9.3 – Dynamical Integrity: Concepts and Tools

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DAY	TIME	LECTURE
Monday 05/11	14.00 -14.45	Historical Framework - A Global Dynamics Perspective in the Nonlinear Analysis of Systems/Structures
	15.00 -15.45	Achieving Load Carrying Capacity: Theoretical and Practical Stability
	16.00 -16.45	**Dynamical Integrity: Concepts and Tools_1**
Wednesday 07/11	14.00 -14.45	Dynamical Integrity: Concepts and Tools_2
	15.00 -15.45	Global Dynamics of Engineering Systems
	16.00 -16.45	Dynamical integrity: Interpreting/Predicting Experimental Response
Monday 12/11	14.00 -14.45	Techniques for Control of Chaos
	15.00 -15.45	A Unified Framework for Controlling Global Dynamics
	16.00 -16.45	Response of Uncontrolled/Controlled Systems in Macro- and Micro-mechanics
Wednesday 14/11	14.00 -14.45	A Noncontact AFM: (a) Nonlinear Dynamics and Feedback Control (b) Global Effects of a Locally-tailored Control
	15.00 -15.45	Exploiting Global Dynamics to Control AFM Robustness
	16.00 -16.45	Dynamical Integrity as a Novel Paradigm for Safe/Aware Design

CONTENTS

INTRODUCTION

- ***Existence*** **of an attractor does** ***not*** **guarantee its practical** ***safety*** *(Thompson and coworkers)*
- **In experiments and practice,** ***disturbances*** **exist, giving** ***uncertainty*** **to the operating initial conditions**
- **If the system is not sufficiently** ***robust*****, dynamical outcome can be totally different from what theoretically predicted**
- ***Robustness*** **of system response to be analyzed**
- ***Dynamic integrity (DI) analysis*** **is able to provide valuable information about the expected dynamics under realistic conditions, which is essential to** ***safely*** **operate a system with the** ***desired behavior*****, depending on** ***expected disturbances*** *(Thompson, 1989; Soliman and Thompson, 1989)*
- **Main underlying** ***concepts*** **and computational** ***tools*** **presented and discussed**

LINES OF ANALYSIS

- ***"Safe basin"*: introducing a proper definition, i.e. stating accurately which are the conditions that we can consider as *safe***

  Involves theoretical and practical issues, e.g. transient vs steady dynamics, whether fractality of the basin can be accepted or must be prevented, etc.

- ***Measures*: introducing an appropriate DI measure, able to assess *quantitatively* how much our safe basin (i.e. our safe condition) is robust**

  Measuring ***accurately and properly*** the DI is a critical point.
  A measure may be appropriate in some particular case-studies but not in others.
  This strongly depends on the ***problem*** to analyze.

- ***Profiles (Charts)*: informing about the basins *evolution* due to variation of system parameters**

  A system must be able to sustain changes in both ***initial conditions*** and ***control parameters*** without changing its desired outcome.

CONTENTS

MAIN ISSUES

Analyzing some *main aspects* of interest for *engineering design*, important

- **for defining properly the *safe basin***
- **for selecting the most appropriate DI *measure***

Depending on the *energy* imparted to the system through initial conditions or parameter values, *two main issues* are of considerable practical importance for a mechanical system:

1) **the possibility to have the system *actually working* within the range of operating conditions for which it is designed;**

2) **the *loss of technical performance* or structural integrity ensuing from the system getting off the foreseen operational regime.**

POTENTIAL WELL (1)

In *dynamical terms*, we restrict ourselves to systems characterized by

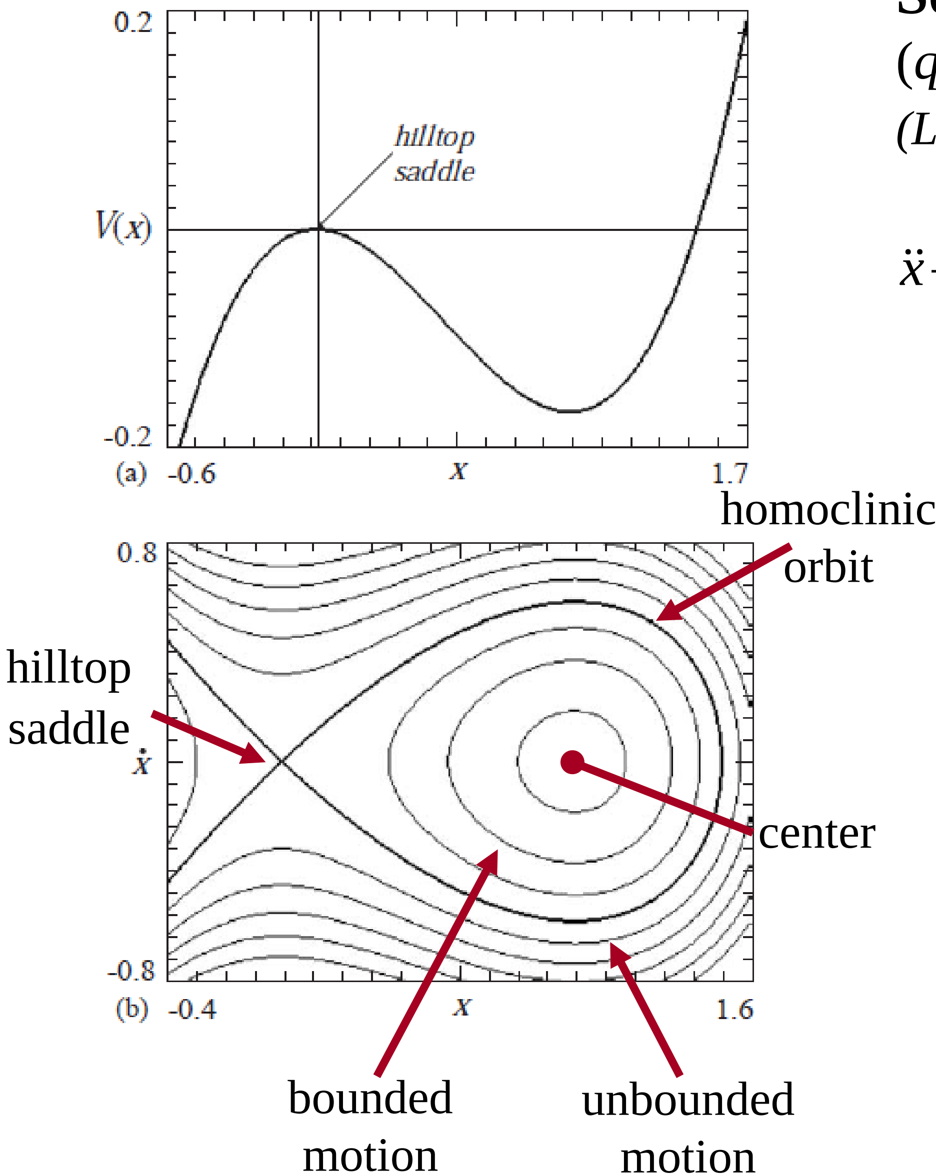
- **a *single-well***
- **a (possibly) *multi-well* potential,**

for which two critical situations may occur:

1) **the motion *does* develop on a *small scale* in phase space, being actually restricted within a *sole* potential well (the foreseen one in multi-well cases);**

2) **the motion *may also* develop on a *large scale* in phase space, i.e. also *beyond* the invariant manifolds of the hilltop saddle which delimit the potential well and organize the whole system dynamics.**

POTENTIAL WELL (2)



Softening Helmholtz oscillator

(*quadratic* nonlinearities)

(Lenci and Rega, 2003)

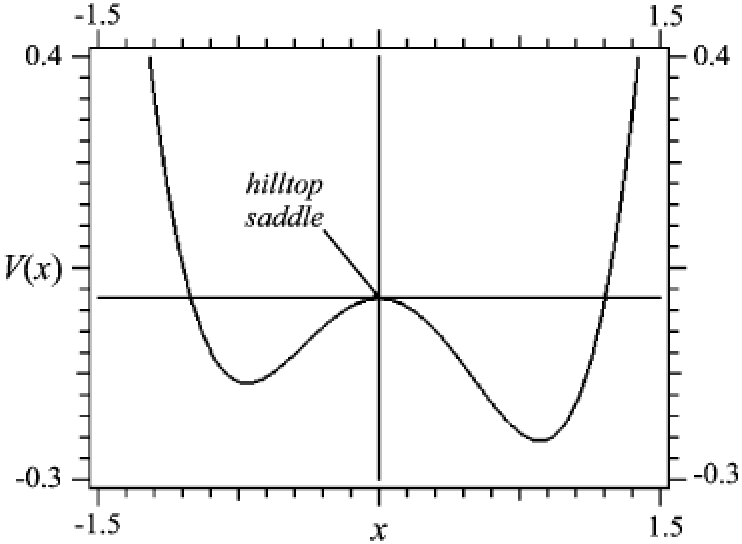
$$\ddot{x}+0.1\dot{x}-x+x^2=\gamma(\omega t)=\gamma_1\sum_{j=1}^{\infty}\frac{\gamma_j}{\gamma_1}\sin(j\omega t+\Psi_j)$$

Single **potential well, asymmetric**

Unperturbed undamped dynamics:

- **one *center***
- **a unique (*hilltop*) saddle**
- **one *homoclinic* orbit (asymmetric) surrounding the potential well**

POTENTIAL WELL (3)



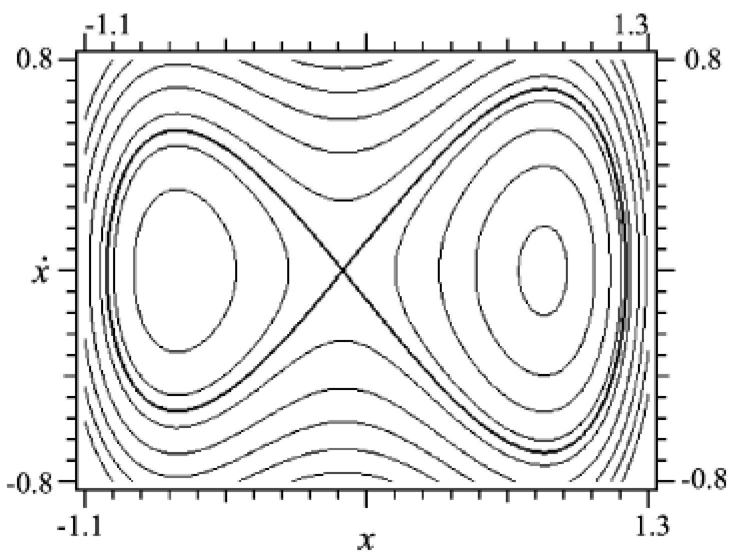
Hardening Helmholtz–Düffing oscillator

(*quadratic* and *cubic* nonlinearities)

(Lenci and Rega, 2004)

$$\ddot{x} + \varepsilon\delta\dot{x} - \sigma x - \tfrac{3}{2}(\sigma - 1)x^2 + 2x^3 = \varepsilon\gamma_1 \sum_{j=1}^{\infty} \frac{\gamma_j}{\gamma_1} \sin(j\omega t + \Psi_j)$$

with: $\sigma = 1.2$

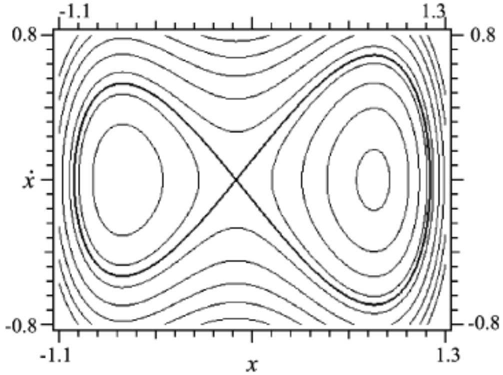


Twin **potential well, asymmetric**

Unperturbed undamped dynamics:

- **two *centers***
- **a unique (*hilltop*) saddle**
- **two *homoclinic* orbits (asymmetric) surrounding the two potential wells**

POTENTIAL WELL (4)



The homoclinic loops separate:

- ***left*** **in-well periodic oscillations**
- ***right*** **in-well periodic oscillations**
- ***large amplitude*** **scattered dynamics**

Due to system asymmetry:

- **the right well is slightly *wider* than the left one,**
- **overall, both wells are considerably *robust***

Adding excitation and damping,

- **each one of the homoclinic loops splits**
- **the scenario becomes increasingly *complex*, modifying the robustness of each well.**

We need to double-check if we can still effectively *rely* on them.

POTENTIAL WELL (5)

Analysing the *robustness* of the *potential well*
is one main issue when investigating a nonlinear system:

- **if the DI of the well is *elevated*,**
  - **there is the possibility to effectively *operate* the system *within* the desired well;**
  - **the system dynamics is worth of further investigations** (e.g. analyzing the individual attractors belonging to the well);
- **if the DI of the well is *residual*,**
  - **it is not worth to further refine the simulations, since the system will *never be able* to operate in these conditions.**

OUT-OF-WELL PHENOMENON (1)

What happens when *exceeding* the potential well?

Investigating the *out-of-well phenomenon*

- **The *homoclinic* bifurcation of the invariant manifolds of the *hilltop* saddle (*) – i.e, of a *globally organizing* saddle – is the main bifurcational event governing the transition of system dynamics from a *small* scale to a *large* scale regime**

- **After the global bifurcation of the manifolds of the *hilltop saddle*, the system overcomes the bounding invariant manifolds and the erosion of the well inevitably starts.**

- **The system's dynamics escape from the potential well and different kinds of *out-of-well* attractor may occur, onto which the system settles down.**

Two basically *different* situations can be distinguished:

- ***hardening* system**
- ***softening* system**

(*) This critical threshold can be analytically computed via the ***Melnikov***'s method

OUT-OF-WELL PHENOMENON (2)

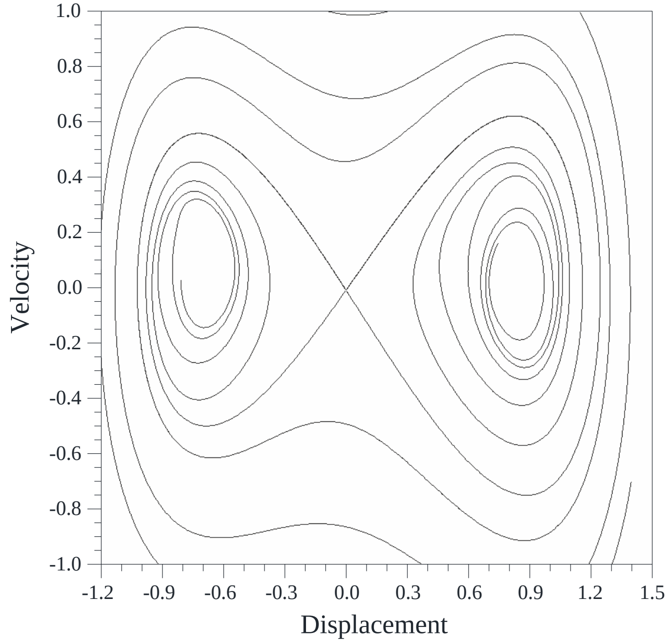
HARDENING SYSTEM

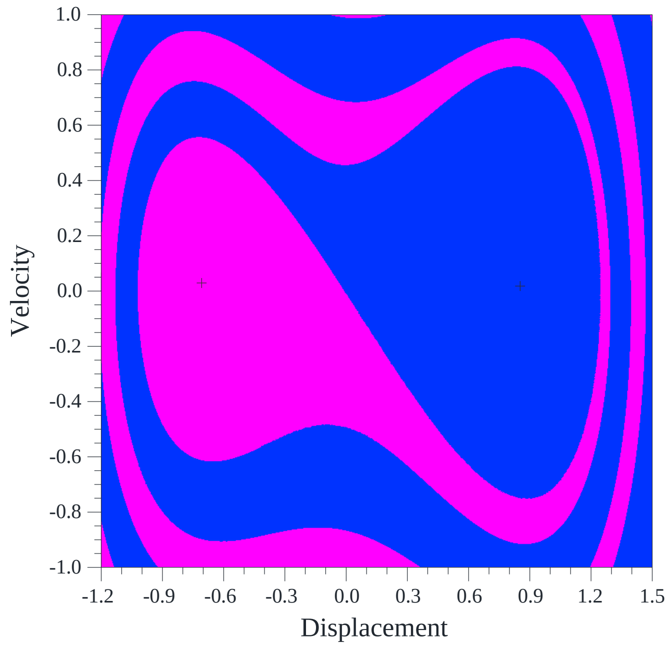
- **The erosion of the well is due to the interpenetration of basins from *adjacent wells*, which basically do not change in magnitude but simply become tangled.**

- **The out-of-well phenomenon that sets on after the erosion is a *scattered (cross-well, usually chaotic) attractor.***
  The motion develops entirely across two neighbouring, bounded, wells, or it wanders around.

- **Though being no more restricted within the reference well, the overall system dynamics still remains *bounded*.**

OUT-OF-WELL PHENOMENON (3)





Hardening Helmholtz–Düffing oscillator

(Lenci and Rega, 2004)

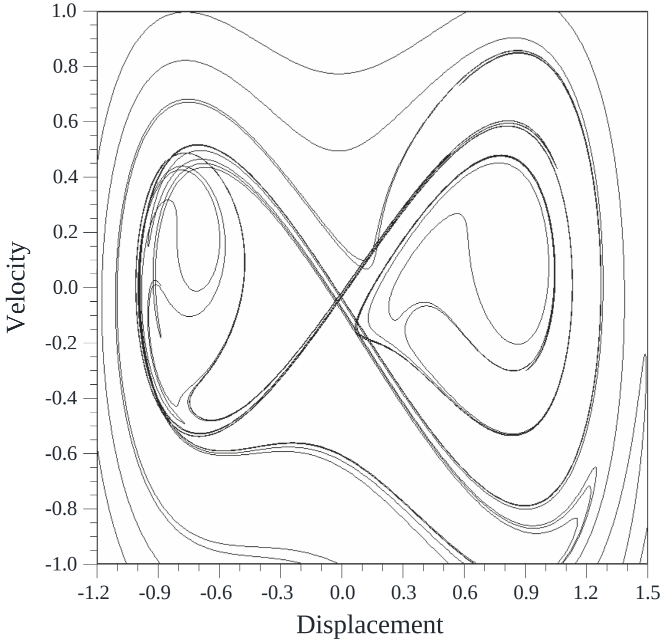
***Before* the homoclinic bifurcation of the hilltop saddle**

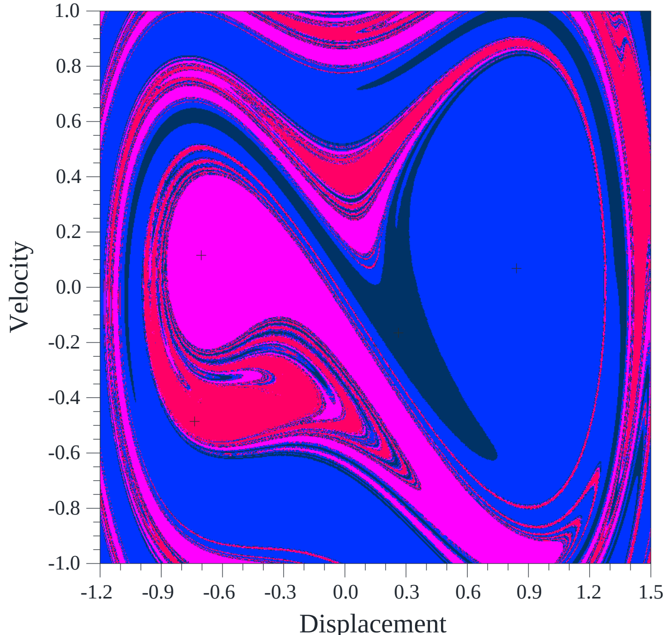
- **Manifolds do *not* intersect**
- **Basins boundaries are *smooth***
- ***No* penetration of basins from adjacent wells**

$\varepsilon\gamma_1 = 0.02 \qquad \varepsilon\delta = 0.1$

$\omega = 1.17 \qquad \Psi_j = 0.0$

OUT-OF-WELL PHENOMENON (4)





After the homoclinic bifurcation of the hilltop saddle

- **Manifolds *do* intersect**
- **Basins become *tangled***
- **The right well *enters* with *fractal* tongues the left well, and viceversa**
- ***Alternating* dynamic regimes between the reference well and the adjacent one**

$\varepsilon\gamma_1 = 0.077 \qquad \varepsilon\delta = 0.1$

$\omega = 1.17 \qquad \Psi_j = 0.0$

OUT-OF-WELL PHENOMENON (5)

- **We can observe *resonance* behavior** (non-resonant and resonant branches) **in both wells, enhancing the *complexity* of system dynamics**
- **The basins of non-resonant and resonant attractors in the left well are *not* adjacent to each other, but *separated* by the basins of the right well. This notably contributes to *reduce* the compactness (i.e. the robustness) of the left well.**

However, despite the out-of-well phenomenon, all possible attractors are *bounded*, i.e. despite the tangling between the wells, only *safe* bounded behaviors can be expected.

<u>Hardening system</u> → the out-of-well attractor is usually *unpleasant* from the application viewpoint, but does *not* usually *destroy* the structure

Depending on the ***application***, this phenomenon:

- may need to be ***avoided*** or ***controlled***
- may be ***desirable***;

OUT-OF-WELL PHENOMENON (6)

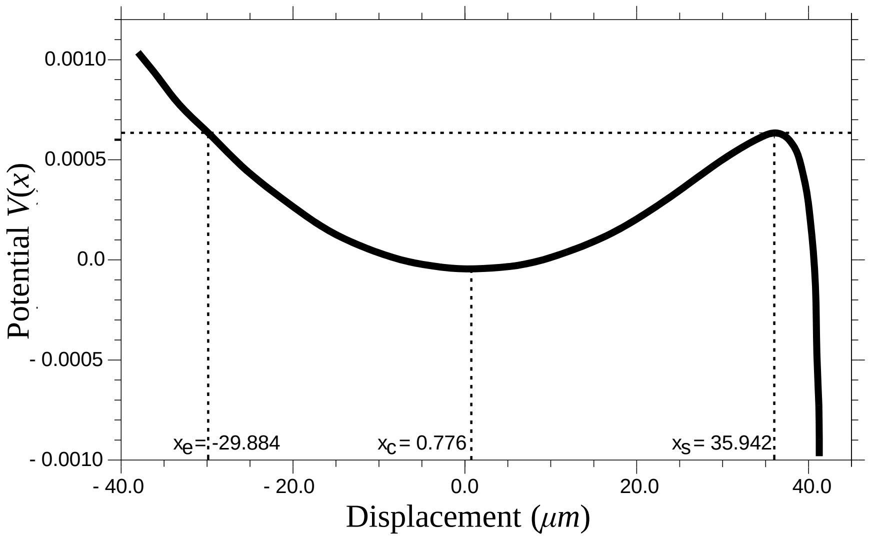
SOFTENING SYSTEM

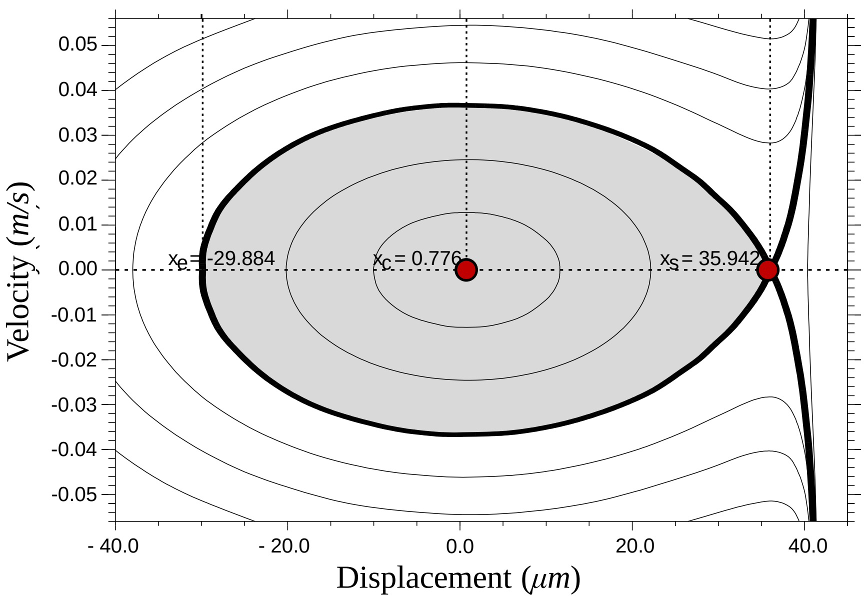
- **The erosion is owed to the penetration of the "*infinity*" attractor 'surrounding' the basin, which is *reduced* in magnitude during erosion.**

- **The system dynamics escape to "*infinity*".**

- **After escape, the motion is theoretically *unbounded*, and practically corresponds to the system settling down onto an attractor far away and completely different from the designed one or, in other terms, to the *definitive system failure*.**

OUT-OF-WELL PHENOMENON (7)





MEMS capacitive accelerometer

(Ruzziconi, Younis, and Lenci, 2010)

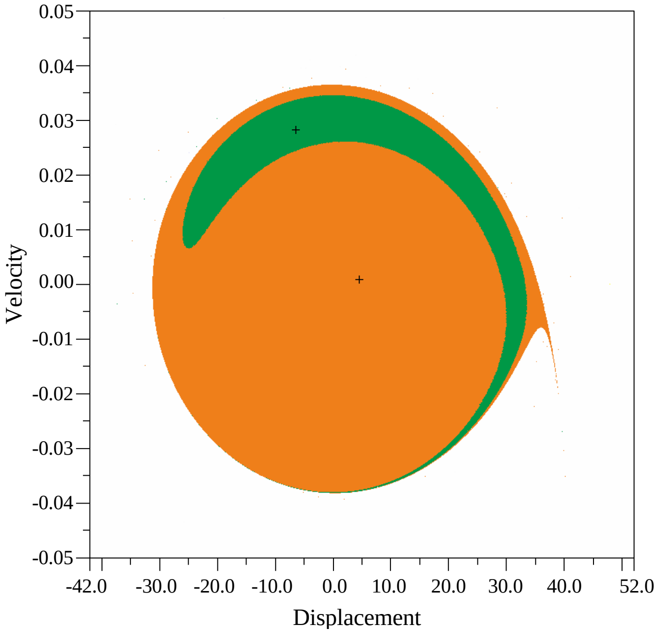
$$\ddot{x}+10.1\dot{x}+1.4629\cdot 10^{6}x=1.2\cdot 10^{-12}\cdot\frac{[40.1+V_{AC}\cos(\Omega t)]^{2}}{(42\cdot 10^{-6}-x)^{2}}$$

Single **asymmetric potential well, with** ***escape*** **direction**

Unperturbed undamped dynamics:

- ***center***
- ***hilltop*** **saddle**
- ***homoclinic*** **orbit separating in-well oscillations and out-of-well escape**

OUT-OF-WELL PHENOMENON (8)

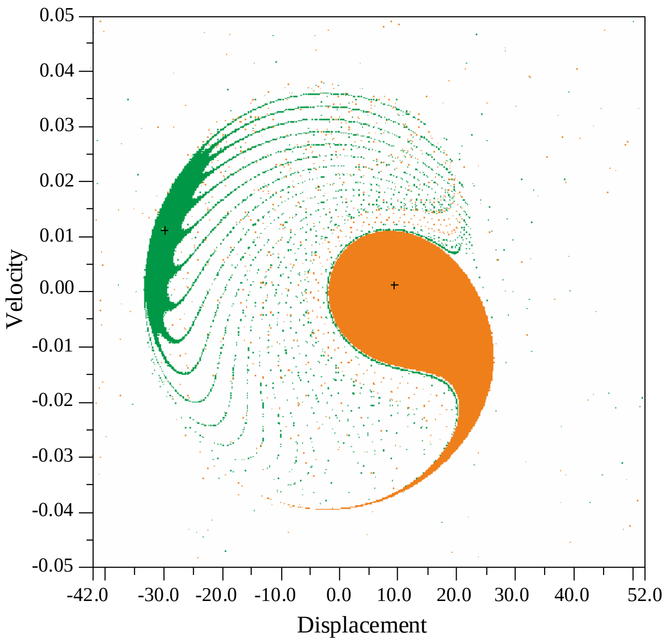


***Before** the homoclinic bifurcation of the hilltop saddle*

- **The *safe* bounded area is wide and compact**
- **The unsafe out-of-well *escape* (white) surrounds it, without entering the well**

$V_{AC} = 3.8\ V$, $\Omega = 185$ Hz

$V_{AC} = 12.5\ V$, $\Omega = 182$ Hz



***After** the homoclinic bifurcation of the hilltop saddle*

- ***Escape* enters the potential well and the *erosion* starts**
- **Fractal tongues of escape *separate* the basins of the non-resonant and resonant attractors, preventing safe *in-well jump***

OUT-OF-WELL PHENOMENON (9)

Softening system $\rightarrow$ **the erosion due to the out-of-well phenomenon is much more *dangerous* from a practical point of view, because it directly leads to *failure* of the system**

Summarizing:

- **Both hardening and softening escape situations are characterized, from a topological viewpoint, by a *progressive erosion process* of**
  - **a *single in-well* basin of attraction**
  - **the *whole collection of in-well* basins.**
- **The occurrence of a system *natural frequency* may meaningfully affect the system sensitivity, in terms of loss of DI**

Monitoring *out-of-well* dynamics and evolution of *erosion* is another fundamental issue when investigating a nonlinear system.

FRACTALITY vs. COMPACTNESS (1)

The degree of *fractality* of phase space is very important for practical applications, because it is strictly linked to the *sensitivity* to initial conditions.

- **Especially in *hardening* systems, basins magnitude can remain unchanged during erosion, but become more and more *tangled*, so the *distance* between attractor and basin boundary decreases dangerously, which**
  - **forewarns the incipient *boundary crisis* triggering the out-of-well phenomenon**
  - **entails practical *unsafety* of the attractor, in spite of its stability.**

- **Even more *dangerous* is the fractality in *softening* systems, where the basin is tangled with the *escape* area, which progressively erodes the safe region leading to unbounded behaviors.**

fractality → ***sensitivity to initial conditions*** → ***unpredictable final behvior***

↓

unpredictability unwanted in practice:

fractal area dangerous from a practical point of view, mostly in softening systems

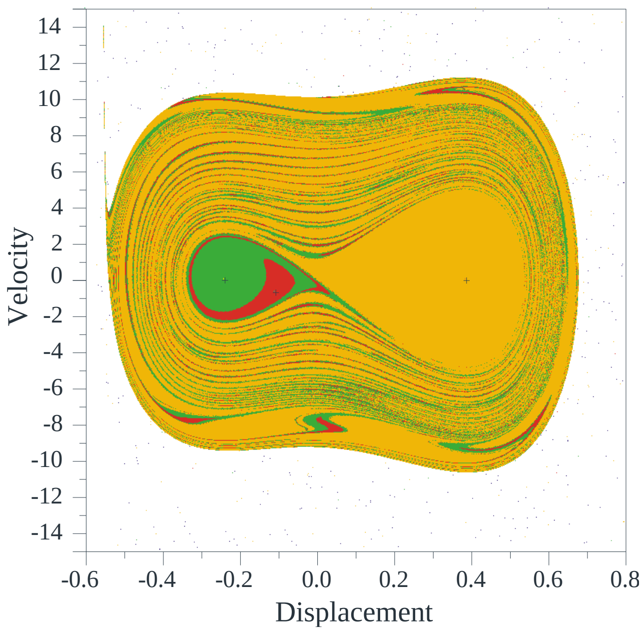
FRACTALITY vs. COMPACTNESS (2)

MEMS imperfect microbeam with axial load

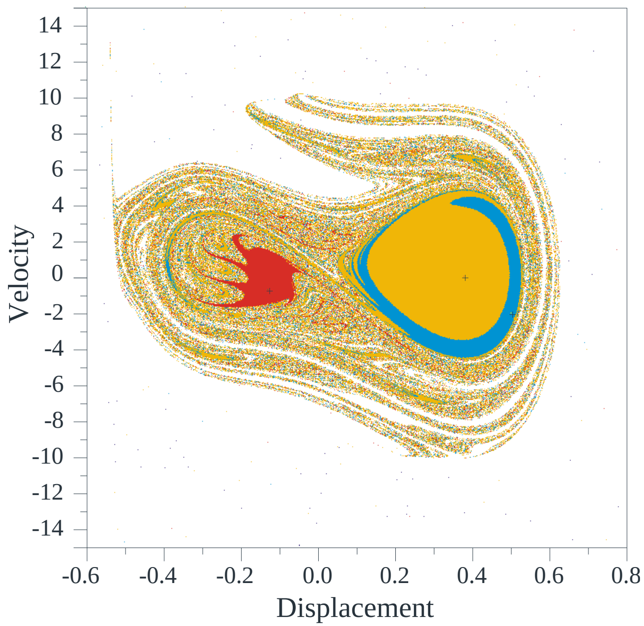
(double asymmetric potential well with escape direction)

(Ruzziconi, Lenci, Younis, 2013)

$V_{AC} = 0.5\ V$, $\Omega = 17$



$V_{AC} = 5.5\ V$, $\Omega = 20$



- **wide compact area (basins of bounded motions) essential to tolerate disturbances since *all* these initial conditions lead to a safe bounded attractor**

- **fractal area sensitive to disturbances: a small *uncertainty* in the initial conditions may lead *in practice* to dynamic pull-in, despite theoretical simulations predict a safe bounded attractor**

FRACTALITY vs. COMPACTNESS (3)

**Since unpredictability is typically unwanted,
for *fractal* basins it may be more expressive to refer to the
compact 'core' of the basin, instead of its wholeness.**

Fractality is a *very critical* issue when investigating a nonlinear system.

A fundamental issue to be taken into *account*:

- **in the *dynamical integrity measures***
- **in the definition of *refined safe basins***

TRANSIENT vs. STEADY DYNAMICS

Importance of the *transient*

The specific mechanical system suggests if it makes more sense to deal with *transient* or *steady dynamics*, i.e., with *short-term* or *long-term* behaviour.

There are situations where a *temporaneous* escape from potential well may be *unessential* for effective system operation, whereas it has to be strictly *avoided* in other practical situations.

This distinction also takes into account the *duration* of the excitation:

- **the *transient* regime is very important and governs the whole system performance in the nonlinear dynamics due to *short-term* excitations** (***impact forces, thermal impulses*** or ***seismic loads***)**;**
- **the *steady state* dynamics are apparently of major interest in the presence of stationary, *long-term*, excitations.**

Moreover, the transient regime is:

- **of minor importance if is *short*** (e.g., in highly damped systems, etc.)
- **critical if is *long*** (e.g., in highly deformable systems, close to bifurcations, etc.)

OTHER ISSUES (1)

INDEPENDENCE OF THE EXCITATION PHASE

- **The *phase* of periodic excitations may play an important role in determining the system response and, accordingly, its integrity.**

  i.e. in the problem of overturning of rigid blocks *(Lenci and Rega, 2003)*.

- **In these cases, one must look for *phase-independent* arguments to correctly measure the loss of integrity of the system, a point of view which is drastically different from those commonly used in non-linear dynamics.**

- **Later on considered when introducing the "*true*" safe basin**

OTHER ISSUES (2)

HOMO/HETEROCLINIC BIFURCATIONS

- **The occurrence of a global, homo/heteroclinic bifurcation of a *hilltop saddle* governs the *whole* system dynamics (or most of it).**
- **Yet, other *local saddles* (up to possibly infinitely many) also exist within each potential well, as a consequence of**
  - **instability of a single in-well periodic solution**
  - **onset of a chaotic saddle born at a previous homo/heteroclinic bifurcation of the coexisting hilltop.**
- **Though being secondary bifurcational events, the intersections of the invariant manifolds of a local saddle may also entail *meaningful changes* of system dynamics, either on the *small* scale (in-well) regime or on the *large* scale (cross-well) regime.**

OTHER ISSUES (3)

SYMMETRIC/ASYMMETRIC SYSTEMS

- **As a rule, symmetric oscillators are *structurally unstable* and can be considered as *particular cases* of asymmetric oscillators, though they are not always a limit, in an appropriate sense, of the latter** (e.g. the softening Helmholtz-Duffing equation *(Lenci and Rega, 2003)*)

SMOOTH/NON-SMOOTH SYSTEMS

- **Non-smooth systems exhibit a further *enriched* pattern of response classes and local/global bifurcations with respect to the already involved scenario of smooth systems**

BOUNDARIES OF THE SAFE BASIN BEING INVARIANT MANIFOLDS OR NOT

etc.

CONTENTS

SAFE BASIN (1)

Starting point of DI analysis:

the correct definition of *safe basin*,

i.e. stating accurately which are the conditions that in our system represent a *safe* situation.

This notion may vary *from case to case*, depending on the *problem*

Many *different* definitions of safe basin have been proposed, according to *which safe condition* we need to consider.

A *general* definition summarizing all of them:

the safe basin as the set of initial conditions sharing a common dynamical property

where such a "*property*" is specified *case by case*, based on the problem we are studying → some examples

SAFE BASIN (2)

POTENTIAL WELL

If we need to analyze

the *robustness* of a given *potential well*,

we can consider as *safe basin*

the set of all initial conditions approaching bounded attractors belonging to a given potential well as $t \rightarrow \infty$

i.e. the union of the basins of attraction of all attractors belonging to a given potential well,

- ***safe condition*** is represented by the basins of all initial conditions leading to attractors in a given well
- ***unsafe condition*** is represented by ***all the other dynamics***

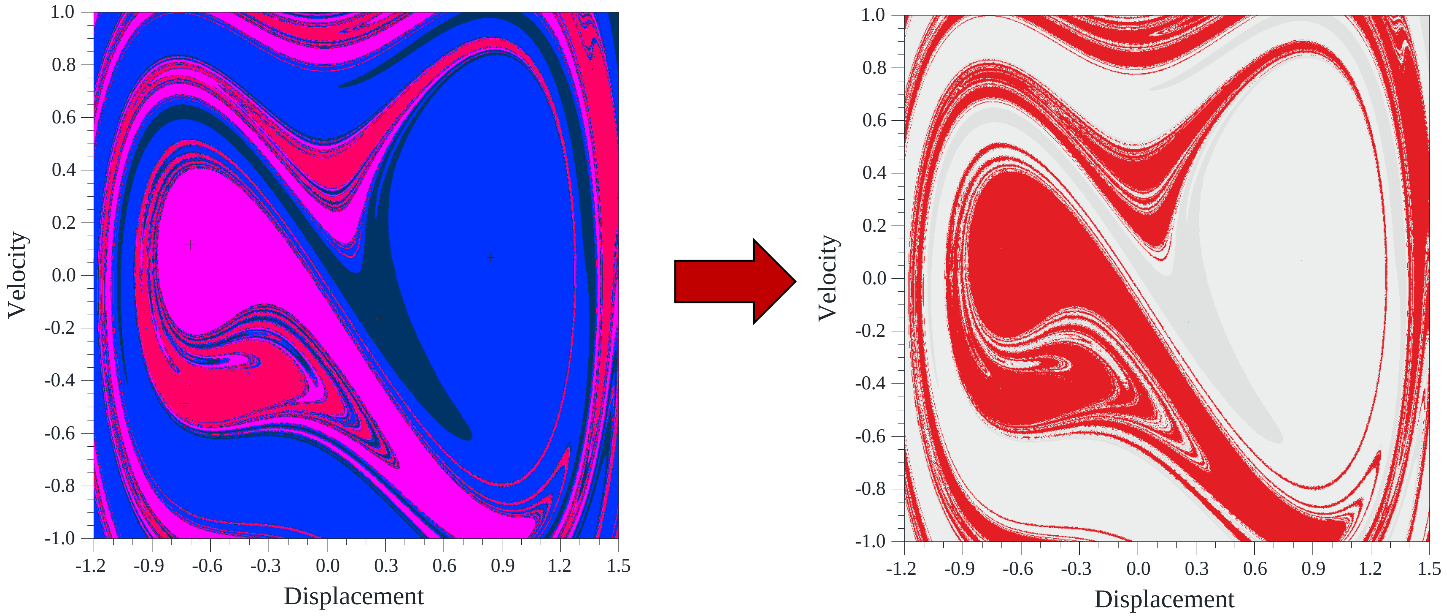
SAFE BASIN (3)

POTENTIAL WELL

$$\varepsilon\gamma_1 = 0.077 \qquad \varepsilon\delta = 0.1$$
$$\omega = 1.17 \qquad \Psi_j = 0.0$$

Hardening Helmholtz–Düffing oscillator

(Lenci and Rega, 2004)



Safe basin → **union of the basins of attraction of *all* attractors belonging to the *left* potential well**

SAFE BASIN (4)

BOUNDED BEHAVIOR

If we need to analyze

the *robustness* of all *bounded attractors*

(regardless which is the well they belong to),

as opposed to the regions leading to escape, we can consider as *safe basin*

the union of all the basins of attraction

- ***safe condition*** is represented by all basins of attraction
- ***unsafe condition*** is represented by the ***escape***

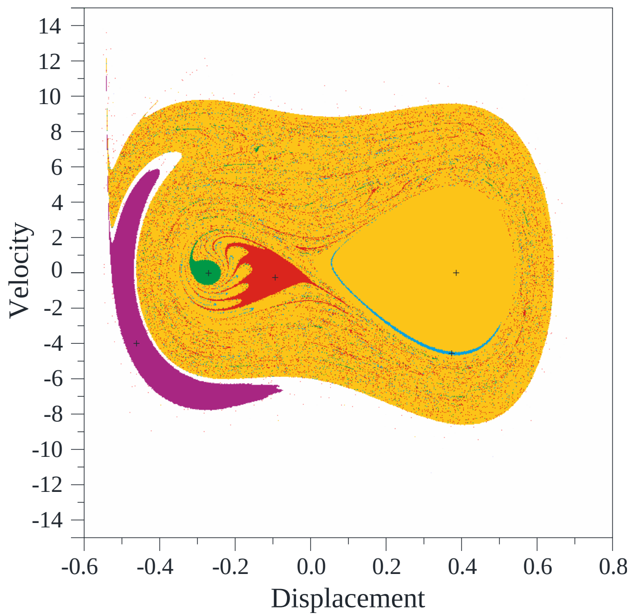
SAFE BASIN (5)

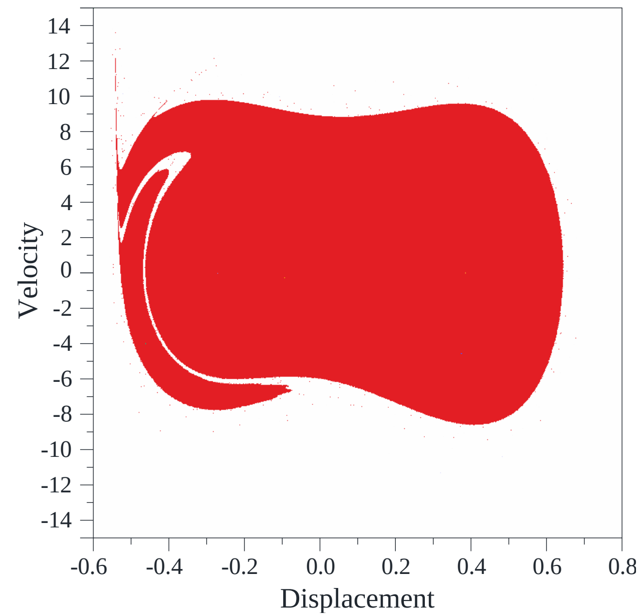
BOUNDED BEHAVIOR

MEMS imperfect microbeam with axial load

(Ruzziconi, Lenci, Younis, 2013)

$V_{AC} = 2.5\ V,\ \Omega = 17.5$





Safe basin → **union of the basins of attraction of *all bounded* attractors**

SAFE BASIN (6)

INDIVIDUAL ATTRACTOR

If we need to distinguish among different attractors and analyze the *robustness* of *each one* of them,

we can consider as *safe basin*

the set of initial conditions leading to a given attractor as $t \rightarrow \infty$,

i.e., the basin of attraction of a given attractor

- ***safe condition*** is represented by only the basin of attraction of the attractor under consideration
- ***unsafe condition*** is represented by all the ***other dynamics*** (***both bounded and unbounded***)

SAFE BASIN (7)

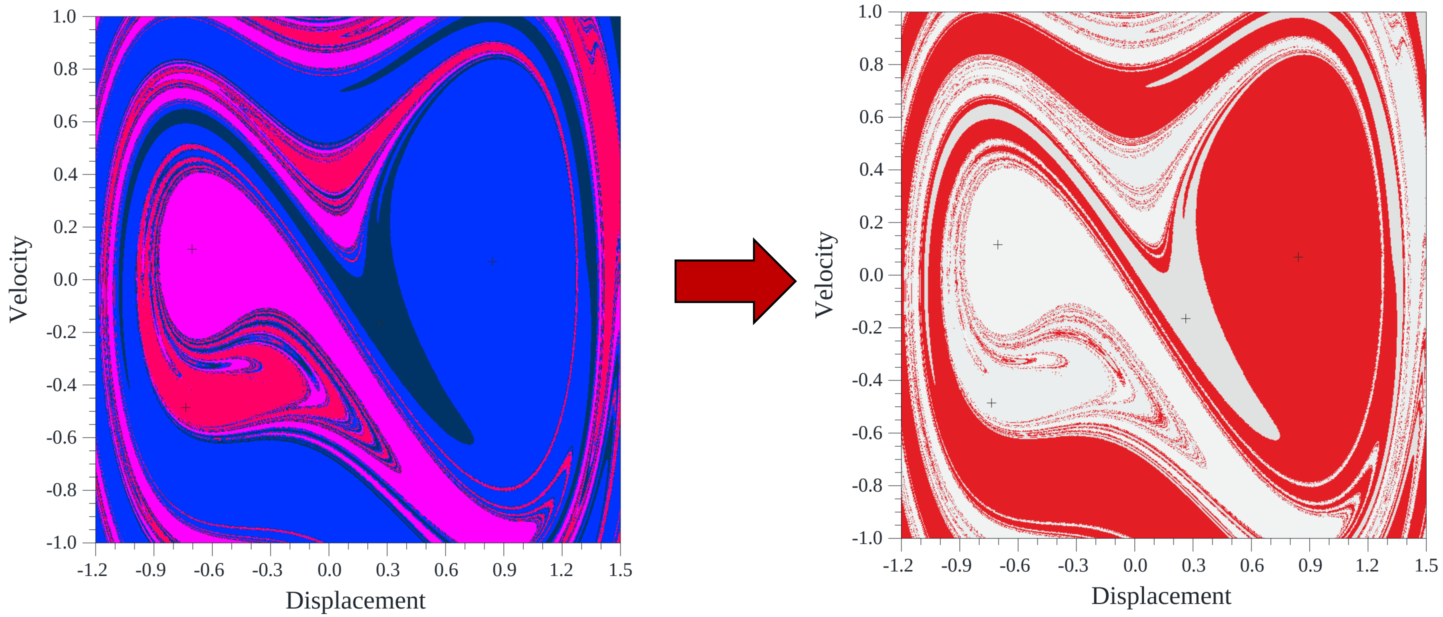
INDIVIDUAL ATTRACTOR

$\varepsilon\gamma_1 = 0.077$ $\varepsilon\delta = 0.1$

$\omega = 1.17$ $\Psi_j = 0.0$

Hardening Helmholtz–Düffing oscillator

(Lenci and Rega, 2004)



Safe basin → **basin of attraction of the *non-resonant* attractor in the *right* well**

SAFE BASIN (8)

TRANSIENT DYNAMICS

All previous definitions ignore *transient dynamics*.

To overcome this point, Lansbury *et al.*, referring to a two-well oscillator, propose the following *alternative* definition:

...We preclude any starting condition which leads to either an attractor or transient which spans both wells of the potential. The set of remaining starting conditions that lead to steady-state motion confined to one well, we define as the safe basin of attraction. In the case ... where more than one attractor coexists within a single well, the safe basin may comprise two, or more competing basins...

This definition ***eliminates*** the initial conditions leading to ***transient out-of-well*** motion.

SAFE BASIN (9)

When the *safe basin* is a *given basin of attraction* or *unions of them*:

- the ***boundaries*** of the safe basin, which play a major role, are ***stable manifolds*** of given saddles;
- they have a ***well-defined*** behaviour and properties which allows the possibility of a ***full inspection*** of erosion in terms of global bifurcations, i.e. their ***evolution*** can be studied in terms of dynamical systems theory;
- they can be computed by ***standard***, efficient and worldwide available numerical techniques.

SAFE BASIN (10)

Transient dynamics-based **definitions of safe basin can also rely on merely *phenomenological* aspects**

e.g., "the initial conditions entailing transient orbits which do not cross a critical line of ***collapse*** in the phase space", with no care of what basins of attraction they belong to

Overall, for them:

- the ***boundaries*** of the safe basin are ***not*** invariant manifolds;
- their behaviour is ***not*** so well-defined;
- they require ***time consuming*** ad-hoc algorithms due to the on-line continuous check on the ***state of the system***;
- statements like "*transient spanning both wells*" or "*crossing a certain line*" require ***further choices***, with also a possible degree of arbitrariness.

SAFE BASIN (11)

The definition of safe basin depends on the '*safe*' condition one wants to address or realize.

By simply *changing* the definition of safe basin, we can analyze *different characteristics* of the system response, i.e. we can investigate the system from

a variety of dynamical integrity *perspectives*

10.1 – Dynamical Integrity: Concepts and Tools (continued)

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Coworkers: S. Lenci, L. Ruzziconi

DAY	TIME	LECTURE
Monday 05/11	14.00 -14.45	Historical Framework - A Global Dynamics Perspective in the Nonlinear Analysis of Systems/Structures
	15.00 -15.45	Achieving Load Carrying Capacity: Theoretical and Practical Stability
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Wednesday 07/11	14.00 -14.45	**Dynamical Integrity: Concepts and Tools_2**
	15.00 -15.45	Global Dynamics of Engineering Systems
	16.00 -16.45	Dynamical integrity: Interpreting/Predicting Experimental Response
Monday 12/11	14.00 -14.45	Techniques for Control of Chaos
	15.00 -15.45	A Unified Framework for Controlling Global Dynamics
	16.00 -16.45	Response of Uncontrolled/Controlled Systems in Macro- and Micro-mechanics
Wednesday 14/11	14.00 -14.45	A Noncontact AFM: (a) Nonlinear Dynamics and Feedback Control (b) Global Effects of a Locally-tailored Control
	15.00 -15.45	Exploiting Global Dynamics to Control AFM Robustness
	16.00 -16.45	Dynamical Integrity as a Novel Paradigm for Safe/Aware Design

CONTENTS

MEASURES (1)

Upon properly defining the 'nominal' safe basin,
an attractor (or the alternative dynamical property)
can be considered as *practically stable* if its safe basin is *large enough*.

But the *meaning* of large enough is not trivial and needs a proper definition of '*magnitude*'.

The need of a suitable *dynamical integrity measure* has been highlighted by Soliman and Thompson (1989).

In their original paper:

- **they *quantify* the dynamical integrity of a system**
- **they intend to *estimate* it properly, according to the requirements of engineering design.**

MEASURES (2)

Choosing an *appropriate* DI measure strongly depends on the problem at hand and may significantly vary from case to case.

Within this framework,

different dynamical integrity measures have been proposed, stating accurately, for each one of them,
both *principal characteristics* and *limitations*.

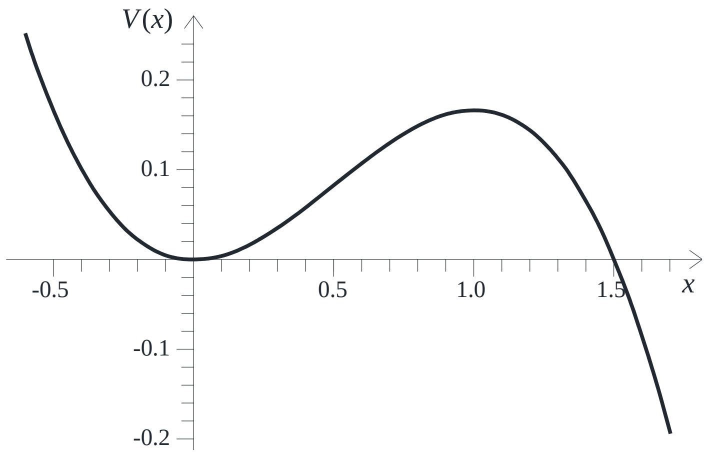
We *analyze* and *compare* those most commonly used.

GIM (1)

GLOBAL INTEGRITY MEASURE (GIM)
is the normalized hypervolume (area in 2D examples) of the safe basin

- **GIM is the *first* dynamical integrity measure introduced by Soliman and Thompson (1989)**
- **GIM is probably the most *intuitive* and *easy* dynamical integrity measure proposed in the literature**

GIM (2)

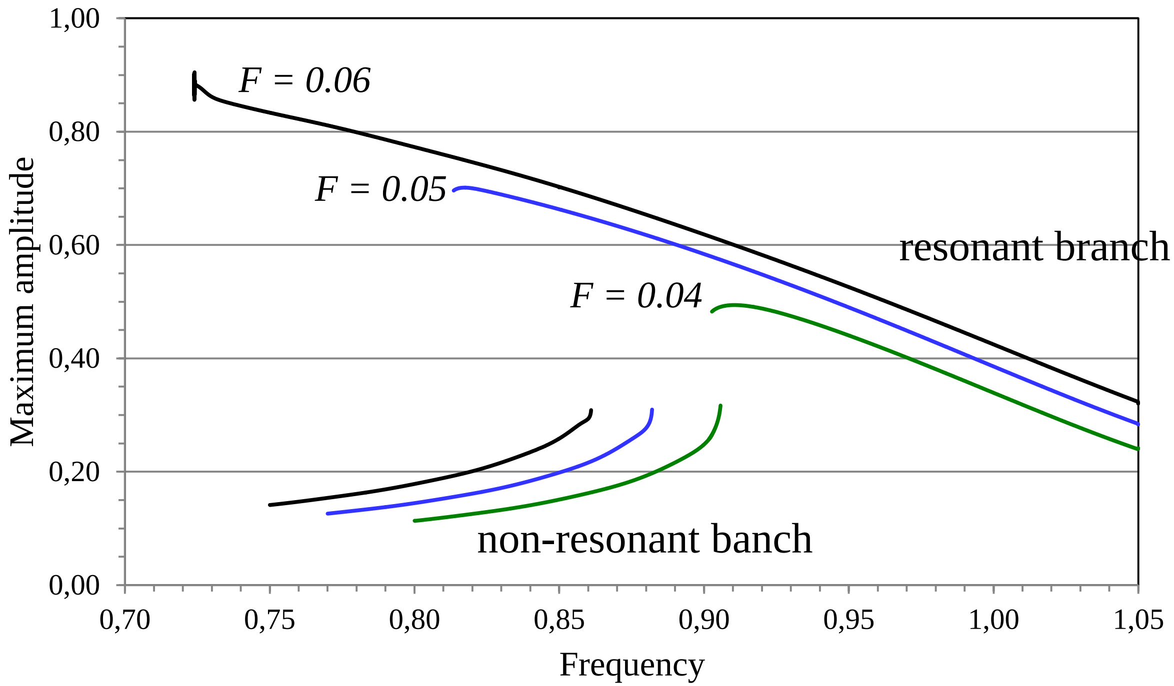


Oscillator with
single asymmetric potential well and escape

(Soliman and Thompson, 1989)

$$\ddot{x} + c\dot{x} + x - x^2 = F\sin(\Omega t)$$

Frequency response diagram



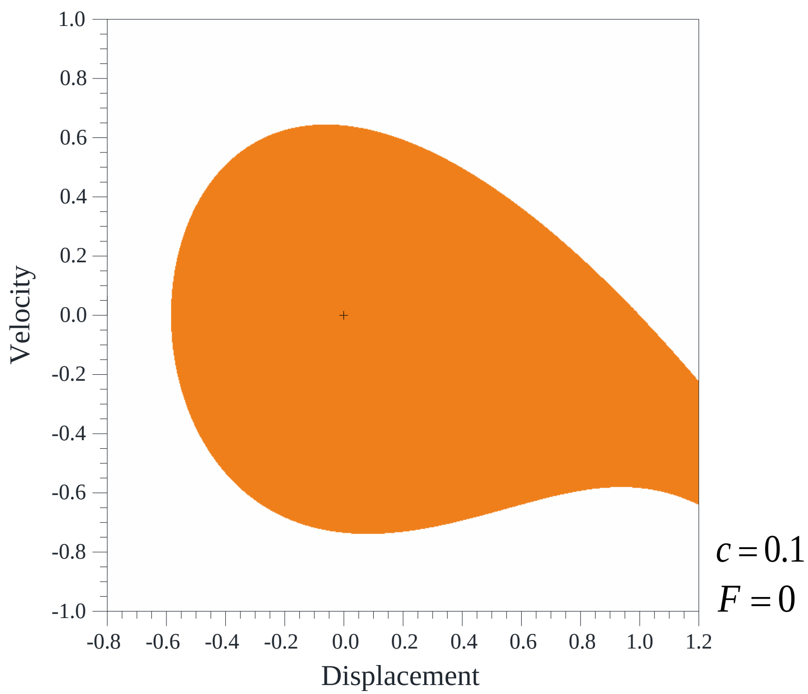
Typical softening behavior at resonance:

- **non-resonant branch,**
- **resonant branch,**
- **bending toward lower frequencies**

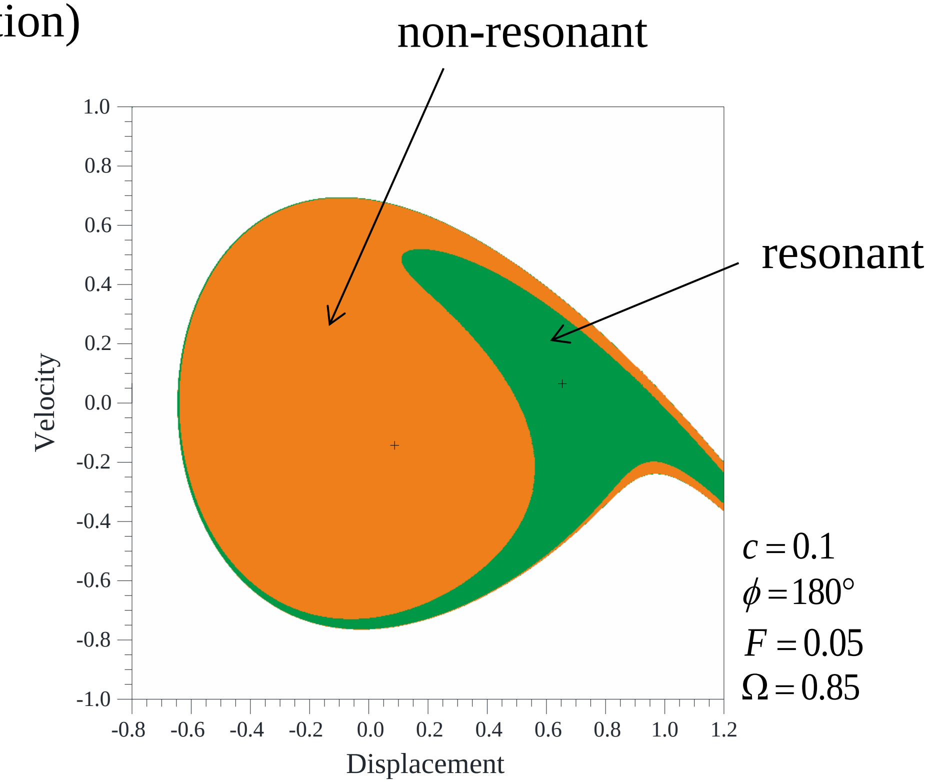
GIM (3)

We analyze the *dynamical integrity* of

- **the entire potential well**
  (safe basin = union of basins of attraction of both resonant and non-resonant attractor)
- **each single attractor**
  (safe basin = each single basin of attraction)



reference case



case-study

GIM (4)

Safe basin	G_{TOT}	G_{REF} (*)	$GIM = G_{TOT}/G_{REF}$
Potential well	232538	233872	99.4 %
Non-resonant attractor	174495	233872	74.6 %
Resonant attractor	58043	233872	24.8 %

GIM is a dimensionless
scalar measure
(usually written in percentage)

Operatively:

- **consider the nominal safe basin**
- **count the total number of dots constituting the safe area**
- **normalize with respect to the total number of dots constituting the safe area in a reference case**

(*) ***Normalization*** is usually performed with respect to the safe basin evaluated at parameter values that are somehow ***meaningful*** for the system.

GIM (5)

Safe basin	GIM (%)
Potential well	99.4
Non-resonant attractor	74.6
Resonant attractor	24.8

GIM – potential well: *elevated*

- **potential well is partially eroded with respect to reference condition, but its *robustness* remains considerable;**
- **despite erosion, this parameter range still offers the possibility to operate the system in *safe conditions***

GIM – attractors: *elevated*

- **Although the escape solution has already eroded the well, the elevated GIM suggests that we may be confident in *catching* both attractors in *real* applications**

GIM is able to provide valuable *quantitative* information about how much DI is still available, compared to the scenario used as reference

GIM (6)

<u>Principal characteristics:</u>

GIM refers only to the *safe basin* and is, conveniently, *independent* of the individual attractors existing inside it

- this is one major differences with respect to LIM (see next), where the reference to the attractors is explicit and directly enters the definition

Once selected the *safe basin* to be analyzed,
GIM measures only the *size* (*magnitude*) of the safe basin,

i.e. it is actually measuring the *probability* to catch the safe basin.

- GIM does not provide information about shape and nature of the basins (e.g., about their fractality)

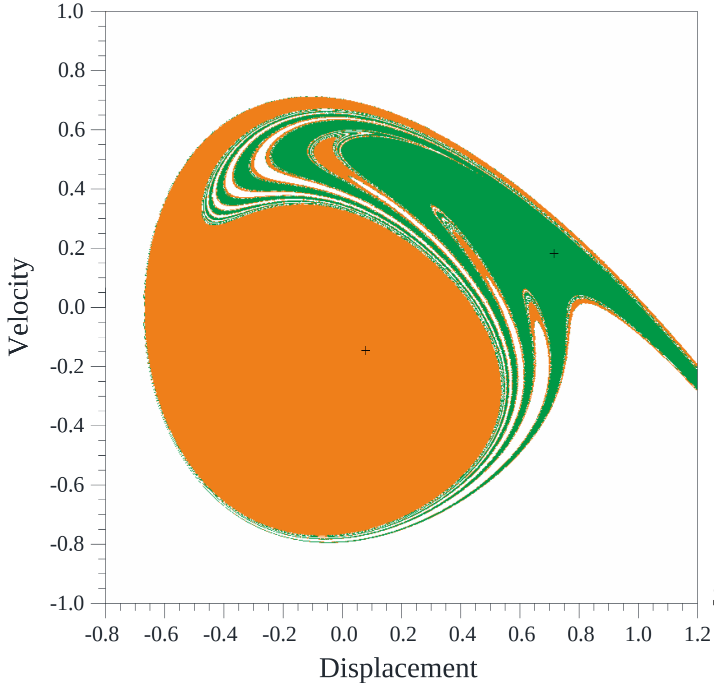
GIM is computationally *easy* and not time consuming

- This is essential for systematic investigations

GIM (7)

<u>Less attractive characteristics:</u>

GIM does not differentiate between *fractal* and *compact* areas but includes in the computation *all* parts constituting the safe basin (both fractal and compact)



$c = 0.1$
$\phi = 180°$
$F = 0.068$
$\Omega = 0.80$

Compact areas represent a safe scenario, since small uncertainties do not affect the final result

Fractal areas are not safe, since small uncertainties may lead to a completely different response, which may be potentially ***dangerous***

GIM (8)

Computing both fractal and compact areas,
GIM may meaningfully *overestimate* the practical stability

Fractality strongly reduces the set of initial conditions whose finite changes are allowed without ending up to another attractor.

Usually ***only*** the compact part (and ***not*** the fractal one) is the area of interest from the ***engineering design*** point of view

GIM does not account for the eccentricity of the attractors

Attractors may be located very close to the safe basin boundary.

Small disturbances can actually shift the response outside the safe basin. GIM is not able to describe this critical scenario.

GIM (9)

Therefore *(Soliman and Thompson, 1989)*:

- **GIM is a useful tool when investigating basin boundaries *metamorphoses*;**
- **however, the *size* of a basin is *just one* of the factors to be considered when analyzing the finite stability of an attractor;**
- ***other aspects* could be of interest for engineering design, e.g. whether the basin is smooth or fractal and the position of the attractor within the basin.**

Hence, the information provided by the GIM is generally incomplete.

This is because the robustness of the system response usually needs to be addressed from a *variety of different perspectives*.

Need of introducing, *in addition* to the GIM,
other different dynamical integrity measures

LIM (1)

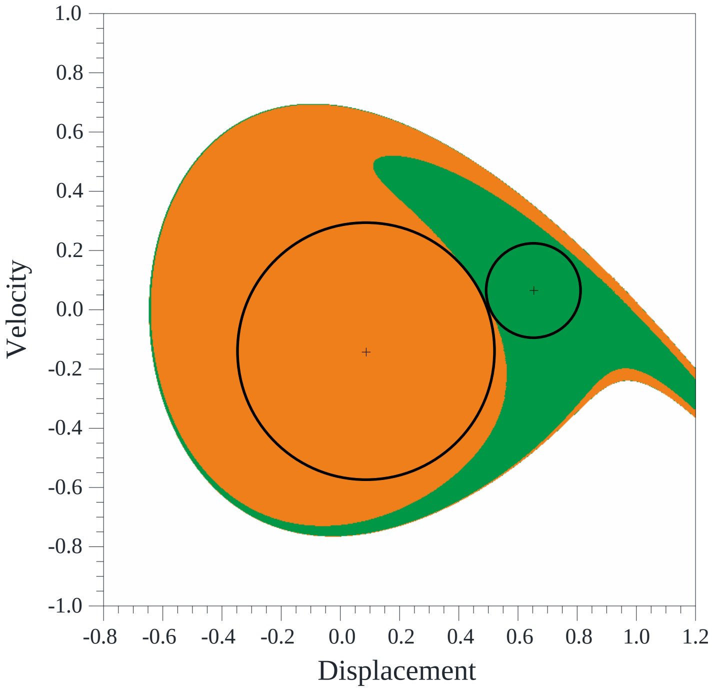
LOCAL INTEGRITY MEASURE (LIM)
is the normalized maximum radius of the hyper-sphere (circle in 2D cases) entirely belonging to the safe basin and centered at the attractor

- **LIM is the *second* dynamical integrity measure introduced by Soliman and Thompson (1989)**
- **The main target of LIM is to study the dynamical integrity aspects where GIM is not satisfactory:**
  - **whether the basin is *smooth* or *fractal***
  - ***position* of the attractor within the basin**

LIM (2)

Operatively:

- for each attractor, consider the nominal safe basin
- draw the largest circle centered at the attractor and entirely belonging to the safe basin and evaluate its radius
- normalize with respect to a reference condition



$c = 0.1$
$\phi = 180°$
$F = 0.05$
$\Omega = 0.85$

LIM analyzes *each single attractor*, *one by one*, *separately*

LIM measures *how close* the attractor is to the *basin boundary*

Safe basin	L_{TOT}	L_{REF}	LIM = L_{TOT}/L_{REF}
Non-resonant attractor	156.1	211.0	74.0 %
Resonant attractor	58.6	211.0	27.8 %

LIM (3)

Principal characteristics:

LIM measures the *compact* – or "*safe*" – part of the safe basin, ruling out all the fractal regions

- since the circle used for the evaluation of the LIM has to be ***entirely belonging*** to the safe basin, only the compact parts may enter its calculation, and not the fractal ones
- LIM is a ***more conservative*** measure, thus being ***safer*** from an engineering point of view; this feature is even more apparent when the safe basin is disconnected, as it may occur for attractors with periodicity greater than one
- this is the ***major*** difference (and usually also the major advantage) with respect to GIM

LIM is a *property* of the attractor

LIM informs on the *eccentricity* of the attractor

LIM (4)

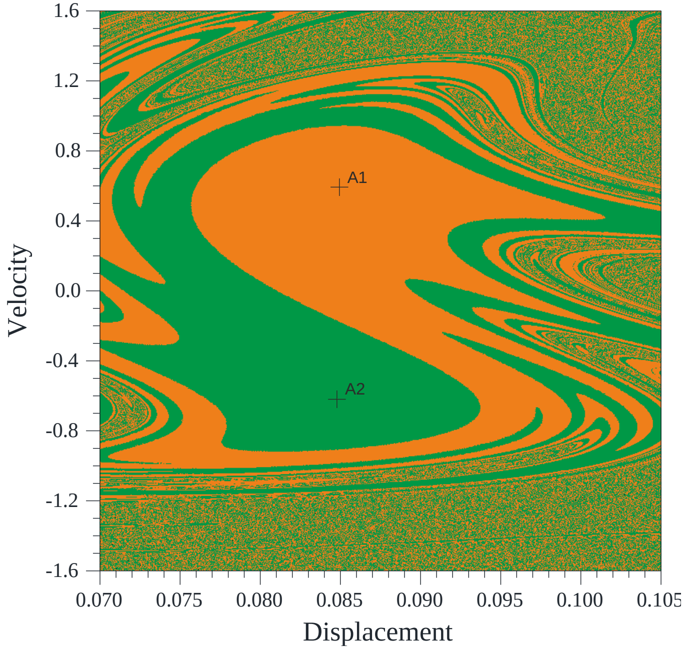
Single-mode model of a cable-supported beam

(Lenci and Ruzziconi, 2009)

$$\ddot{x} + 0.1\dot{x} + 26.75x + 130.43x^2 + 6688.96x^3 = F\cos(\Omega t)$$

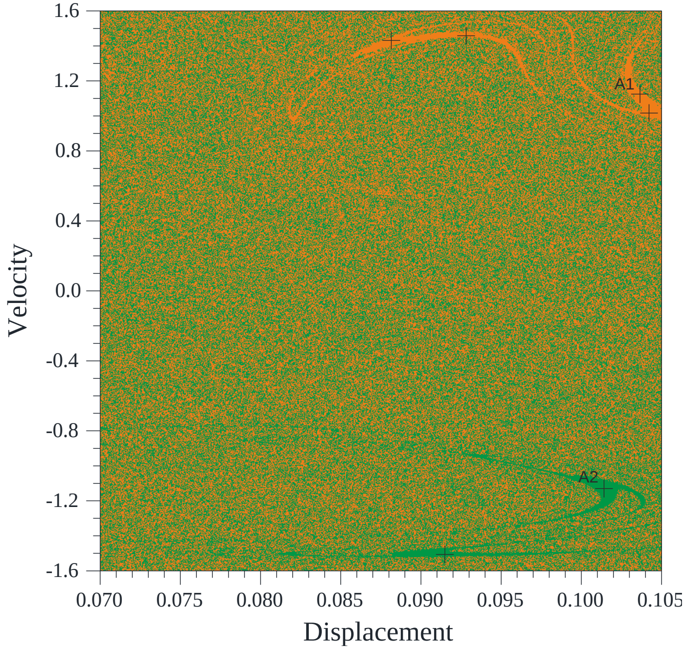
There are only two periodic attractors,
but, increasing the excitation amplitude, the entire phase space becomes completely fractal

F = 18.35, Ω = 4



reference case

F = 21.0, Ω = 4



case-study

LIM (5)

Safe basin	G_{TOT}	G_{REF}	**GIM (%)**	L_{TOT}	L_{REF}	**LIM (%)**
Attractor 1	276718	262467	**105 %**	6.1	79.0	**7.7 %**
Attractor 2	241676	255927	**94 %**	9.9	64.9	**15.2 %**

The larger is the magnitude of a basin
the larger is the *probability* of approaching the final response;

yet, due to the extended fractality, almost all the phase space is *unpredictable*,
i.e. we cannot know *exactly* in advance the final outcome

In this parameter range the system, despite being *deterministic*,
acts like a *probabilistic* one due to the difficulty in predicting the
type of motion it may follow

LIM (6)

Safe basin	GIM (%)	LIM (%)
Attractor 1	**105 %**	**7.7 %**
Attractor 2	**94 %**	**15.2 %**

GIM: elevated; LIM: small

we have *reliable* information on the *probability* of approaching a given motion

However, since unpredictability is unwanted in practice,
DI analysis recommends considering this parameter range as *dangerous* from a practical point of view,
although there are only two apparently innocuous periodic attractors

GIM analysis is important, but usually not exhaustive (in this case, totally *misleading*!)

GIM and LIM are two *complementary* measures for the dynamical integrity analysis of a nonlinear system

LIM (7)

<u>Less attractive characteristics:</u>

LIM is really effective to study each single attractor, but becomes *unclear* - or cumbersome - when we need to study the *potential well* (LIM is a property of the *attractor* and not of the potential well)

LIM is not always computationally easy,
and may be *numerically onerous*,
especially when the in-well attractor is chaotic.

LIM has not a clear *theoretical background*
permitting an in-depth investigation, although is certainly somehow linked to classical dynamical phenomena

IF (1)

INTEGRITY FACTOR (IF)
is the normalized radius of the largest hyper-sphere (circle in 2D cases) entirely belonging to the safe basin

- **The aim of IF is to overcome the *drawbacks* of GIM and LIM, without losing their advantages. In particular:**
  - **accounting for the only *compact* part of the safe basin, ruling out the fractal one (as LIM)**
  - **being able to investigate not only the attractors, but also and *remarkably* the *potential well* (as GIM)**
- **IF is introduced by** *Lenci and Rega (2003)*

IF (2)

Düffing oscillator – Twin potential well, asymmetric

(Lenci and Rega, 2004)

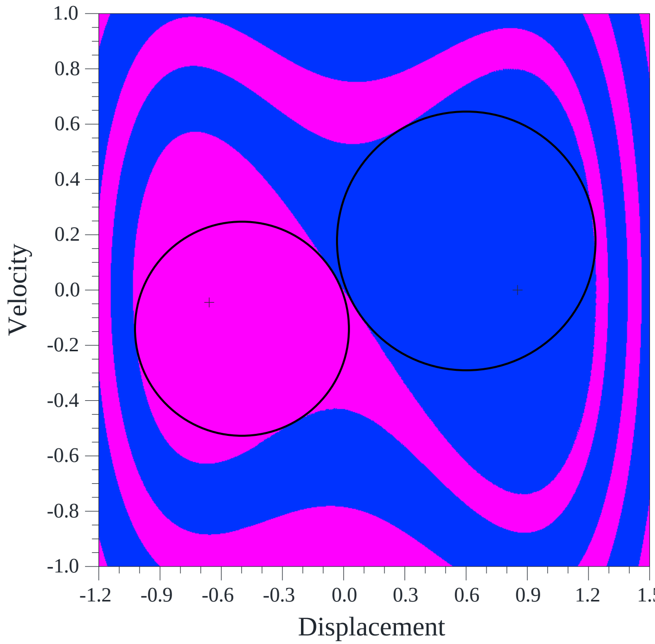
$$\ddot{x} + \varepsilon\delta\dot{x} - \sigma x - \tfrac{3}{2}(\sigma - 1)x^2 + 2x^3 = \varepsilon\gamma_1 \sum_{j=1}^{\infty} \frac{\gamma_j}{\gamma_1} \sin(j\omega t + \Psi_j)$$

$\varepsilon\delta = 0.1$

$\Psi_j = 0.0$

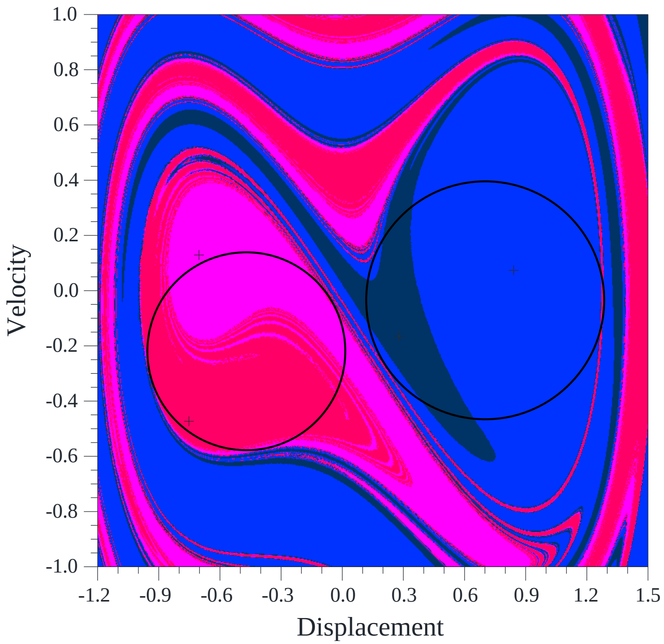
$\sigma = 1.2$

The aim is to analyze the *robustness* of *each* potential well



$\varepsilon\gamma_1 = 0.0$

reference case



$\varepsilon\gamma_1 = 0.075$

$\omega = 1.2$

case-study

IF (3)

We consider as *safe basin* the union of basins of all attractors belonging to the same potential well

Accordingly, in this case-study there are two safe basins:

1) **one consists of all basins of attraction leading to the *right* well**
2) **the other one, of all basins of attraction leading to the *left* well**

For each safe basin, we draw the *largest circle* entirely belonging to it and compute its radius

- IF does not distinguish between different in-well attractors (thus being able to analyze the ***potential well***)
- Only the ***compact*** part of the safe basin is taken into account

We finally *normalize* with respect to a reference value

IF (4)

Safe basin	I_{TOT}	I_{REF}	IF = I_{TOT}/I_{REF}
Potential well - left	128.8	139.1	92.6 %
Potential well - right	154.6	167.9	92.1 %

The larger are the circles (IF),
the larger is the *cross-well integrity* of the basin,
the smaller is the *cross-well fractality* and the related sensitivity to initial conditions

The definition of IF is exactly *equal* to the definition of LIM
(desirable to **remove fractal tongues**),
except that does not specify the hyper-sphere to be "*centered at the attractor*"
(desirable to **investigate the potential well**)

IF (5)

Principal characteristics:

IF investigates only the compact '*core*' of the safe basin, ruling out the fractal tongues from the DI evaluation

- IF is a rather ***conservative*** measure,
  which is appropriate for engineering purposes

IF is a property of the *safe basin* (and not of the attractors)

- IF investigates the ***safe basin***, and is independent of the attractors inside it
- able to investigate the ***potential well***

IF is really *easy* in the evaluation

Less attractive characteristics:

IF has not a clear *theoretical background*

IF does not inform about the *eccentricity* of the attractors which may be relevant in many practical case-studies.

COMPARISON

By definition:

$$\boxed{\text{LIM} < \text{IF} < \text{GIM}}$$

- **in absence of *fractal* basins boundaries GIM, LIM and IF are somehow *equivalent***
- **only in presence of fractality the use of LIM or IF may become necessary**

LIM and IF differ significantly from each other when the attractor is not "*centered*" in its basin.

ACTUAL SAFE BASIN (1)

MOTIVATION

We keep addressing the issue of analyzing the sole *compact* part of the basin, which also corresponds to ruling *fractality* out of calculation

The methods analyzed so far are based on the definition of DI *measures* which are able to eliminate the fractal parts

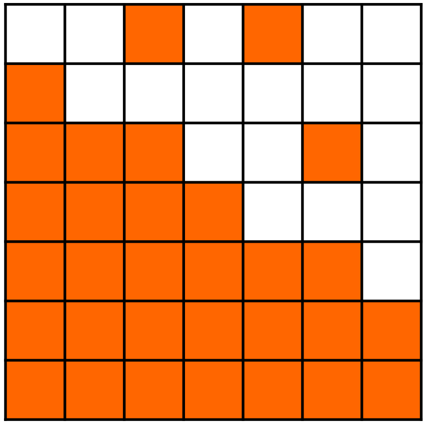
Now, we pursue the same target via a *conceptually* different approach. The *idea* is:

- **keeping using the same DI *measures* already existing in the literature, *but***
- **modifying properly the *definition* of *safe* basin, so that fractal parts are eliminated (or strongly reduced)**

Within this framework, the *Actual Safe Basin* is introduced
(Lenci and Rega, 2013)

ACTUAL SAFE BASIN (2)

The actual safe basin moves from the observation that, in practice, safe basins are always obtained in a *discrete* way, i.e. by approximating the *continuous phase space* with a *finite number of cells*, which are pixels in any graphical representation



Each cell contains a number
(colour in the picture)
making reference to the considered dynamical property
(kind of attractor, for basins of attraction)

The "*actual*" safe basin is defined as that obtained by eliminating from the *nominal* safe basin all cells which are not surrounded by cells of initial conditions having the same number

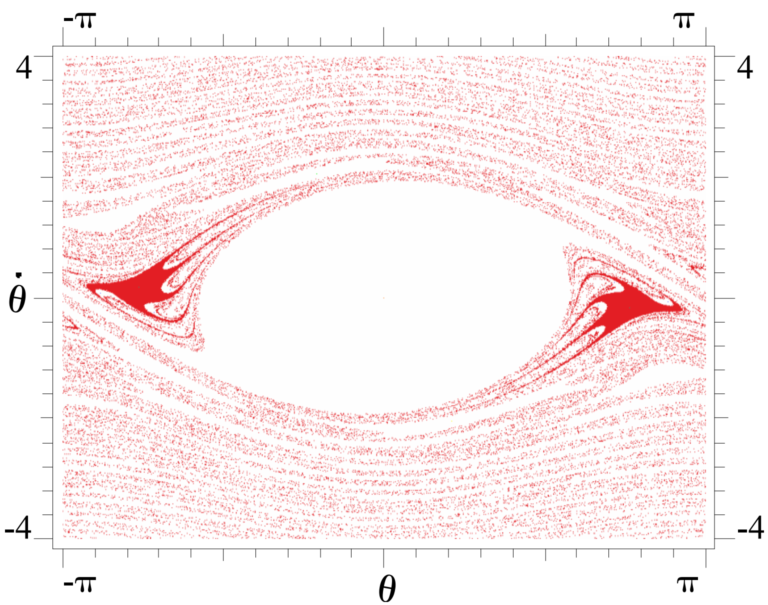
ACTUAL SAFE BASIN (3)

Rotating pendulum *(Lenci et al., 2013)*

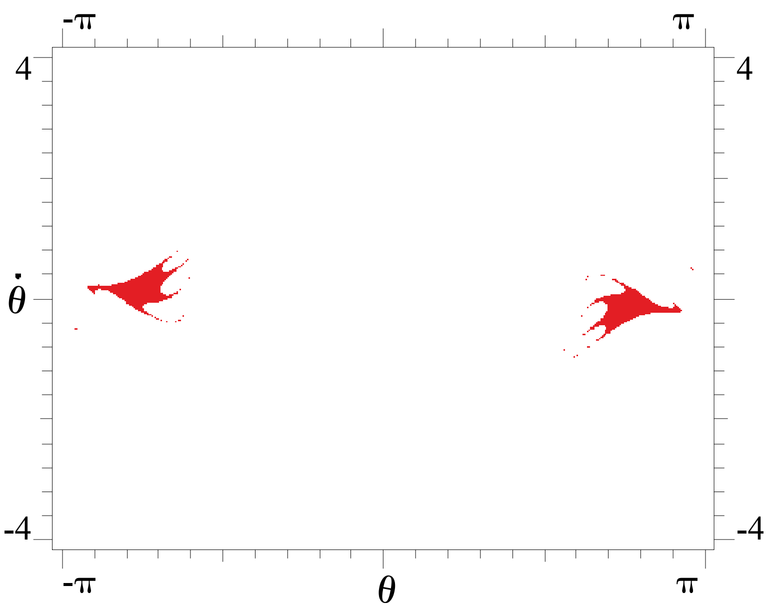
$$\ddot{\theta} + h\dot{\theta} + [1 + p\cos(\omega t)]\sin\theta = 0$$

$$h = 0.015 \quad \omega = 1.3 \quad p = 0.12$$

we consider the *nominal* safe basin constituted by the basin of period-2 oscillation, and construct the associated *actual* safe basin



nominal* *safe basin



actual* *safe basin

cells *not* surrounded by other cells of the same safe basin are eliminated

ACTUAL SAFE BASIN (4)

Principal characteristics:

fractal parts **no longer present (strongly reduced)**
and the actual safe basin has a *substantially compact shape*
(however, it may be disconnected)

conceptually *simple* and *easy* to implement

Less attractive characteristics:

the procedure also eliminates a *stripe*, of one cell width,
along the entire boundary of the compact part of the nominal safe basin

- if the accuracy of the basin discretization (i.e. the number of cells) is large enough, this does ***not*** entail meaningful underestimations

ACTUAL SAFE BASIN (5)

Having preliminarily eliminated (reduced) the fractal parts, we no longer need to pay attention to the definition of the DI *measure*

Using the GIM (which is the easiest one), and evaluating it for the Actual Safe Basin, we get the

Actual Global Integrity Measure (AGIM).

By definition: **AGIM < GIM**

Rotating pendulum

Safe basin	Pixel
Nominal safe basin	43519
Actual safe basin	5786

AGIM is 13.3 % of GIM

due to the extended fractality

basin safety is 752 % more reliable

Anyway, AGIM is much less conservative than LIM or IF whenever the though compact part of the basin of attraction is disconnected

“TRUE” SAFE BASIN (1)

MOTIVATION

In previous analyses we did *not* care about the *phase* of the excitation, which may be *free* and/or *unknown* in engineering applications

**To address this issue,
instead of evaluating the DI at each phase and/or introducing a new ad hoc measure, an *approach* similar to the “actual” safe basin has been proposed.**

The *same* DI measures already existing in the literature are used, *but* the considered definition of safe basin is *properly* modified, by introducing the *“True” Safe Basin* *(Lenci and Rega, 2004)*

The *“True”* Safe Basin is the *intersection* of all safe basins when the excitation *phase* ranges over the time interval of interest

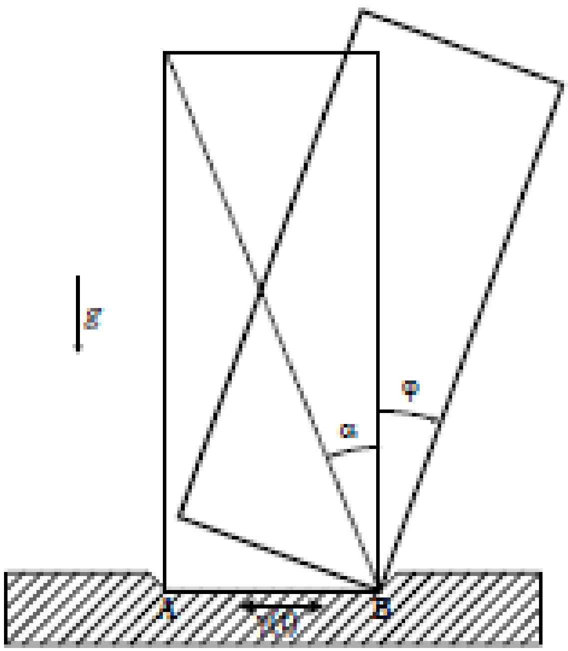
The *interval of interest* may be:

- **the period, for periodic excitations**
- **the period of free vibration, for impulsive excitation**
- **etc.**

"TRUE" SAFE BASIN (2)

Overturning of the rest position of rigid blocks

(Lenci and Rega, 2004)



$$\ddot{\phi}+\delta\dot{\phi}-\phi+\alpha+\gamma(t)=0 \qquad \phi>0$$

$$\ddot{\phi}+\delta\dot{\phi}-\phi-\alpha+\gamma(t)=0 \qquad \phi<0$$

$$\dot{\phi}(t^{+})=r\dot{\phi}(t^{-}) \qquad \phi=0$$

where:

$$\gamma(t)=\sum_{j=1}^{N}\gamma_j\cos(j\omega t+\psi_j)$$

generic 2π/ω periodic external excitation representing the horizontal dimensionless acceleration of the foundation

δ **damping coefficient**

α **block shape parameter**

r **restitution coefficient**

$\gamma_1=0.20$ $\quad r=0.95$

$\omega=3.5$ $\quad \alpha=0.2$

$\delta=0.02$

“TRUE” SAFE BASIN (3)

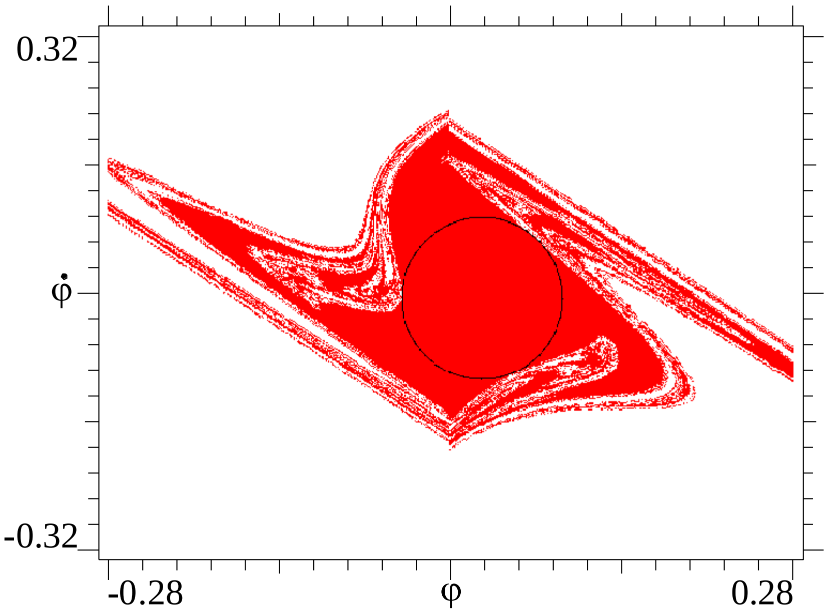
- **In the problem of *overturning* of the rest position, the initial condition is fixed and the excitation phase is *unknown***
- **It can, and *does*, occur that for a given phase ψ the block does not overturn, while it topples down for a different ψ**

➡ important role played by the *excitation phase* ψ

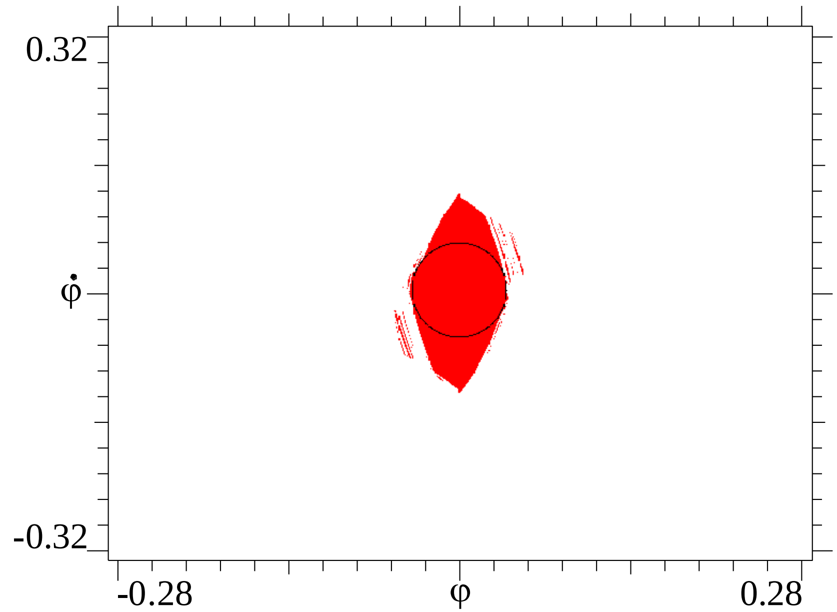
- **Safe basins of attraction in the classical sense (the union of basins of all in-well attractors), related to a *fixed* ψ, do *not* provide adequate information.**
- **Hence, we have to look for *phase-independent* arguments:**

➡ defining the *“true” safe basin*

"TRUE" SAFE BASIN (4)



Classical safe basin



"True" safe basin

- **Very different *magnitude*:** classical safe basin largely overestimates the integrity
- **More *regular*:** fractal parts are strongly reduced
- ***Compact* and "*closer*" to the circle involved in the definition of IF**

“TRUE” SAFE BASIN (5)

Construction

- **boundaries of classical safe basins are the *stable sets* of the hilltop saddles of the 2D stroboscopic Poincaré map associated with the 3D flow**

- **varying the excitation phase is equivalent to *change* the *time position* of the Poincaré section**

- **looking for the union of all these sections corresponds to *project* the 2D stable manifolds, in the 3D phase space $(\varphi, \dot{\varphi}, t)$, onto the plane $(\varphi, \dot{\varphi})$**

- **the “*true*” safe basin will then be the area out of this projection surrounding the rest position (0,0).**

“TRUE” SAFE BASIN (6)

Principal characteristics:

Is the smallest *phase-independent* set of initial conditions leading to “*safe*” dynamics

Less attractive characteristics:

Its construction is computationally *expensive*

- time consuming
- needs of skill numerical algorithm

Once the *“true” safe basin* has been defined,
GIM, LIM, IF or other *measures* can be applied to study
its dynamical integrity

OTHER MEASURES (1)

Other *distance-based measures* of DI have been proposed in the literature:

- **Impulsive Integrity Measure (IIM)** – *(Soliman and Thompson, 1989)*
- **Stochastic Integrity Measure (SIM)** – *(Soliman and Thompson, 1989)*
- **Protection Thickness** – *(Sun, 1994)*
- **Maximum Speed of Erosion (σ)** – *(de Souza Jr and Rodrigues, 2002)*
- **Ratio of Safe Initial Points (RSIP)** – *(Gan and He, 2007)*

OTHER MEASURES (2)

IMPULSIVE INTEGRITY MEASURE (IIM)

is the normalized attractor-basin boundary minimum distance in the direction of the *velocity* coordinate

- meaningful measure in ***impulsive*** *problems*
- the aim is to analyze the sensitivity of an attractor subjected to ***impact loading***, to inform about the size of impulse that could be sustained without failure
- moves from the observation that, when subjected to an impulse, the system could be thought to experience an ***instantaneous step change in velocity***

OTHER MEASURES (3)

STOCHASTIC INTEGRITY MEASURE (SIM)

mean escape time when the attractor is subjected to white noise of prescribed intensity

- meaningful measure in the ***stochastic*** *framework*
- quantifies the effects of a ***noise excitation*** superimposed to a basic deterministic (harmonic) excitation, by correlating them with ***geometric changes*** experienced by the deterministic basin of attraction

OTHER MEASURES (4)

MAXIMUM SPEED OF EROSION (σ)

is an indicator of loss of *global safety* and quantifies the *swiftness* with which safe basins are lost as the driving parameter (excitation amplitude, in the original paper) **is increased.**

- is defined as:

$$\sigma = \max\left\{ \frac{abs(S_{i+1} - S_i)}{F_{i+1} - F_i}, i = 1,2,\ldots, N-1 \right\}$$

where:

S_i	=	ratio of safe to total starting conditions within the window used
F_i	=	driving parameter
$S_{i+1} - S_i$		is usually negative, thus the absolute value is taken

OTHER MEASURES (5)

PROTECTION THICKNESS of a period-K attractor with K disjoint subsets of cells in the state space

is the minimum of the *K* distances between each subset and the set of multi-domicile cells

- formulated in the generalized cell mapping terminology, with a view to the effect of random disturbances

- is nearly identical to the LIM

RATIO OF SAFE INITIAL POINTS (RSIP)

estimates the probability that the system works satisfactorily in a given limited domain within a *specified time interval*

- meaningful measure in the ***stochastic*** *framework*

COMMENTS

Other **dynamical integrity measures relying on alternative** ***geometric*** **(e.g., distances of different hyper-volumes) or** ***mechanical*** **(e.g., energy-based) criteria could be introduced**

All the reported DI measures are ***conceptually*** **different from each other, since each one has been proposed in the literature to address** ***different*** **issues.**

It stands to the ***designer*** **selecting** ***which one should be used*****, according to the information needed, i.e. to the** ***problem to be addressed*****.**

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SELECTING MEASURES

Using one measure or the other to describe system global dynamics depends on the safe condition one wants to achieve or analyze

- **GIM accounts for also the transient, i.e. for thin fractal and/or smooth tongues of the basins of two *competing attractors*, and is thus able to provide information on the probability to catch either one of them.**
- **But an elevated probability does not entail *actual robustness* in system operation. In this respect, LIM – which is much lower than GIM – provides more reliable information about possible practical disappearance of the attractor. Also, LIM provides a good estimate of the *attractor robustness* if it is *eccentric* with respect to its safe basin.**
- **Nevertheless, LIM becomes unclear (or cumbersome) when analyzing the robustness of the whole potential well. In this case, IF should preferably be used, since it does not refer to an attractor.**

CASE STUDY (1)

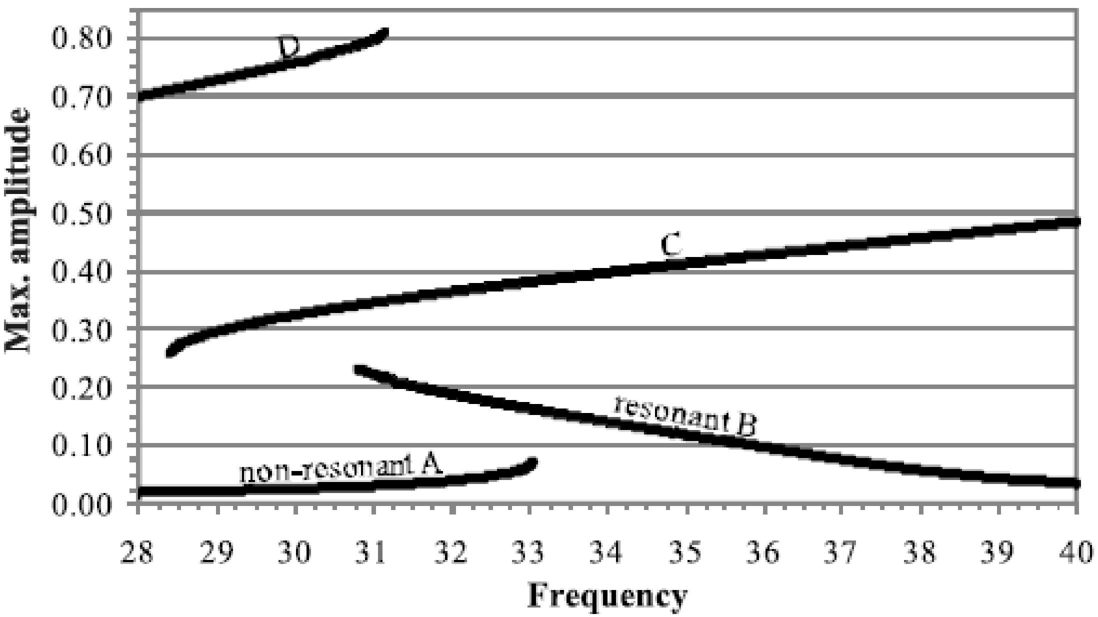
Slacked carbon nanotube (CNT)

(Ruzziconi, Younis, Lenci, 2013)

$$\ddot{Y} + 0.01\dot{Y} + 1390.2 - 11570.8\,Y^2 + 36392\,Y^3 - \frac{0.2658}{(0.5687 - Y)^2}(V_{DC} + V_{AC}\cos(\Omega t))^2 = 0$$

In this case-study, the scenario is rather rich, due to the coexistence of *several* competing attractors with different characteristics:

$V_{AC} = 3\ V$



resonant and non-resonant branches with bending toward lower frequencies

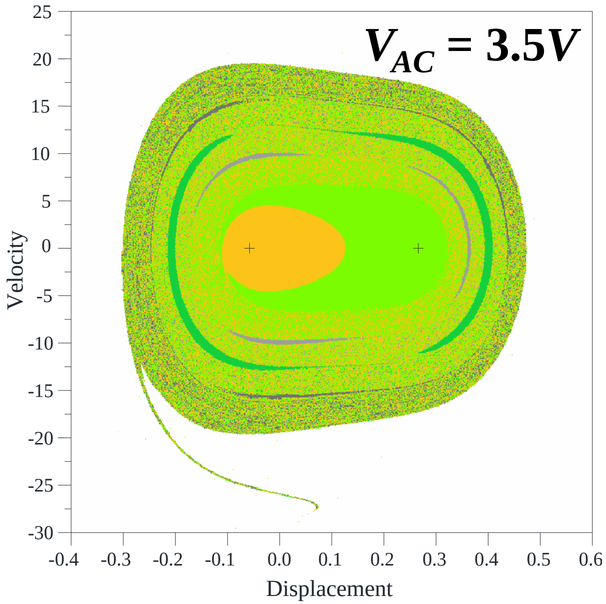
attractor C
oscillation enlarging at increasing frequencies

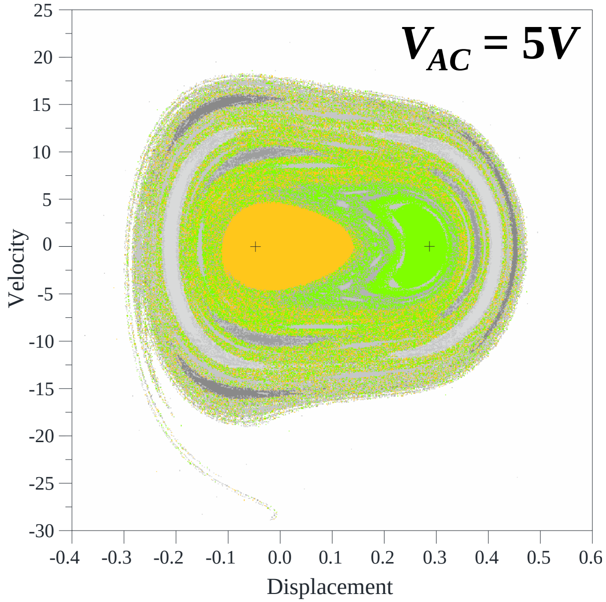
attractor D
large oscillations

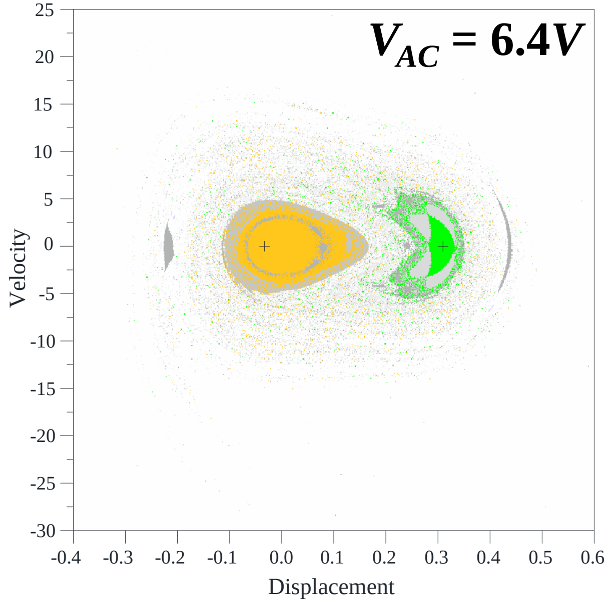
CASE STUDY (2)

non-resonant and large oscillations disappeared
resonant and C survive

$\Omega = 33$ **increasing V_{AC}**







- **principal attractors with large basin**
- **minor attractors with small basin**
- **large fractality (but without escape)**

- **minor attractors enlarge their basins**
- **they reduce the basin of principal ones**
- **escape still outside the fractal region**

- **escape enters the well (dangerous) and erodes large parts of the basins**

CASE STUDY (3)

The system has a rich nonlinear behavior;
thus, to have a complete and detailed description of the device response,
we need to combine information coming from
different* DI *perspectives

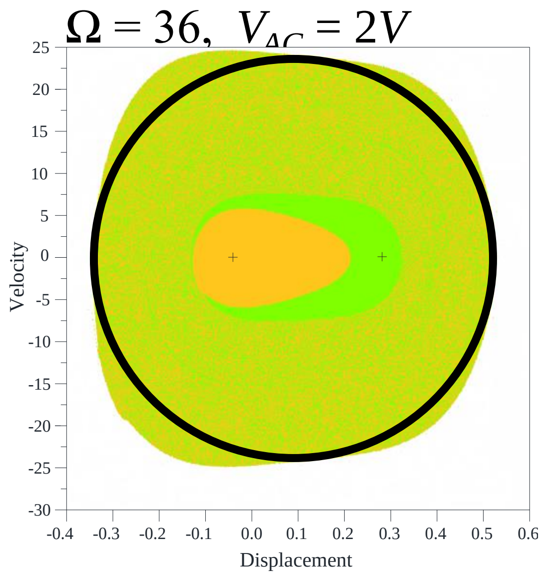
In the following we analyze:

- **Robustness of the whole potential well**
- **For each single attractor**
  - **probability**
  - **disappearance**

by looking at

- **Integrity profiles: how a DI measure varies with a varying control parameter**
  (usually, frequency *or* amplitude of excitation)
- **Integrity charts: providing contour lines of DI in a control parameter space**
  (usually frequency *and* amplitude of excitation)

POTENTIAL WELL (1)

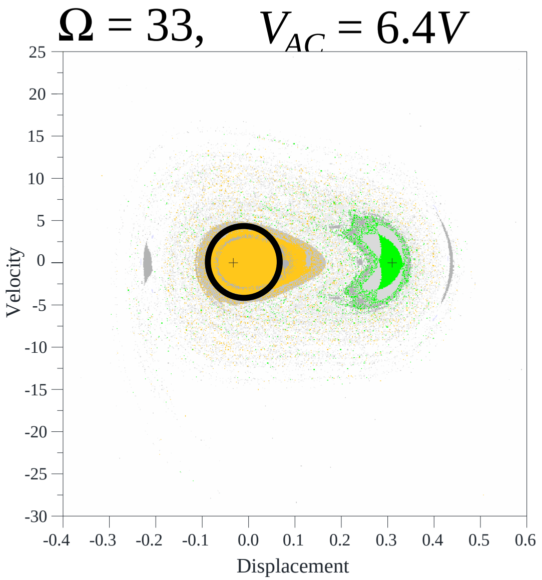


Robustness of the whole POTENTIAL WELL:

detect the parameter range where the well is wide enough to tolerate disturbances

***Safe basin*: union of all basins of attraction**

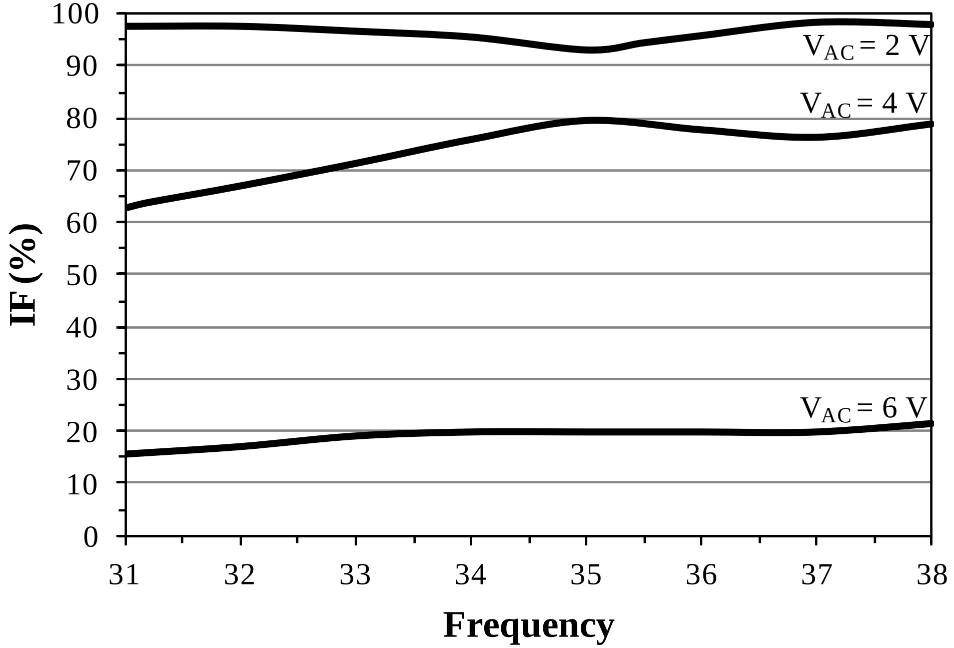
Integrity measure*: *IF



- **IF is very *suitable* to investigate the dynamical integrity of the potential well.**
- **In fact, IF is *easy* and *simple* to be computed, and immediately *suggests* if the well is robust or not, i.e. if it is worth to further investigate a certain parameter range**

POTENTIAL WELL (2)

IF integrity *profile*



IF dynamical integrity chart,
to summarize the ***overall*** scenario
when both frequency and voltage are
varying

IF is *large*:

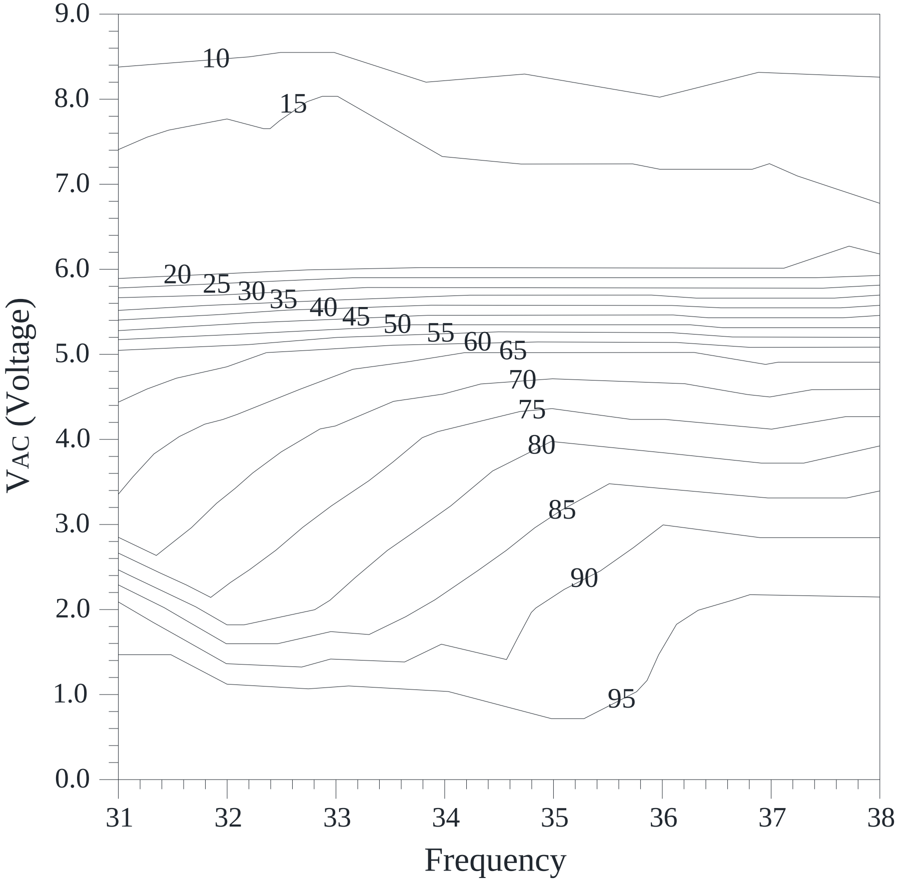
- **the well is robust**
- **guarantees the possibility of *safe* scenario, far from the danger of dynamic pull-in**

IF completely *drops*:

- ***danger* of dynamic pull-in arises much *before* the theoretical inevitable escape**
- **this range is *not worth* to further refine simulations, since each single attractor is equally or *even more* vulnerable than the *well***

POTENTIAL WELL (3)

IF integrity *chart*



***residual* integrity:**
CNT totally *vulnerable* to pull-in

sudden *fall*:

- **escape enters and *destroys* the well *suddenly*, within a small interval**
- **we are required to assume in the *design* adequate *coefficients* of safety, to keep any application far from this unsafe threshold**

***wide* integrity:**
safe conditions (no pull-in)

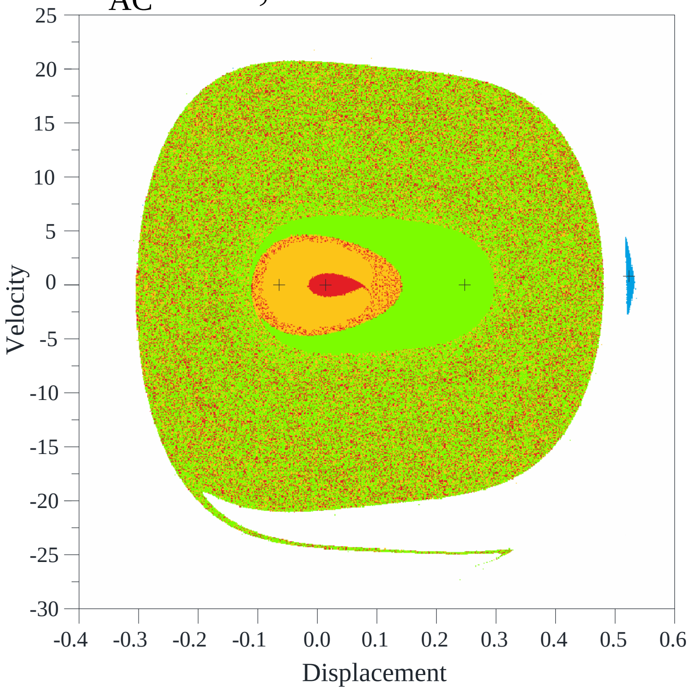
In the following we analyze *only this range*, since the *IF* dynamical integrity analysis highlights that this is the only range *worth* for further investigations

PROBABILITY (1)

**Once detected the *safe* parameter range,
we analyze the possible *behaviors* existing in the well by exploring *separately* the DI of the principal attractors**

PROBABILITY to catch each single attractor

$V_{AC} = 2V$, $\Omega = 33$

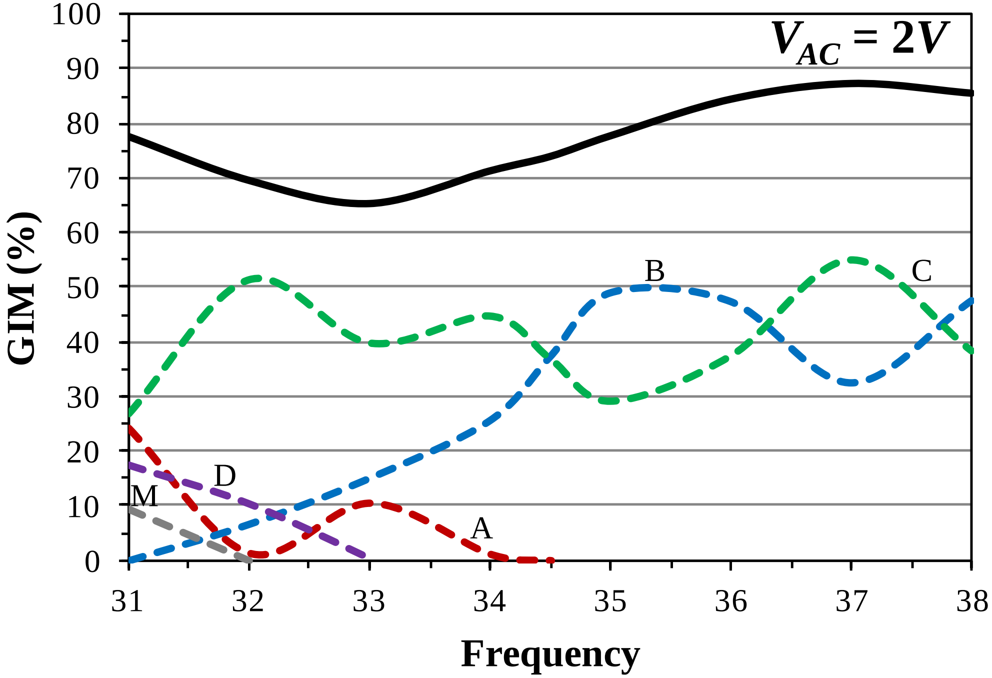


- **if the magnitude of its basin of attraction is very *narrow*, from a practical point of view the corresponding attractor does *not* exist;**
- **otherwise, it may effectively *operate* the device and may deserve further investigations**

***Safe basin*: each basin of attraction**
Integrity measure*: *GIM

PROBABILITY (2)

GIM **integrity profile**



total **dots of the well**

large **probability to catch the principal attractors B and C**

small **probability for minor attractors (M) and large oscillations (D)**

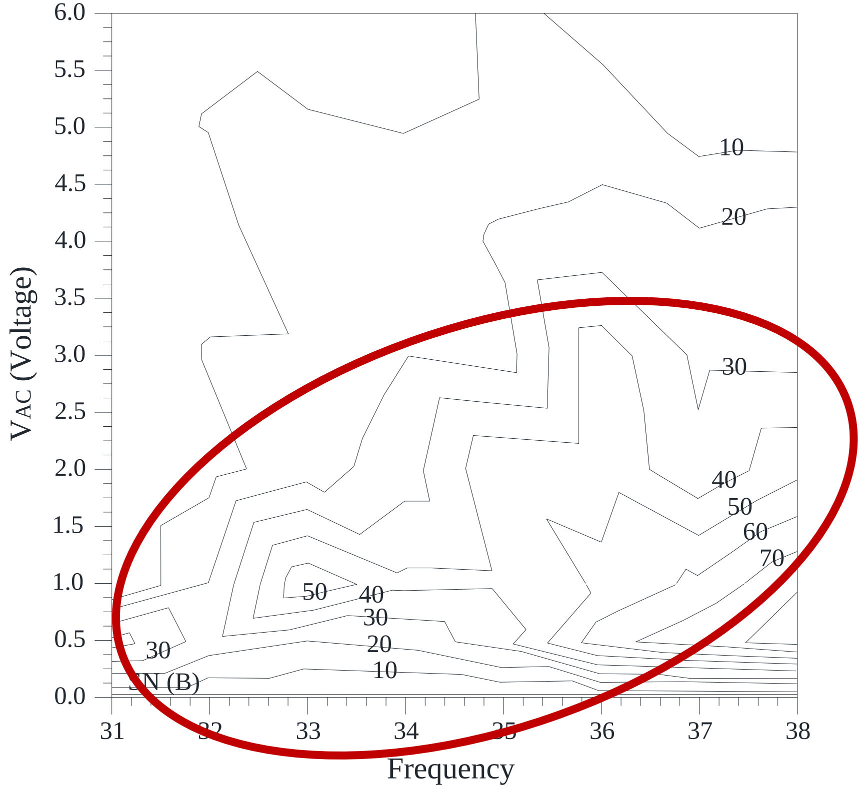
In the safe range (of the potential well) , the motion is *dominated* by the *principal* attractors (especially B and C). They are *by far* the most probable ones.

The secondary minor attractors practically do *not* exist (not feasible in practice).

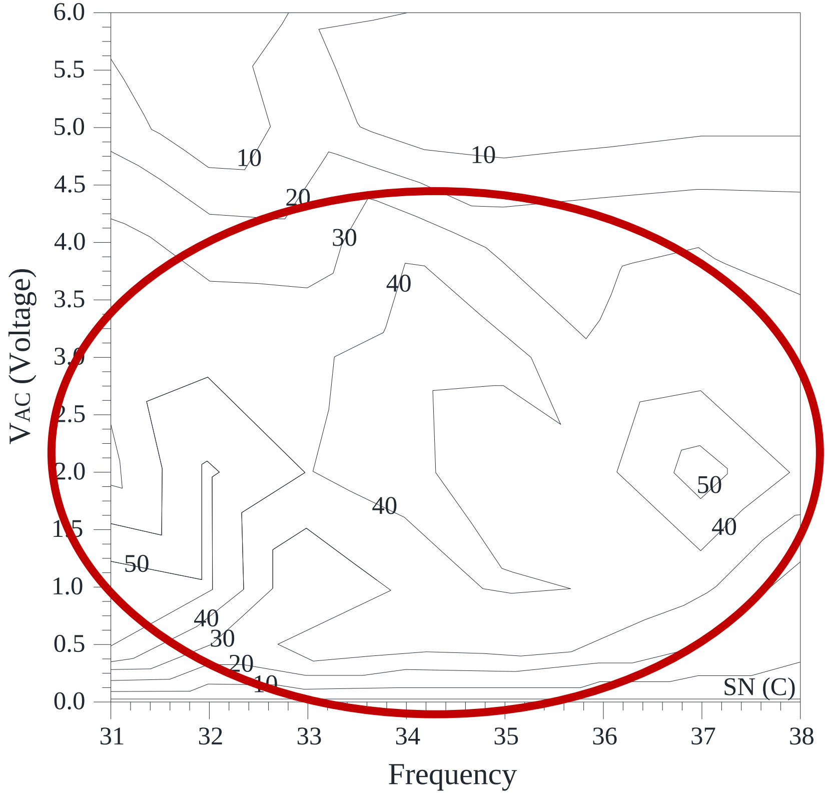
PROBABILITY (3)

GIM integrity charts
confirm a large parameter range where both attractor C and the resonant branch B can be effectively caught in practice

resonant branch



attractor C



PROBABILITY (4)

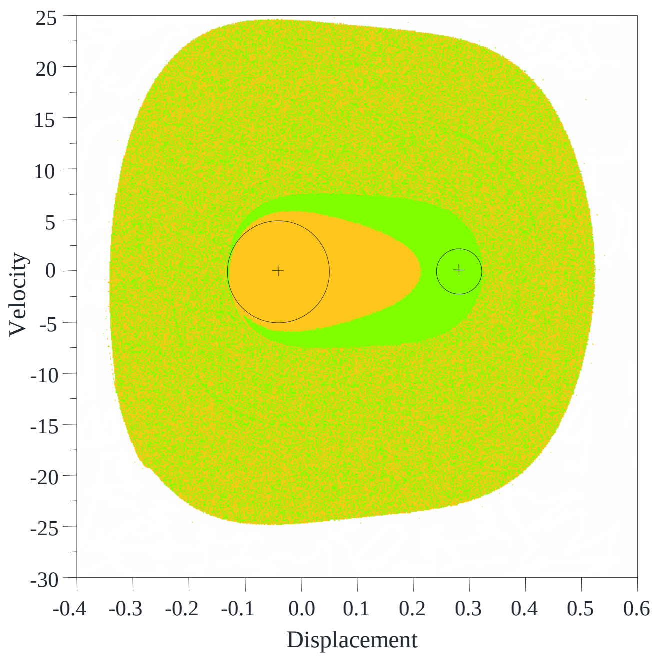
Hence, despite the extensive fractality and the multistability *GIM* analysis clearly indicates which attractors can be *effectively* observed (in this case, only B and C).
Their number is *smaller* than the number of theoretical attractors.

- **However, the high *probability* to catch these attractors does not guarantee to operate the device with them.**
- **This is because in many parameter ranges these attractors are paralleled with a *small* compact part of their basins of attraction, which may be not robust enough to tolerate the inevitable uncertainties in the initial conditions.**
- **Consequently, the attractors may practically *disappear*.**

DISAPPEARANCE (1)

To have a more detailed description of the scenario that may occur, we need to examine also the *practical disappearance* of the most probable attractors, in order to have information about the range where each one of them can be *reliably* observed under realistic conditions.

DISAPPEARANCE of each single attractor

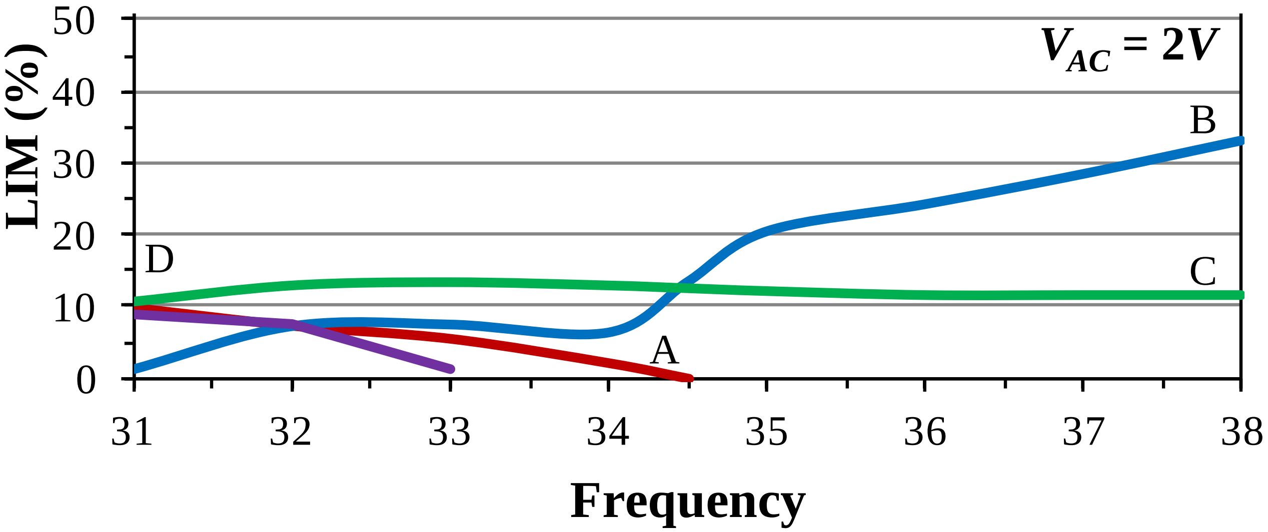


***Safe basin*: each basin of attraction**

Integrity measure*: *LIM

DISAPPEARANCE (2)

LIM **integrity profile**



LIM of B is at maximum the 35 %

Similarly, C never exceeds 15%

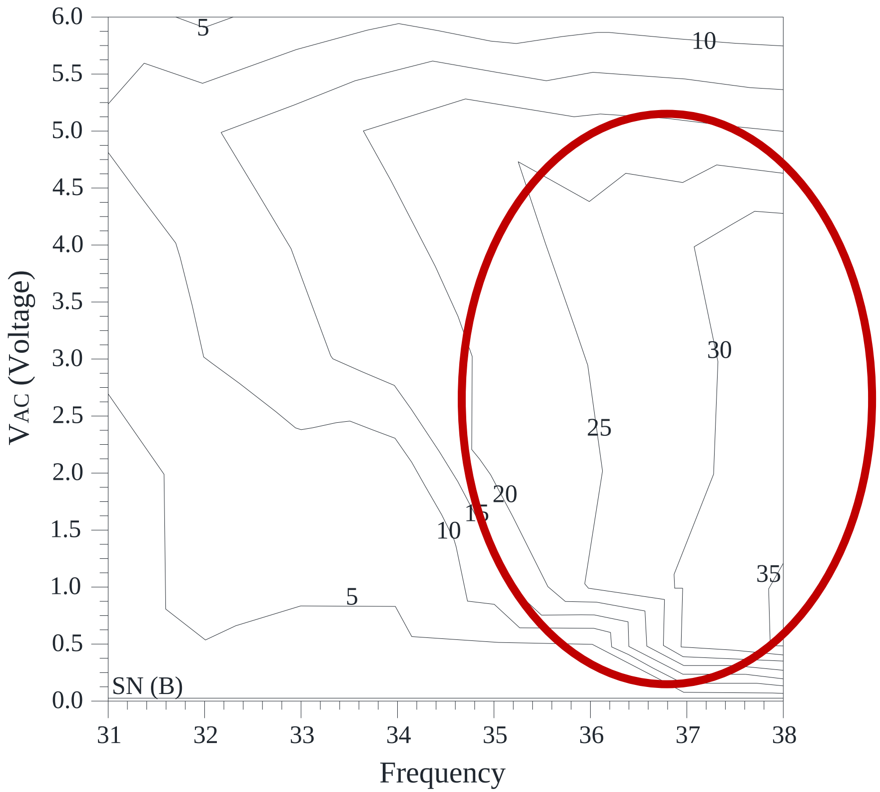
Even if there is a large probability to catch both B and C, the compact area around each attractor is not so wide, except in small parameter ranges. So, each attractor is very sensitive to disturbances:

- **a small disturbance may make the attractor *disappear***
- **the response may *jump* to another one of the most probable motions**

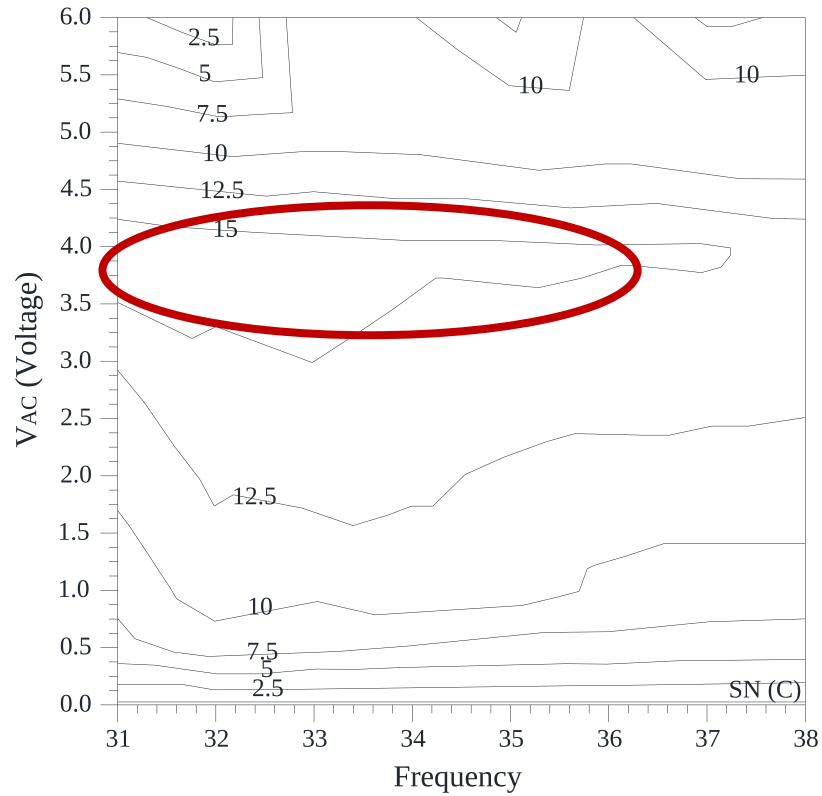
DISAPPEARANCE (3)

LIM integrity charts
confirm the sensitivity to disturbances (small LIM) along large part of the parameter range, both for the resonant branch and for C

resonant branch



attractor C



SUMMARY

Combined use of integrity charts provides a detailed description of the device safety under realistic conditions:

- **overall safe (i.e., *bounded*) behavior occurs up to V_{AC}=5 V, after which there is vulnerability to pull-in, i.e. the safe range is *smaller* (considerably smaller !) than the theoretical one**
  (analysis of the **potential well robustness** through **IF**)

- **in the safe range, *large probability* to catch B and C**
  (analysis of the **attractor probability** through **GIM**

- ***but* both B and C are very sensitive to disturbances: if large disturbances are expected, they may *practically vanish*, ending up to a coexisting bounded attractor or to pull-in depending on the amount of the disturbance**
  (analysis of **practical disappearance** through **LIM**)