ESCOLA POLITÉCNICA DA UNIVERSIDADE DE SÃO PAULO

DEPARTAMENTO DE ENGENHARIA DE ENERGIA E AUTOMAÇÃO ELÉTRICAS

Equações de Estado

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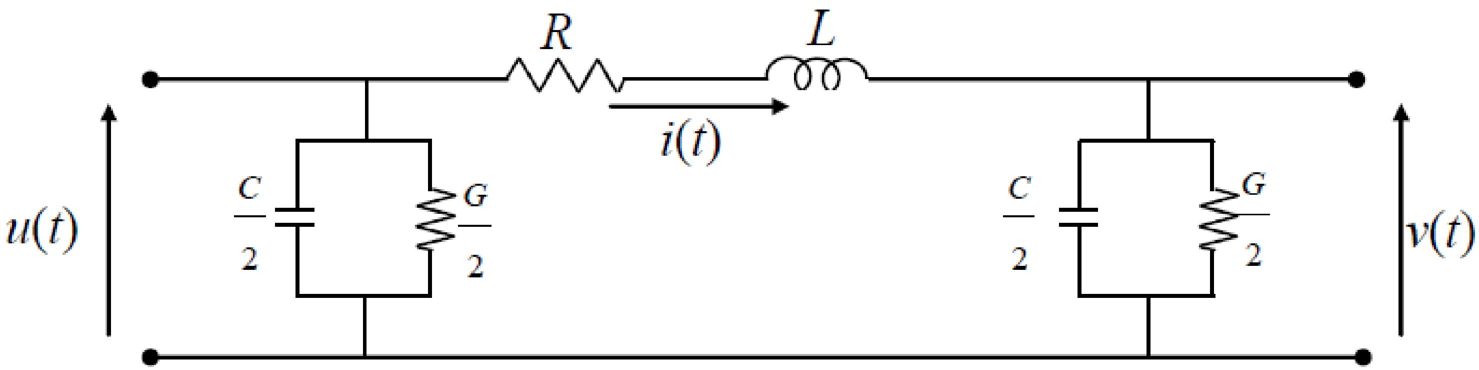
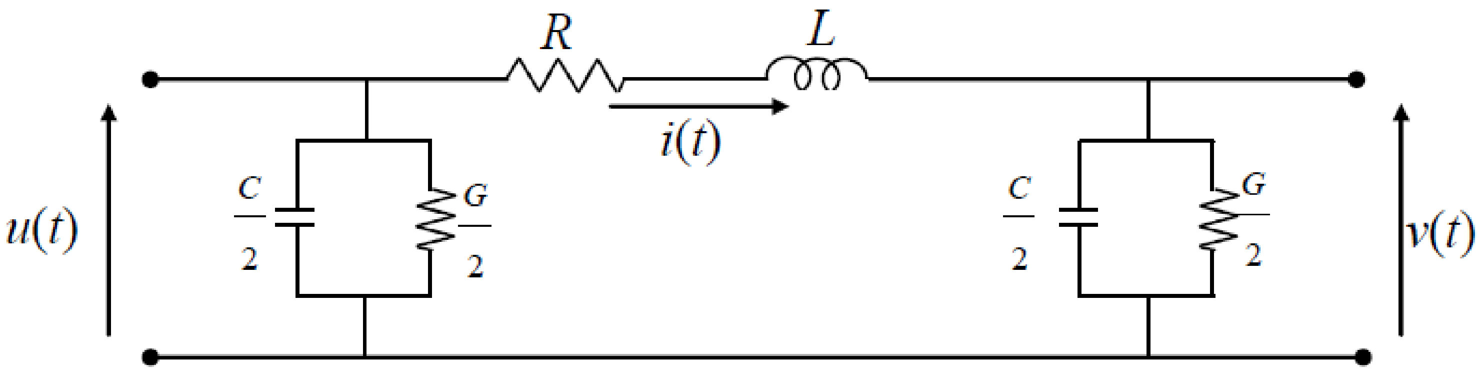
Circuito π



$$u(t) - Ri(t) - L\frac{di(t)}{d(t)} - v(t) = 0$$

$$L\frac{di(t)}{dt} = - Ri(t) - v(t) + u(t) \longrightarrow \frac{di(t)}{dt} = -\frac{R}{L}i(t) - \frac{1}{L}v(t) + \frac{1}{L}u(t)$$

Circuito π



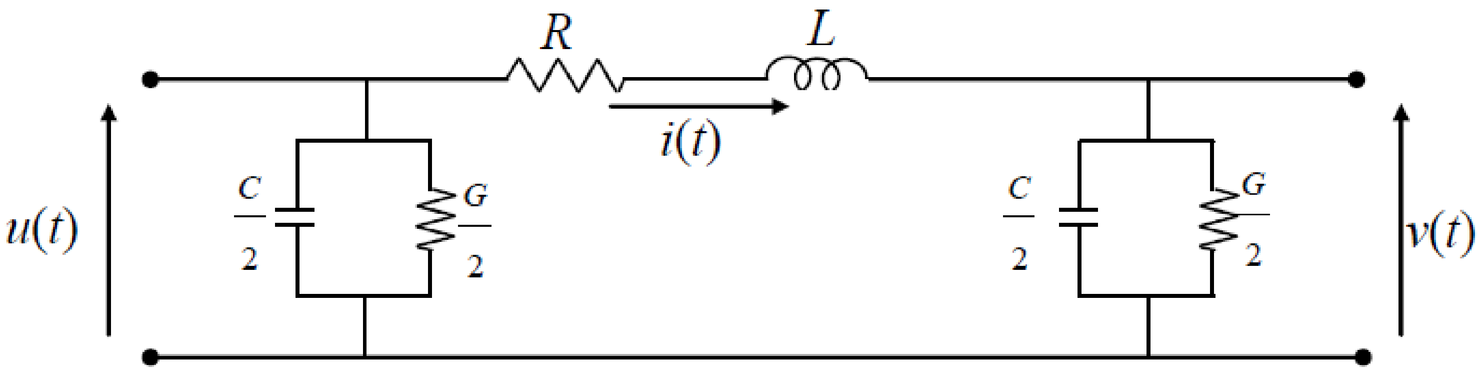
$$i(t) = i_G(t) + i_C(t)$$

$$i_G = v(t)\frac{G}{2}$$

Portanto:

$$i(t) = v(t)\frac{G}{2} + i_C(t)$$

Circuito π



Sendo a tensão no capacitor $v(t) = \frac{1}{C}\int i_C(t)dt$ *e* $i(t) = v(t)\frac{G}{2} + i_C(t)$

$$v(t) = \frac{1}{\frac{C}{2}}\int\left[i(t) - v(t)\frac{G}{2}\right]dt = \frac{2}{C}\int\left[i(t) - v(t)\frac{G}{2}\right]dt$$

$$\frac{dv(t)}{dt} = \frac{d\left[\frac{2}{C}\int\left[i(t) - \frac{G}{2}v(t)\right]dt\right]}{dt} \longrightarrow \frac{dv(t)}{dt} = \frac{2}{C}\left[i(t) - \frac{G}{2}v(t)\right] = \frac{2}{C}i(t) - \frac{G}{C}v(t)$$

Circuito π

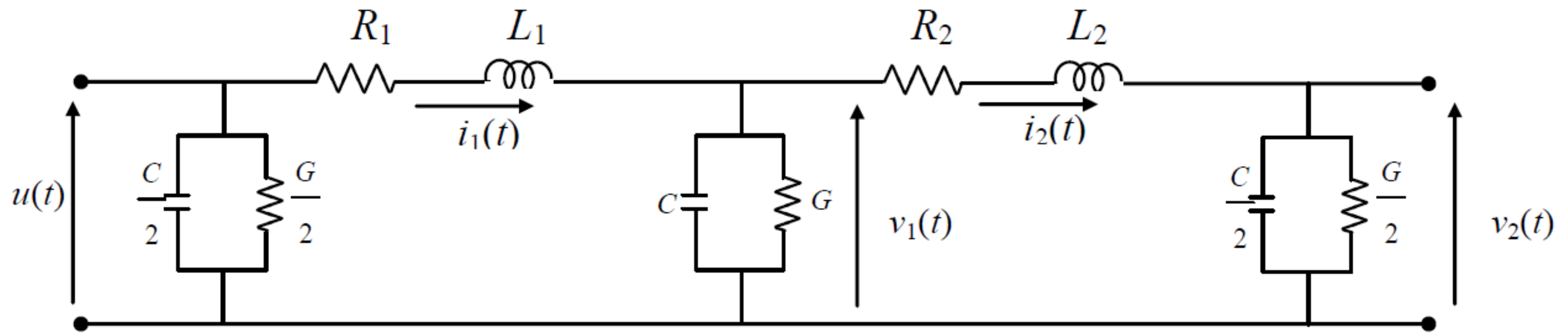
$$\begin{cases} \dfrac{di(t)}{dt} = -\dfrac{R}{L}i(t) - \dfrac{1}{L}v(t) + \dfrac{1}{L}u(t) \\ \dfrac{dv(t)}{dt} = \dfrac{2}{C}i(t) - \dfrac{G}{C}v(t) \end{cases}$$

$$[\dot{x}] = [A][x] + [B]u(t)$$

$$[x] = \begin{bmatrix} i(t) & v(t) \end{bmatrix}^T$$

$$[A] = \begin{bmatrix} -\dfrac{R}{L} & -\dfrac{1}{L} \\ \dfrac{2}{C} & -\dfrac{G}{C} \end{bmatrix} \qquad [B] = \begin{bmatrix} \dfrac{1}{L} & 0 \end{bmatrix}^T$$

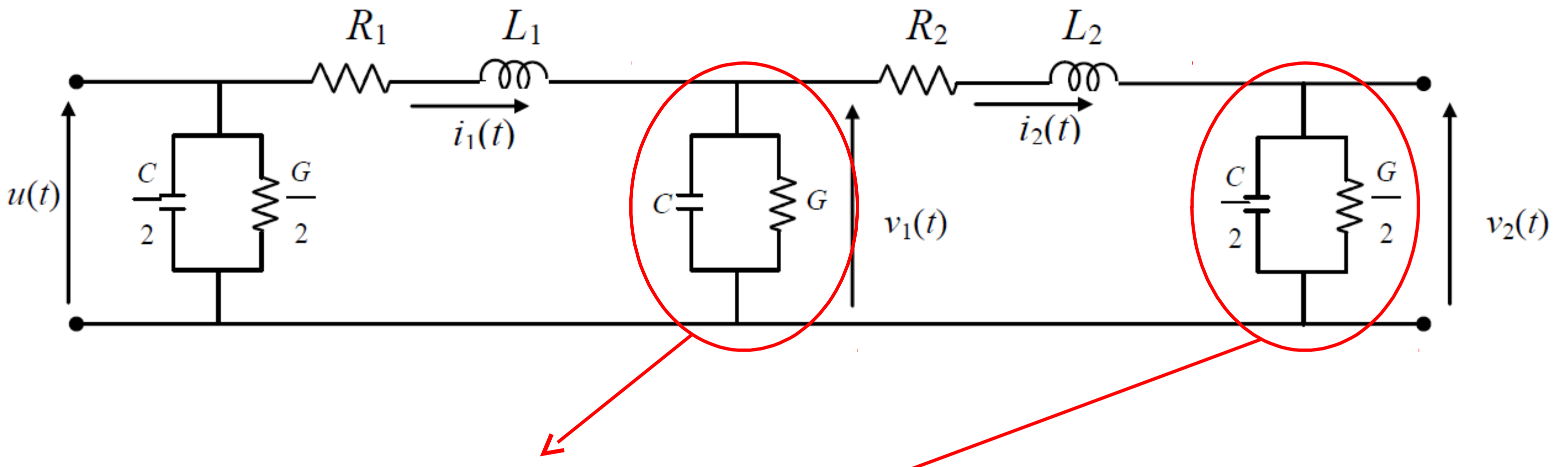
Dois Circuitos π



$$u(t) - i_1(t)R_1 - L_1\frac{di_1(t)}{dt} - v_1(t) = 0 \quad \longrightarrow \quad \frac{di_1(t)}{dt} = -\frac{R_1}{L_1}i_1(t) - \frac{1}{L_1}v_1(t) + \frac{1}{L_1}u(t)$$

$$v_1(t) - i_2(t)R_2 - L_2\frac{di_2(t)}{dt} - v_2(t) = 0 \quad \longrightarrow \quad \frac{di_2(t)}{dt} = -\frac{R_2}{L_2}i_1(t) - \frac{1}{L_2}v_1(t) + \frac{1}{L_2}v_2(t)$$

Dois Circuitos π



$$i_C(t) = i_1(t) - v_1(t)G - i_2(t)$$

$$i_C(t) = i_2(t) - v_2(t)\frac{G}{2}$$

Dois Circuitos π

$$i_C(t) = i_1(t) - v_1(t)G - i_2(t)$$

$$v_1(t) = \frac{1}{C}\int (i_C(t)dt)$$

$$v_1(t) = \frac{1}{C}\int [i_1(t) - i_2(t) - v_1(t)G]dt$$

$$\frac{dv_1(t)}{dt} = \frac{1}{C} i_1(t) - \frac{1}{C} i_2(t) - \frac{G}{C} v_1(t)$$

$$i_C(t) = i_2(t) - v_2(t)\frac{G}{2}$$

$$v_2(t) = \frac{1}{C}\int i_C(t)dt$$

$$v_2(t) = \frac{2}{C}\int [i_2(t) - \frac{G}{2} v_2(t)]dt$$

$$\frac{dv_2(t)}{dt} = \frac{2}{C} i_2(t) - \frac{G}{C} v_2(t)$$

Dois Circuitos π

$$
\begin{cases}
\dfrac{di_1(t)}{dt} = -\dfrac{R_1}{L_1} i_1(t) - \dfrac{1}{L_1} v_1(t) + \dfrac{1}{L_1} u(t) \\[2ex]
\dfrac{di_2(t)}{dt} = -\dfrac{R_2}{L_2} i_1(t) - \dfrac{1}{L_2} v_1(t) + \dfrac{1}{L_2} v_2(t) \\[2ex]
\dfrac{dv_1(t)}{dt} = \dfrac{1}{C} i_1(t) - \dfrac{1}{C} i_2(t) - \dfrac{G}{C} v_1(t) \\[2ex]
\dfrac{dv_2(t)}{dt} = \dfrac{2}{C} i_2(t) - \dfrac{G}{C} v_2(t)
\end{cases}
$$

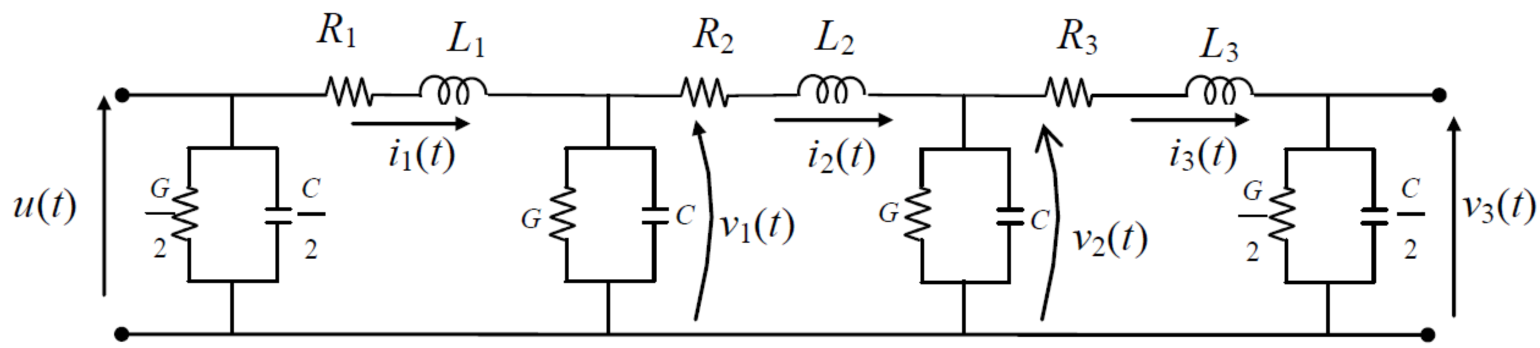
$$[\dot{x}] = [A][x] + [B]u(t)$$

$$[x] = \begin{bmatrix} i_1(t) & i_2(t) & v_1(t) & v_2(t) \end{bmatrix}^T$$

$$[A] = \begin{bmatrix} -\dfrac{R_1}{L_1} & 0 & -\dfrac{1}{L_1} & 0 \\[2ex] 0 & -\dfrac{R_2}{L_2} & \dfrac{1}{L_2} & -\dfrac{1}{L_2} \\[2ex] \dfrac{1}{C} & -\dfrac{1}{C} & -\dfrac{G}{C} & 0 \\[2ex] 0 & \dfrac{2}{C} & 0 & -\dfrac{G}{C} \end{bmatrix}$$

$$[B] = \begin{bmatrix} \dfrac{1}{L_1} & 0 & 0 & 0 \end{bmatrix}^T$$

Três Circuitos π

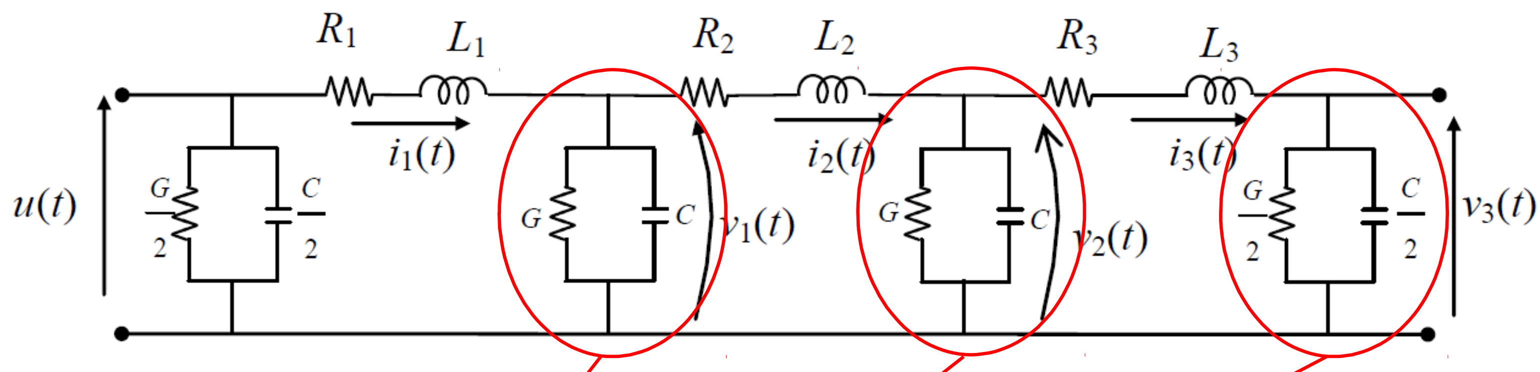


$$u(t) - R_1 i_1(t) - L_1 \frac{di_1(t)}{dt} - v_1(t) = 0 \longrightarrow \frac{di_1(t)}{dt} = -\frac{R_1}{L_1} i_1(t) - \frac{1}{L_1} v_1(t) + \frac{1}{L_1} u(t)$$

$$v_1(t) - R_2 i_2(t) - L_2 \frac{di_2(t)}{dt} - v_2(t) = 0 \longrightarrow \frac{di_2(t)}{dt} = -\frac{R_2}{L_2} i_2(t) + \frac{1}{L_2} v_1(t) - \frac{1}{L_2} v_2(t)$$

$$v_2(t) - R_3 i_3(t) - L_3 \frac{di_3(t)}{dt} - v_3(t) = 0 \longrightarrow \frac{di_3(t)}{dt} = -\frac{R_3}{L_3} i_3(t) + \frac{1}{L_3} v_2(t) - \frac{1}{L_3} v_3(t)$$

Três Circuitos π



$$i_{C1}(t) = i_1(t) - i_2(t) - Gv_1(t)$$

$$i_{C2}(t) = i_2(t) - i_3(t) - Gv_2(t)$$

$$i_{C3}(t) = i_3(t) - \frac{G}{2}v_3(t)$$

Três Circuitos π

$$i_{C1}(t) = i_1(t) - i_2(t) - Gv_1(t)$$

$$v_1(t) = \frac{1}{C}\int i_{C1}(t)dt = \frac{1}{C}\int [i_1(t) - i_2(t) - Gv_1(t)]dt$$

$$\frac{dv_1(t)}{dt} = \frac{1}{C}i_1(t) - \frac{1}{C}i_2(t) - \frac{G}{C}v_1(t)$$

$$i_{C2}(t) = i_2(t) - i_3(t) - Gv_2(t)$$

$$v_2(t) = \frac{1}{C}\int i_{C2}(t)dt = \frac{1}{C}\int [i_2(t) - i_3(t) - Gv_2(t)]dt$$

$$\frac{dv_2(t)}{dt} = \frac{1}{C}i_2(t) - \frac{1}{C}i_3(t) - \frac{G}{C}v_2(t)$$

Três Circuitos π

$$i_{C2}(t) = i_2(t) - i_3(t) - Gv_2(t)$$

$$v_2(t) = \frac{1}{C}\int i_{C2}(t)dt = \frac{1}{C}\int [i_2(t) - i_3(t) - Gv_2(t)]dt$$

$$\frac{dv_2(t)}{dt} = \frac{1}{C}i_2(t) - \frac{1}{C}i_3(t) - \frac{G}{C}v_2(t)$$

$$i_{C3}(t) = i_3(t) - \frac{G}{2}v_3(t)$$

$$v_3(t) = \frac{1}{C}\int i_{C3}(t)dt = \frac{2}{C}\int [i_3(t) - \frac{G}{2}v_3(t)]dt$$

$$\frac{dv_3(t)}{dt} = \frac{2}{C}i_3(t) - \frac{G}{C}v_3(t)$$

Três Circuitos π

$$
\begin{cases}
\dfrac{di_1(t)}{dt} = -\dfrac{R_1}{L_1} i_1(t) - \dfrac{1}{L_1} v_1(t) + \dfrac{1}{L_1} u(t) \\[2ex]
\dfrac{di_2(t)}{dt} = -\dfrac{R_2}{L_2} i_2(t) + \dfrac{1}{L_2} v_1(t) - \dfrac{1}{L_2} v_2(t) \\[2ex]
\dfrac{di_3(t)}{dt} = -\dfrac{R_3}{L_3} i_3(t) + \dfrac{1}{L_3} v_2(t) - \dfrac{1}{L_3} v_3(t) \\[2ex]
\dfrac{dv_1(t)}{dt} = \dfrac{1}{C} i_1(t) - \dfrac{1}{C} i_2(t) - \dfrac{G}{C} v_1(t) \\[2ex]
\dfrac{dv_2(t)}{dt} = \dfrac{1}{C} i_2(t) - \dfrac{1}{C} i_3(t) - \dfrac{G}{C} v_2(t) \\[2ex]
\dfrac{dv_3(t)}{dt} = \dfrac{2}{C} i_3(t) - \dfrac{G}{C} v_3(t)
\end{cases}
$$

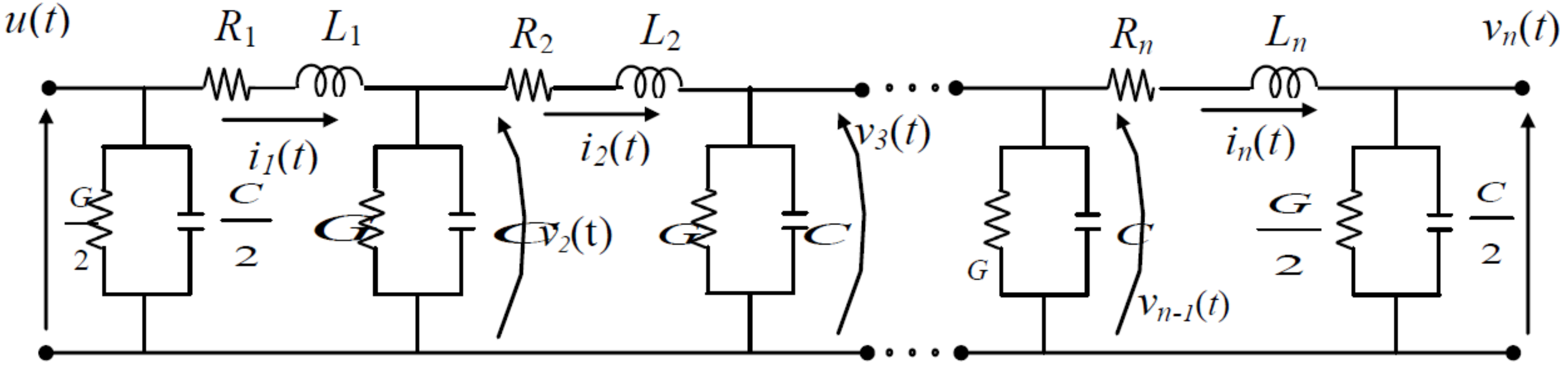
$$[\dot{x}] = [A][x] + [B]u(t)$$

$$[x] = \begin{bmatrix} i_1(t) & i_2(t) & i_3(t) & v_1(t) & v_2(t) & v_3(t) \end{bmatrix}^T$$

$$[A] = \begin{bmatrix}
-\dfrac{R_1}{L_1} & 0 & 0 & -\dfrac{1}{L_1} & 0 & 0 \\
0 & -\dfrac{R_2}{L_2} & 0 & \dfrac{1}{L_2} & -\dfrac{1}{L_2} & 0 \\
0 & 0 & -\dfrac{R_3}{L_3} & 0 & \dfrac{1}{L_3} & -\dfrac{1}{L_3} \\
\dfrac{1}{C} & -\dfrac{1}{C} & 0 & -\dfrac{G}{C} & 0 & 0 \\
0 & \dfrac{1}{C} & -\dfrac{1}{C} & 0 & -\dfrac{G}{C} & 0 \\
0 & 0 & \dfrac{2}{C} & 0 & 0 & -\dfrac{G}{C}
\end{bmatrix}$$

$$[B] = \begin{bmatrix} \dfrac{1}{L_1} & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T$$

n Circuitos π



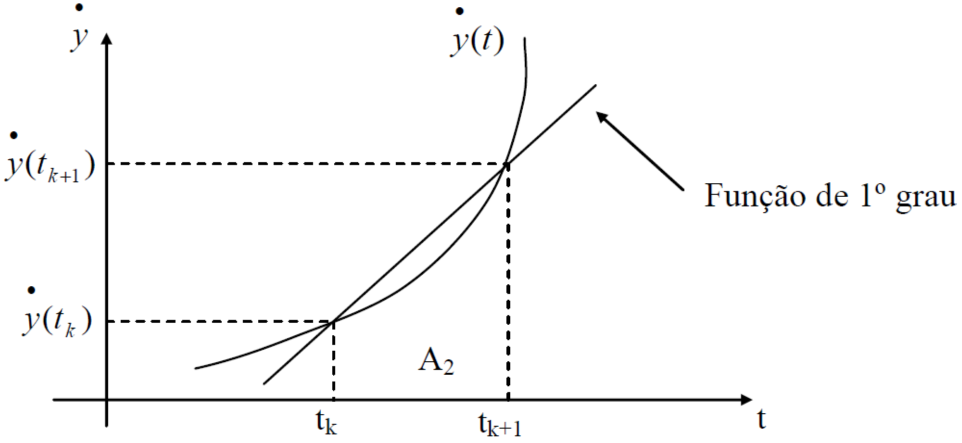
$$[x] = \begin{bmatrix} i_1(t) & i_2(t) & i_3(t) & \cdots & i_n(t) & v_1(t) & v_2(t) & v_3(t) & \cdots & v_n(t) \end{bmatrix}^T$$

$$[B] = \begin{bmatrix} \frac{1}{L_1} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

n Circuitos π

$$
[A] = \left[\begin{array}{ccccc|ccccc}
-\frac{R_1}{L_1} & 0 & 0 & \cdots & 0 & -\frac{1}{L_1} & 0 & 0 & \cdots & 0 \\
0 & -\frac{R_2}{L_2} & 0 & \cdots & 0 & \frac{1}{L_2} & -\frac{1}{L} & 0 & \cdots & 0 \\
0 & 0 & -\frac{R_3}{L_3} & \cdots & 0 & 0 & \frac{1}{L} & -\frac{1}{L_3} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & -\frac{R_n}{L_n} & 0 & 0 & \frac{1}{L_n} & \cdots & -\frac{1}{L_n} \\
\hline
\frac{1}{C} & -\frac{1}{C} & 0 & \cdots & 0 & -\frac{G}{C} & 0 & 0 & \cdots & 0 \\
0 & \frac{1}{C} & -\frac{1}{C} & \cdots & 0 & 0 & -\frac{G}{C} & 0 & \cdots & 0 \\
0 & 0 & \frac{1}{C} & \cdots & -\frac{1}{C} & 0 & 0 & -\frac{G}{C} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \frac{2}{C} & 0 & 0 & 0 & \cdots & -\frac{G}{C}
\end{array} \right]
$$

Método de Integração Trapezoidal



$$h = t_{k+1} - t_k$$

$$\int_{t_k}^{t_{k+1}} \dot{y}(t).dt = A_2 \longrightarrow \int_{t_k}^{t_{k+1}} \dot{y}(t)dt = \frac{1}{2}[\dot{y}(t_k) + \dot{y}(t_{k+1})]h$$

Integrando a parte esquerda da integral definida acima:

$$y(t_{k+1}) = y(t_k) + \frac{1}{2}[\dot{y}(t_k) + \dot{y}(t_{k+1})]h \longrightarrow x(t_{k+1}) = x(t_k) + \frac{1}{2}[\dot{x}(t_k) + \dot{x}(t_{k+1})].h$$

Método de Integração Trapezoidal

Sendo $\dot{x}(t)$ *uma equação de estado do sistema, pode-se adaptar a equação anterior para a seguinte forma:*

$$\dot{x}(t_{k+1}) = [A]x(t_{k+1}) + [B]u(t_{k+1})$$

Substituindo a equação acima na equação anterior, obtem-se a seguinte equação matricial:

$$x(t_{k+1}) = \left[I - \frac{1}{2}hA\right]^{-1}\left[\left[I + \frac{1}{2}hA\right]x(t_k) + \frac{1}{2}hB(u(t_k) + u(t_{k+1}))\right]$$

- R. M. Nelms et al. "Using a personal computer to teach power system transients" *IEEE Trans. Power Systems*, v. 4, n. 3, 1293-1297, 1989.

- J. A. R. Macías et al. "A comparison of techniques for state-space transient analysis of transmission lines", v. 20, n. 2, 894-903, 2005.