{vk}]$ (= k-th column of $[T_v]$) is found by solving the system of linear equations

$$\{[Z'_{phase}] [Y'_{phase}] - \lambda_k [U]\} [t_{vk}] = 0 \tag{4.77}$$

with [U] = unit or identity matrix. Eq. (4.77) shows that the eigenvectors are not uniquely defined in the sense that they can be multiplied with any nonzero (complex) constant and still remain proper eigenvectors[13], in contrast to the eigenvalues which are always uniquely defined.

Floating-point overflow may occur in eigenvalue/eigenvector subroutines if the matrix is not properly scaled. Unless the subroutine does the scaling automatically, $[Z'_{phase}][Y'_{phase}]$ should be scaled before the subroutine call, by dividing each element by $-(\omega^2\varepsilon_0\mu_0)$, as suggested by Galloway, Shorrows and Wedepohl [39]. This division brings the matrix product close to unit matrix, because $[Z'_{phase}][Y'_{phase}]$ is a diagonal matrix with elements $-\omega^2\varepsilon_0\mu_0$ if resistances, internal reactances and Carson's correction terms are ignored in Eq. (4.7) and (4.8), as explained in Section 4.1.5.2. The eigenvalues from this scaled matrix must of course be multiplied with $-\omega^2\varepsilon_0\mu_0$ to obtain the eigenvalues of the original matrix. In [39] it is also suggested to subtract 1.0 from the diagonal elements after the division; the eigenvalues of this modified matrix would then be the p.u. deviations from the eigenvalues of the lossless high-frequency approximation of Section 4.1.5.2, and would be much more separated from each other than the unmodified eigenvalues which lie close together. Using subroutines based on [67] gave identical results with and without this subtraction of 1.0, however.

In general, a different transformation must be used for the currents,

$$[I_{phase}] = [T_i] [I_{mode}] \tag{4.78a}$$

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[13]This is important if matrices $[T_v]$ obtained from different programs are compared. The ambiguity can be removed in a number of ways, e.g., by agreeing that the elements in the first row should always be 1.0, or by normalizing the columns to a Euclidean vector length of 1.0, that is, by requiring $t_{v1}t_{v1}^* + t_{v2}t_{v2}^* + \ldots = 1.0$, with t^* = conjugate complex of t. In the latter case, there is still ambiguity in the sense that each column could be multiplied with a rotation constant $e^{j\alpha}$ and still have vector length = 1.0.

and

$$[I_{mode}] = [T_i]^{-1} [I_{phase}] \tag{4.78b}$$

because the matrix products in Eq. (4.73a) and (4.73b) have different eigenvectors, though their eigenvalues are identical. Therefore, Eq. (4.73b) is transformed to

$$\left[\frac{d^2 I_{mode}}{dx^2}\right] = [\Lambda] [I_{mode}] \tag{4.79}$$

with the same diagonal matrix as in Eq. (4.74). While $[T_i]$ is different from $[T_v]$, both are fortunately related to each other [58],

$$[T_i] = [T_v^t]^{-1} \tag{4.80}$$

where "t" indicates transposition. It is therefore sufficient to calculate only one of them.

Modal analysis is a powerful tool for studying power line carrier problems [59-61] and radio noise interference [62, 63]. Its use in the EMTP is discussed in Section 4.1.5.3. It is interesting to note that the modes in single-circuit three-phase lines are almost identical with the α, β, 0-components of Section 4.1.3.1 [58]. Whether the matrix products in Eq. (4.73) can always be diagonalized was first questioned by Pelissier in 1969 [64]. Brandao Faria and Borges da Silva have shown in 1985 [65] that cases can indeed be constructed where the matrix product cannot be diagonalized. It is unlikely that such situations will often occur in practice, because extremely small changes in the parameters (e.g., in the 8th significant digit) seem to be enough to make it diagonalizable again. Paul [66] has shown that diagonalization can be guaranteed under simplifying assumptions, e.g., by neglecting conductor resistances.

The physical meaning of modes can be deduced from the transformation matrices $[T_v]$ and $[T_i]$. Assume, for example, that column 2 of $[T_i]$ has entries of (-0.6, 1.0, -0.4). From Eq. (4.78a) we would then know that mode-2 current flows into phase B in one way, with 60% returning in phase A and 40% returning in phase C.

4.1.5.1 Line Equations in Modal Domain

With the decoupled equations of (4.74) and (4.79) in modal quantities, each mode can be analyzed as if it were a single-phase line. Comparing the modal equation

$$\frac{d^2 V_{mode-k}}{dx^2} = \lambda_k V_{mode-k}$$

with the well-known equation of a single-phase line,

$$\frac{d^2 V}{dx^2} = \gamma^2 V$$

with the propagation constant γ defined in Eq. (1.15), shows that the modal propagation constant γ_{mode-k} is the square root of the eigenvalue,

$$\gamma_{mode-k} = \alpha_k + j\beta_k = \sqrt{\gamma_k} \tag{4.81}$$

with

α_k = attenuation constant of mode k (e.g., in Np/km),

β_k = phase constant of mode k (e.g., in rad/km).

The phase velocity of mode k is

$$phase\ \ velocity = \frac{\omega}{\beta_k} \tag{4.82a}$$

and the wavelength is

$$wave\ \ length = \frac{2\pi}{\beta_k} \tag{4.82b}$$

While the modal propagation constant is always uniquely defined, the modal series impedance and shunt admittance as well as the modal characteristic impedance are not, because of the ambiguity in the eigenvectors. Therefore, modal impedances and admittances only make sense if they are specified together with the eigenvectors used in their calculation. To find them, transform Eq. (4.72a) to modal quantities

$$-\left[\frac{dV_{mode}}{dx}\right] = [T_v]^{-1}\,[Z'_{phase}]\,[T_i]\,[I_{mode}] \tag{4.83}$$

The triple matrix product in Eq. (4.83) is diagonal, and the modal series impedances are the diagonal elements of this matrix

$$[Z'_{mode}] = [T_v]^{-1}\,[Z'_{phase}]\,[T_i] \tag{4.84a}$$

or with Eq. (4.80),

$$[Z'_{mode}] = [T_i^t]\,[Z'_{phase}]\,[T_i] \tag{4.84b}$$

Similarly, Eq. (4.72b) can be transformed to modal quantities, and the modal shunt admittances are then the diagonal elements of the matrix

$$[Y'_{mode}] = [T_i]^{-1}\,[Y'_{phase}]\,[T_v] \tag{4.85a}$$

or with Eq. (4.80),

$$[Y'_{mode}] = [T_v^t]\,[Y'_{phase}]\,[T_v] \tag{4.85b}$$

The proof that both [Z'_{mode}] and [Y'_{mode}] are diagonal is given by Wedepohl [58]. Finally, the modal characteristic impedance can be found from the scalar equation

$$Z_{char-mode-k} = \frac{\sqrt{Z'_{mode-k}}}{\sqrt{Y'_{mode-k}}} \tag{4.86a}$$

or from the simpler equation

$$Z_{char-mode-k} = \frac{\gamma_{mode-k}}{Y'_{mode-k}} \tag{4.86b}$$

A good way to obtain the modal parameters may be as follows: First, obtain the eigenvalues λ_k and the eigenvector matrix [T_v] of the matrix product [Z'_{phase}][Y'_{phase}]. Then find [Y'_{mode}] from Eq. (4.85b), and the modal series impedance from the scalar equation

$$Z'_{mode-k} = \frac{\lambda_k}{Y'_{mode-k}} \tag{4.86c}$$

The modal characteristic impedance can then be calculated from Eq. (4.86a), or from Eq. (4.86b) if the propagation constant from Eq. (4.81) is needed as well. If [T_i] is needed, too, it can be found efficiently from Eq. (4.85a)

$$[T_i] = [Y'_{phase}]\,[T_v]\,[Y'_{mode}]^{-1} \tag{4.85c}$$

because the product of the first two matrices is available anyhow when [Y'_{mode}] is found, and the post-multiplication with [Y'_{mode}]$^{-1}$ is simply a multiplication of each column with a constant (suggested by Luis Marti). Eq. (4.85c) also establishes the link to an alternative formula for [T_i] mentioned in [57],

$$[T_i] = [Y'_{phase}]\,[T_v]\,[D]$$

with [D] being an arbitrary diagonal matrix. Setting [D] = [Y'_{mode}]$^{-1}$ leads us to the desirable condition [T_i] = [T_v^t]$^{-1}$ of Eq. (4.80). If the unit matrix were used for [D], all modal matrices in Eq. (4.84) and (4.85) would still be diagonal, but with the strange-looking result that all modal shunt admittances become 1.0 and that the modal series impedances become λ_k. Eq. (4.80) would, of course, no longer be fulfilled. For a lossless line, the modal series impedance would then become a negative resistance, and the modal shunt admittance would become a shunt conductance with a value of 1.0 S. As long as the case is solved in the frequency domain, the answers would still be correct, but it would obviously be wrong to associate such modal parameters with

$$-\frac{\partial v}{\partial x} = R'i \qquad and \qquad -\frac{\partial i}{\partial x} = G'v$$

(with R' negative and G' = 1.0) in the time domain.