11.2 –
A Unified Framework for Controlling Global Dynamics

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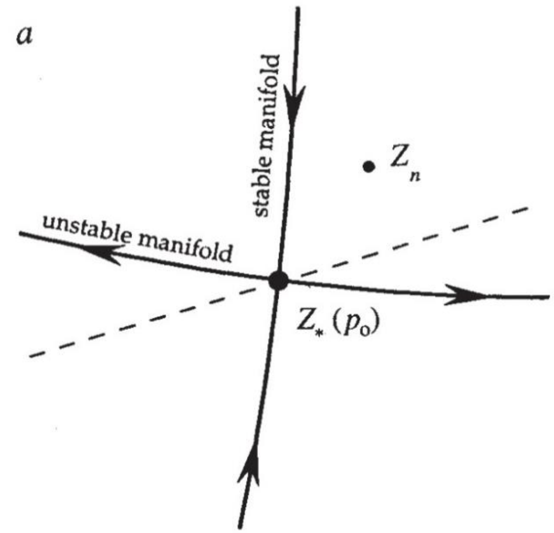
DAY	TIME	LECTURE
Monday 05/11	14.00 -14.45	Historical Framework - A Global Dynamics Perspective in the Nonlinear Analysis of Systems/Structures
	15.00 -15.45	Achieving Load Carrying Capacity: Theoretical and Practical Stability
	16.00 -16.45	Dynamical Integrity: Concepts and Tools_1
Wednesday 07/11	14.00 -14.45	Dynamical Integrity: Concepts and Tools_2
	15.00 -15.45	Global Dynamics of Engineering Systems
	16.00 -16.45	Dynamical integrity: Interpreting/Predicting Experimental Response
Monday 12/11	14.00 -14.45	Techniques for Control of Chaos
	15.00 -15.45	**A Unified Framework for Controlling Global Dynamics**
	16.00 -16.45	Response of Uncontrolled/Controlled Systems in Macro- and Micro-mechanics
Wednesday 14/11	14.00 -14.45	A Noncontact AFM: (a) Nonlinear Dynamics and Feedback Control (b) Global Effects of a Locally-tailored Control
	15.00 -15.45	Exploiting Global Dynamics to Control AFM Robustness
	16.00 -16.45	Dynamical Integrity as a Novel Paradigm for Safe/Aware Design

Invariant (stable and unstable) manifolds

Stable manifold of a **saddle** point: the set of initial conditions that approach the saddle forward in time along its stable eigenvector forward in time

Practically: backward iteration of stable eigenvector



Unstable manifold of a **saddle** point: the set of initial conditions that approach the saddle backward in time along its unstable eigenvector forward in time

Practically: forward iteration of unstable eigenvector

If stable and unstable manifolds intersect in one point, they intersect in infinitely many points (forward and backward in time)

Relevance of invariant manifolds

Invariant manifolds "provide a useful stepping stone in the understanding of the overall system dynamics" [Katz & Dowell, 1994] even if they are structurally unstable sets that cannot be "seen" directly

- **stable manifold** (insets) are **boundaries of basins of attractions** (this is why they are so important for dynamical integrity)

- ***responsible for fractal basin boundaries*** (when stable and unstable manifolds intersect)
- ***involved in many topological phenomena***

The Smale-Birkhoff (or Moser) homoclinic theorem

Let a (2D) map (i.e. the Poincaré map of a continuous system) has a saddle. Let the stable and unstable manifolds of the saddle intersect transversally. Then

an iteration of the map has an invariant Cantor set on which it is topologically conjugate to a full shift on N symbols

- Apart from technicalities, this theorem proves that ***homoclinic intersection is responsible for "chaos"***
- Smale horseshoe (shift) map: at the heart of chaos
- Similar results for heteroclinic orbits (manifolds of different saddles)

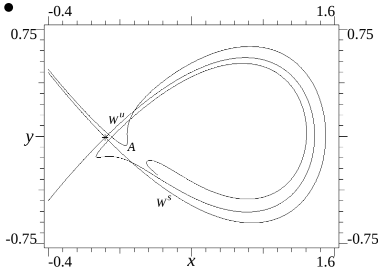
Relevance of invariant manifolds

"…it is not an exaggeration to claim that in virtually every manifestation of chaotic behaviour known thus far, some type of homoclinic behaviour is lurking in the background…" [Kovacic & Wiggins, 1992]

- ***skeleton of chaotic attractors*** (maybe)
- ***responsible for chaotic transient*** (certainly)

So, now the question is: how to check if the stable and unstable manifolds intersect (transversally)?

- Graphically
- By measuring their distance



Manifolds distance

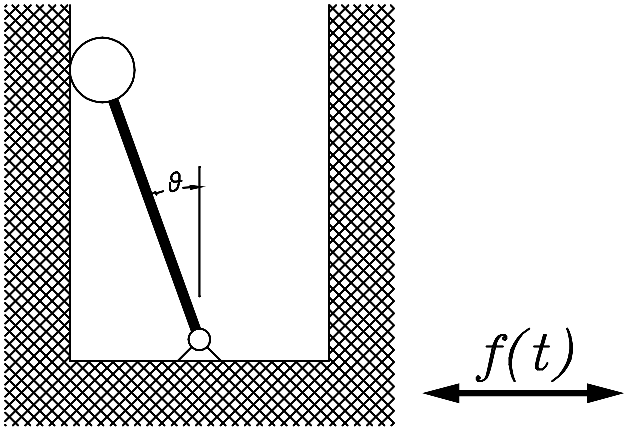
- How to measure the distance?

(i) ***Exactly***. The best solution, but unfortunately this is possible only in few cases

(ii) *Approximately*, for example by the perturbative Melnikov method. Most commonly used, but valid only in certain circumstances (e.g. only hilltop saddle)

(iii) *Numerically*, when no other options are available

An example: inverted pendulum



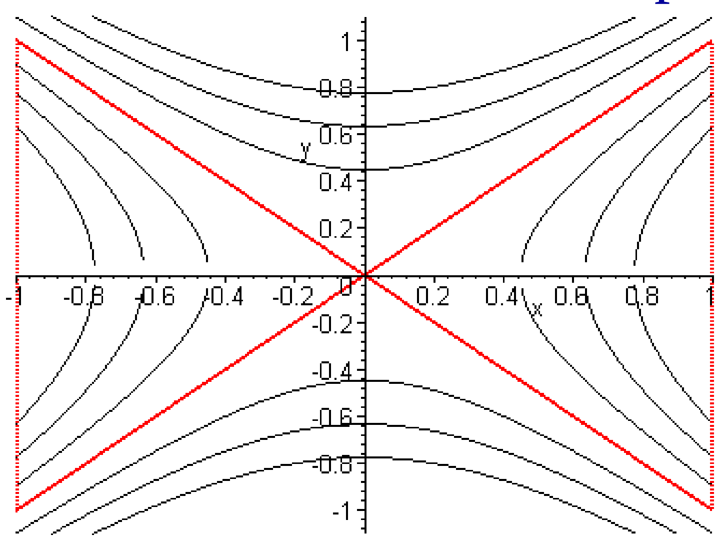
Equation of motion

$$\ddot{x}+\delta\dot{x}-x=\gamma_1\sum_{j=1}^{\infty}\frac{\gamma_j}{\gamma_1}\sin(j\omega t+\Psi_j),\qquad |x|<1$$

$$\dot{x}(t^{+})=-r\dot{x}(t^{-}),\qquad |x|=1$$

generic periodic excitation

Phase space
(without excitation and damping)



Manifold distance

$$d_{r,l}(\tau)=\pm\frac{C_0}{2}+\gamma_1 C_1 h(\tau)$$

$$h(\tau)=sin(\tau+\Phi_1)+\sum_{j=2}^{\infty}\frac{\gamma_j C_j}{\gamma_1 C_1}sin(j\tau+\Phi_j)$$

C_j known parameters

Manifolds distance

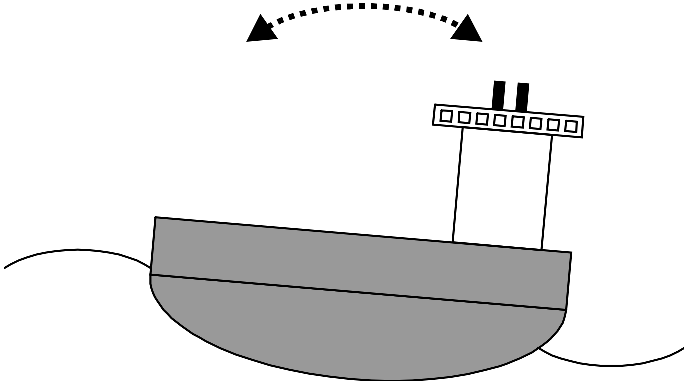
- How to measure the distance?

(i) *Exactly*. The best solution, but unfortunately this is possible only in few cases, e.g. piece-wise linear systems

(ii) ***Approximately***, for example by the perturbative Melnikov method. Most commonly used, but valid only in certain circumstances (e.g. only hilltop saddle)

(iii) *Numerically*, when no other options are available

An example: Helmholtz

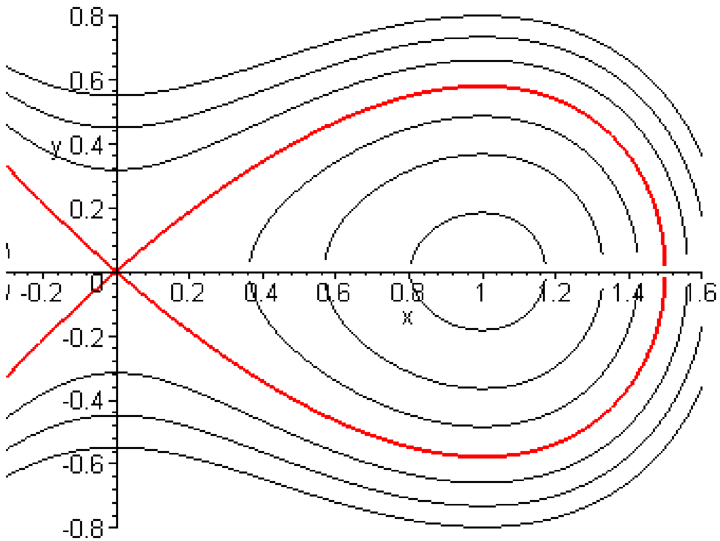


Equation of motion

$$\ddot{x}+\varepsilon\delta\dot{x}-x+x^2=\varepsilon\gamma_1\sum_{j=1}^{\infty}\frac{\gamma_j}{\gamma_1}\sin(j\omega t+\Psi_j)$$

generic periodic excitation

Phase space
(without excitation and damping)



Manifold distance
(up to the first order)

$$d(\tau)=c\left[\delta+\gamma_1\frac{5\pi\omega^2}{\sinh(\omega\pi)}h(\tau)\right]=c\left[1+\frac{\gamma_1}{\delta\frac{\sinh(\omega\pi)}{5\pi\omega^2}}h(\tau)\right]$$

$$h(\tau)=\sum_{j=1}^{\infty}h_j\cos(j\tau+\Psi_j)$$

$$h_j=\frac{\gamma_j}{\gamma_1}\frac{j^2\sinh(\omega\pi)}{\sinh(j\omega\pi)}$$

[Wiggins, 2003]
[Guckenheimer & Holmes, 1983]

Manifolds distance

- How to measure the distance?

(i) *Exactly. The best solution, but unfortunately this is possible* only in few cases, e.g. piece-wise linear systems

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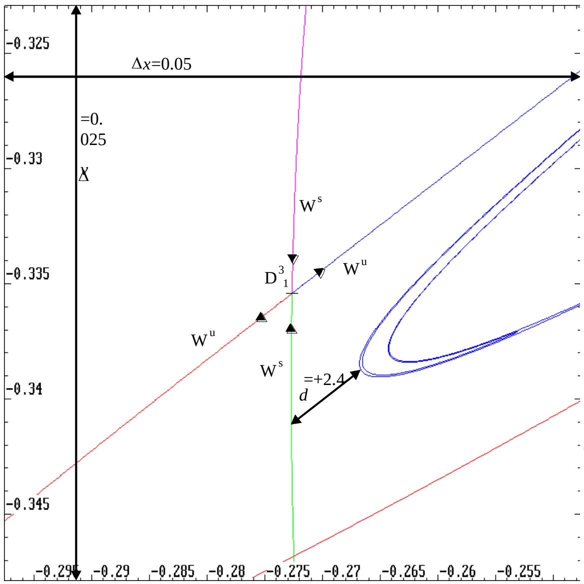
(iii) Numerically, when no other options are available

An example: Duffing

Non hilltop saddle
(Melnikov does not apply)



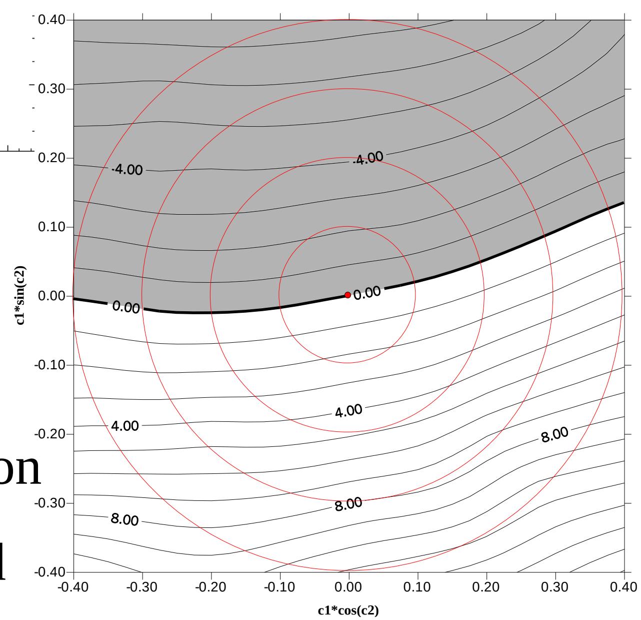
Equation of motion

$$\ddot{x}+0.164\dot{x}-0.2x+x^3=A[\sin(t)+c_1\sin(nt+c_2)]$$

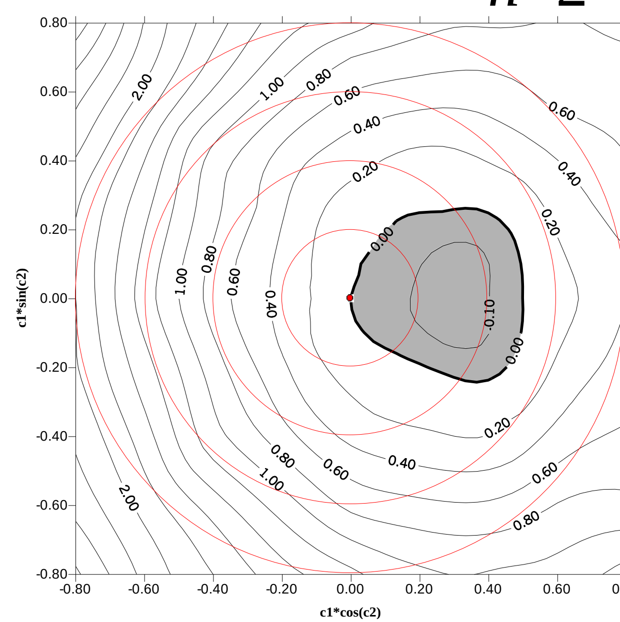
Multi-harmonic excitation

Manifold distance

n=3



n=2



Grey: manifolds intersection

White: manifolds detached

Manifolds distance: summary

- The distance can "always" be written in the form

$$d(m) = c\left[1 + \frac{\gamma_1}{\gamma^h_{1,cr}(\omega)} h(m)\right] \qquad h(m) = \cos(m) + \sum_{j=2}^{\infty} h_j \cos(jm + \Psi_j)$$

the difference between the various systems is due to the

1) different definition of $\gamma^h_{1,cr}(\omega)$
2) different relations between the h_j (amplitudes of the superharmonics in the distance) and γ_j (amplitudes of the superharmonics in the excitation)

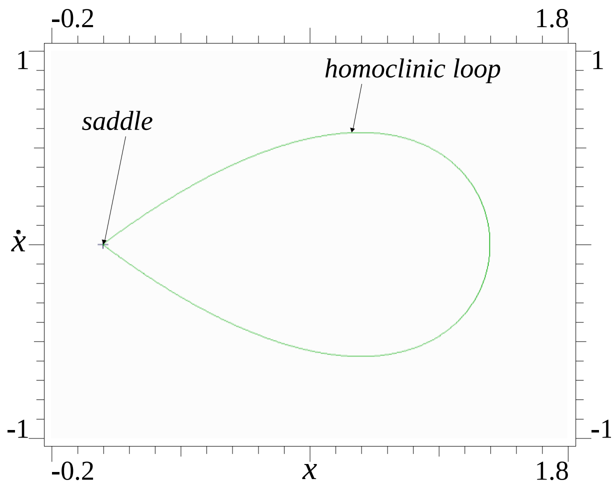
- The structure of the distance is ***system-independent***
- The relations between h_j and γ_j, and the function $\gamma^h_{1,cr}(\omega)$, are ***system-dependent***

Stable and unstable manifolds distance: an example

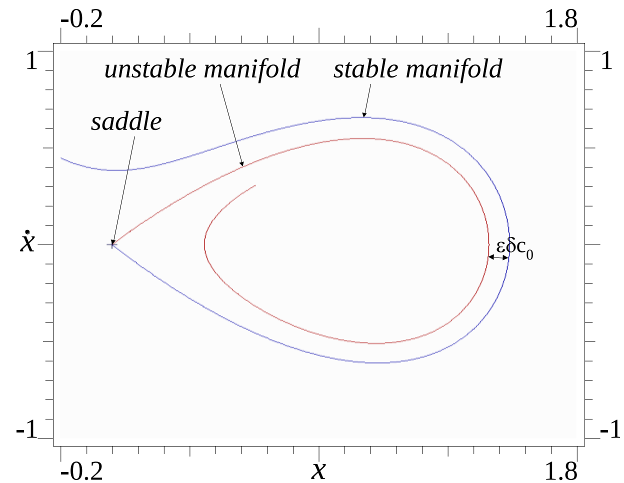
Helmholtz oscillator: $x'' + \varepsilon\delta x' - x + x^2 = \varepsilon\gamma_1 \sin(\omega t)$ $\qquad m = \omega t$

distance(m) = constant part ($\varepsilon\delta a_0$) + oscillating part ($\varepsilon\gamma_1 \cos(m) a_1(\omega)$)

unperturbed

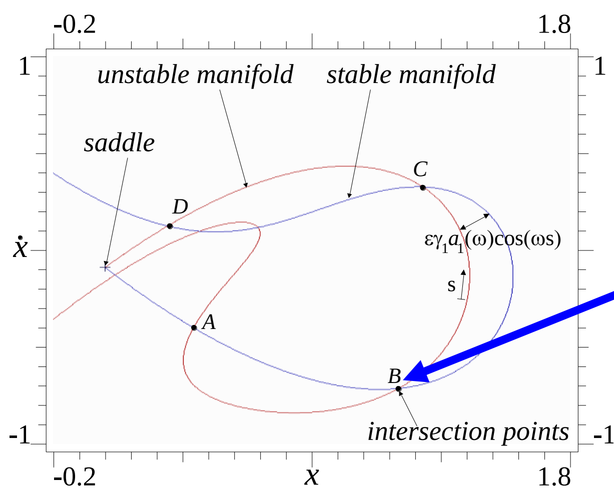


adding **damping**



keeps manifolds disjont

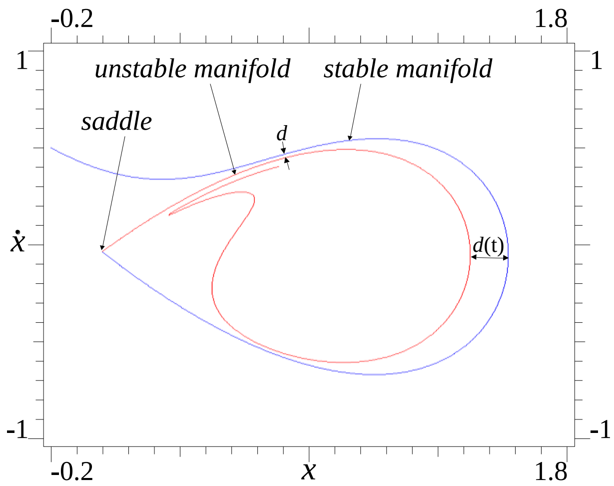
adding excitation



homoclinic intersection, root to chaos and fractality

enforces manifolds intersection

Manifolds distance



$$d(m) = c\left[1 + \frac{\gamma_1}{\gamma^h_{1,cr}(\omega)} h(m)\right]$$

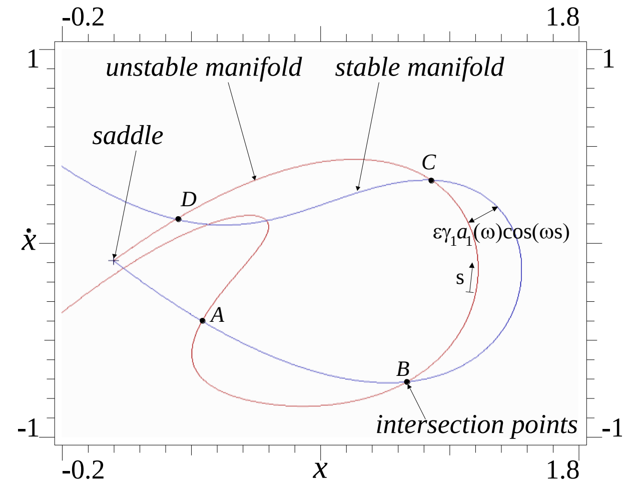
$$\boxed{d = \min_m \{d(m)\}} = c\left[1 + \frac{\gamma_1}{\gamma^h_{1,cr}(\omega)} \min_m \{h(m)\}\right]$$

- No intersection if the distance does not change sign, i.e. *d* is positive
- Intersection if the distance changes sign, i.e. if the minimum distance *d* is negative
- $\gamma^h{}_{1,cr}(\omega)$ critical excitation amplitude with harmonic excitation

  (for the Helmholtz oscillator $\gamma^h{}_{1,cr}(\omega) = \delta \frac{\sinh(\omega\pi)}{5\pi\omega^2}$)
- For harmonic excitation $h(m)=\cos(m)$

Basic idea of control (1)

- Suppose to have *homoclinic intersections*
- Then there is fractal basin boundaries (bad for dynamical integrity), chaotic transient and possibly chaotic attractor



- Suppose to have an ***harmonic excitation***. The distance is then

$$d(m)=c\left[1+\frac{\gamma_1}{\gamma^h_{1,cr}(\omega)}\cos(m)\right]\rightarrow d=\min_m\{d(m)\}=c\left[1-\frac{\gamma_1}{\gamma^h_{1,cr}(\omega)}\right]$$

and *d* is negative for some values of *m* (since we have intersection)

Basic idea of control (2)

- What to do to get out from this situation? Or, in other words, how to detach the manifolds?

$$d = c\left[1 - \frac{\gamma_1}{\gamma_{1,cr}^h(\omega)}\right] < 0$$

1) Increasing the damping, which entails increasing $\gamma^h_{1,cr}(\omega)$
(for the Helmholtz oscillator $\gamma^h_{1,cr}(\omega) = \delta \frac{\sinh(\omega\pi)}{5\pi\omega^2}$)

2) Reducing the excitation amplitude γ_1

3) Changing the system parameters (e.g. ω)

- All good, but "trivial" (while of course useful, if possible)

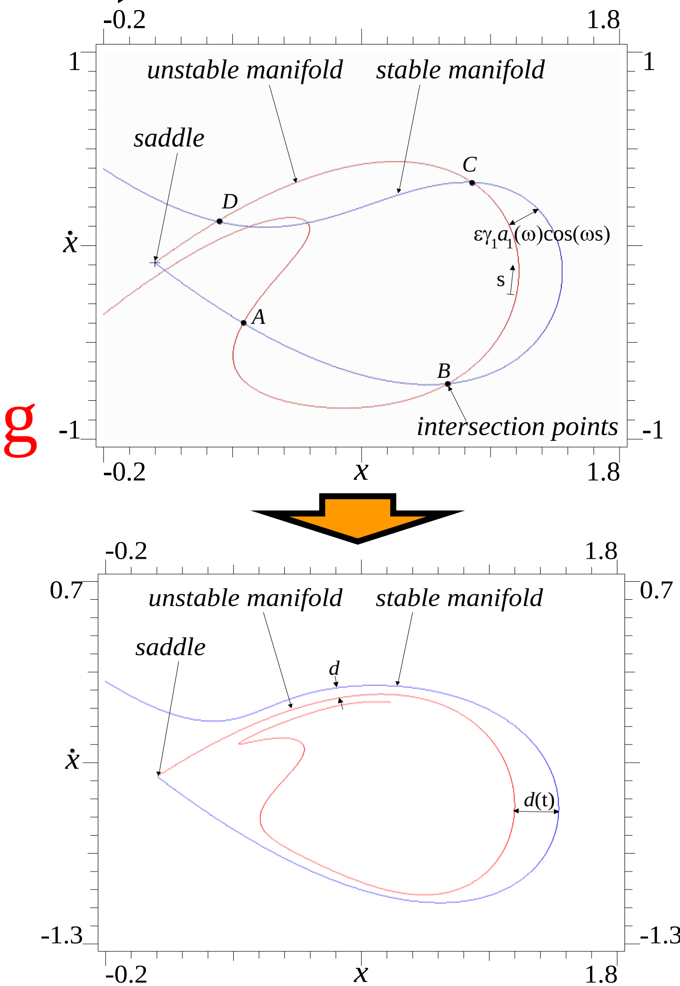
Basic idea of control (3)

- It is possible to do better, by **varying the excitation** (keeping fixed the amplitude, of course)
- HOW?

→ Adding external/parametric excitation

→ ***Adding superharmonic***, i.e. keeping fixed the period but ***changing the shape*** of the excitation

→ Adding subharmonic (which entail reshaping the excitation *and* changing its period)



Basic idea of control (4)

- Let us try by adding a superharmonic in the excitation
- With given (harmonic excitation) we have

$$d(m)=c\left[1+\frac{\gamma_1}{\gamma_{1,cr}^h(\omega)}\cos(m)\right]\rightarrow d=\min_m\{d(m)\}=c\left[1-\frac{\gamma_1}{\gamma_{1,cr}^h(\omega)}\right]<0$$

- Adding a single superharmonic (to fix ideas) we get

$$d(m)=c\left[1+\frac{\gamma_1}{\gamma_{1,cr}^h(\omega)}\{\cos(m)+\boxed{h_2\cos(2m+\Psi_2)}\}\right]$$

Added (controlling) superharmonic

- h_2 and Ψ_2 can be chosen so that $d=\min_m\{d(m)\}$ becomes positive, which corresponds to ***detached manifolds*** !

A main point

- The control method just illustrated is ***system-independent*** (we have chosen h_2 = 0.4 and Ψ_2 =0 without referring to a specific system), and thus general, "**universal**"
- The practical implementation of control, which require computing γ_2 from h_2 is instead ***system-dependent***, since the function $\gamma_2(h_2)$ changes from system to system (for the Helmholtz oscillator we have $h_j = \frac{\gamma_j}{\gamma_1} \frac{j^2 \sinh(\omega\pi)}{\sinh(j\omega\pi)}$)

Generalizations

- More superharmonics
- Subharmonics
- Adding parametric/external excitations
- Optimization
- More homo/heteroclinic bifurcations
- ….

We are going to see some of them

A step ahead: homoclinic bifurcation

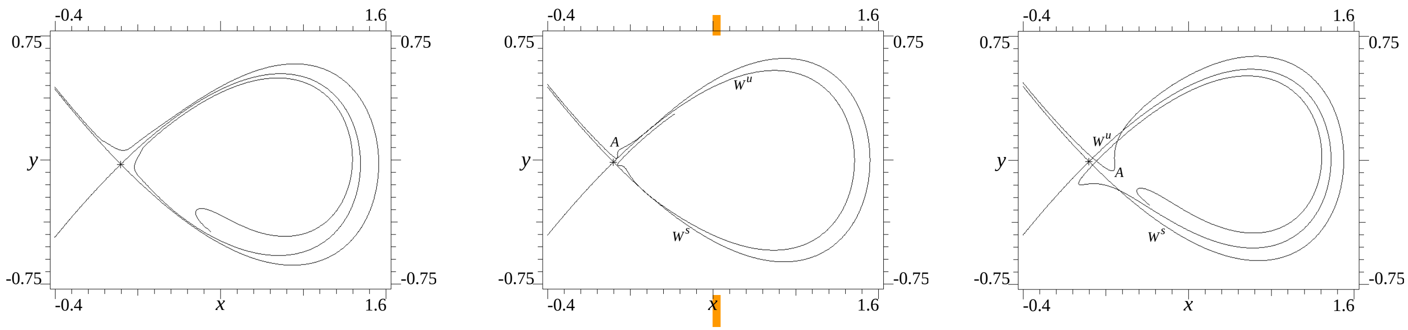
- If, by varying a parameter of the system, the stable and unstable manifolds pass from intersection to detachment (or viceversa), we have a **homoclinic bifurcation**
- The same for the heteroclinic bifurcation
- At the bifurcation the manifolds are tangent (i.e they intersect NON transversally), so that the Smale-Birkhoff (or Moser) homoclinic theorem does not apply

Relevance of homo/heteroclinic bifurcations

Homo/heteroclinic bifurcations of selected saddles are the mechanisms responsible for:

- starting of ***fractalization of basin boundaries*** and sensitivity to initial conditions
- appearance/disappearance of chaotic attractors or their sudden enlargement/reduction
- ***triggering phenomena*** of basins erosion suddenly leading to ***out-of-well dynamics***:
  - transition from single-well to cross-well chaos in multi-well systems
  - escape from potential well in single-well systems

Homoclinic bifurcation and basins of attraction

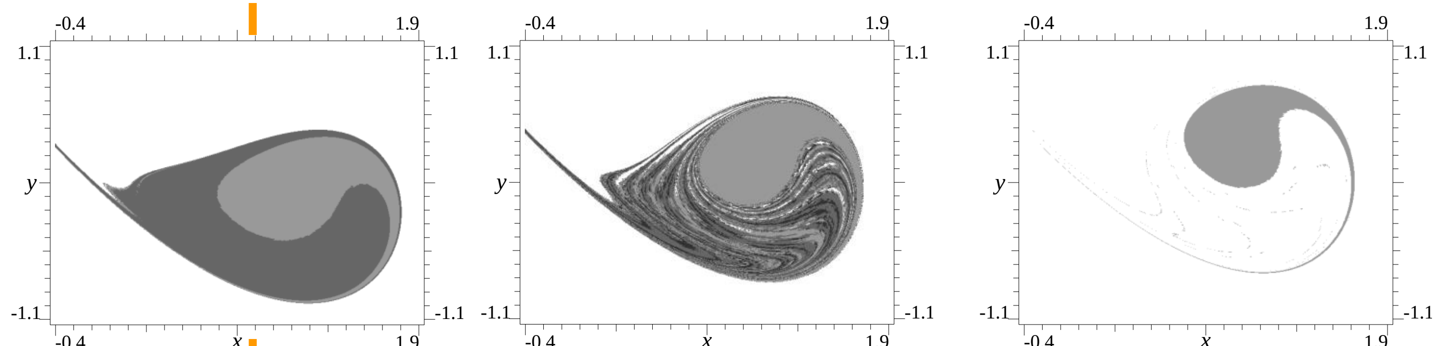


detached manifolds | manifolds tangency | manifolds intersection

homoclinic bifurcation

varying one parameter (e.g., increasing excitation amplitude)

associated basins of attraction erosion



uneroded basin | erosion starts | toward complete erosion and inevitable escape

example of erosion

Out-of-well dynamics: Escape from a potential well

- effects of overcoming a potential hill:

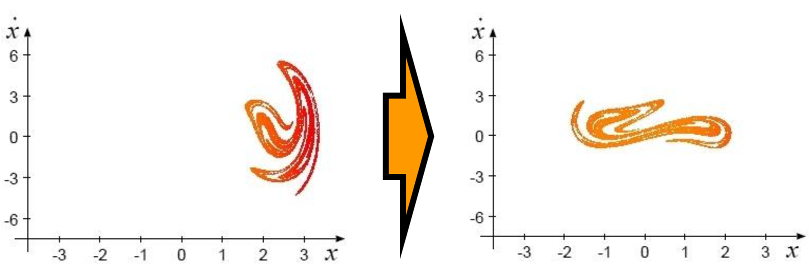
increasing dangerousness

- scattered periodic motions
- scattered chaotic motions
- unbounded motions

- destroying the structure by fatigue
- failure of the structure

dynamical effects

practical effects



capsizing

overturning

Homoclinic bifurcation

- The homoclinic bifurcation occurs when

$$d=\min_m\{d(m)\}$$

passes from negative to positive values, i.e. when

$d = 0 \rightarrow$

$$\gamma_{1,cr} = \frac{\gamma_{1,cr}^h(\omega)}{-\min_m\{h(m)\}}$$

homoclinic bifurcation threshold

- With harmonic excitation $h(m)=\cos(m) \rightarrow$ $-\min_m\{h(m)\}=1 \rightarrow \gamma_{1,cr} = \gamma_{1,cr}^h(\omega)$
- $\gamma_{1,cr}^h(\omega)$ is the homoclinic bifurcation threshold for *harmonic excitation*

Control of homoclinic bifurcation: Basic idea

- ***Act on the excitation*** to control the occurrence of homoclinic bifurcation (the same for heteroclinic)
- But
$$\gamma_{1,cr} = \frac{\gamma^{h}_{1,cr}(\omega)}{-\min_{m}\{h(m)\}}$$

thus, controlling the homoclinic bifurcation threshold requires changing

$$M = -\min_{m}\{h(m)\},$$

which is system-independent

- In particular, increasing $\gamma_{1,cr}$ entails decreasing M

Control of homo/heteroclinic bifurcation

key idea: **controlling homo/heteroclinic bifurcations**

([Lima & Pettini, 90], [Cicogna & Fronzoni, 93], [Kivshar et al., 94], [Chacón & Díaz Berjarano, 93], [Sanjuán, 98], [Shaw, 90], [Lenci & Rega, 98a,b; 00, 03], [Cao et al., 03], [Nana Nbendjo et al., 03])

specific objective versus practical strategies

modifications of the system

modifications of the excitation

- modifying frequency/amplitude [Blazejczyk *et al.*, 93]
- adding parametric/external excitations [Lima & Pettini, 90]
- **modifying the shape** ([Shaw, 90], [Lenci & Rega, 98a,b,00,03])

The performance of control

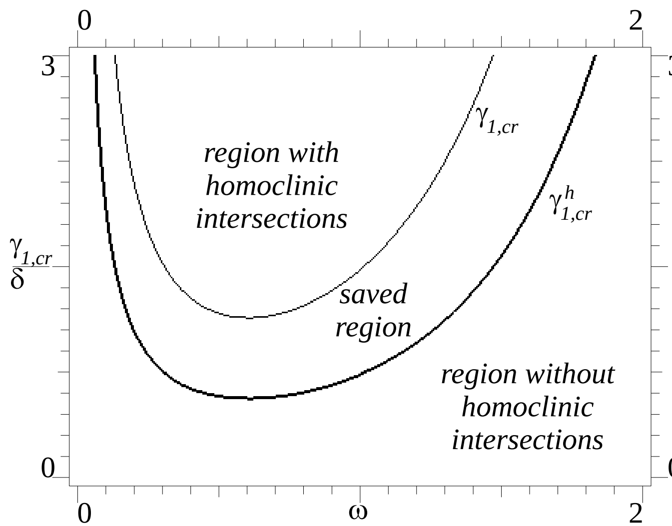
- How better the control excitation is with respect to the reference harmonic excitation?

$$\frac{\gamma_{1,cr}}{\gamma_{1,cr}^{h}(\omega)} = \frac{1}{-\min_{m}\{h(m)\}} = \frac{1}{M} = G$$

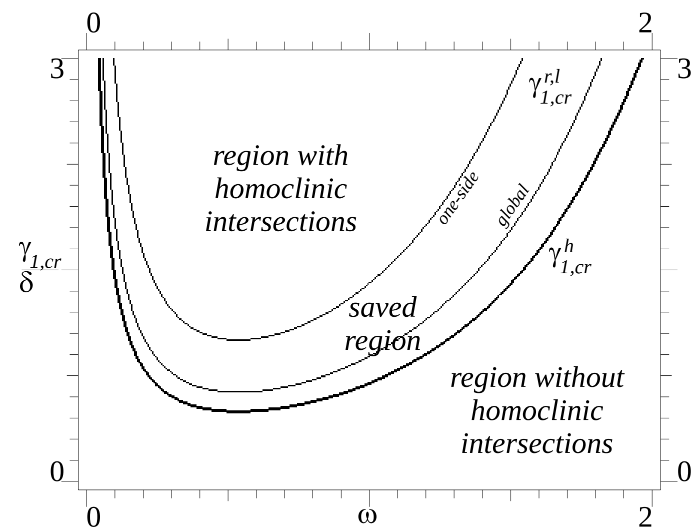
- G is called the ***gain***
- Other reference excitations can be chosen, without any conceptual difference
- **Saved region**: homoclinic intersection with harmonic excitation, homoclinic detachment with control excitation, i.e where the control is effective

Example of saved regions

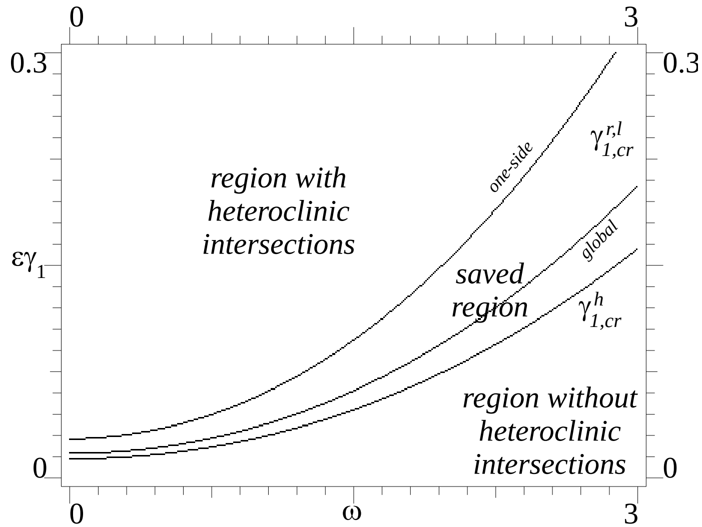
Helmholtz



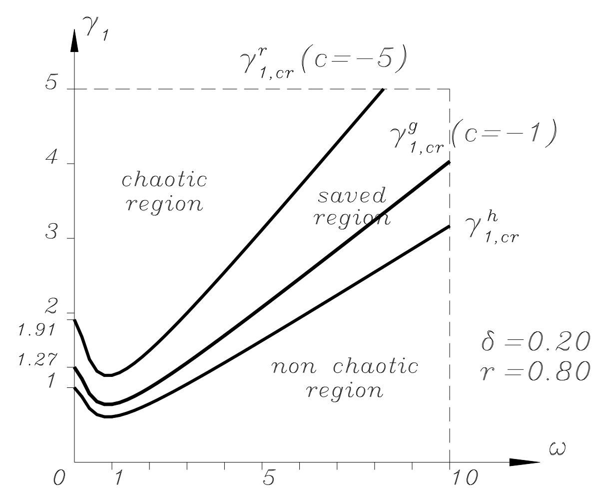
Duffing



Rigid block



Inverted pendulum



Optimal control (1)

key idea: **Choosing the "optimal" excitation**

What is "optimal"?

- For ***fixed*** system parameters ($\varepsilon\delta$, ω, γ_1), maximize the distance between stable and unstable manifolds

(i) **dynamical system point of view**, (ii) used in numerical approach to control

- For ***varying*** system parameters (in particular γ_1), shift as much as possible the bifurcation threshold

(i) **engineering point of view**, (ii) used in analytical approach to control,
(iii) equivalent to enlarging as much as possible the saved region

Optimal control (2)

- In any case, optimization entails **maximing *G* by varying *h*(*m*)**
- The optimization problem is then

Maximize G

by varying the coefficients h_j and Ψ_j

- Is equivalent to maximize M by varying $h(m)$
- The optimization problem is then

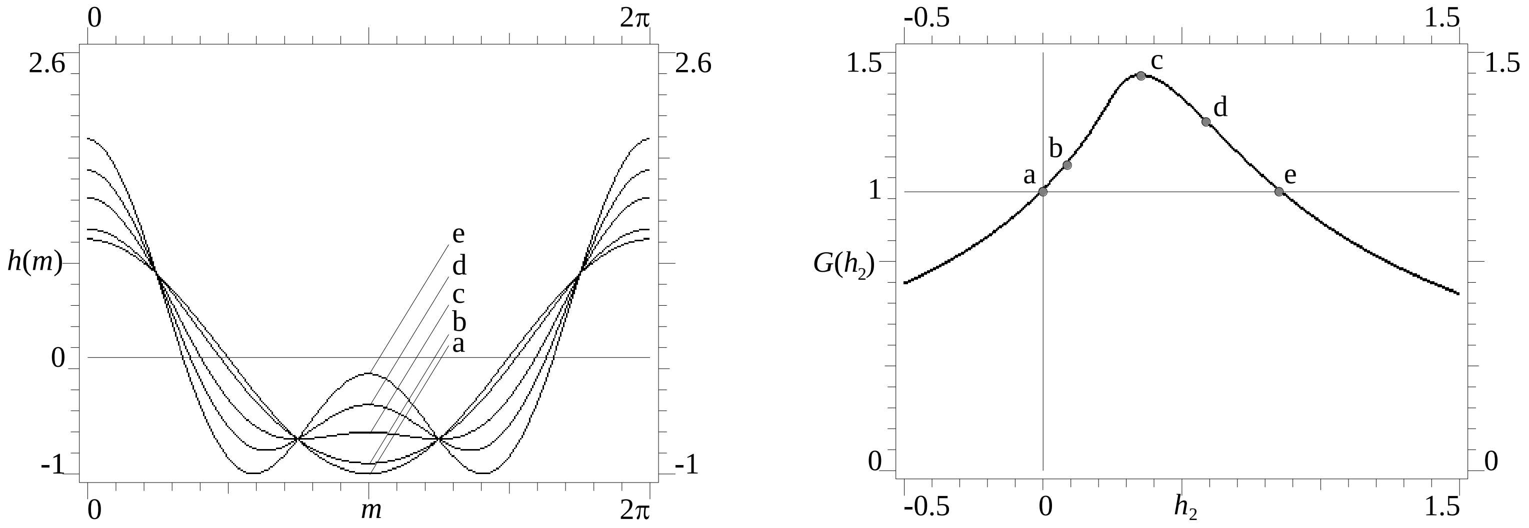
$$\text{Maximize } \min_m \left\{ \cos(m) + \sum_{j=2}^{N} h_j \cos(jm + \Psi_j) \right\}$$

by varying the coefficients h_j and Ψ_j

- This problem is system-independent

Optimal solution (1)

- To fix ideas, let us start with the simple example N=2, i.e. only one superharmonic added to the reference harmonic excitation ($\Psi_2 = 0$)

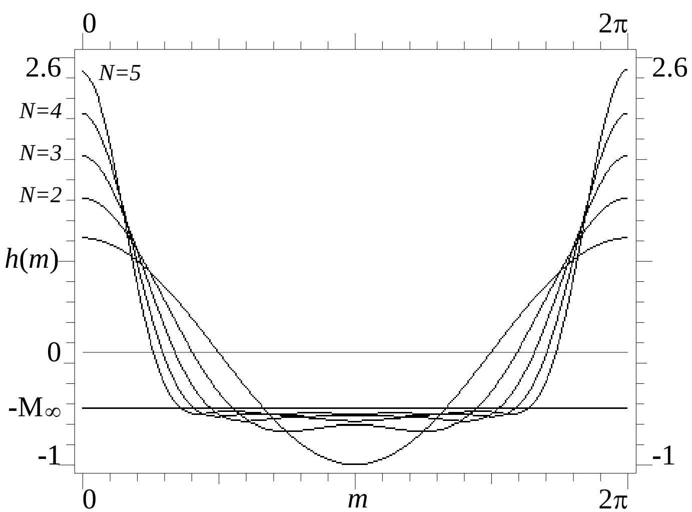


a) h_2=0 (harmonic excitation), b) h_2=0.0875, c) h_2=0.3535 (optimal), d) h_2=0.5875, e) h_2=0.8535

Optimal solution (2)

- The "universal" optimal solution is given by $\Psi_j = 0$ and

N	G_N	M_N	h_2	h_3	h_4	h_5	h_6	h_7	h_8	h_9
2	1.4142	0.7071	0.353553							
3	1.6180	0.6180	0.552756	0.170789						
4	1.7321	0.5773	0.673525	0.333274	0.096175					
5	1.8019	0.5550	0.751654	0.462136	0.215156	0.059632				
6	1.8476	0.5412	0.807624	0.567084	0.334898	0.153043	0.042422			
7	1.8794	0.5321	0.842528	0.635867	0.422667	0.237873	0.103775	0.027323		
8	1.9000	0.5263	0.872790	0.706011	0.527198	0.355109	0.205035	0.091669	0.024474	
9	1.9130	0.5227	0.877014	0.705931	0.518632	0.341954	0.195616	0.091497	0.031316	0.005929
…	…	…	…	…	…	…	…	…	…	…
∞	2	0.5	1	1	1	1	1	1	1	1



Remark. The coincidence of the minima of the optimal $h(m)$ has important consequences in terms of homoclinic bifurcation. In fact, while in the case of harmonic excitation there is only one minimum of $h(m)$ and there is only one homoclinic point at the bifurcation value, in the case of optimal excitations there are more minima of $h(m)$ and more *distinct* homoclinic points, so that the Birkhoff signature is different and the corresponding homoclinic bifurcation is degenerate and structurally unstable.

Unified framework for control

Investigation of how a ***generic*** **dynamical property** (homo/heteroclinic bifurcations) entails a ***generic*** **approach to control**:

- *System-independent* structure of the distance between stable and unstable manifolds
- *System-independent* optimization problems
- *System-independent* solutions

The "core" of control is *generic*

The differences between various systems are of technical nature and are due to different values of relevant coefficients, i.e. from the h_j to the γ_j

Some considerations

- From the previous developments we obtain the physical optimal excitation $f_{opt}(\omega t)$, which is system-dependent
- If in the uncontrolled case the excitation is harmonic, $\gamma_1 \sin(\omega t)$, the control excitation is

$$f_{con}(\omega t) = f_{opt}(\omega t) - \gamma_1 \sin(\omega t)$$

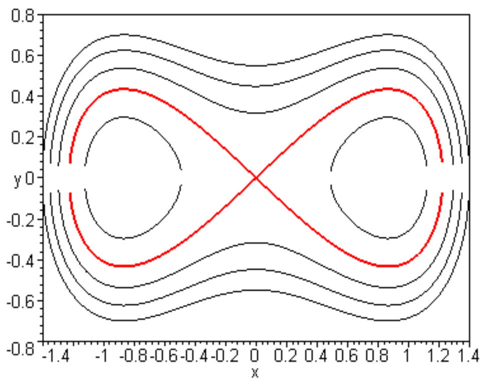
- More generally, if the uncontrolled excitation is generic, the control excitation is simply given by

$$f_{con}(\omega t) = f_{opt}(\omega t) - f_{uncon}(\omega t)$$

- *Open-loop* control method
- Only periodic excitations considered

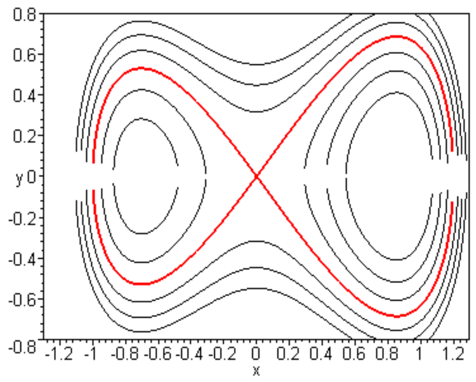
Controlling more homoclinic intersections

- What happens when there are more (e.g. two) possible homoclinic intersections to be controlled?
- Example: the symmetric Duffing oscillator



$$\ddot{x}-\frac{x}{2}+\frac{x^3}{2}=0$$

- More involved example: the asymmetric Helmholtz-Duffing oscillator



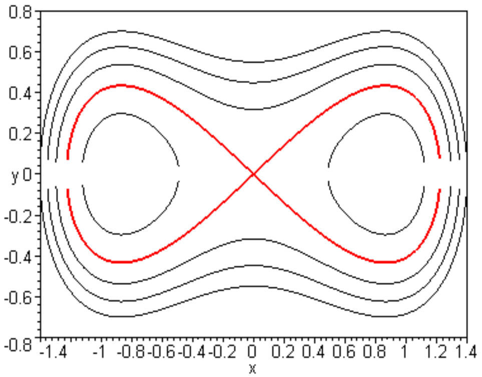
$$\ddot{x}-\sigma x-\frac{3}{2}(\sigma-1)x^2+2x^3=0$$

Distances

- There are more (e.g. two) distances, one per homoclinic intersection to be controlled

$$d^{r,l}(m) = c\left[1 + \frac{\gamma_1^{r,l}}{\gamma_{1,cr}^{h;r,l}(\omega)} h^{r,l}(m)\right]$$

$$h^{r,l}(m) = \cos(m) + \sum_{j=2}^{\infty} h_j^{r,l} \cos(jm + \Psi_j)$$

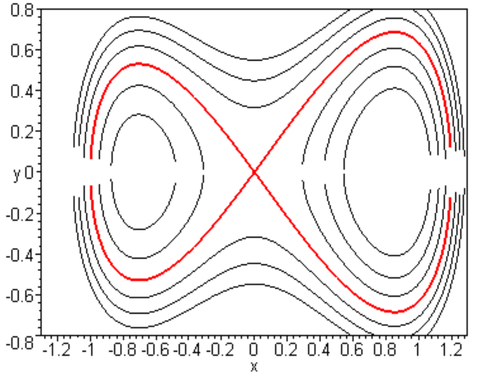


- Symmetric Duffing

$$h_j^{r,l} = \pm \frac{\gamma_j}{\gamma_1} j \frac{\cosh\left(\frac{\omega\pi}{\sqrt{2}}\right)}{\cosh\left(\frac{j\omega\pi}{\sqrt{2}}\right)}$$

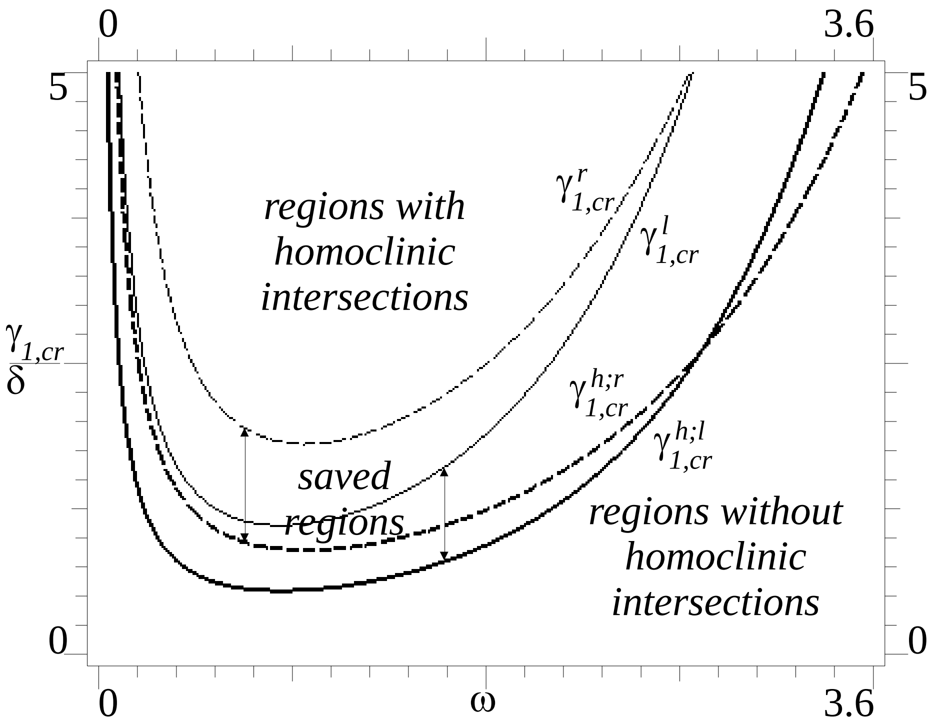
- Asymmetric Helmholtz-Duffing

$$h_j^{r,l} = \frac{\gamma_j}{\gamma_1} j \frac{\sinh\left(\frac{\omega\pi}{\sqrt{\sigma}}\right)}{\sinh\left(\frac{j\omega\pi}{\sqrt{\sigma}}\right)} \frac{\sinh\left[\frac{j\omega}{\sqrt{\sigma}} \operatorname{acos}\left(\pm\frac{\sigma-1}{\sigma+1}\right)\right]}{\sinh\left[\frac{\omega}{\sqrt{\sigma}} \operatorname{acos}\left(\pm\frac{\sigma-1}{\sigma+1}\right)\right]}$$



Saved regions

- There are more (e.g. two) saved regions, one per homoclinic intersection to be controlled
- Asymmetric Helmholtz-Duffing



Gains

- There are more (e.g. two) gains, one per homoclinic intersection to be controlled

$$\frac{\gamma_1^r}{\gamma_{1,cr}^{h;r}(\omega)} = \frac{1}{-\min_m \{h^r(m)\}} = \frac{1}{M^r} = G^r$$

$$\frac{\gamma_1^l}{\gamma_{1,cr}^{h;l}(\omega)} = \frac{1}{-\min_m \{h^l(m)\}} = \frac{1}{M^l} = G^l$$

$$G^i$$

$$\ldots$$

The optimization problem (1)

- We can apply the previous control separately to each G^i, giving up to control the other possible homoclinic intersections → **"local" (or "one-side") control**
- Or, we can try to control *all the gains simultaneously* → **"global" control**
- In the latter case the (new) optimization problem is

Maximize $\min\{G^1, G^2, G^3, \ldots.\}$

$$G^i = \frac{1}{-\min_m \{h^i(m)\}}$$

by varying the functions $h^1(m)$, $h^2(m)$, $h^3(m)$,….

- Practically corresponds to increasing the lowest gain up to the second lowest, than increasing both up to the third last, etc.

The optimization problem (2)

- The global optimization problem is system-dependent, contrarily to the local one
- This is due to the fact that all $h^1(m)$, $h^2(m)$, $h^3(m)$,…. are related to the same excitations (with system-dependent equations), and thus cannot be varied independently
- For example, for the asymmetric Duffing-Helmholtz:

$$h_j^{r,l} = \frac{\gamma_j}{\gamma_1} j \frac{\sinh\left(\frac{\omega\pi}{\sqrt{\sigma}}\right)}{\sinh\left(\frac{j\omega\pi}{\sqrt{\sigma}}\right)} \frac{\sinh\left[\frac{j\omega}{\sqrt{\sigma}} \operatorname{acos}\left(\pm\frac{\sigma-1}{\sigma+1}\right)\right]}{\sinh\left[\frac{\omega}{\sqrt{\sigma}} \operatorname{acos}\left(\pm\frac{\sigma-1}{\sigma+1}\right)\right]} \longrightarrow \frac{h_j^l}{h_j^r} = \frac{\sinh\left[\frac{j\omega}{\sqrt{\sigma}} \operatorname{acos}\left(-\frac{\sigma-1}{\sigma+1}\right)\right]}{\sinh\left[\frac{\omega}{\sqrt{\sigma}} \operatorname{acos}\left(-\frac{\sigma-1}{\sigma+1}\right)\right]} \frac{\sinh\left[\frac{\omega}{\sqrt{\sigma}} \operatorname{acos}\left(+\frac{\sigma-1}{\sigma+1}\right)\right]}{\sinh\left[\frac{j\omega}{\sqrt{\sigma}} \operatorname{acos}\left(+\frac{\sigma-1}{\sigma+1}\right)\right]}$$

and we can vary only h^r_j (for example), and then h^l_j varies accordingly (with system-dependent law)

The optimization problem (3)

- The “global” optimization problem is much more complicated than the “local” one (not only because it is no longer system-independent)
- The “local” optimization problem can be seen as a particular case of the “global” optimization problem, where the restraints coming from the other gains are removed
- This implies that the “global” optimum is lesser than (or equal to) the “local” optimum

The optimization problem (4)

- Only in special cases also the "global" optimization problem is system-independent
- This happens when there are system-independent relations between homoclinic bifurcations to be controlled
- A remarkable case is that of symmetric systems, for example the Duffing oscillator
- In this example we have $h_j^{r,l} = \pm \frac{\gamma_j}{\gamma_1} j \frac{\cosh\left(\frac{\omega\pi}{\sqrt{2}}\right)}{\cosh\left(\frac{j\omega\pi}{\sqrt{2}}\right)}$, i.e. $h^r{}_j = -\,h^l{}_j$

Global control for symmetric systems (1)

- For symmetric systems $h^r(m) = -h^l(m)$
- $-\min_m\{h^r(m)\} = -\min_m\{-h^l(m)\} = \max_m\{h^l(m)\}$

$$G^l = \frac{1}{-\min_m\{h^l(m)\}} \qquad G^r = \frac{1}{\max_m\{h^l(m)\}}$$

- The optimization problem is then

Maximize $\min\{G^l, G^r\}$

by varying the function $h^l(m)$

- The optimal is obtained when $G^l = G^r$, namely when $-\min_m\{h^l(m)\} = \max_m\{h^l(m)\}$

Global control for symmetric systems (2)

- The optimization problem can then be reformulated as

$$\text{Maximize } \min_m\{h(m)\},$$

$$h(m) = \cos(m) + \sum_{j=2}^{N} h_j \cos(jm + \Psi_j),$$

by varying the coefficients h_j and Ψ_j

under the constraint – $\min_m\{h^l(m)\} = \max_m\{h^l(m)\}$

- The constraint is automatically satisfied by considering only odd harmonics, i.e. $h_2=h_4=h_6=\ldots=0$
- This is again a system-independent optimization problem, since the constraint is system-independent

Generalization

- When the asymmetry of the system is given, we arrive at the following optimization problem

$$\text{Maximize } \min_m\{h(m)\},$$

$$h(m) = \cos(m) + \sum_{j=2}^{N} h_j \cos(jm + \Psi_j),$$

by varying the coefficients h_j and Ψ_j

under the constraint $-\min_m\{h^l(m)\} = \alpha \max_m\{h^l(m)\}$

- $\alpha \neq 1$ is the asymmetry parameter
- This is again a system-independent optimization problem, since the constraint is system-independent, as α is assumed to be system-independent

Optimal solutions

- The "universal" optimal solution is given by $\Psi_j = 0$ and

the symmetric case α=1 (only odd superharmonics!)

N	G_N	M_N	h_2	h_3	h_4	h_5	h_6	h_7	h_8	h_9
2	1	1	0							
3	1.1547	0.8660	0	-0.166667						
4	1.1547	0.8660	0	-0.166667	0					
5	1.2071	0.8284	0	-0.232259	0	0.060987				
6	1.2071	0.8284	0	-0.232259	0	0.060987	0			
7	1.2310	0.8123	0	-0.264943	0	0.100220	0	-0.028897		
8	1.2310	0.8123	0	-0.264943	0	0.100220	0	-0.028897	0	
9	1.2440	0.8038	0	-0.284314	0	0.125257	0	-0.053460	0	0.0163649
…	…	…	…	…	…	…	…	…	…	…
∞	1.2732	0.7854	0	-0.333333	0	0.200000	0	-0.142857	0	0.1111111

an asymmetric case, α=0.5

N	G_N	M_N	h_2	h_3	h_4	h_5	h_6	h_7	h_8	h_9
2	1.4142	0.7071	0.353553							
3	1.4472	0.6910	0.352786	0.029180						
4	1.5000	0.6667	0.388672	0.000000	-0.055339					
5	1.5669	0.6382	0.441927	0.007496	-0.123186	-0.061735				
6	1.5771	0.6340	0.431789	-0.007810	-0.135928	-0.078875	-0.016550			
7	1.5834	0.6315	0.438843	-0.007077	-0.142507	-0.080418	-0.009175	0.006199		
8	1.5903	0.6288	0.443799	-0.001783	-0.140034	-0.079330	-0.005682	0.012261	0.004333	
9	1.5935	0.6275	0.446347	-0.002890	-0.145983	-0.083588	-0.004504	0.015251	0.005398	-0.000152
…	…	…	…	…	…	…	…	…	…	…
∞	1.6540	0.6046	0.500000	0.000000	-0.250000	-0.200000	0.000000	0.142857	0.125000	0.000000

Numerical control

- What to do when it is not possible to have an analytical (not even approximate, by the Melnikov method) expression for the distances between the stable and unstable manifolds?
- We have to measure it numerically
- Of course, even if the idea of the control method remains the same, its application, including the optimization problem, are system-dependent

Numerical control

- What to do when it is not possible to have an analytical (not even approximate, by the Melnikov method) expression for the distances between the stable and unstable manifolds?
- We have to measure it numerically
- Of course, even if the idea of the control method remains the same, its application, including the optimization problem, are system-dependent
- A "guideline" for application of (numerical) control illustrated: more important than the specific example

The mechanical model

- The Duffing equation

$$\ddot{x} + 0.164\dot{x} - 0.2x + x^3 = A[\sin(t) + c_1 \sin(nt + c_2)]$$

A: excitation amplitude
c_1: *relative* amplitude of the control superharmonic (physical amplitude is Ac_1)
c_2: the phase

- only a single control superharmonic is added to the basic harmonic excitation to perform reasonable analyses

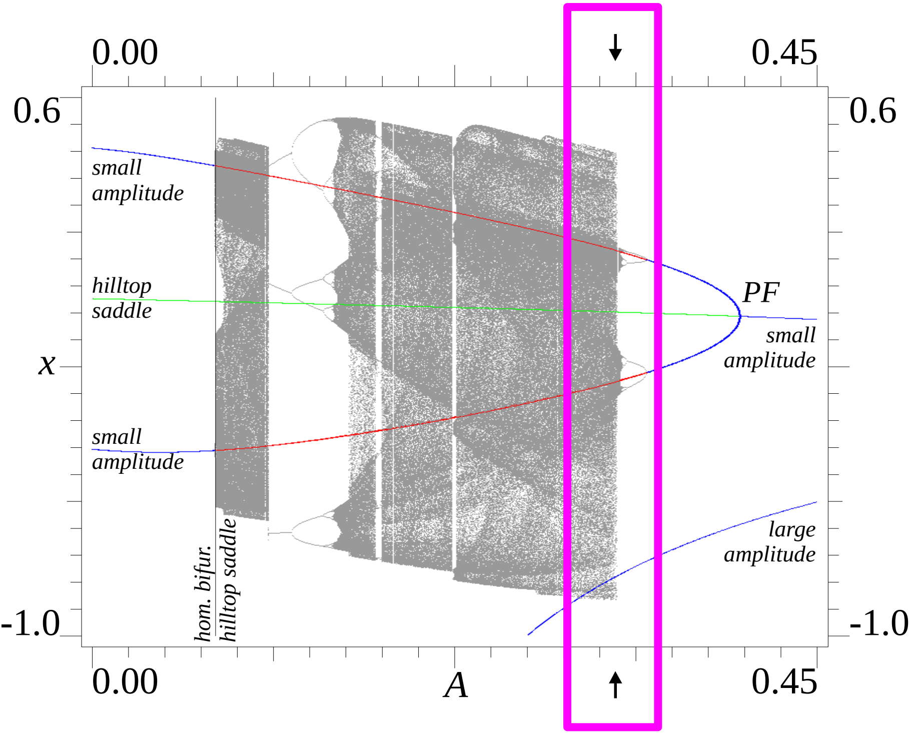
- practical interest: archetype of the 1 d.o.f. smooth hardening nonlinear oscillators with two potential wells, buckled beams, magnetoelastic pendulum, etc. [Moon, 1992]

Tools and dynamical phenomena of interest

- Saddles are determined by a modified Newton method [Nusse & Yorke, 98]
- Invariant manifolds are detected by standard numerical algorithms based on forward and backward iterations of the unstable and stable eigenvectors, respectively ([You *et al.*, 91], [Hobson, 93], [Chan & Wang, 00])
- Specific dynamical phenomena to be controlled: transitions from single-well chaos to cross-well chaos, due to a homoclinic bifurcation of a P3 saddle ([Katz & Dowell, 94], [Ueda *et al.*, 90])

System response under harmonic excitation

- Preliminary analysis needed to understand the dynamical behaviour of the system



➔ A^{hom}=0.0765 (numerical), A^{hom}=0.0738 (Melnikov), error 3.6% due to high damping

➔ rest positions $x_{1,3}$ get back their initial stability for large value of A, were they are period 1, confined and small amplitude oscillations

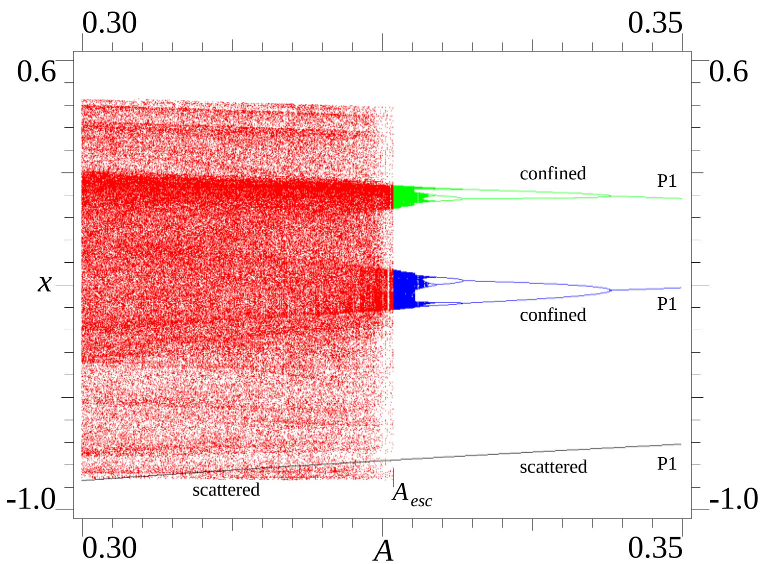
➔ they coalesce by a pitchfork (PF) symmetry breaking bifurcation leaving a unique P1, scattered and ***small amplitude*** attractor

➔ coexistence with another P1, scattered and ***large amplitude*** attractor

➔ in the range of instability of P1 oscillations the scenario is quite involved (sequences of confined/scattered periodic/chaotic attractors)

➔ we focus attention on the last scattered to confined transition, at $A=A_{esc}\cong 0.3252$

Bifurcation diagram around the interval of interest

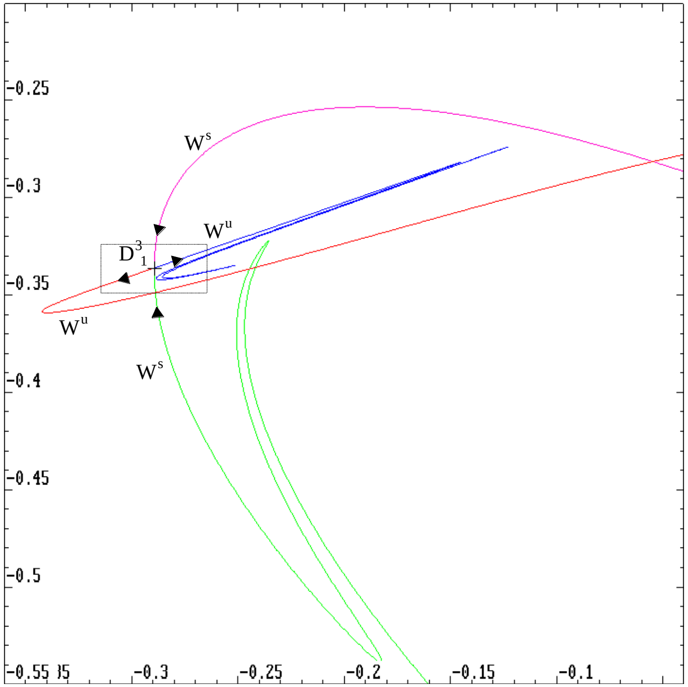


- it has been studied by Katz & Dowell [94] and Ueda et al. [90]

- for $A>A_{esc}$ confined periodic/chaotic and scattered periodic attractors
- for $A<A_{esc}$ scattered periodic and chaotic attractors
- the scattered periodic solution is not affected by the crisis

Dynamical event triggering the transition

- homoclinic bifurcation of a P3 saddle – non hilltop, that’s why we cannot use Melnikov method

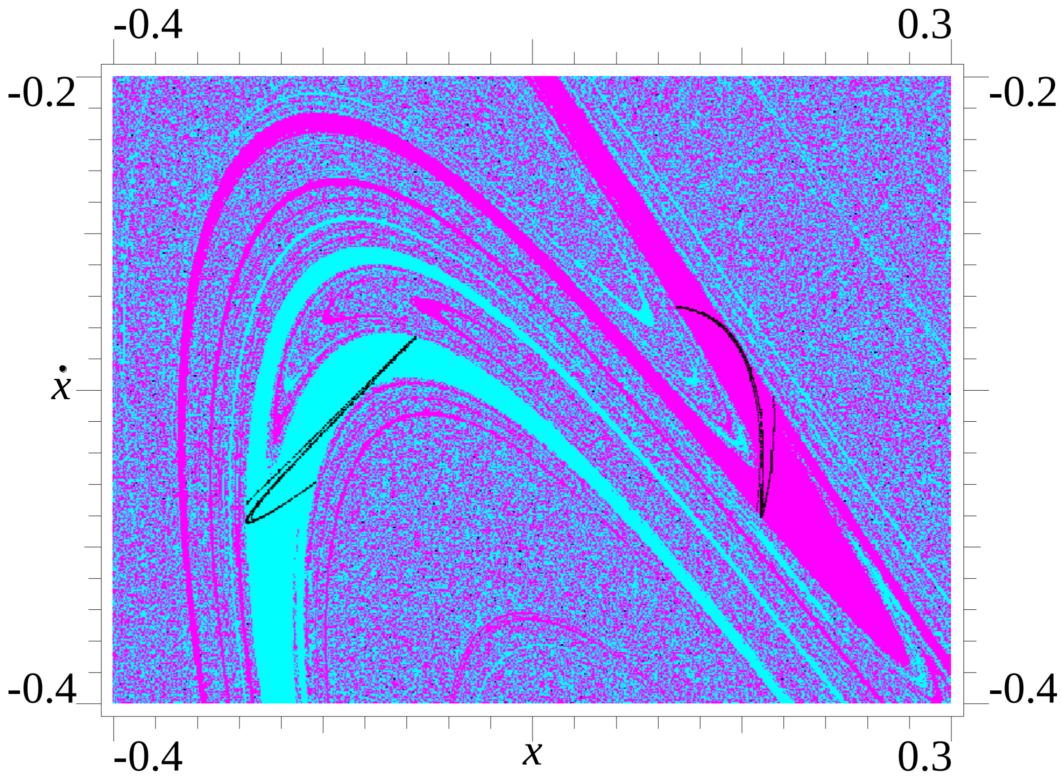


large view

zoom

Boundary crises

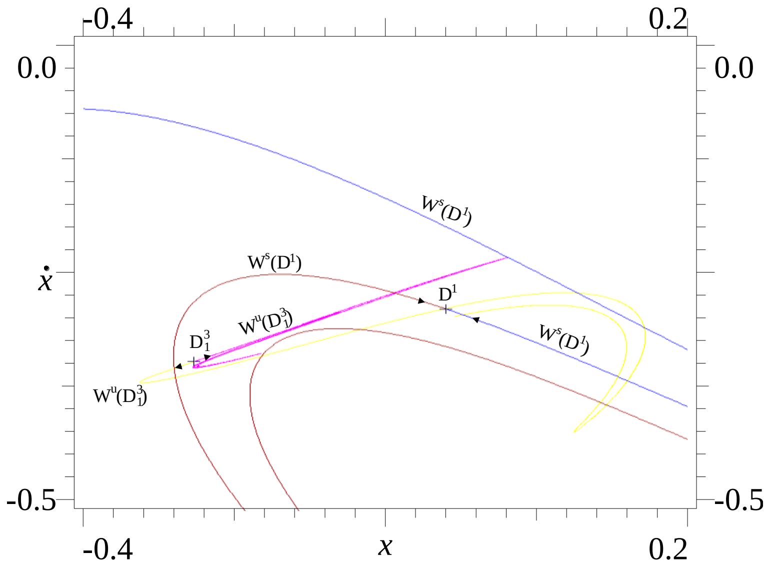
- the illustrated phenomenon corresponds to a boundary crisis



Dynamical behaviour

- Topological mechanism of transition from confined to scattered dynamics: connection with the heteroclinic intersection of the unstable manifold of $D^3{}_1$ and the stable manifold of the hilltop D_1;

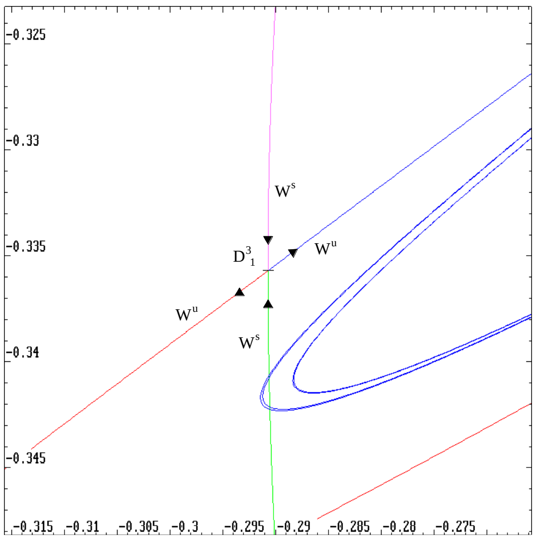
- The neighborhood of the formerly confined chaotic attractor is mapped
  (1) along the unstable manifold of $D^3{}_1$,
  (2) along the stable manifold of D^1,
  (3) up to near D^1, and finally
  (4) along the branch of the unstable manifold of D^1,
  (5) up to entering the other potential well

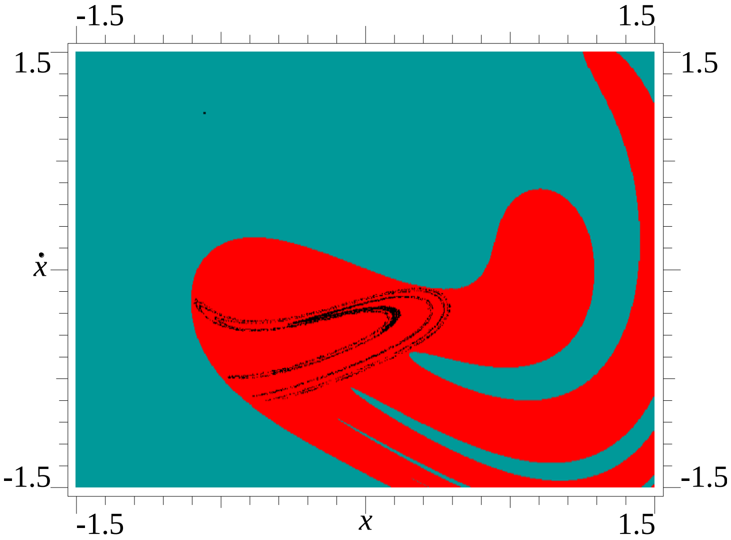


Preliminary euristic example (1)

- Fixed excitation amplitude $A=0.3246<A_{esc}\cong 0.3252$

Harmonic excitation

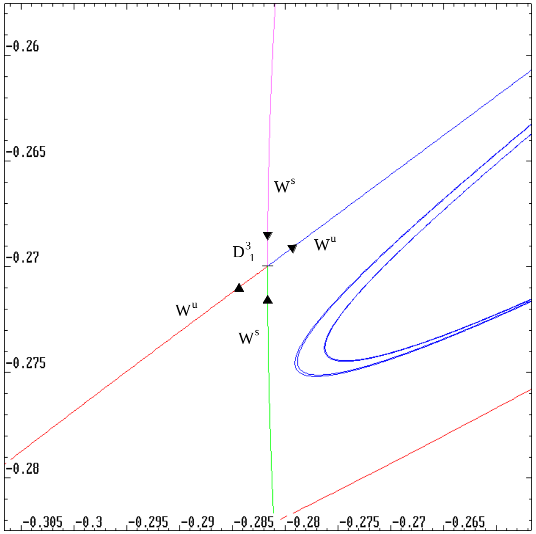


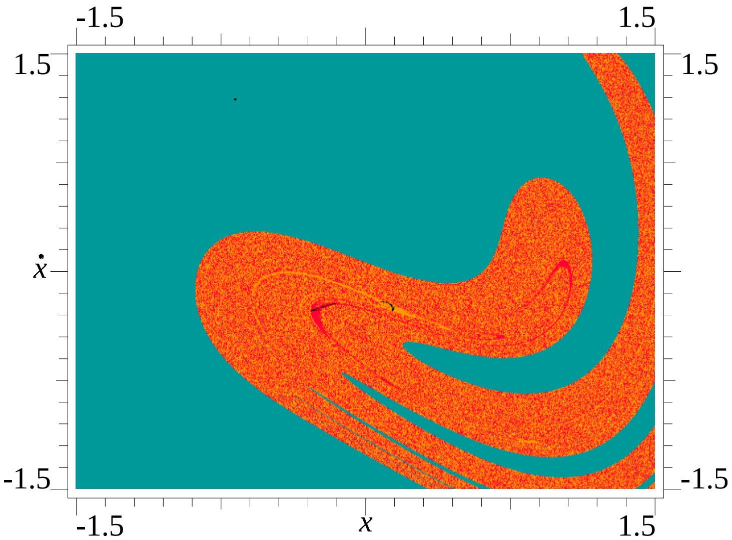


- $A<A_{esc}$ → manifold intersection → cross-well chaos (+ scattered P1)

Preliminary euristic example (2)

Symmetric control excitation (c_1=0.6, c_2=π, n=3)

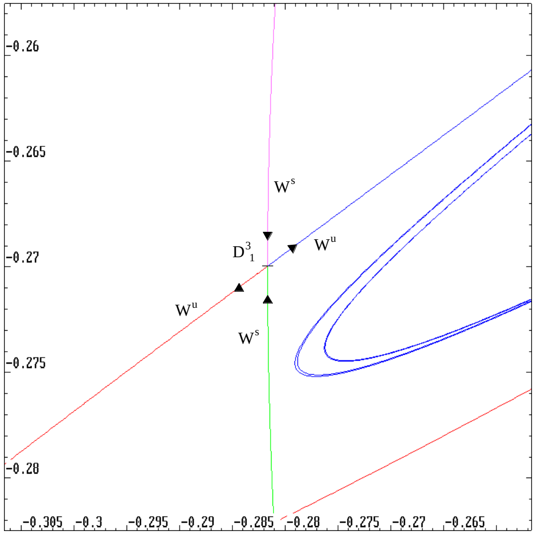


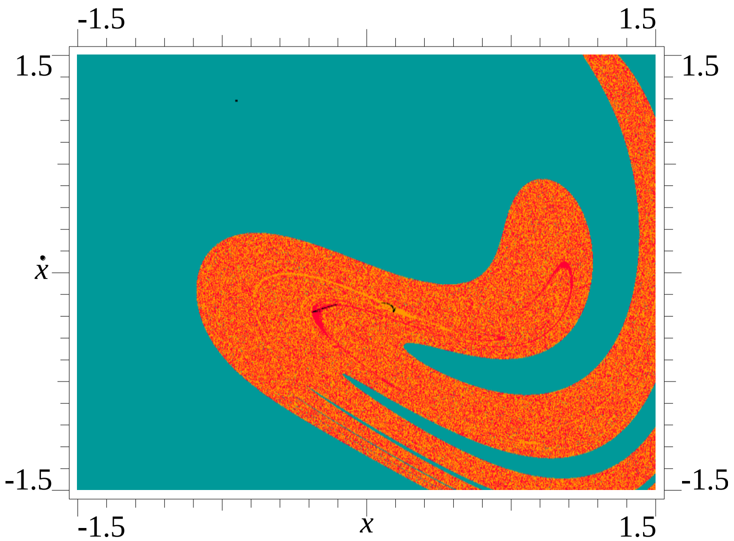


- Elimination of manifold intersection (theoretical effectiveness of control) → two confined chaotic attractors (+ scattered P1) (practical effectiveness: confinement of the dynamics)

Preliminary euristic example (2)

Symmetric control excitation (c_1=0.6, c_2=π, n=3)

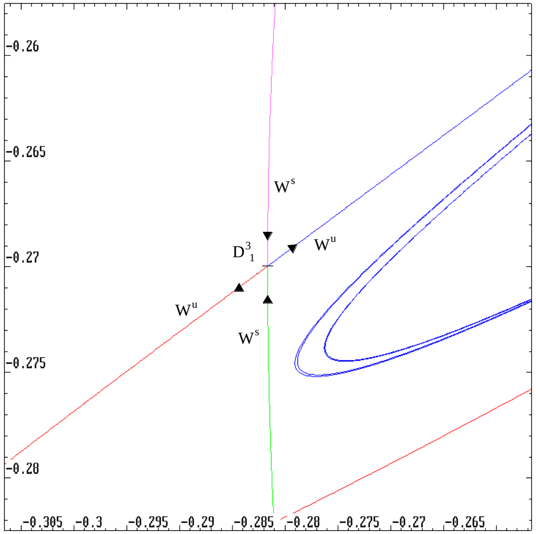


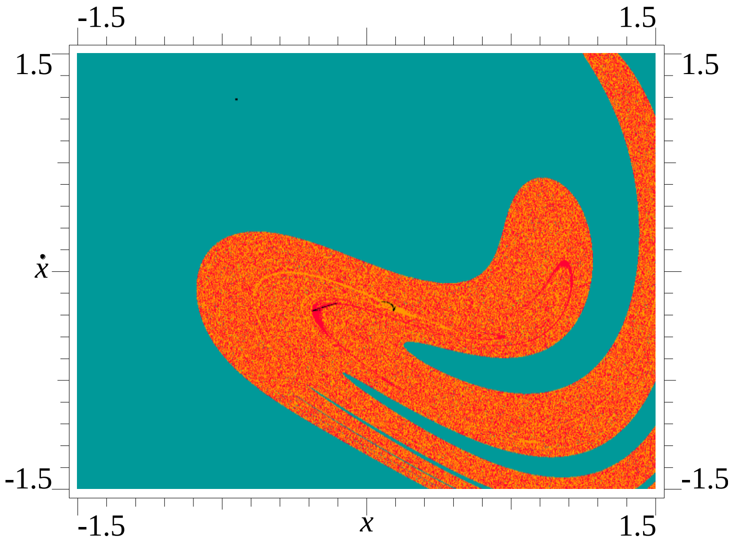


- Cross-well chaos has been eliminated, though chaoticity survives to a minor extent and can be possibly eliminated by better calibration of control (see the following)

Preliminary euristic example (2)

Symmetric control excitation (c_1=0.6, c_2=π, n=3)

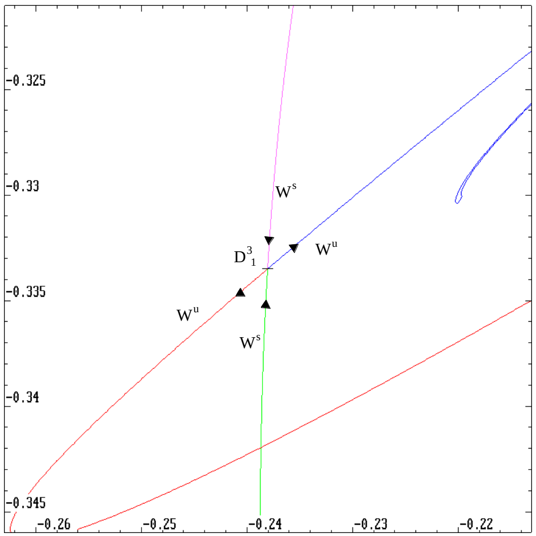


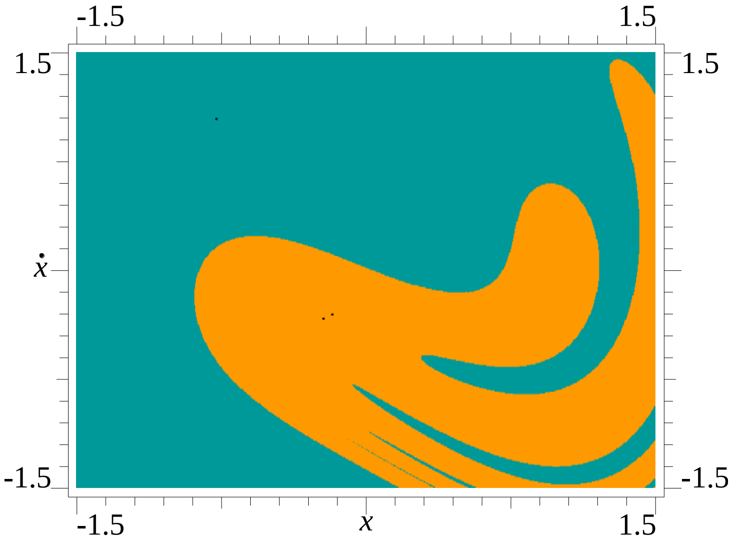


- The basin boundaries of the two confined attractors are very intertwined and are numerically fractal (due to homoclinic intersection of D^1)

Preliminary euristic example (3)

Asymmetric control excitation (c_1=0.3, c_2=3π/2, n=2)

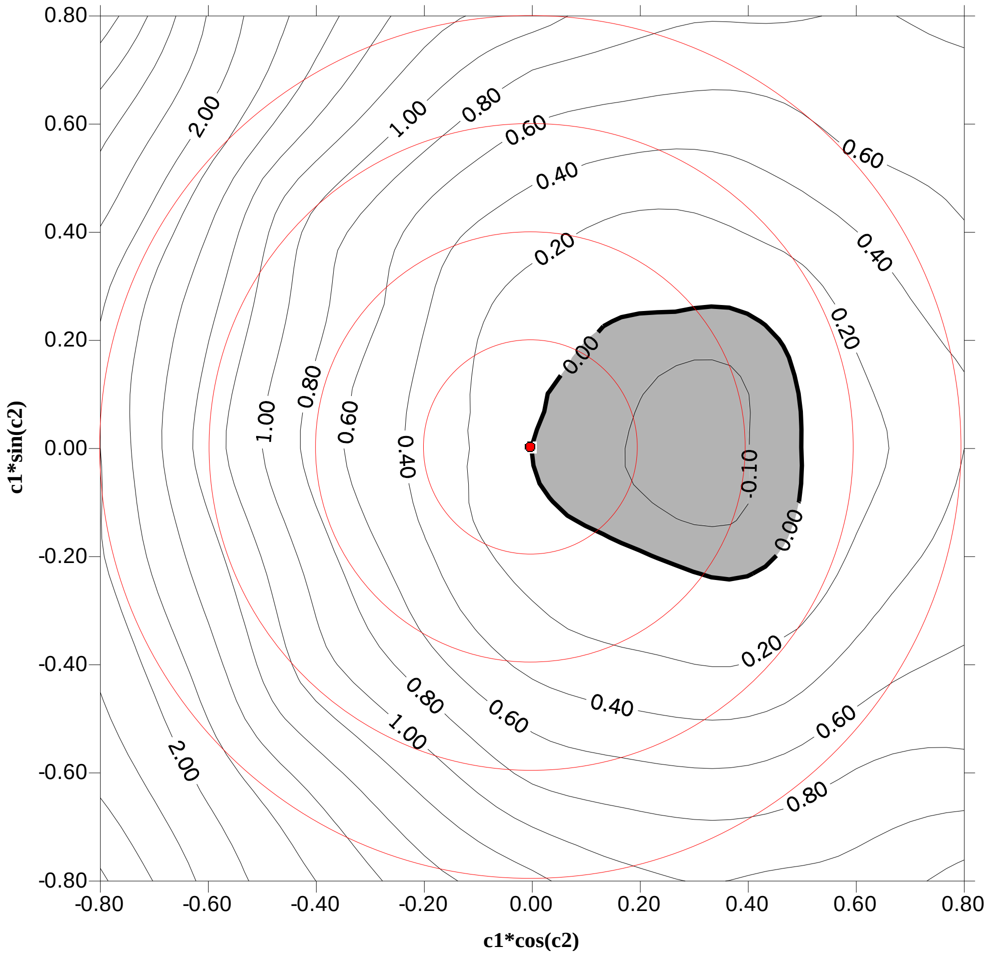




- elimination of manifold intersection (theoretical effectiveness of control) → one confined P2 attractors (+ scattered P1) (practical effectiveness: confinement and regularization of the dynamics)

Optimal control with symmetric excitation (1)

- The distance between stable and unstable manifolds



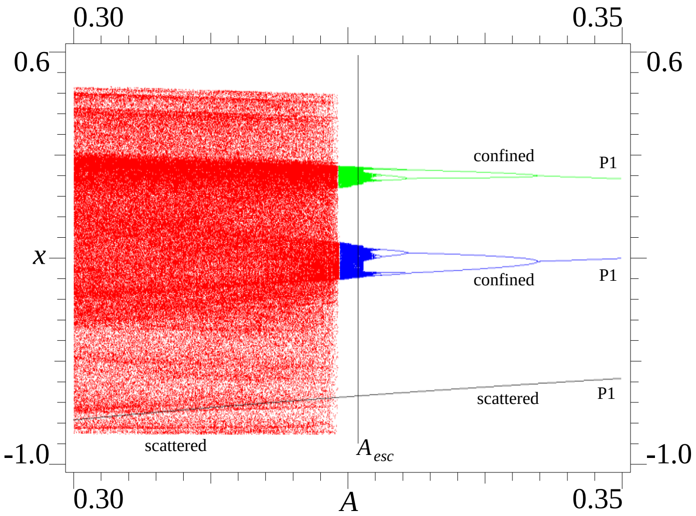
Grey: manifolds intersection

White: manifolds detached

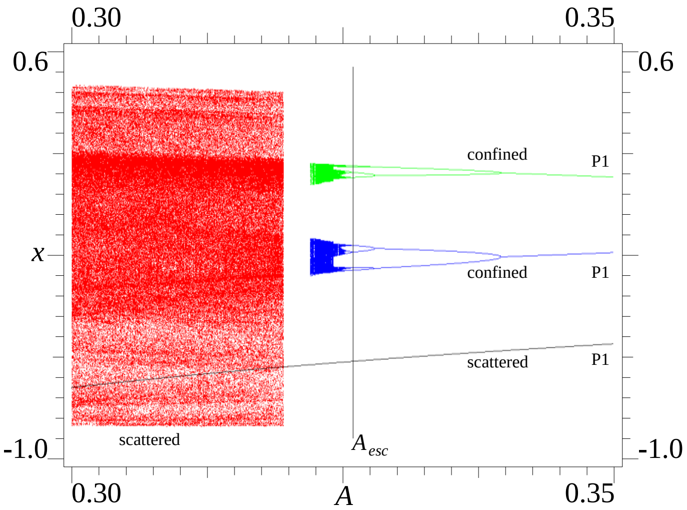
the *optimal excitation* has $c_2 \approx 1.08\pi$ and the largest possible c_1

Optimal control with symmetric excitation (2)

c_2=1.08π and $\underline{c_1=0.4}$



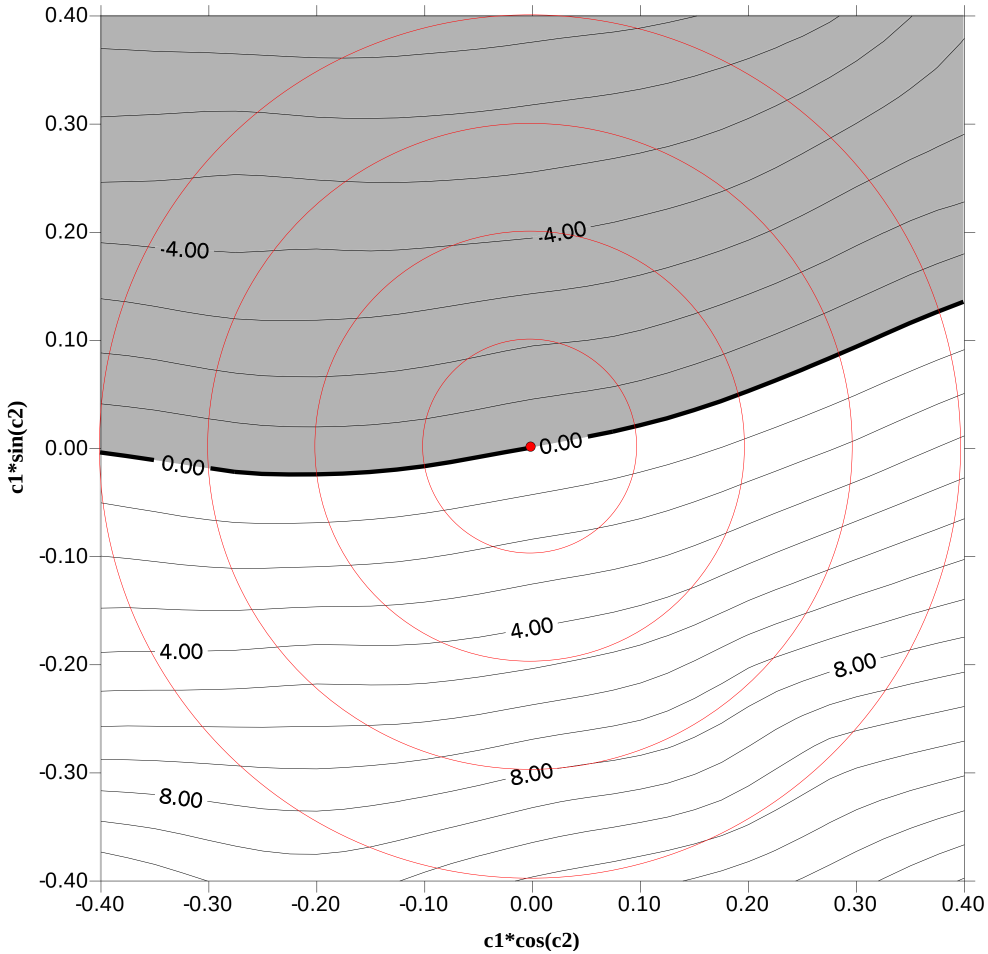
c_2=1.08π and $\underline{c_1=0.8}$



- Effectiveness of control, that increases for increasing c_1

Optimal control with asymmetric excitation (1)

- The distance between stable and unstable manifolds



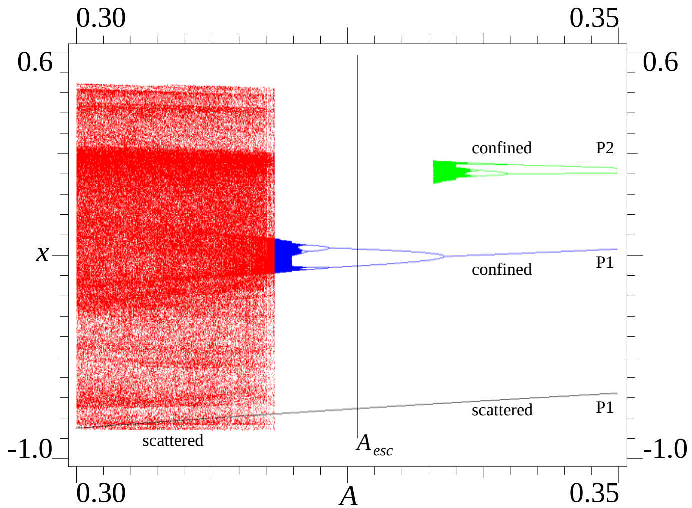
Grey: manifolds intersection

White: manifolds detached

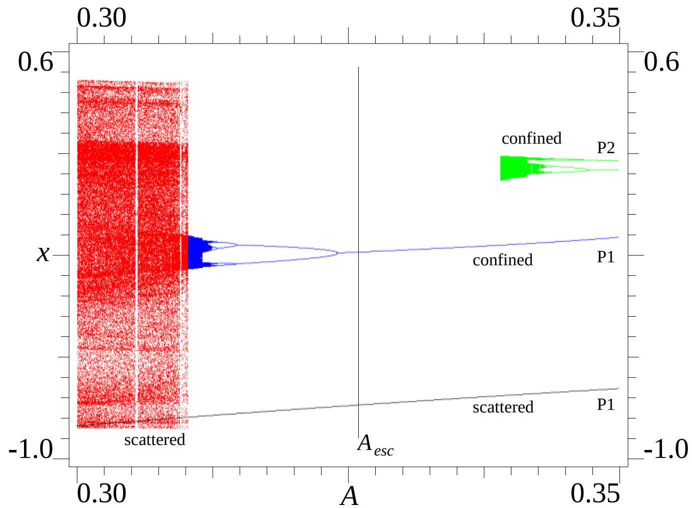
- the *left optimal excitation* has $c_2 \approx 1.65\pi$ and the largest possible c_1
- the *right optimal excitation* has $c_2 \approx 0.65\pi$ and the largest possible c_1

Optimal control with asymmetric excitation (2)

c_2=1.65π and $\underline{c_1=0.2}$



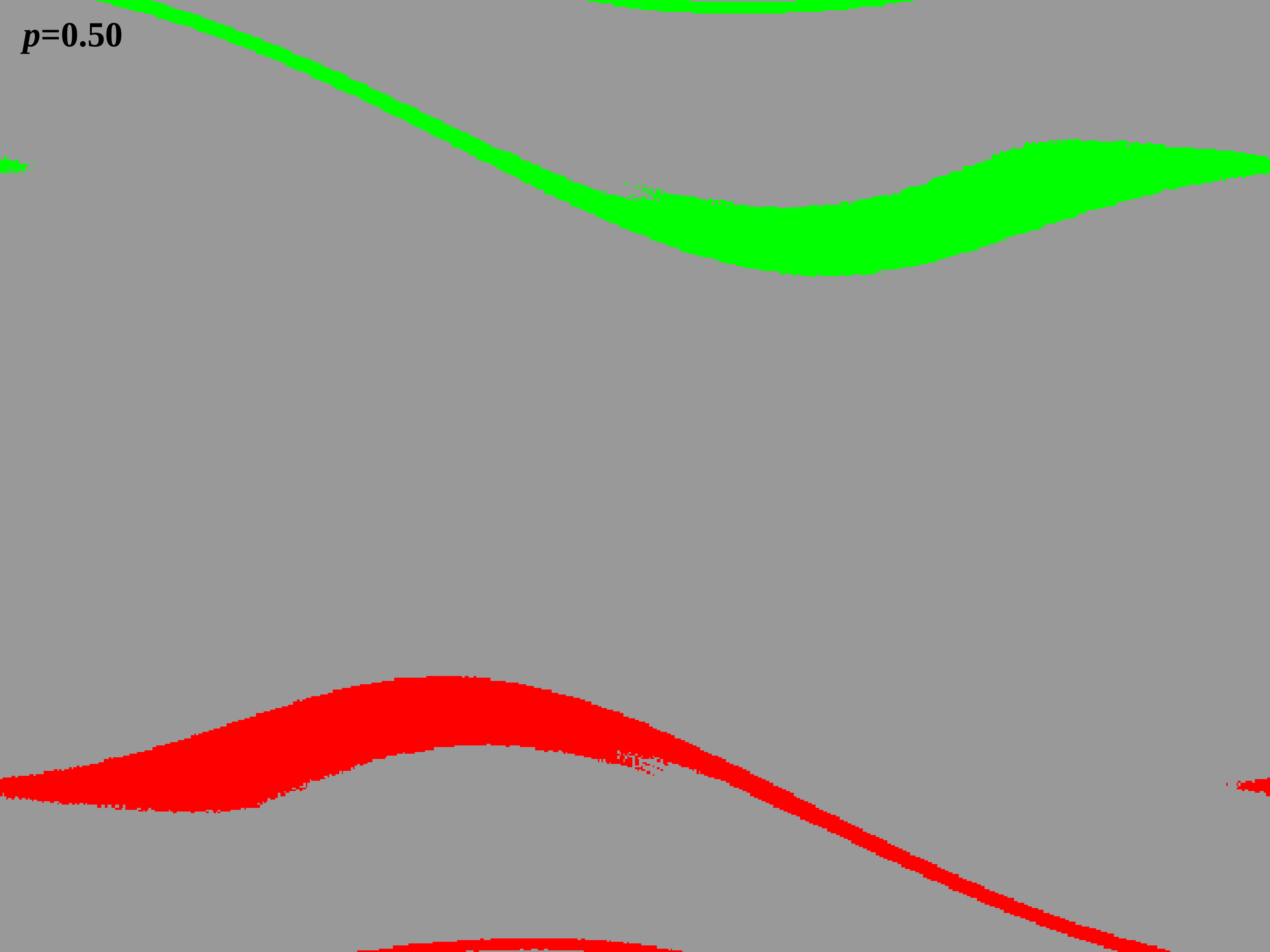
c_2=1.65π and $\underline{c_1=0.4}$

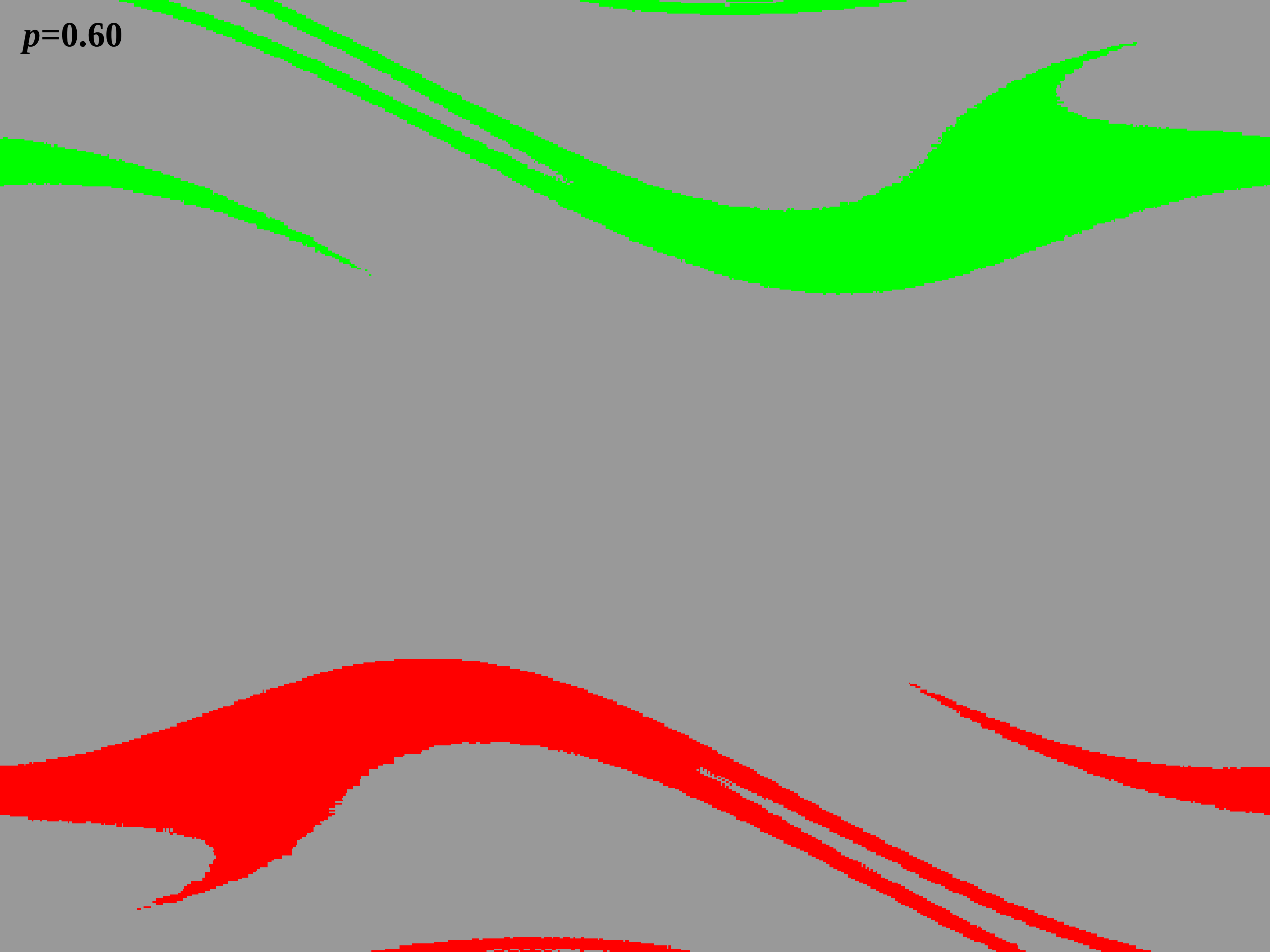


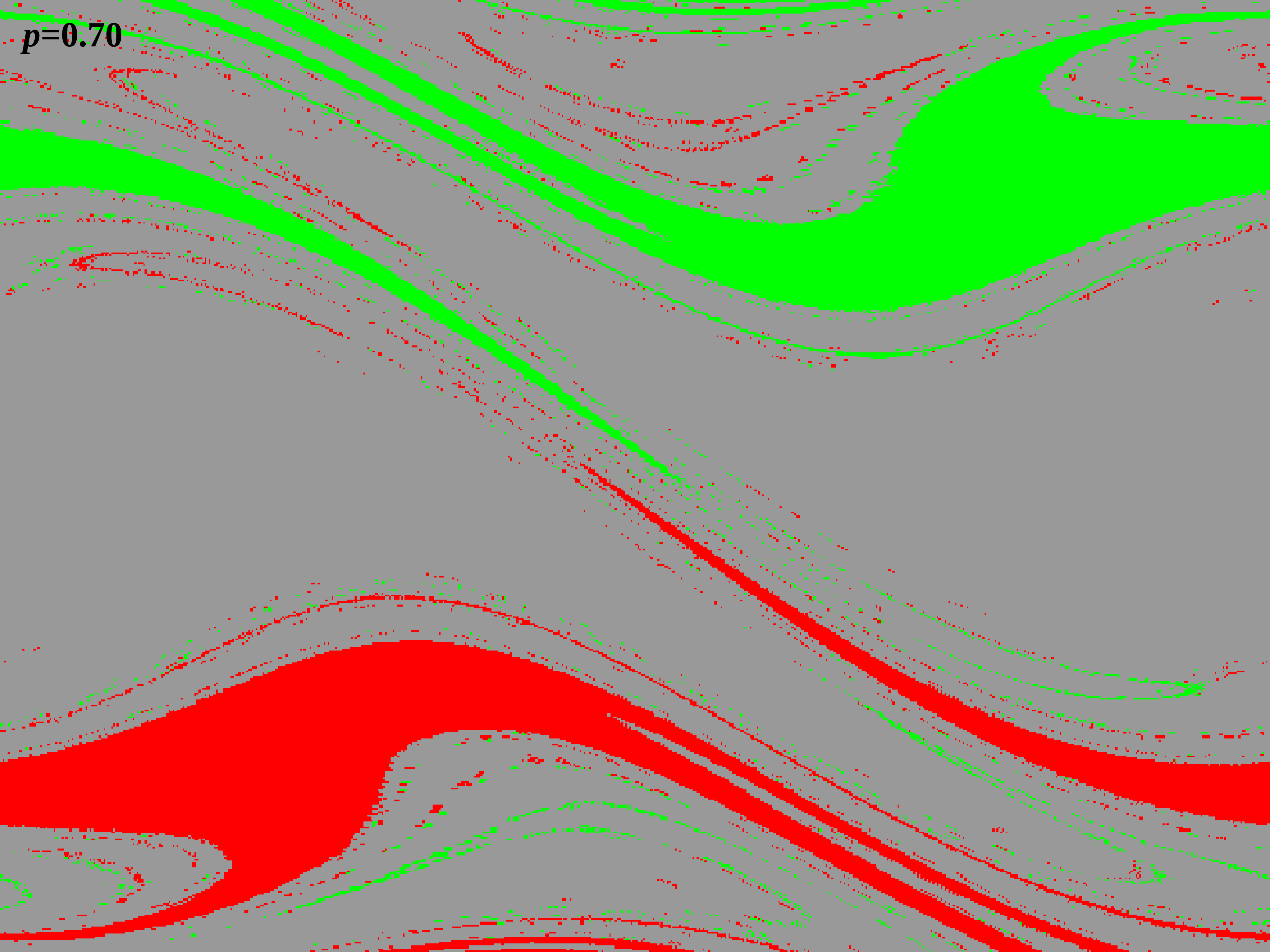
- Left optimal excitation
- Better performances that the symmetric excitation: asymmetric excitations ≈15 times more efficient
- Same results for the right optimal excitation

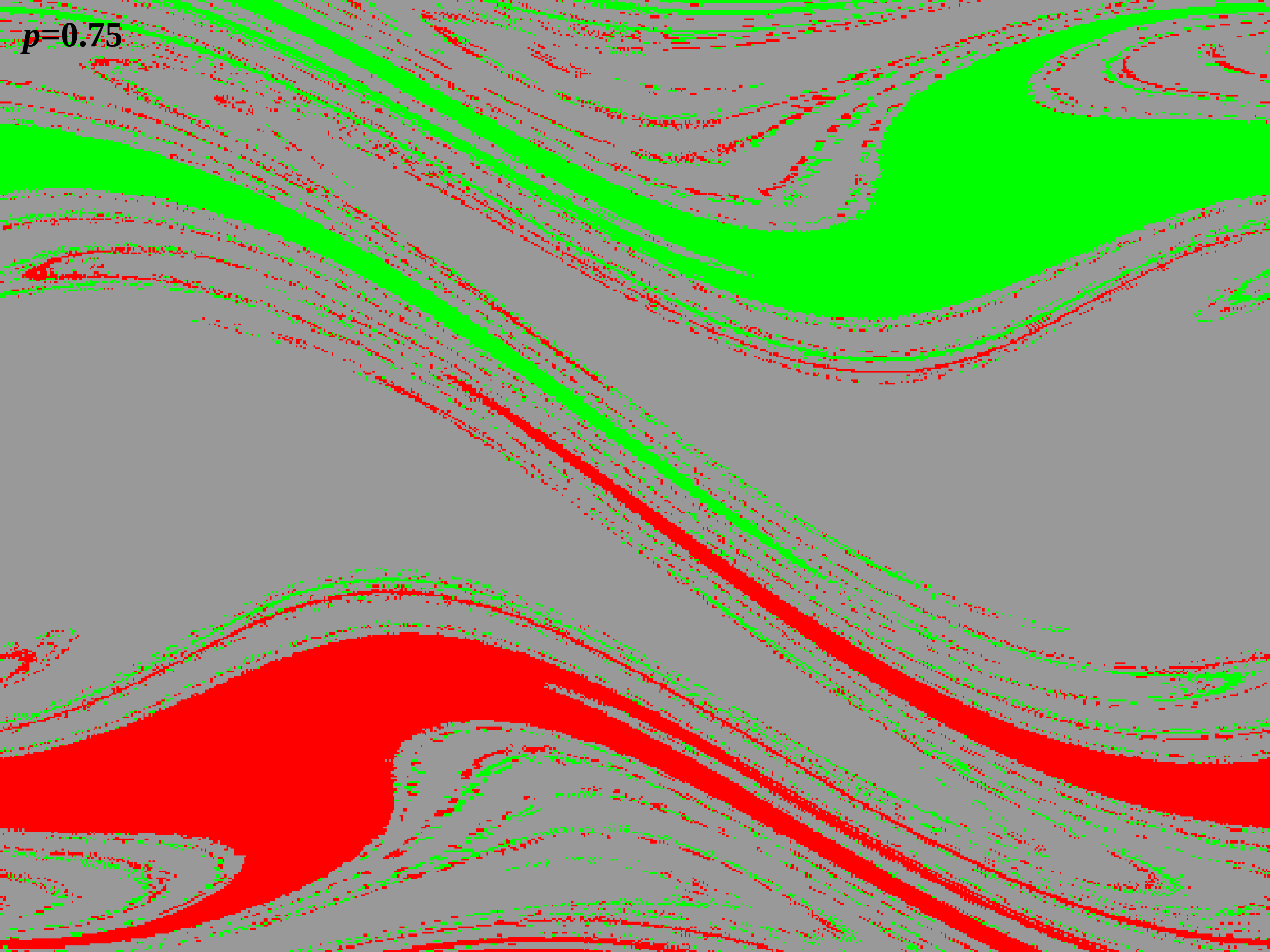
p=0.40

p=0.45
rotating clockwise
oscillating
rotating anti-clockwise

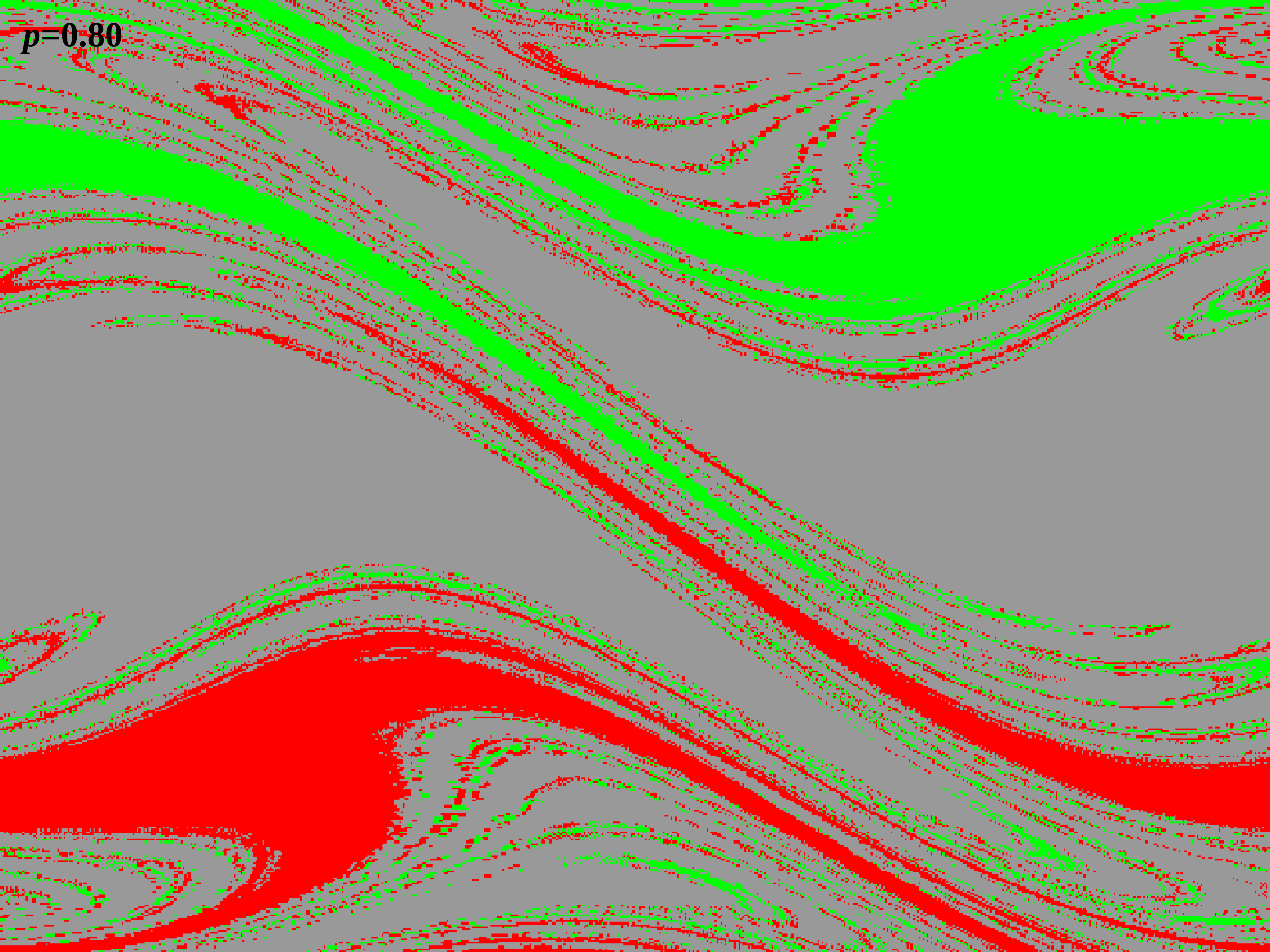

p=0.50

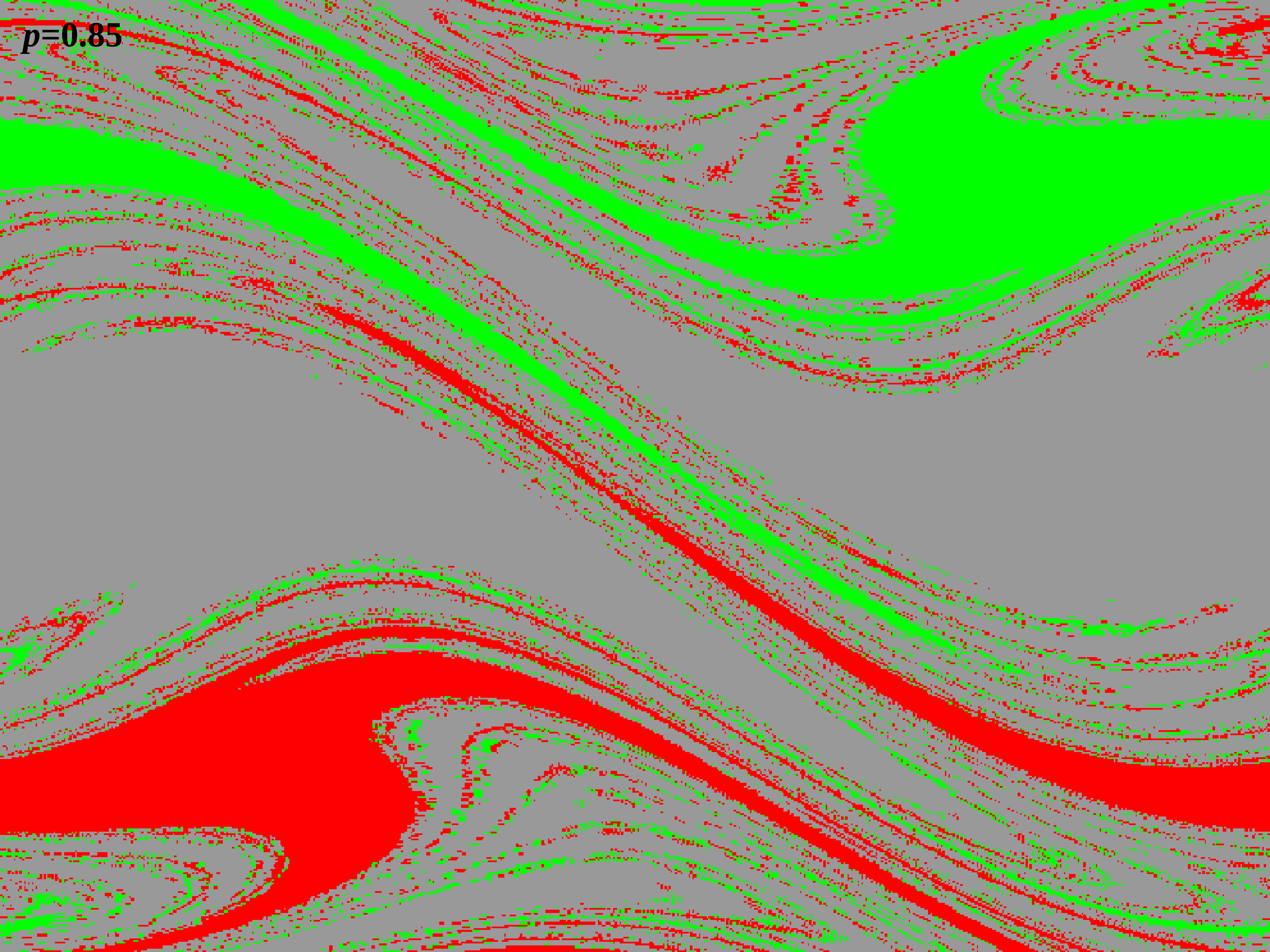

p=0.60

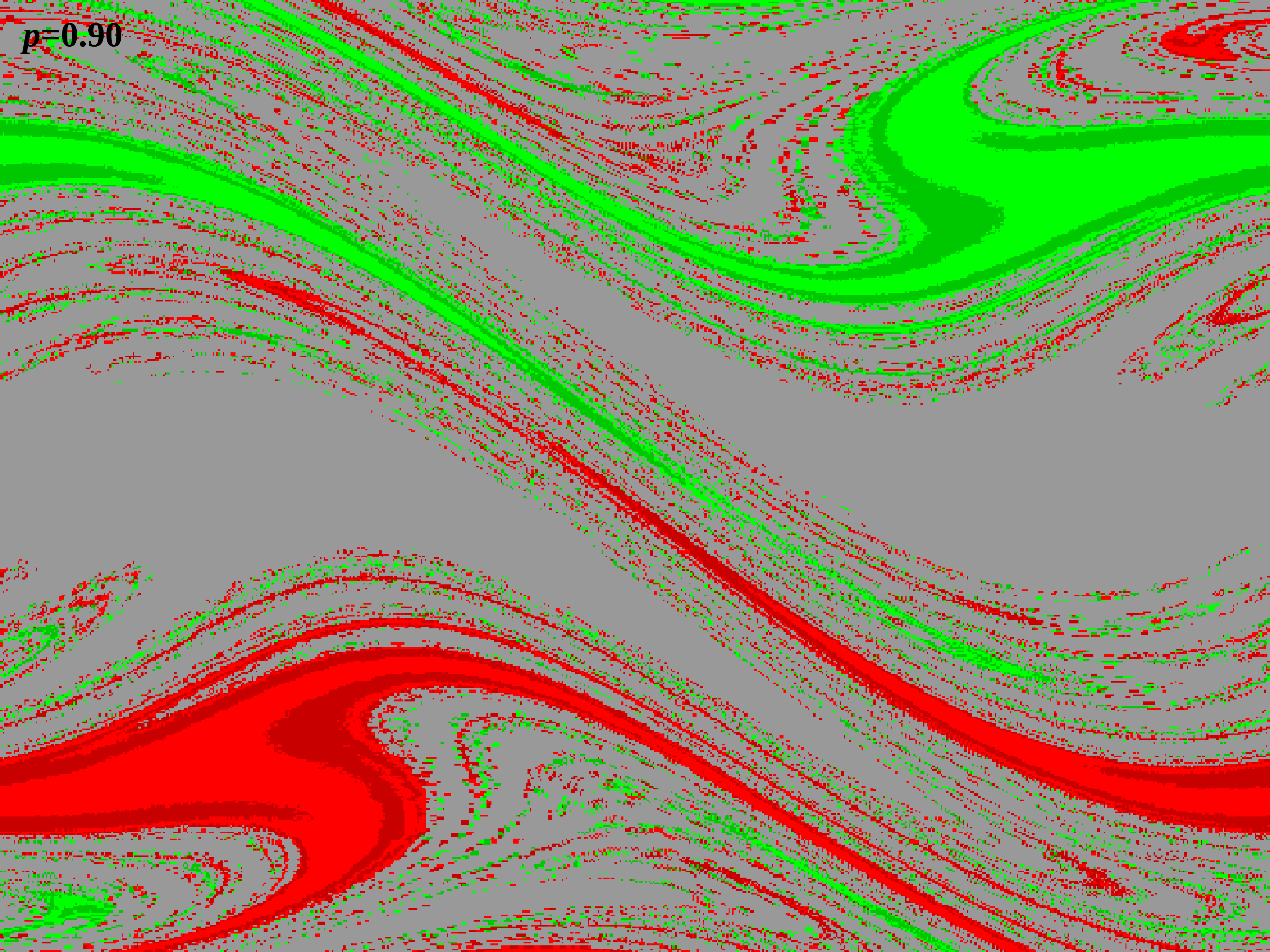

p=0.70

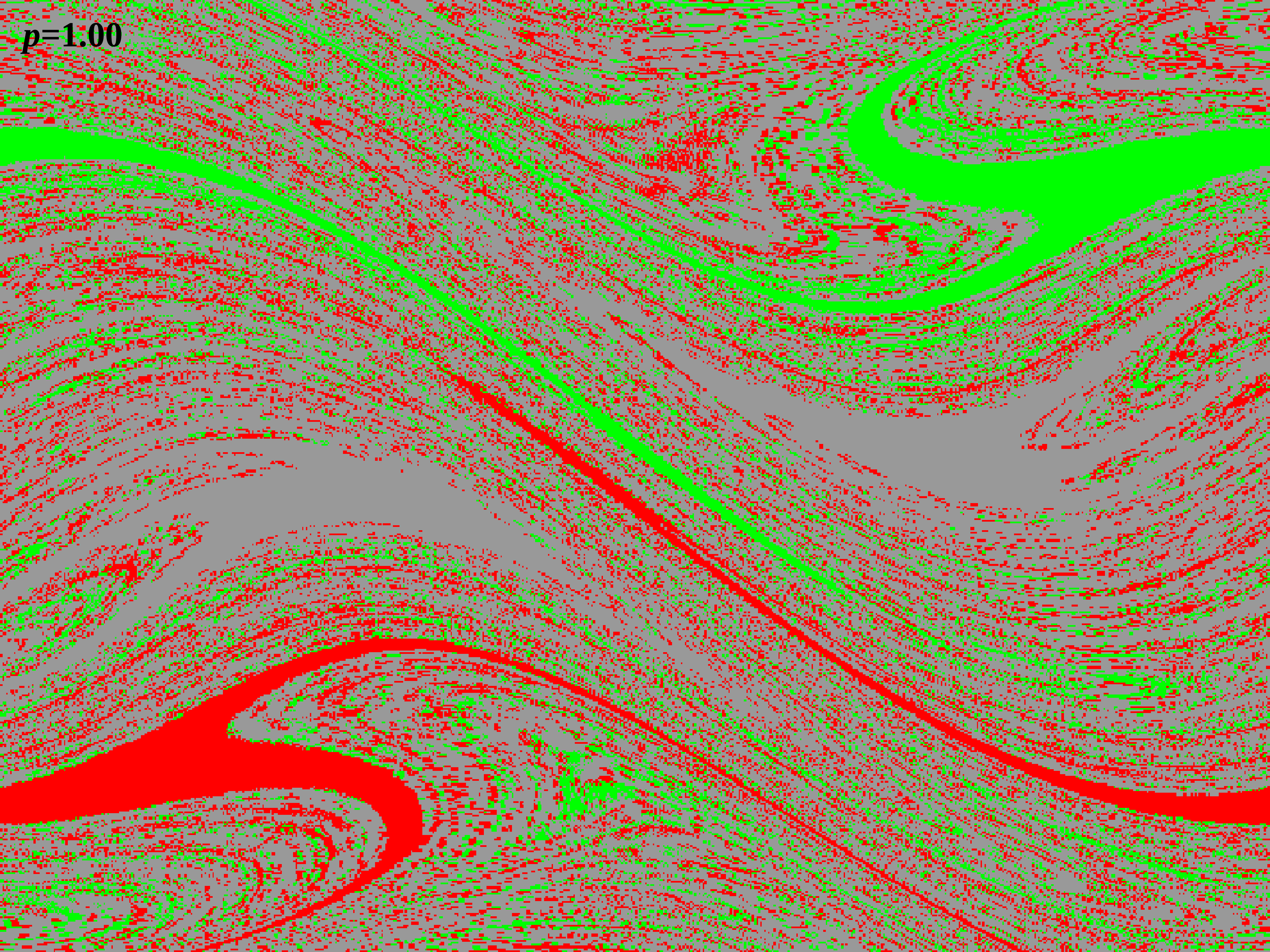


p=0.75

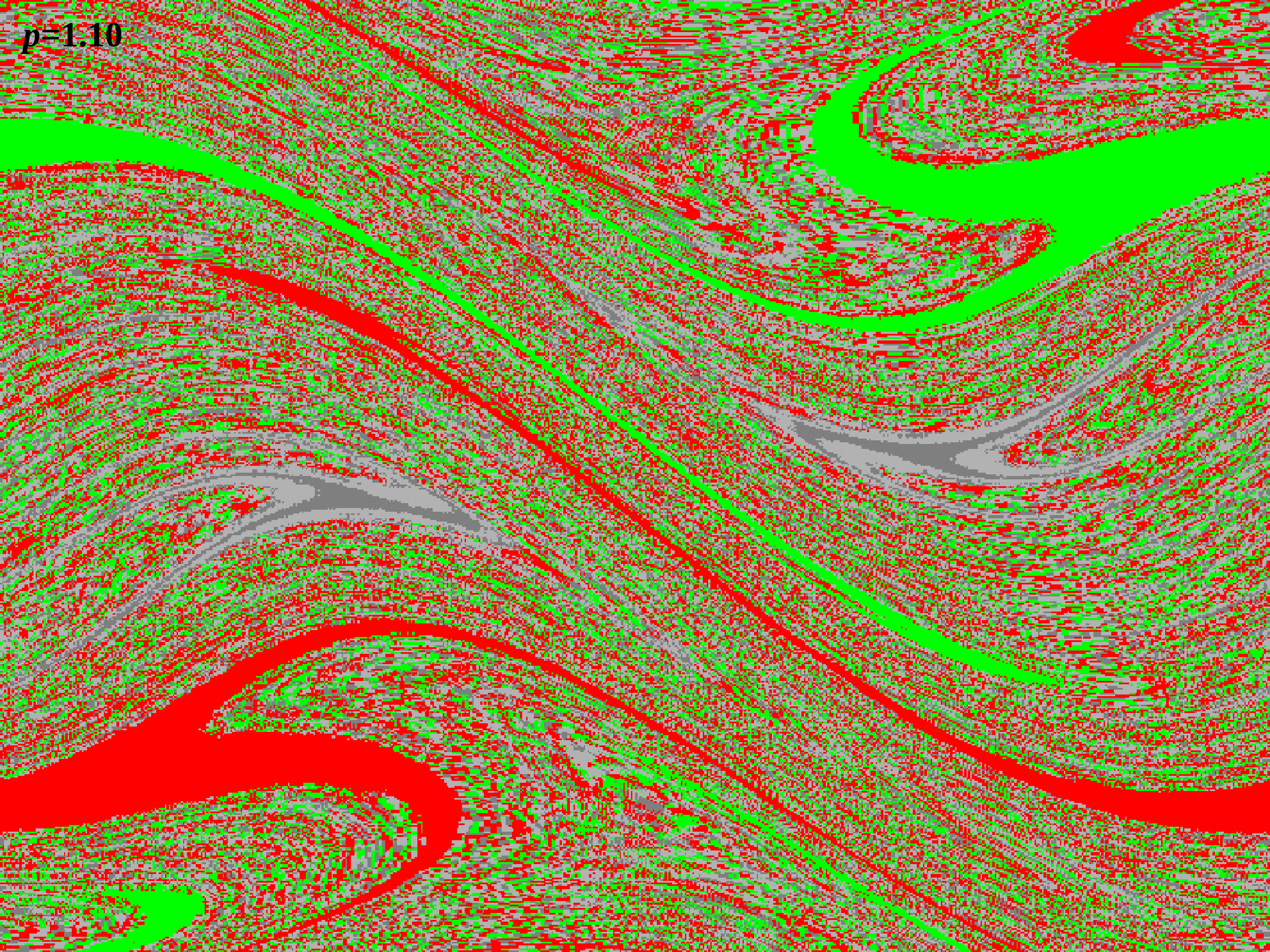

p=0.80

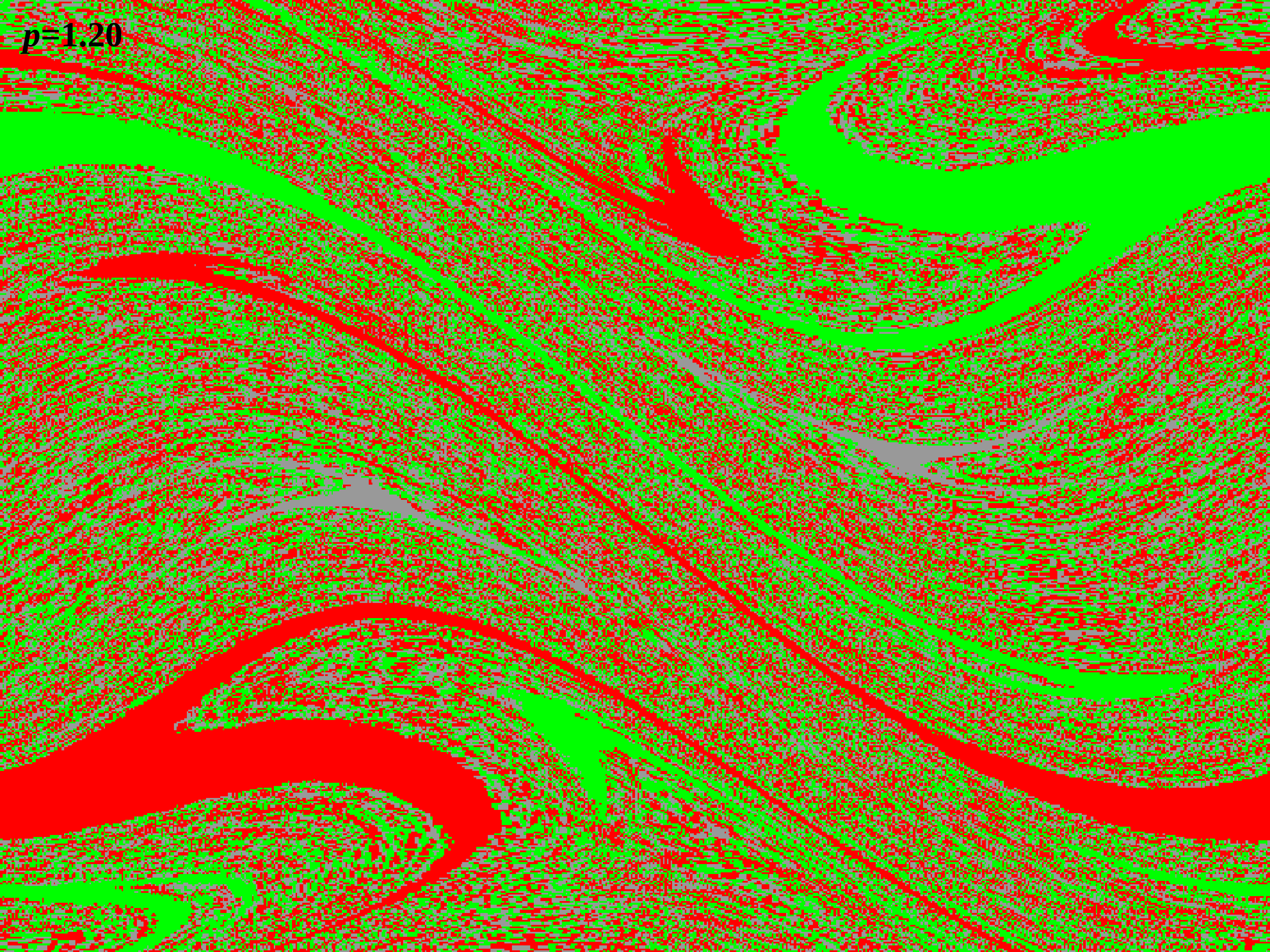

p=0.85


p=0.90



p=1.00


p=1.10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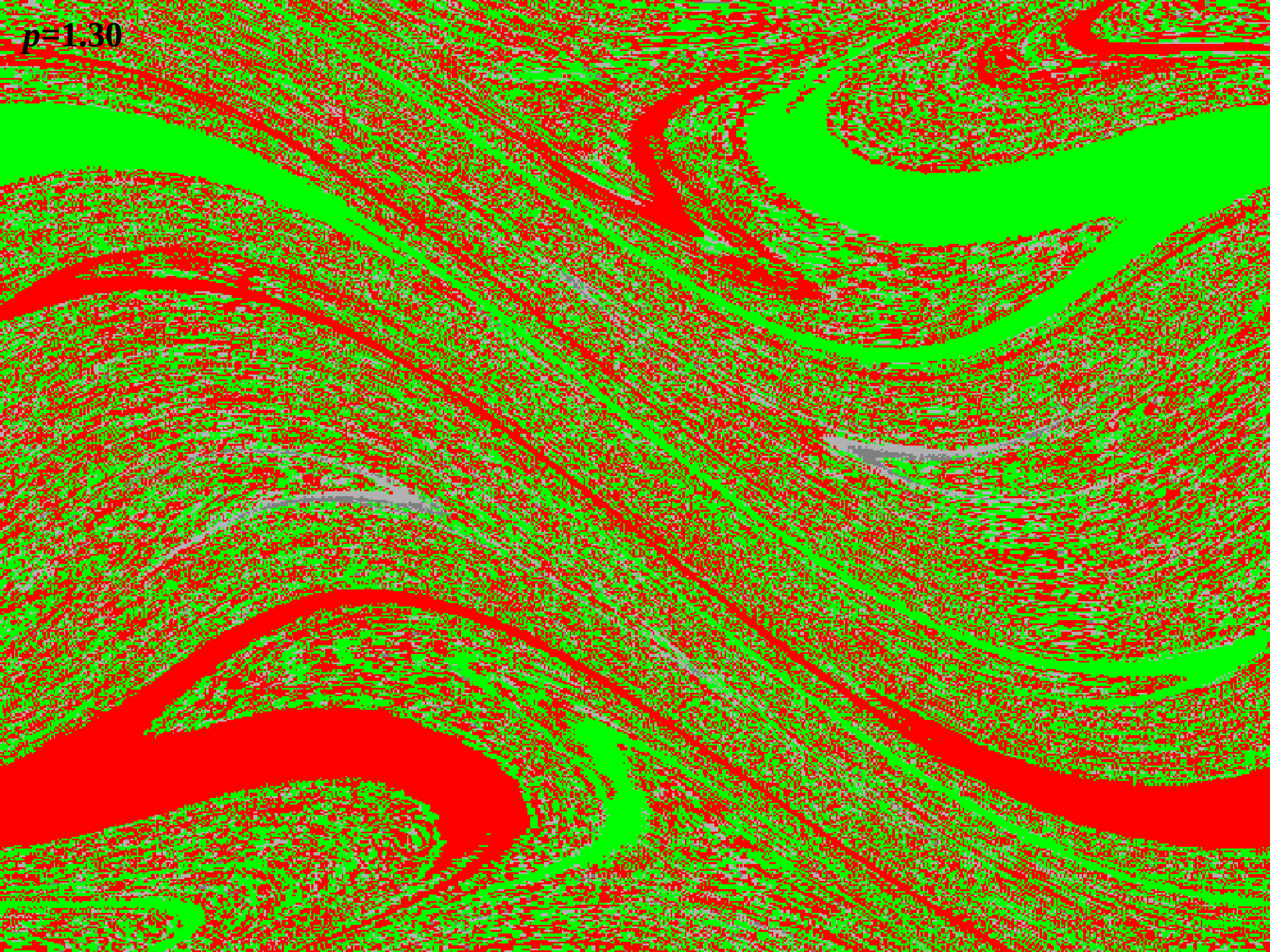
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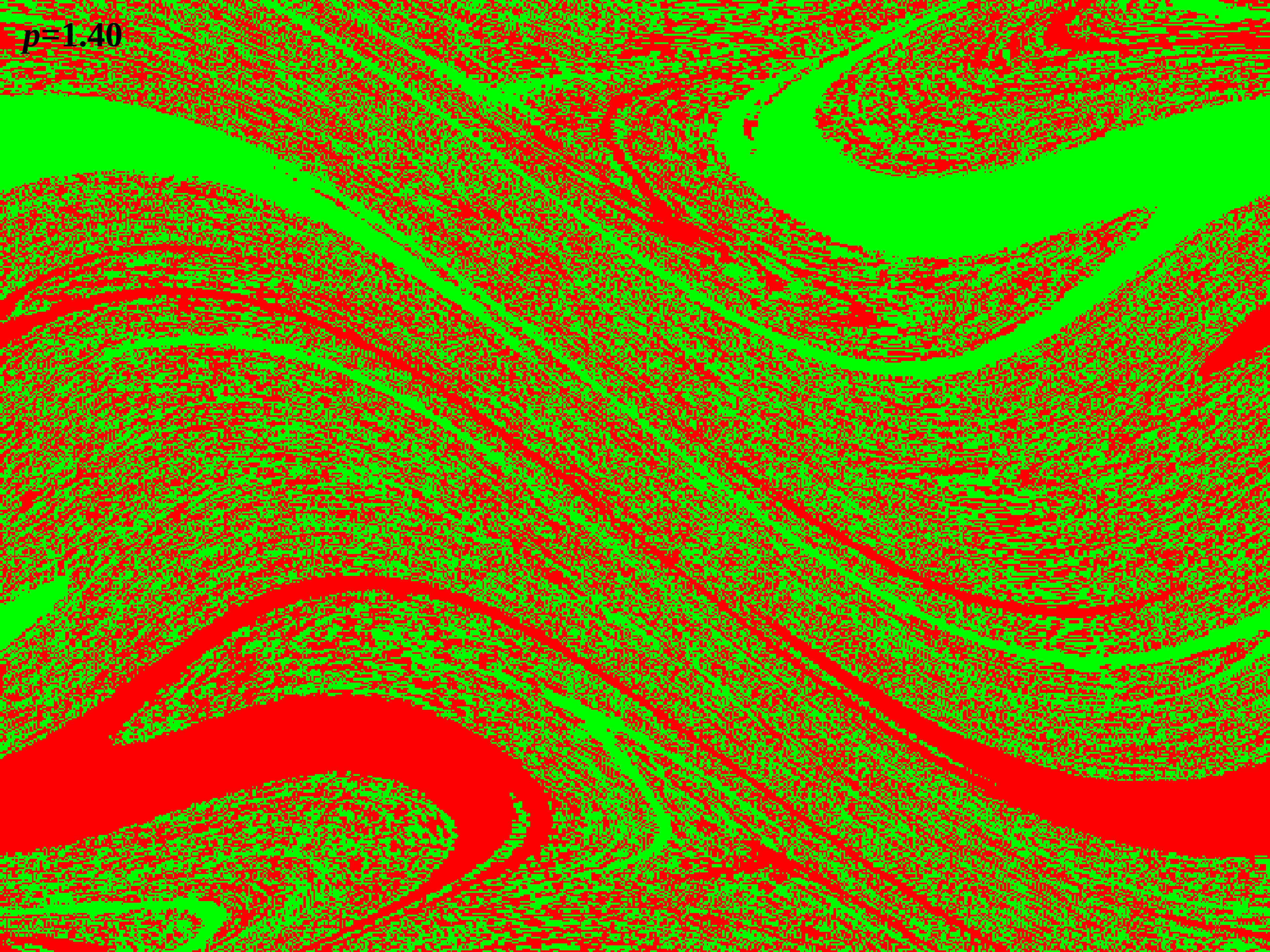

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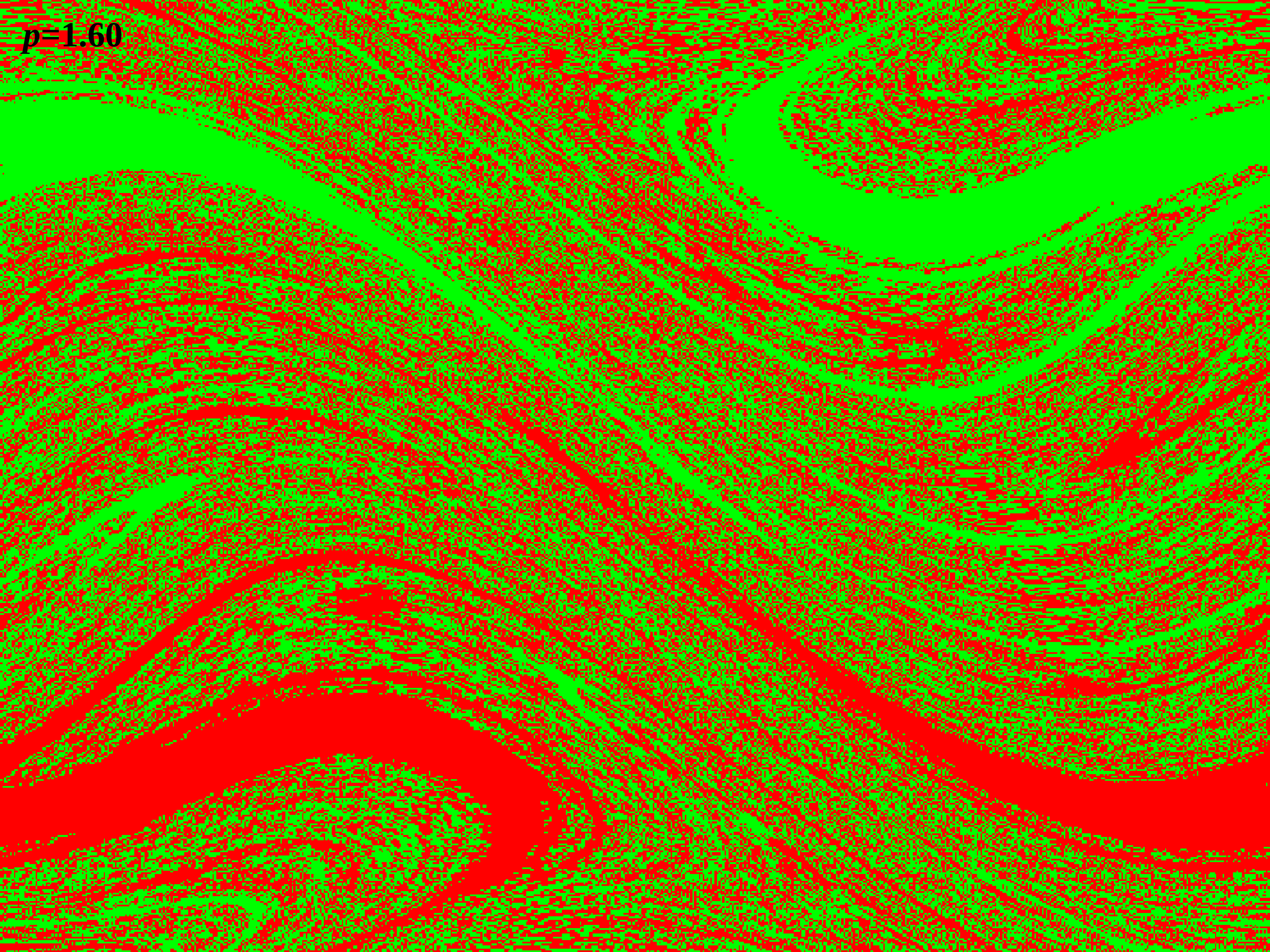
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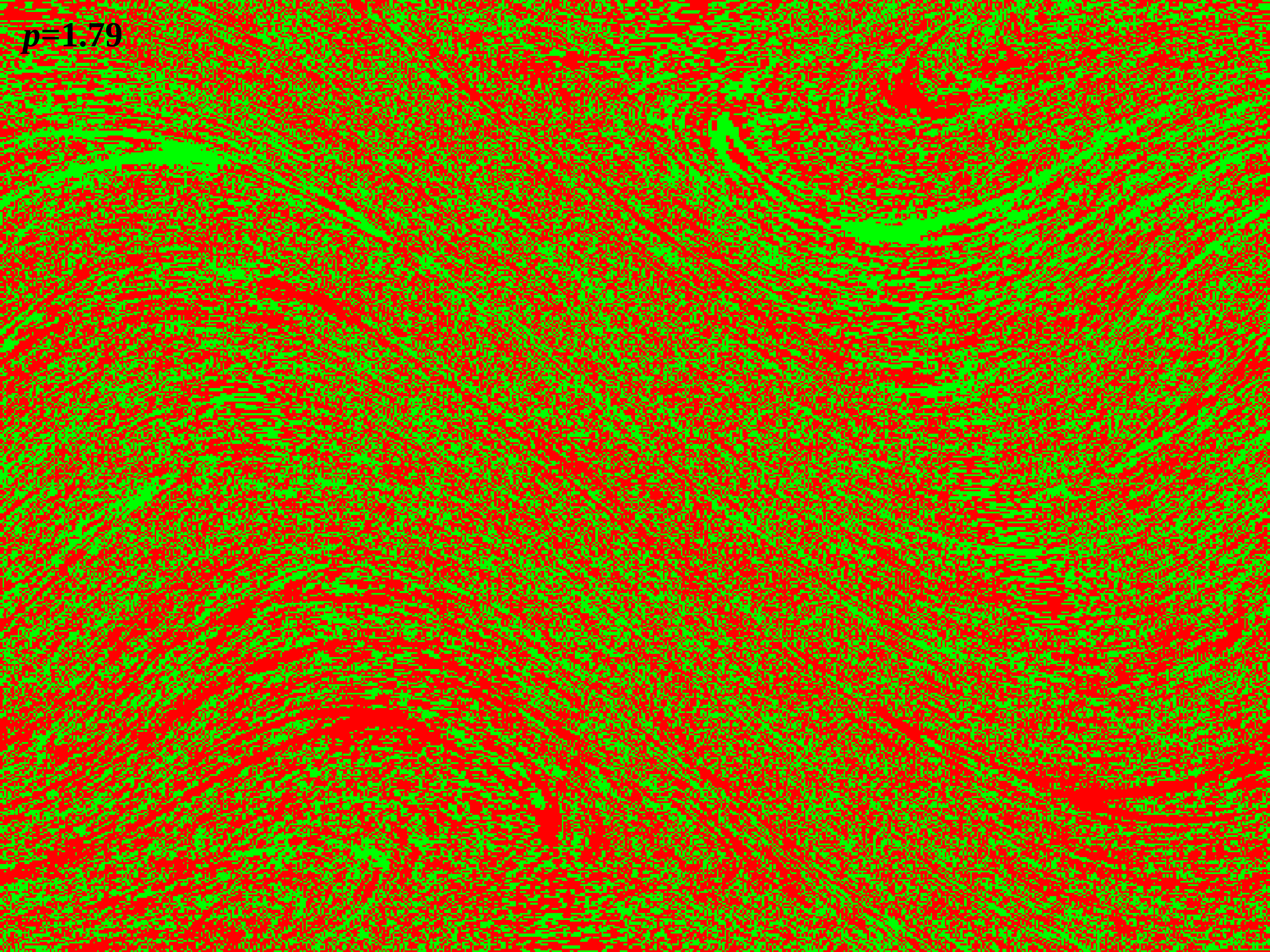
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