Notes on the replica symmetric solution of the classical and quantum SK model, including the matrix of second derivatives and the spin glass susceptibility.

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A review of the replica symmetric solution of the classical and quantum, infinite-range, Sherrington-Kirkpatrick spin glass is presented.

I.   INTRODUCTION

These notes assemble together many results on the replica symmetric (RS) solution of the infinite-range Sherrington-Kirkpatrick [1] (SK) model. The quantum version, in which a transverse field is added, will be discussed in detail, as well as the original classical version. Little here is original, and the bibliography indicates original sources. Some of the material is taken from an old review article [2].

In fact, the (RS) solution is unstable below the critical point, and the correct solution, which is much more complicated, was found by Parisi [3, 4], several years after the model was originally proposed. In a magnetic field, there is a line of transitions, first found by de Almeida and Thouless [5], in the temperature-field plane below which the RS solution is unstable. This is known as the AT line. Almeida and Thouless obtained this line by looking at the stability of the RS solution with respect to fluctuations in the order parameters. Here, we shall discuss this stability matrix, both in the quantum and classical cases. At the point where the RS solution goes unstable, a response function called the spin glass susceptibility, χ_{SG}, diverges. We shall compute χ_{SG} for both the classical and quantum case. Our expression for χ_{SG} in the quantum case, seems to be new; probably the *only* new result in these notes.

II.   THE CLASSICAL SK MODEL

The Sherrington Kirkpatrick (SK) [1] model aims to provide a mean field solution of the spin glass problem as the exact solution of an infinite range model. The Hamiltonian is

$$\mathcal{H} = -\sum_{\langle i,j \rangle} J_{ij} S_i S_j \,, \tag{1}$$

where the S_i are Ising spins, $s_i = \pm 1$, and the J_{ij} are independent random variables with the *same* distribution for all pairs i and j,

$$P(J_{ij}) = \frac{1}{J}\sqrt{\frac{N}{2\pi}} \exp\left(-NJ_{ij}^2/2J^2\right), \tag{2}$$

so the mean is zero and the standard deviation is J. The spin glass transition temperature is at

$$T_c = J. \tag{3}$$

The model is solved by the replica trick, see e.g. [6], according to which one calculates the average free energy, i.e. the average of the *log* of the partition function Z, from the average of the n-th power of Z in the limit $n \to 0$. One has

$$[Z^n]_{\text{av}} = \prod_{\langle i,j \rangle} \left[\int_{-\infty}^{\infty} P(J_{ij})\, dJ_{i,j}\right] \sum_{\{S_i^\alpha = \pm 1\}} \exp\left[\beta \sum_{\langle i,j \rangle} J_{ij} \sum_{\alpha=1}^{n} S_i^\alpha S_j^\alpha\right], \tag{4a}$$

$$= \sum_{\{S_i^\alpha = \pm 1\}} \exp\left[\frac{(\beta J)^2}{2N} \sum_{\langle i,j \rangle} \sum_{\alpha,\beta=1}^{n} S_i^\alpha S_j^\alpha S_i^\beta S_j^\beta\right], \tag{4b}$$

where $[\cdots]_{\text{av}}$ denotes an average over the quenched bond disorder. Separating out the $\alpha = \beta$ terms, and dropping some $1/N$ corrections gives

$$[Z^n]_{\text{av}} = \exp\left[\tfrac{1}{4}(\beta J)^2 Nn\right] \sum_{\{S_i^\alpha = \pm 1\}} \exp\left[\frac{(\beta J)^2}{2N} \sum_{\alpha<\beta} \left(\sum_i S_i^\alpha S_i^\beta\right)^2\right]. \tag{5}$$

We decouple the square using the Hubbard-Stratonovich transformation for each pair of indices $\alpha < \beta$,

$$e^{\lambda a^2/2} = \sqrt{\frac{\lambda}{2\pi}} \int_{-\infty}^{\infty} dx\, \exp\left[-\lambda\frac{x^2}{2} + a\lambda x\right] \tag{6}$$

with

$$\lambda = N, \qquad a = \frac{\beta J}{N} \sum_i S_i^\alpha S_i^\beta, \qquad x = (\beta J)\, q_{\alpha\beta} \tag{7}$$

which gives

$$[Z^n]_{\text{av}} = \exp\left[\tfrac{1}{4}(\beta J)^2 Nn\right] \prod_{\alpha<\beta} \left[\sqrt{\frac{N}{2\pi}}\,(\beta J) \int_{-\infty}^{\infty} dq_{\alpha\beta}\right] \times$$

$$\exp\left[-N\frac{(\beta J)^2}{2} \sum_{\alpha<\beta} q_{\alpha\beta}^2 + N \ln \text{Tr}\exp[-H]\right], \tag{8}$$

where

$$H = -(\beta J)^2 \sum_{\alpha < \beta} q_{\alpha\beta} S^\alpha S^\beta , \tag{9}$$

and the trace is now over the n Ising spins S^α. The N sites i have been decoupled and so the trace over each spin gives the same result, namely $\mathrm{Tr}\exp(-H)$. We can write Eqs. (8) and (9) as

$$[Z^n]_{\mathrm{av}} = \prod_{\alpha < \beta} \left[\sqrt{\frac{N}{2\pi}} (\beta J) \int_{-\infty}^{\infty} dq_{\alpha\beta} \right] \exp\left[-Nn\beta f(q) \right] , \tag{10}$$

where

$$-\beta f(q) = \lim_{n \to 0} \left[\frac{(\beta J)^2}{4} - \frac{(\beta J)^2}{4n} \sum_{(\alpha,\beta)} q_{\alpha\beta}^2 + \frac{1}{n} \log \mathrm{Tr} e^{-H} \right] , \tag{11}$$

where the notation (α, β) means sum over all distinct replicas (so each pair is counted *twice*). Because of the overall factor of N in the exponent in Eq. (10), we will evaluate the integrals by the method of steepest descent. Neglecting subleading terms, the answer is just the exponential in Eq. (10) with the $q_{\alpha\beta}$ evaluated at the saddle point, i.e. the $q_{\alpha\beta}$ are given by a self-consistent solution of

$$q_{\alpha\beta} = \frac{\mathrm{Tr}\, S^\alpha S^\beta\, e^{-H}}{\mathrm{Tr}\, e^{-H}} \qquad \left(= \langle S^\alpha S^\beta \rangle \right) , \tag{12}$$

where the average $\langle \cdots \rangle$ is with respect to the weight e^{-H} with the $q_{\alpha\beta}$ taking their saddle point values.

We look for the replica-symmetric solution where each of the $n(n-1)/2$ order parameters $q_{\alpha\beta}$ takes the same value q. In this case

$$H = -\tfrac{1}{2}(\beta J)^2 q \sum_{(\alpha,\beta)} S^\alpha S^\beta = \tfrac{1}{2}(\beta J)^2 q \left[\left(\sum_\alpha S^\alpha \right)^2 - n \right] , \tag{13}$$

so

$$-\beta f = \lim_{n \to 0} \left[\frac{(\beta J)^2}{4}(1-q)^2 + \frac{1}{n} \ln \mathrm{Tr} \exp\left[\frac{(\beta J)^2}{2} q \left(\sum_\alpha S^\alpha \right)^2 \right] \right] . \tag{14}$$

We decouple the term quadratic in the spins by another Hubbard-Stratonovich transformation, Eq. (6), with $\lambda = 1, a = \beta J q^{1/2} \sum_\alpha S^\alpha, x = z$, i.e.

$$\exp\left[\frac{(\beta J)^2}{2} q \left(\sum_\alpha S^\alpha \right)^2 \right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dz\, e^{-z^2/2}\, e^{\beta J q^{1/2} z \sum_\alpha S^\alpha} . \tag{15}$$

Hence

$$\text{Tr} \exp\left[\frac{(\beta J)^2}{2} q \left(\sum_\alpha S^\alpha\right)^2\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dz\, e^{-z^2/2} \, \text{Tr}\, e^{\beta J q^{1/2} z \sum_\alpha S^\alpha} \tag{16a}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dz\, e^{-z^2/2} \left(2\cosh \beta J q^{1/2} z\right)^n \tag{16b}$$

$$= 1 + n \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dz\, e^{-z^2/2} \ln(2\cosh \beta J q^{1/2} z) + O(n^2)\,, \tag{16c}$$

where the last line expands the result in powers of n. Substituting into Eq. (14) and taking the limit $n \to 0$ gives the free energy of the SK model in the replica symmetric ansatz as

$$-\beta f = \frac{(\beta J)^2}{4}(1-q)^2 + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dz\, e^{-z^2/2} \ln(2\cosh \beta J q^{1/2} z)\,. \tag{17}$$

The order parameter q is obtained by finding an extremal value of f. This gives

$$\frac{(\beta J)^2}{2}(1-q) = \frac{1}{\sqrt{2\pi}} \frac{\beta J}{2q^{1/2}} \int_{-\infty}^{\infty} dz\, e^{-z^2/2} \, z \, \tanh(\beta J q^{1/2} z)\,. \tag{18}$$

Integrating by parts gives the final self-consistent equation for q.

$$q = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dz\, e^{-z^2/2} \, \tanh^2(\beta J q^{1/2} z)\,. \tag{19}$$

This can also be derived by noting that $q\ (= q_{\alpha\beta}) = \langle S^\alpha S^\beta \rangle$, where the average is over the weight e^{-H}, see Eq. (12). Using Eqs. (13) and (15), one readily obtains Eq. (19).

Next we consider fluctuations about the saddle point, i.e.

$$q_{\alpha\beta} = q + \delta q_{\alpha\beta}\,. \tag{20}$$

The first derivative of f with respect to $q_{\alpha\beta}$ is zero so we go to the second derivative of f in Eq. (11), i.e.

$$\beta f[\{q\}] = \beta f[q^c] + \lim_{n\to 0} \frac{1}{n} \frac{1}{2} \sum_{\alpha<\beta,\gamma<\delta} A^{\alpha\beta,\gamma\delta} \delta q_{\alpha\beta}\, \delta q_{\gamma\delta}\,. \tag{21}$$

where, from Eq. (11),

$$A^{\alpha\beta,\gamma\delta} \equiv \frac{\partial^2(\beta f)}{\partial q_{\alpha\beta} \partial q_{\gamma\delta}} \quad (\times n)\,,$$

$$= (\beta J)^2 \delta_{\alpha\beta,\gamma\delta} - (\beta J)^4 \left[\langle S^\alpha S^\beta S^\gamma S^\delta \rangle - \langle S^\alpha S^\beta \rangle \langle S^\gamma S^\delta \rangle\right]\,, \tag{22}$$

Firstly consider $T > T_c$, where $H = 0$, so $\langle S^\alpha S^\beta \rangle = 0$ and hence only the $(\alpha, \beta) = (\gamma, \delta)$ term contributes. Thus

$$A^{\alpha\beta,\gamma\delta} = \delta_{\alpha\beta,\gamma\delta} (\beta J)^2 \left(1 - (\beta J)^2\right)\,. \tag{23}$$

Now A is a matrix of size $n(n-1)2$ so above T_c all $n(n-1)/2$ eigenvalues are equal and given by

$$\lambda = (\beta J)^2 \left(1 - (\beta J)^2\right) \quad (T > T_c).$$

(24)

Now we consider $T < T_c$. There are three types of term:

- $(\alpha\beta)(\alpha\beta)$.

  Here we have $S^\alpha S^\beta S^\gamma S^\delta = 1$, and $\langle S^\alpha S^\beta \rangle = \langle S^\gamma S^\delta \rangle = q$. Hence

  $$(\beta J)^{-2} A^{\alpha\beta,\alpha\beta} = 1 - (\beta J)^2 (1 - q^2) \qquad (= P, \text{ say}).$$

  (25)

- $(\alpha\beta)(\alpha\gamma)$ with $\beta \neq \gamma$.

  Now $\langle S^\alpha S^\beta S^\alpha S^\gamma \rangle = \langle S^\beta S^\gamma \rangle = q$. Also $\langle S^\alpha S^\beta \rangle = \langle S^\gamma S^\delta \rangle = q$. Hence

  $$(\beta J)^{-2} A^{\alpha\beta,\alpha\gamma} = -(\beta J)^2 (q - q^2) \qquad (= Q, \text{ say}).$$

  (26)

- $(\alpha\beta)(\gamma\delta)$ with all indices different.

  As before $\langle S^\alpha S^\beta \rangle = \langle S^\gamma S^\delta \rangle = q$. What about $\langle S^\alpha S^\beta S^\gamma S^\delta \rangle$? From Eq. (16a) we see that the (unnormalized) probability distribution for the $\{S^\alpha\}$ is

  $$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dz\, e^{-z^2/2}\, e^{\beta J q^{1/2} z \sum_\alpha S^\alpha}.$$

  Hence

  $$
  \begin{aligned}
  \langle S^\alpha S^\beta S^\gamma S^\delta \rangle &= \lim_{n\to 0} \left[\frac{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dz\, e^{-z^2/2} \operatorname{Tr} S^\alpha S^\beta S^\gamma S^\delta\, e^{\beta J q^{1/2} z \sum_\alpha S^\alpha}}{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dz\, e^{-z^2/2} \operatorname{Tr} e^{\beta J q^{1/2} z \sum_\alpha S^\alpha}} \right] \\
  &= \lim_{n\to 0} \left[\frac{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dz\, e^{-z^2/2} \sinh^4(\beta J q^{1/2} z)\, \cosh^{n-4}(\beta J q^{1/2} z)}{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dz\, e^{-z^2/2} \cosh^n(\beta J q^{1/2} z)} \right] \\
  &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dz\, e^{-z^2/2} \tanh^4(\beta J q^{1/2} z) \qquad (= r \text{ say}).
  \end{aligned}
  $$

  (27)

  Hence, for all indices different,

  $$(\beta J)^{-2} A^{\alpha\beta,\gamma\delta} = -(\beta J)^2 (r - q^2) \qquad (= R, \text{ say}).$$

  (28)

According to de Almeida and Thouless [5], the important eigenvalue, the one which goes negative, is the "replicon" mode, λ_r, where

$$\lambda_r = (\beta J)^2 \left[P - 2Q + R\right].$$

(29)

Hence

$$\lambda_r = (\beta J)^2 \left[1 - (\beta J)^2 (1 - q^2) + 2(\beta J)^2 (q - q^2) - (\beta J)^2 (r - q^2) \right]$$

$$= (\beta J)^2 \left[1 - (\beta J)^2 (1 - 2q + r) \right] , \tag{30}$$

$$= (\beta J)^2 \left\{ 1 - (\beta J)^2 \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dz \, e^{-z^2/2} \left[1 - \tanh^2(\beta J q^{1/2} z) \right]^2 \right\} . \tag{31}$$

Next we compute the spin glass susceptibility χ_{SG} defined by

$$\chi_{SG} = \frac{1}{T^2} \frac{1}{N} \sum_{i,j=1}^{N} \left[(\langle S_i S_j \rangle - \langle S_i \rangle \langle S_j \rangle)^2 \right]_{\text{av}} . \tag{32}$$

Frequently the factor of $1/T^2$ is omitted because it varies smoothly near a classical transition at finite T but here we will eventually consider a quantum transition at $T = 0$ so we include it.

Considering the terms separately, it is standard to show, see e.g. Binder and Young [2], that they can be expressed as

$$\left[\langle S_i S_j \rangle^2 \right]_{\text{av}} = \lim_{n \to 0} \frac{1}{n(n-1)} \sum_{(\alpha,\beta)} \langle S_i^\alpha S_j^\alpha S_i^\beta S_j^\beta \rangle, \tag{33a}$$

$$\left[\langle S_i S_j \rangle \langle S_i \rangle \langle S_j \rangle \right]_{\text{av}} = \lim_{n \to 0} \frac{1}{n(n-1)(n-2)} \sum_{(\alpha,\beta,\gamma)} \langle S_i^\alpha S_j^\alpha S_i^\beta S_j^\gamma \rangle, \tag{33b}$$

$$\left[\langle S_i \rangle^2 \langle S_j \rangle^2 \right]_{\text{av}} = \lim_{n \to 0} \frac{1}{n(n-1)(n-2)(n-3)} \sum_{(\alpha,\beta,\gamma,\delta)} \langle S_i^\alpha S_j^\beta S_i^\gamma S_j^\delta \rangle, \tag{33c}$$

where the averages on the RHS are with respect to the weight factor in Eq. (4b) and the notation (α, β) etc. means all distinct sets of replicas are to be summed over. Note that each thermal average on the LHS of Eqs. (33) corresponds to a distinct replica on the RHS.

To calculate these averages we add a set of fictitious fields $\Delta_{\alpha\beta}$ which couple to $\sum_i S_i^\alpha S_i^\beta$, i.e.

$$[Z^n]_{\text{av}} = \sum_{\{S_i^\alpha = \pm 1\}} \exp \left[\frac{(\beta J)^2}{2N} \sum_{\langle i,j \rangle} \sum_{\alpha,\beta=1}^{n} S_i^\alpha S_j^\alpha S_i^\beta S_j^\beta + \sum_{\alpha < \beta} \Delta_{\alpha\beta} \sum_i S_i^\alpha S_i^\beta \right] . \tag{34}$$

Note that for $n \to 0$ there is no normalizing denominator so

$$\sum_i \langle S_i^\alpha S_i^\beta \rangle = \lim_{n \to 0} \frac{\partial}{\partial \Delta_{\alpha\beta}} [Z^n]_{\text{av}} , \tag{35a}$$

$$\sum_{i,j} \langle S_i^\alpha S_i^\beta S_j^\gamma S_j^\delta \rangle = \lim_{n \to 0} \frac{\partial^2}{\partial \Delta_{\alpha\beta} \Delta_{\gamma\delta}} [Z^n]_{\text{av}} , \tag{35b}$$

in which the replicas $\alpha, \beta, \gamma, \delta$ can take any values subject to the restrictions $\alpha < \beta, \gamma < \delta$. We note that $[Z^n]_{\text{av}}$ is still given by Eq. (10), with $\beta f(q)$ still given by (11), but now

$$H = -(\beta J)^2 \sum_{\alpha < \beta} \left[q_{\alpha\beta} - (\beta J)^{-2} \Delta_{\alpha\beta} \right] S^\alpha S^\beta . \tag{36}$$

We define shifted variables $u_{\alpha\beta}$ by

$$u_{\alpha\beta} = q_{\alpha\beta} - (\beta J)^{-2}\Delta_{\alpha\beta} \,, \tag{37}$$

so the $\Delta_{\alpha\beta}$ no longer appear in H but rather in the quadratic term in Eq. (11), so

$$-\beta f(q) = \lim_{n\to 0}\left[\frac{(\beta J)^2}{4} - \frac{1}{4n}\sum_{(\alpha,\beta)}\left\{(\beta J)^2 u_{\alpha\beta}^2 + 2\Delta_{\alpha\beta}u_{\alpha\beta} + (\beta J)^{-2}\Delta_{\alpha\beta}^2\right\} + \frac{1}{n}\log\mathrm{Tr}e^{-H}\right], \tag{38}$$

where now

$$H = -(\beta J)^2\sum_{\alpha<\beta}u_{\alpha\beta}S^\alpha S^\beta \,, \tag{39}$$

and the $u_{\alpha\beta}$ have to be integrated like the $q_{\alpha\beta}$ in Eq. (10). Performing the derivatives in Eqs. (35) we get

$$\frac{1}{N}\sum_i\langle S_i^\alpha S_i^\beta\rangle = \lim_{n\to 0}\langle q_{\alpha\beta}\rangle, \tag{40a}$$

$$\frac{1}{N}\sum_{i,j}\langle S_i^\alpha S_j^\beta S_i^\gamma S_j^\delta\rangle = \lim_{n\to 0}\left[-(\beta J)^{-2}\delta_{\alpha\beta,\gamma\delta} + \langle q_{\alpha\beta}\, q_{\gamma\delta}\rangle\right], \tag{40b}$$

where the averages are to be evaluated with $\Delta_{\alpha\beta} = 0$.

Hence, from Eqs. (32), (33) and (40b) we have[1]

$$\chi_{SG} = \frac{1}{T^2}\left[-(\beta J)^{-2} + \langle q_{\alpha\beta}^2\rangle - 2\langle q_{\alpha\beta}q_{\alpha\gamma}\rangle + \langle q_{\alpha\beta}q_{\gamma\delta}\rangle\right]. \tag{41}$$

We write

$$q_{\alpha\beta} = q_c + \delta q_{\alpha\beta} \tag{42}$$

where q_c is the value of the $q_{\alpha\beta}$ at the (replica-symmetric) saddle point. Inserting Eq. (42) into Eq. (41) the factors of q_c cancel and so we have

$$\chi_{SG} = -\frac{1}{J^2} + \frac{1}{T^2}\left[\langle\delta q_{\alpha\beta}^2\rangle - 2\langle\delta q_{\alpha\beta}\delta q_{\alpha\gamma}\rangle + \langle\delta q_{\alpha\beta}\delta q_{\gamma\delta}\rangle\right]. \tag{43}$$

The averages involve Gaussian integrals which come from the weight given by Eq. (10) in which $f[\{q\}]$ is given by the quadratic expression in Eq. (21).

We will need the result that if a set of variables x_i have Gaussian distribution, i.e.

$$P(\{x\}) \propto \exp[-\tfrac{1}{2}x_i A_{ij}x_j] \tag{44}$$

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[1] Unfortunately, in Eq. (4.47) of the review of Binder and Young [2], which is the equation corresponding to our Eq. (41), the term $-(\beta J)^{-2}$ is missing.

then correlation functions of the x_i are given by

$$\langle x_i x_j \rangle = \left(A^{-1}\right)_{ij}. \tag{45}$$

The combination of averages in Eq. (43) corresponds to the "replicon" eigenvector of the matrix A, see Eqs. (25)–(29). Hence, from Eq. (45), these averages just give the inverse of the replicon eigenvalue λ_r in Eq. (30) so

$$\chi_{SG} = -\frac{1}{J^2} + \frac{1}{T^2}\frac{1}{\lambda_r}, \tag{46}$$

$$= -\frac{1}{J^2} + \frac{1}{J^2}\frac{1}{1 - (\beta J)^2(1 - 2q + r)}, \tag{47}$$

$$= \beta^2 \frac{1 - 2q + r}{1 - (\beta J)^2(1 - 2q + r)}. \tag{48}$$

If we define

$$\boxed{\chi_{SG}^0 = \beta^2(1 - 2q + r),} \tag{49}$$

then

$$\boxed{\chi_{SG} = \frac{\chi_{SG}^0}{1 - J^2\chi_{SG}^0},} \tag{50}$$

a result which is very reminiscent of the random phase approximation.

Equations (49) and (50) are also valid in the presence of a field, either uniform or random, provided the expressions for q, r and λ_r, in Eqs. (19),(27) and (31) are modified appropriately. For example, for a Gaussian random field with standard deviation h, the factor of $Jq^{1/2}$ is replaced by $(J^2q + h^2)^{1/2}$, see Refs. [7, 8], and for a unform field, $Jq^{1/2}z$ is replaced by $Jq^{1/2} + h$, see e.g. Ref. [2].

In the paramagnetic phase, where $q = r = 0$ and so $\chi_{SG}^0 = \beta^2$, we see that χ_{SG} has the simple form

$$\chi_{SG} = \frac{1}{T^2 - J^2}, \qquad (T > T_c = J), \tag{51}$$

which shows that the transition occurs when $T = J$, as is well known [1], see also Eq. (3).

III. THE QUANTUM SK MODEL

Now we make the model **quantum** by adding a transverse field.

$$\mathcal{H} = -\sum_{\langle i,j \rangle} J_{ij}\sigma_i^z\sigma_j^z - h^T \sum_i \sigma_i^x. \tag{52}$$

This model has been studied in many works, including Refs. [9–17], and these notes will use their methods.

The standard approach is to use the imaginary time path integral formulation [18], where imaginary time, τ, is in the range $0 \leq \tau \leq \beta$ and there are periodic boundary conditions in the τ direction. Imaginary time is divided into M time slices, each of width

$$\Delta\tau = \frac{\beta}{M}.\tag{53}$$

The partition function is then given by the action

$$Z = \mathrm{Tr}\exp\left[\sum_{l=1}^{M}\left(\sum_{\langle i,j\rangle} J_{ij}S_i(l)S_j(l)\Delta\tau + K^\tau \sum_i S_i(l)S_i(l+1)\right)\right],\tag{54}$$

where

$$e^{-2K^\tau} = \tanh(h^T\Delta\tau),\tag{55}$$

and the $S_i(l)$ are Ising variables at each site i and time slice l. The K^τ term is a ferromagnetic coupling along the imaginary time direction. Now we replicate, in order to average over disorder. Disorder averaging does not alter the K^τ term because it is not random, so averaging over the J_{ij} term goes through as for the classical case, but with the addition of the imaginary time indices. The analog of Eq. (4b) is

$$\begin{aligned}
[Z^n]_{\mathrm{av}} = \sum_{\{S_i^\alpha(l)\}=\pm 1}\exp\Bigg[&\frac{(\Delta\tau J)^2}{2N}\sum_{l,l'=1}^{M}\sum_{\langle i,j\rangle}\sum_{\alpha,\beta=1}^{n} S_i^\alpha(l)S_j^\alpha(l)S_i^\beta(l')S_j^\beta(l') + \\
&K^\tau\sum_i\sum_{l=1}^{M}\sum_{\alpha=1}^{n} S_i^\alpha(l)S_i^\alpha(l+1)\Bigg].
\end{aligned}\tag{56}$$

In the first term in the exponential we consider separately the $\alpha = \beta$ and $\alpha \neq \beta$ terms.

- $\alpha = \beta$ terms.

$$\sum_{l,l'=1}^{M}\sum_{\langle i,j\rangle}\sum_{\alpha=1}^{n} S_i^\alpha(l)S_j^\alpha(l)S_i^\alpha(l')S_j^\alpha(l')\tag{57}$$

$$= \frac{1}{2}\sum_{\alpha=1}^{n}\sum_{l,l'=1}^{M}\left(\sum_i S_i^\alpha(l)S_i^\alpha(l')\right)^2,\tag{58}$$

where we have neglected terms of order $1/N$. We will decouple the square using a Hubbard-Stratonovich (HS) transformation as we did to go from Eq. (5) to (8).

- $\alpha \neq \beta$ terms.

$$\sum_{\alpha<\beta} \sum_{l,l'=1}^{M} \left(\sum_i S_i^\alpha(l) S_i^\beta(l') \right)^2 , \tag{59}$$

We will do a HS transformation for this too.

The result of the HS transformations is

$$[Z^n]_{\text{av}} = \prod_{\alpha,l,l'} \left[\sqrt{\frac{N}{4\pi}} (\Delta\tau J) \int_{-\infty}^{\infty} dr_\alpha(l,l') \right] \prod_{\alpha<\beta,l,l'} \left[\sqrt{\frac{N}{2\pi}} (\Delta\tau J) \int_{-\infty}^{\infty} dq_{\alpha\beta}(l,l') \right]$$
$$\exp\left[-\frac{N}{4}(\Delta\tau J)^2 \sum_{l,l',\alpha} r_\alpha^2(l,l') - \frac{N}{2}(\Delta\tau J)^2 \sum_{l,l',\alpha<\beta} q_{\alpha\beta}^2(l,l') \right] \left(\text{Tr}\, e^{-H} \right)^N , \tag{60}$$

where

$$H = -(\Delta\tau J)^2 \left[\frac{1}{2} \sum_\alpha \sum_{l,l'} r_\alpha(l,l') S^\alpha(l) S^\alpha(l') + \sum_{\alpha<\beta} \sum_{l,l'} q_{\alpha\beta}(l,l') S^\alpha(l) S^\beta(l') \right]$$
$$-K^\tau \sum_\alpha \sum_l S^\alpha(l) S^\alpha(l+1) . \tag{61}$$

We have time translational invariance so $q_{\alpha\beta}(l,l')$ is only a function of $\Delta l \equiv l - l'$, and similarly for $r_\alpha(l,l')$. Hence

$$[Z^n]_{\text{av}} = \prod_{\alpha,\Delta l} \left[\sqrt{\frac{N}{2\pi}} (\Delta\tau J) \int_{-\infty}^{\infty} dr_\alpha(\Delta l) \right] \prod_{\alpha<\beta,\Delta l} \left[\sqrt{\frac{N}{2\pi}} (\Delta\tau J) \int_{-\infty}^{\infty} dq_{\alpha\beta}(\Delta l) \right]$$
$$\exp\left[-\frac{N}{4}(\Delta\tau J)^2 M \sum_{\Delta l,\alpha} r_\alpha^2(\Delta l) - \frac{N}{2}(\Delta\tau J)^2 M \sum_{\Delta l,\alpha<\beta} q_{\alpha\beta}^2(\Delta l) \right] \left(\text{Tr}\, e^{-H} \right)^N , \tag{62}$$

where

$$H = -(\Delta\tau J)^2 \left[\frac{1}{2} \sum_\alpha \sum_{\Delta l} r_\alpha(\Delta l) \sum_l S^\alpha(l) S^\alpha(l+\Delta l) + \sum_{\alpha<\beta} \sum_{\Delta l} q_{\alpha\beta}(\Delta l) \sum_l S^\alpha(l) S^\beta(l+\Delta l) \right]$$
$$-K^\tau \sum_\alpha \sum_l S^\alpha(l) S^\alpha(l+1) . \tag{63}$$

We minimize w.r.t. $q_{\alpha\beta}(\Delta l)$ and $r_\alpha(\Delta l)$. This gives

$$r_\alpha(\Delta l) = \langle S^\alpha(0) S^\alpha(\Delta l) \rangle , \tag{64a}$$

$$q_{\alpha\beta}(\Delta l) = \langle S^\alpha(0) S^\beta(\Delta l) \rangle, \tag{64b}$$

where the average is with respect to e^{-H} and we have used time translational invariance.

We write Eq. (62) as

$$[Z^n]_{\text{av}} = \prod_{\alpha,\Delta l} \left[\sqrt{\frac{N}{2\pi}}(\Delta\tau J) \int_{-\infty}^{\infty} dr_\alpha(\Delta l) \right] \prod_{\alpha<\beta,\Delta l} \left[\sqrt{\frac{N}{2\pi}}(\Delta\tau J) \int_{-\infty}^{\infty} dq_{\alpha\beta}(\Delta l) \right] \exp[-Nn\beta f] \quad (65)$$

where

$$-\beta f = \lim_{n\to 0} \left[-\frac{1}{4n}\beta\Delta\tau J^2 \sum_{\alpha,\Delta l} r_\alpha(\Delta l)^2 - \frac{1}{2n}\beta\Delta\tau J^2 \sum_{\alpha<\beta}\sum_{\Delta l} q_{\alpha\beta}^2(\Delta l) + \frac{1}{n} \ln \text{Tr}\, e^{-H} \right]. \quad (66)$$

We evaluate the integrals in Eq. (65) by steepest descent and look for the replica symmetric solution:

$$r_\alpha(\Delta l) = r(\Delta l)\,, \quad (67\text{a})$$

$$q_{\alpha\beta}(\Delta l) = q(\Delta l)\,. \quad (67\text{b})$$

This yields

$$-\beta f = \lim_{n\to 0} \left[-\frac{1}{4}\beta\Delta\tau J^2 \sum_{\Delta l} r(\Delta l)^2 + \frac{1}{4}\beta\Delta\tau J^2 \sum_{\Delta l} q^2(\Delta l) + \frac{1}{n} \ln \text{Tr}\, e^{-H} \right], \quad (68)$$

where

$$H = -(\Delta\tau J)^2 \left[\frac{1}{2} \sum_{\Delta l} r(\Delta l) \sum_\alpha \sum_l S^\alpha(l) S^\alpha(l+\Delta l) + \sum_{\Delta l} q(\Delta l) \sum_{\alpha<\beta} \sum_l S^\alpha(l) S^\beta(l+\Delta l) \right]$$
$$-K^\tau \sum_\alpha \sum_l S^\alpha(l) S^\alpha(l+1)\,. \quad (69)$$

Now we need to think about the physics. The parameters $q(\Delta l)$ are order parameters, corresponding to a product of a single spin in two replicas. We expect these to be independent of time. Hence we will assume that $q(\Delta l) = q$. Consequently

$$-\beta f = \lim_{n\to 0} \left[-\frac{1}{4}\beta\Delta\tau J^2 \sum_{\Delta l} r(\Delta l)^2 + \frac{1}{4}(\beta J)^2 q^2 + \frac{1}{n} \ln \text{Tr}\, e^{-H} \right]. \quad (70)$$

We write the second term in the expression for H in Eq. (69) as follows

$$\frac{1}{2}(\Delta\tau J)^2 q \sum_{l_1,l_2}\sum_{(\alpha,\beta)} S^\alpha(l_1) S^\beta(l_2) = \frac{1}{2}(\Delta\tau J)^2 q \left[\left(\sum_l \sum_\alpha S_\alpha(l) \right)^2 - \sum_\alpha \sum_{l,\Delta l} S^\alpha(l) S^\alpha(l+\Delta l) \right]. \quad (71)$$

The second term on the RHS of Eq. (71) can be combined with the r term in Eq. (69). The K^τ term in Eq. (69) can also be combined so we define

$$\bar{r}(\Delta l) = r(\Delta l) - q\,, \qquad\qquad (\Delta l \neq 1)\,,$$

$$\bar{r}(\Delta l) = r(\Delta l) - q + K^\tau/(\Delta\tau J)^2\,. \qquad (\Delta l = 1)\,, \quad (72)$$

The square in the first term on the RHS of Eq. (71) is decoupled by a Hubbard-Stratonovich transformation, as in the classical case, see Eqs. (16), i.e.

$$\text{Tr} \exp\left[\frac{(\Delta\tau J)^2}{2}\left(\sum_{\Delta l}\overline{r}(\Delta l)\sum_\alpha\sum_l S^\alpha(l)S^\alpha(l+\Delta l) + q\left(\sum_l\sum_\alpha S^\alpha(l)\right)^2\right)\right]$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}dz e^{-z^2/2}\text{Tr}\exp\left[\sum_\alpha\left(\frac{(\Delta\tau J)^2}{2}\sum_{\Delta l}\overline{r}(\Delta l)\sum_l S^\alpha(l)S^\alpha(l+\Delta l) + (\Delta\tau J)q^{1/2}z\sum_l S^\alpha(l)\right)\right]$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}dz e^{-z^2/2}\left(\text{Tr}\, e^{-\overline{H}(z)}\right)^n \tag{73}$$

in which $\overline{H}(z)$ is given by

$$\boxed{\overline{H}(z) = -(\Delta\tau J)^2\sum_{\langle l_1,l_2\rangle}\left\{r(|l_1 - l_2|) - q\right\}S(l_1)S(l_2) - K^\tau\sum_l S(l)S(l+1) - (\Delta\tau J)q^{1/2}z\sum_l S(l),}$$
$$\tag{74}$$

where we used Eq. (72). We see that $\overline{H}(z)$ is the Hamiltonian of a one-dimensional chain with long-range interactions, in which there is a (uniform) field proportional to $q^{1/2}z$.

Expanding Eq. (73) in powers of n and substituting into Eq. (68) gives

$$\boxed{-\beta f = \lim_{n\to 0}\left[-\frac{1}{4}\beta\Delta\tau J^2\sum_{\Delta l=1}^{M}r(\Delta l)^2 + \frac{1}{4}(\beta J)^2 q^2 + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}dz\, e^{-z^2/2}\ln\text{Tr}\, e^{-\overline{H}(z)}\right].} \tag{75}$$

Now we determine the self-consistent equations for $r(\Delta\tau)$ and q.

- $r(\Delta l)$.

$$\frac{1}{2}\beta\Delta\tau J^2 r(\Delta l) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}dz\, e^{-z^2/2}\left\{\frac{(\Delta\tau J)^2}{2}\frac{\text{Tr}\sum_l S(l)S(l+\Delta l)e^{-\overline{H}(z)}}{\text{Tr}\, e^{-\overline{H}(z)}},\right\} \tag{76}$$

so

$$\boxed{r(\Delta l) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}dz\, e^{-z^2/2}\langle\, S(0)\, S(\Delta l)\,\rangle_{\overline{H}},} \tag{77}$$

where we have used time translational invariance and

$$\boxed{\langle\cdots\rangle_{\overline{H}}\text{ indicates an average over the spins with weight } e^{-\overline{H}}.}$$

- q.

$$\frac{1}{2}(\beta J)^2 q = -\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}dz\, e^{-z^2/2}\times$$

$$\left\{\frac{\frac{(\Delta\tau J)}{2q^{1/2}}z\,\text{Tr}\left(\sum_l S(l)e^{-\overline{H}(z)}\right) - \frac{(\Delta\tau J)^2}{2}\text{Tr}\sum_{l,\Delta l}S(l)S(l+\Delta l)e^{-\overline{L}(z)}}{\text{Tr}\, e^{-\overline{H}(z)}}\right\} \tag{78}$$

As in the classical case, we integrate by parts with respect to z in the first term in curly brackets. This gives

$$- (\beta J)^2 q = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dz\, e^{-z^2/2} \times$$

$$\left\{ \frac{(\Delta\tau J)}{q^{1/2}} (\Delta\tau J) q^{1/2} \sum_{l_1,l_2} \left(\langle S(l_1)S(l_2)\rangle_{\overline{H}} - \langle S(l_1)\rangle_{\overline{H}} \langle S(l_2)\rangle_{\overline{H}} \right) - (\Delta\tau J)^2 \sum_{l_1,l_2} \langle S(l_1)S(l_2)\rangle_{\overline{H}} \right\},$$

(79)

which simplifies to

$$\boxed{q = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dz\, e^{-z^2/2} \langle\, S(0)\,\rangle_{\overline{H}}^2 \,,}$$

(80)

where the average of a single spin $\langle S(0)\rangle_{\overline{H}}$ could be evaluated at any time slice because of time translational invariance.

We now check that we recover the standard results for the SK model when we take the classical limit $h^T \to 0$. From Eq. (55) this corresponds to $K^\tau \to \infty$. Hence all spins along the time direction are fully correlated. It follows that

$$\langle S(0)S(\Delta l)\rangle = 1, \qquad \text{for all } \Delta l,$$

(81)

and so, from Eq. (77),

$$r(\Delta l) = 1, \qquad \text{for all } \Delta l.$$

(82)

In $\overline{H}(z)$ in Eq. (74) the first two terms are constants which cancel when computing averages. In the third term all the $S(l)$ are equal and so we can write $\sum_l S(l) = MS$ where $S = \pm 1$, so

$$(\Delta\tau J)q^{1/2} z \sum_l S(l) = (\beta J)q^{1/2} zS,$$

(83)

exactly in the classical case, Eq. (15). Hence

$$\langle S(l)\rangle_{\overline{H}} = \langle S\rangle_{\overline{H}} = \tanh(\beta J q^{1/2} z),$$

(84)

and the self-consistent equation of q, Eq. (80), reduces to the result for the SK model, Eq. (19).

We now consider the stability of the replica symmetric solution in the quantum case. The free energy is given by Eq. (66) (but with $q_{\alpha\beta}$ now independent of Δl) and H given by Eq. (63). The stability has to be determined with respect to the $n(n-1)/2$ static order parameters $q_{\alpha\beta}$ and the nM correlation functions $r_\alpha(\Delta l)$. The matrix of second derivatives therefore has the form

$$T = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix},$$

(85)

where A is a matrix of dimension $n(n-1)/2 \times n(n-1)/2$ which describes fluctuations in the $q_{\alpha\beta}$ sector, C is of size $nM \times nM$ and describes fluctuations in the $r_\alpha(\Delta l)$ sector, and B, which is of dimension $n(n-1)/2 \times nM$, describes the mixed second derivatives.

For the classical case, i.e. the SK model, de Almeida and Thouless (AT) [5] showed that the instability comes in the "replicon" eigenvector of A, see Eq. (30). AT also showed that, in cases where one also has to consider terms diagonal in replica indices (such as the r terms here) the replicon eigenvector has zero values for these components. Here we assume that the replicon mode will still be the important one, and so we will neglect the sector involving the r terms [13]. Thus we just need to consider the $n(n-1)/2 \times n(n-1)/2$ matrix A, and so write the expansion of the free energy as in Eq. (21).

Taking the derivatives in Eq. (66), we find that $A^{\alpha\beta,\gamma\delta}$, the matrix of coefficients in the expansion in Eq. (21), is given by

$$A^{\alpha\beta,\gamma\delta} \equiv \frac{\partial^2(\beta f)}{\partial q_{\alpha\beta} \partial q_{\gamma\delta}} \quad (\times n) ,$$

$$= J^2 \beta \Delta\tau M \delta_{\alpha\beta,\gamma\delta} - (\beta\Delta\tau)^4 \sum_{l_1,l_2,l_3,l_4} \left[\langle S^\alpha(l_1)S^\beta(l_2)S^\gamma(l_3)S^\delta(l_4) \rangle - \langle S^\alpha(l_1)S^\beta(l_2) \rangle \langle S^\gamma(l_3)S^\delta(l_4) \rangle \right]$$

$$= (\beta J)^2 \delta_{\alpha\beta,\gamma\delta} - (\beta J)^4 \left[\left(\frac{1}{M^4} \sum_{l_1,l_2,l_3,l_4} \langle S^\alpha(l_1)S^\beta(l_2)S^\gamma(l_3)S^\delta(l_4) \rangle \right) - q^2 \right] , \tag{86}$$

where we used that $\langle S^\alpha(l_1)S^\beta(l_2) \rangle$ for $\alpha \neq \beta$ is independent of the values of τ and is just order the order parameter $q_{\alpha\beta}(= q$ here since we expand about the replica symmetric solution). Eq. (86) is the generalization to the quantum case of Eq. (22).

As for the classical case, we have to consider three cases depending on which replica indices are equal, see Eq. (25), (26) and (27). The averages are to be evaluated in the replica symmetric solution, so the spin averages in each replica are to be evaluated with weight $e^{-\overline{H}(z)}$, where $\overline{H}(z)$ is given by Eq. (74), and finally the combined average over the different replicas is to be averaged over the Gaussian random field z which has zero mean and standard deviation unity, see e.g. Eq. (80) which is for an average over two different replicas.

- $(\alpha\beta)(\alpha\beta)$.

$$\left(\frac{1}{M^4} \sum_{l_1,l_2,l_3,l_4} \langle S^\alpha(l_1)S^\beta(l_2)S^\alpha(l_3)S^\beta(l_4) \rangle \right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dz\, e^{-z^2/2} \left(\frac{1}{M^2} \sum_{l_1,l_2} \langle S(l_1)S(l_2) \rangle_{\overline{H}} \right)^2 .$$
$$\tag{87}$$

- $(\alpha\beta)(\alpha\gamma)$ with $\beta \neq \gamma$.

$$
\left(\frac{1}{M^4} \sum_{l_1,l_2,l_3,l_4} \langle S^\alpha(l_1)S^\beta(l_2)S^\alpha(l_3)S^\gamma(l_4) \rangle \right) =
$$
$$
\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dz \, e^{-z^2/2} \left(\frac{1}{M^2} \sum_{l_1,l_2} \langle S(l_1)S(l_2) \rangle_{\overline{H}} \right) \left(\frac{1}{M} \sum_l \langle S(l) \rangle_{\overline{H}} \right)^2 . \tag{88}
$$

- $(\alpha\beta)(\gamma\delta)$ with all indices different.

$$
\left(\frac{1}{M^4} \sum_{l_1,l_2,l_3,l_4} \langle S^\alpha(l_1)S^\beta(l_2)S^\gamma(l_3)S^\delta(l_4) \rangle \right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dz \, e^{-z^2/2} \left(\frac{1}{M} \sum_l \langle S(l) \rangle_{\overline{H}} \right)^4 . \tag{89}
$$

As in the classical case, defining $(\beta J)^{-2} A^{\alpha\beta,\alpha\beta} = P$, $(\beta J)^{-2} A^{\alpha\beta,\alpha\gamma} = Q$, and $(\beta J)^{-2} A^{\alpha\beta,\gamma\delta} = R$, the replicon eigenvalue is given by

$$
(\beta J)^{-2} \lambda_r = P - 2Q + R
$$

$$
\boxed{= 1 - J^2 \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dz \, e^{-z^2/2} \left[\sum_l \left(\Delta\tau \{ \langle S(0)S(l) \rangle_{\overline{H}} - \langle S(0) \rangle_{\overline{H}} \langle S(l) \rangle_{\overline{H}} \} \right) \right]^2 .} \tag{90}
$$

The correspondence with the classical (SK model) result in Eq. (31) is clear.

To summarize, averages denoted by $\langle \cdots \rangle_{\overline{H}}$ are with respect to weight $e^{-\overline{H}(z)}$ where $\overline{H}(z)$ is given by Eq. (74). The $M - 1$ values of $r(\Delta l)$, as well as q, are to be determined self-consistently from Eqs. (77) and (80).

In the continuum limit the expression corresponding to Eq. (90) is clearly

$$
(\beta J)^{-2} \lambda_r = 1 - J^2 \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dz \, e^{-z^2/2} \left[\int_0^\beta d\tau \left(\langle S(0)S(\tau) \rangle_{\overline{H}} - \langle S(0) \rangle_{\overline{H}} \langle S(\tau) \rangle_{\overline{H}} \right) \right]^2 . \tag{91}
$$

In the paramagnetic phase, single spin expectation values vanish, and so does q, and the integral over z gives unity, so

$$
\boxed{\lambda_r = (\beta J)^2 \left\{ 1 - \left[\sum_l \left(\Delta\tau \langle S(0)S(\tau) \rangle_{\overline{H}} \right) \right]^2 \right\}} \qquad \text{(where } q = 0 \text{).} \tag{92}
$$

The phase boundary is where $\lambda_r = 0$, i.e.

$$
\boxed{1 = J\Delta\tau \sum_l \langle S(0)S(l) \rangle_{\overline{H}}} \qquad \text{(on phase boundary) ,} \tag{93}
$$

where now, since $q = 0$,

$$\overline{H} = -(\Delta\tau J)^2 \sum_{\langle l_1, l_2 \rangle} r(|l_1 - l_2|) S(l_1) S(l_2) - K^\tau \sum_l S(l) S(l+1), \qquad \text{(if } q = 0), \qquad (94)$$

see Eq. (74).

As for the classical case we want to calculate the spin glass susceptibility, since the divergence of this quantity is governed by the vanishing of the replicon eigenvalue. In the quantum case, χ_{SG} is given by

$$\chi_{SG} = \frac{1}{N} \sum_{i,j=1}^{N} \left[\left(\int_0^\beta [\langle \sigma_i^z(\tau) \sigma_j^z(0) \rangle - \langle \sigma_i^z \rangle \langle \sigma_j^z \rangle] \, d\tau \right)^2 \right]_{av}. \qquad (95)$$

In the path integral formulation, this becomes

$$\chi_{SG} = \frac{1}{N} \sum_{i,j=1}^{N} \left[\left(\sum_{l=1}^{M} [\langle S_i(l_0 + l) S_j(l_0) \rangle - \langle S_i(l_0) \rangle \langle S_j(l_0) \rangle] \Delta\tau \right)^2 \right]_{av}, \qquad (96)$$

where we have discretized imaginary time as before.

We consider each term separately, and average over all possible time slices, which means sum over four time labels and have to divide by M^2. We follow similar steps to those in the classical case in Sec. II.

$$\frac{1}{M^2} \sum_{l_1, l_2, l_3, l_4 = 1}^{M} \langle S_i(l_1) S_j(l_2) \rangle \langle S_i(l_3) S_j(l_4) \rangle =$$

$$\frac{1}{M^2} \lim_{n \to 0} \frac{1}{n(n-1)} \sum_{(\alpha, \beta)} \sum_{l_1, l_2, l_3, l_4 = 1}^{M} \langle S_i^\alpha(l_1) S_j^\alpha(l_2) S_i^\beta(l_3) S_j^\beta(l_4) \rangle, \quad (97a)$$

$$\frac{1}{M^2} \sum_{l_1, l_2, l_3, l_4 = 1}^{M} \langle S_i(l_1) S_j(l_2) \rangle \langle S_i(l_3) \rangle \langle S_j(l_4) \rangle =$$

$$\frac{1}{M^2} \lim_{n \to 0} \frac{1}{n(n-1)(n-2)} \sum_{(\alpha, \beta, \gamma)} \sum_{l_1, l_2, l_3, l_4 = 1}^{M} \langle S_i^\alpha(l_1) S_j^\alpha(l_2) S_i^\beta(l_3) S_j^\gamma(l_4) \rangle, \quad (97b)$$

$$\frac{1}{M^2} \sum_{l_1, l_2, l_3, l_4 = 1}^{M} \langle S_i(l_1) \rangle \langle S_j(l_2) \rangle \langle S_i(l_3) \rangle \langle S_j(l_4) \rangle =$$

$$\frac{1}{M^2} \lim_{n \to 0} \frac{1}{n(n-1)(n-2)(n-3)} \sum_{(\alpha, \beta, \gamma, \delta)} \sum_{l_1, l_2, l_3, l_4 = 1}^{M} \langle S_i^\alpha(l_1) S_j^\beta(l_2) S_i^\gamma(l_3) S_j^\delta(l_4) \rangle, \quad (97c)$$

where the sums (α, β, γ) etc. are over all distinct pairs of replicas.

How do we do the averages in Eqs. (97)? We refer to Eq. (56) which we reproduce again here, in a slightly modified form.

$$[Z^n]_{\text{av}} = \sum_{\{S_i^\alpha(l)\}=\pm 1} \exp\left[\frac{(\Delta\tau J)^2}{2N} \sum_{l,l'=1}^{M} \sum_{\langle i,j\rangle} \left(\sum_{(\alpha,\beta)} S_i^\alpha(l)S_j^\alpha(l)S_i^\beta(l')S_j^\beta(l')+\right.\right.$$

$$\left.\left. + \sum_\alpha S_i^\alpha(l)S_j^\alpha(l)S_i^\alpha(l')S_j^\alpha(l')\right) + K^\tau \sum_i \sum_{l=1}^{M} \sum_{\alpha=1}^{n} S_i^\alpha(l)S_i^\alpha(l+1)\right]. \quad (98)$$

We now add fictitious fields $\Delta_{\alpha\beta}$ in in Eq. (98), i.e. we add

$$\frac{1}{M^2} \sum_{\alpha<\beta} \Delta_{\alpha\beta} \sum_{i=1}^{N} \sum_{l_1,l_2=1}^{M} S_i^\alpha(l_1)S_i^\beta(l_2) \quad (99)$$

in the exponent. Remembering that, for $n \to 0$, there is no normalizing denominator, we have

$$\frac{1}{M^2} \sum_i \sum_{l_1,l_2=1}^{M} \langle S_i^\alpha(l_1)S_i^\beta(l_2)\rangle = \lim_{n\to 0} \frac{\partial}{\partial\Delta_{\alpha\beta}}[Z^n]_{\text{av}}\,, \quad (100a)$$

$$\frac{1}{M^4} \sum_{i,j} \sum_{l_1,l_2,l_3,l_4=1}^{M} \langle S_i^\alpha(l_1)S_i^\beta(l_2)S_j^\gamma(l_3)S_j^\delta(l_4)\rangle = \lim_{n\to 0} \frac{\partial^2}{\partial\Delta_{\alpha\beta}\Delta_{\gamma\delta}}[Z^n]_{\text{av}}\,, \quad (100b)$$

in which the replicas $\alpha,\beta,\gamma,\delta$ can take any values subject to the restrictions $\alpha < \beta, \gamma < \delta$. Proceeding as earlier in this section, we get to Eq. (65) but with the $\Delta_{\alpha\beta}$ added, and also assume that $q_{\alpha\beta}(\Delta l)$ is independent of Δl for $\alpha \neq \beta$, a result which we assumed above but at a later stage. This gives

$$[Z^n]_{\text{av}} = \prod_{\alpha,\Delta l} \left[\sqrt{\frac{N}{2\pi}}(\Delta\tau J)\int_{-\infty}^{\infty} dr_\alpha(\Delta l)\right] \prod_{\alpha<\beta} \left[\sqrt{\frac{N}{2\pi}}(\Delta\tau J)\int_{-\infty}^{\infty} dq_{\alpha\beta}\right]$$

$$\exp\left[-\frac{N}{4}(\Delta\tau J)^2 M \sum_{\Delta l,\alpha} r_\alpha^2(\Delta l) - \frac{N}{2}(\Delta\tau J)^2 M^2 \sum_{\alpha<\beta} q_{\alpha\beta}^2\right] \left(\text{Tr}\, e^{-H}\right)^N\,, \quad (101)$$

where

$$H = -(\Delta\tau J)^2 \left[\frac{1}{2}\sum_\alpha \sum_{\Delta l} r_\alpha(\Delta l) \sum_l S^\alpha(l)S^\alpha(l+\Delta l)+\right.$$

$$\left. \sum_{\alpha<\beta} \sum_{l_1,l_2=1}^{M} \left(q_{\alpha\beta} - (\beta J)^{-2}\Delta_{\alpha\beta}\right)S^\alpha(l_1)S^\beta(l_2)\right] - K^\tau \sum_\alpha \sum_l S^\alpha(l)S^\alpha(l+1)\,. \quad (102)$$

As in the classical case we define shifted variables by

$$u_{\alpha\beta} = q_{\alpha\beta} - (\beta J)^{-2}\Delta_{\alpha\beta}\,, \quad (103)$$

in terms of which

$$[Z^n]_{\mathrm{av}} = \prod_{\alpha,\Delta l}\left[\sqrt{\frac{N}{2\pi}}(\Delta\tau J)\int_{-\infty}^{\infty}dr_\alpha(\Delta l)\right]\prod_{\alpha<\beta}\left[\sqrt{\frac{N}{2\pi}}(\Delta\tau J)\int_{-\infty}^{\infty}dq_{\alpha\beta}\right]$$
$$\exp\left[-\frac{N}{4}(\Delta\tau J)^2 M\sum_{\Delta l,\alpha}r_\alpha^2(\Delta l) - \frac{N}{2}\sum_{\alpha<\beta}\left\{(\beta J)^2 u_{\alpha\beta}^2 + 2\Delta_{\alpha\beta}u_{\alpha\beta} + (\beta J)^{-2}\Delta_{\alpha\beta}^2\right\}\right]\times \quad (104)$$
$$\left(\mathrm{Tr}\,e^{-H}\right)^N ,$$

where

$$H = -(\Delta\tau J)^2\left[\frac{1}{2}\sum_\alpha\sum_{\Delta l}r_\alpha(\Delta l)\sum_l S^\alpha(l)S^\alpha(l+\Delta l)+\right.$$
$$\left.\sum_{\alpha<\beta}\sum_{l_1,l_2=1}^{M}u_{\alpha\beta}S^\alpha(l_1)S^\beta(l_2)\right] - K^\tau\sum_\alpha\sum_l S^\alpha(l)S^\alpha(l+1) . \quad (105)$$

Doing the derivatives in Eqs. (100) gives

$$\frac{1}{N}\frac{1}{M^2}\sum_i\sum_{l_1,l_2=1}^{M}\langle S_i^\alpha(l_1)S_i^\beta(l_2)\rangle = \lim_{n\to 0}\langle q_{\alpha\beta}\rangle, \quad (106\mathrm{a})$$

$$\frac{1}{N}\frac{1}{M^4}\sum_{i,j}\sum_{l_1,l_2,l_3,l_4=1}^{M}\langle S_i^\alpha(l_1)S_i^\beta(l_2)S_j^\gamma(l_3)S_j^\delta(l_4)\rangle = \lim_{n\to 0}\left[-(\beta J)^{-2}\delta_{\alpha\beta,\gamma\delta} + \langle q_{\alpha\beta}\,q_{\gamma\delta}\rangle\right], \quad (106\mathrm{b})$$

where the averages are to be evaluated with $\Delta_{\alpha\beta} = 0$. Notice the strong similarity between Eq. (106b) and the corresponding one for the classical case, Eqs. (40b).

Referring to Eqs. (96), (97) and (106b), we see that χ_{SG} is given by

$$\chi_{SG} = \beta^2\left[(-\beta J)^{-2} + \langle q_{\alpha\beta}^2\rangle - 2\langle q_{\alpha\beta}\,q_{\alpha\gamma}\rangle + \langle q_{\alpha\beta}\,q_{\gamma\delta}\rangle\right] , \quad (107)$$

where we used that $M\Delta\tau = \beta$. Note that Eq. (107) is identical to the corresponding classical result, Eq. (41). Separating out the replica symmetric saddle point value q_c as in Eq. (41) we get

$$\chi_{SG} = -\frac{1}{J^2} + \frac{1}{T^2}\left[\langle\delta q_{\alpha\beta}^2\rangle - 2\langle\delta q_{\alpha\beta}\,\delta q_{\alpha\gamma}\rangle + \langle\delta q_{\alpha\beta}\,\delta q_{\gamma\delta}\rangle\right] . \quad (108)$$

The averages in Eq. (108) are over Gaussian integrals given by the weight in Eq. (65) in which f is given by the quadratic expression in Eq. (21). The combination of averages in Eq. (107) just corresponds to the "replicon" eigenvector of the matrix A, see Eq. (90) and Eqs. (86)–(89). Hence, according to Eq. (45), these averages just give the inverse of the replicon eigenvalue in Eq. (90).

Consequently the spin glass susceptibility is given, in the quantum case, by

$$\chi_{SG} = -\frac{1}{J^2} + \frac{1}{T^2}\frac{1}{\lambda_r}, \tag{109}$$

$$= -\frac{1}{J^2} + \frac{1}{J^2}\frac{1}{1 - J^2\chi_{SG}^0}, \tag{110}$$

$$\boxed{= \frac{\chi_{SG}^0}{1 - J^2\chi_{SG}^0},} \tag{111}$$

where

$$\boxed{\chi_{SG}^0 = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dz\, e^{-z^2/2}\left[\sum_l \Delta\tau\Big(\langle\, S(0)S(l)\,\rangle_{\overline{H}} - \langle\, S(0)\,\rangle_{\overline{H}}\langle\, S(l)\,\rangle_{\overline{H}}\Big)\right]^2.} \tag{112}$$

In the paramagnetic phase, the single spin averages in Eq. (112) vanish, and $q = 0$ so the integral over z gives unity. We therefore have

$$\chi_{SG}^0 = \left[\sum_l\Big(\Delta\tau\langle\, S(0)S(l)\,\rangle_{\overline{H}}\Big)\right]^2, \qquad \text{(for } q = 0). \tag{113}$$

Equations (111) and (113) have recently been verified by comparing with series expansions [19].

IV.   CONCLUSIONS

We have derived in detail the RS solution of the classical and quantum SK model. This solution is unstable at low temperatures, longitudinal fields, and transverse fields, but the expressions derived here, in particular the results for the replicon eigenvalue and the spin glass susceptibility, are those use to *show* the instability of the RS state. These notes are largely an assembly of existing results; one result which seems to be new is that for χ_{SG} in the quantum case, Eqs. (111) and (112).

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