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Critical behavior of the Ising model on a hierarchical lattice with aperiodic interactions

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Abstract

We write the exact renormalization-group recursion relations for nearest-neighbor ferromagnetic Ising models on Migdal–Kadanoff hierarchical lattices with a distribution of aperiodic exchange interactions according to a class of substitutional sequences. For small geometric fluctuations, the critical behavior is unchanged with respect to the uniform case. For large fluctuations, as in the case of the Rudin–Shapiro sequence, the uniform fixed point in the parameter space cannot be reached from any physical initial conditions. We derive a criterion to check the relevance of the geometric fluctuations.

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There are a number of recent investigations on the critical behavior of ferromagnetic spin systems with the inclusion of aperiodic exchange interactions [1]. In particular, there are detailed studies of the critical behavior in the ground state of the quantum Ising chain in a transverse field [2–5] (which is known to be related to the transition at finite temperatures of the two-dimensional Ising model). The nearest-neighbor ferromagnetic exchange interactions are chosen according to some substitution sequences, and the geometric fluctuations are gauged by a wandering exponent ω associated with the eigenvalues of the substitution matrix of the sequence. According to a heuristic criterion proposed by Luck [3], the critical behavior remains unchanged (that is, of Onsager type) for bounded fluctuations (small values of ω), but large fluctuations should induce much weaker singularities, similar to the case of a disordered Ising ferromagnet.

In a very recent publication [6], we took advantage of the simplifications brought about by a Migdal–Kadanoff hierarchical (MKH) lattice [7] to perform some exact calculations for characterizing the critical behavior of aperiodic ferromagnetic Ising

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models. The (layered) exchange interactions between nearest neighbors were chosen according to a certain class of generalized Fibonacci sequences [8]. In this paper, we review some of these calculations, discuss the well-known case of the Rudin–Shapiro sequence, and present an exact derivation of an analog of Luck's criterion to check whether the geometric fluctuations are strong enough to change the critical behavior of the uniform system.

Consider a particular two-letter generalized Fibonacci sequence given by the substitutions

$$A \to AB\,, \qquad B \to AA\,. \tag{1}$$

If we start with letter A, the successive application of this inflation rule produces the sequences

$$A \to AB \to ABAA \to ABAAABAB \to \cdots\,. \tag{2}$$

At each stage of this construction, the numbers N_A and N_B, of letters A and B, can be obtained from the recursion relations

$$\begin{pmatrix} N'_A \\ N'_B \end{pmatrix} = \mathbf{M} \begin{pmatrix} N_A \\ N_B \end{pmatrix}, \tag{3}$$

with the substitution matrix

$$\mathbf{M} = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}. \tag{4}$$

The eigenvalues of this matrix, $\lambda_1 = 2$ and $\lambda_2 = -1$, govern most of the geometrical properties. For any sequence, the total number of letters, at a large order n of the construction, depends asymptotically on λ_1^n. The fluctuations are of the order $|\lambda_2|^n$. It is interesting to define the wandering exponent [3],

$$\omega = \frac{\ln|\lambda_2|}{\ln \lambda_1}\,, \tag{5}$$

that expresses the asymptotic dependence of the fluctuations with the total number of letters, $\Delta N^{(n)} \sim N^{\omega}$.

The nearest-neighbor Ising model is given by the Hamiltonian

$$\mathscr{H} = -\sum_{(i,j)} J_{i,j}\sigma_i\sigma_j\,, \tag{6}$$

with the spin variables $\sigma_i = \pm 1$ on the sites of a hierarchical diamond structure. In Fig. 1 , which is suitable for the period-doubling Fibonacci rule of Eq. (1), we draw the first stages of the construction of a diamond lattice with a basic polygon of four bonds (that is, of a MKH lattice with cell length $b = 2$, and number of branches $q = 2$, which amounts to 4 bonds in the diamond unit cell). As indicated in this figure, we simulate a layered system by the introduction of the interactions $J_A > 0$ and $J_B > 0$ along

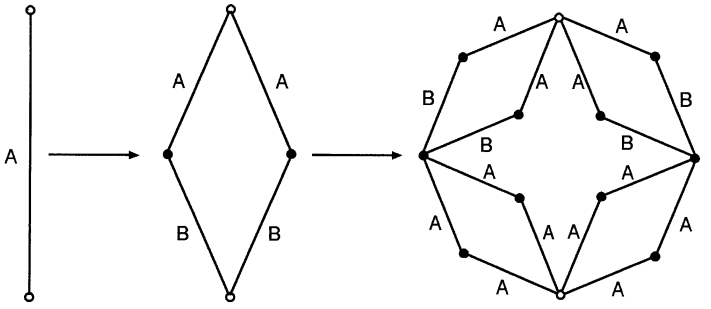

Fig. 1. Some stages of the construction of a Migdal–Kadanoff hierarchical lattice with bond length $b=2$ and $q=2$ branches (which is called a diamond lattice) for the period-doubling sequence $A \to AB$ and $B \to AA$ (letters A and B indicate the exchange interactions, J_A and J_B).

the branches of the structure. If we keep in mind the rules of Eq. (1), and decimate the intermediate spins, it is straightforward [6] to establish the recursion relations

$$x'_A = \frac{2x_A x_B}{1 + x_A^2 x_B^2} \tag{7}$$

and

$$x'_B = \frac{2x_A^2}{1 + x_A^4}\,, \tag{8}$$

where $x_A = \tanh K_A$, $x_B = \tanh K_B$, $K_A = \beta J_A$, $K_B = \beta J_B$, and β is the inverse of the temperature.

It should be remarked that a similar procedure can be used to consider much more general substitutional sequences. However, to avoid any changes in the topology of the hierarchical lattice, we restrict the analysis to period-multiplying substitutions. In these cases, the largest eigenvalue of the inflation matrix, λ_1, gives the multiplication factor of the period. Therefore, $b = \lambda_1$ for all cases under consideration (and, in particular, $b = \lambda_1 = 2$, for the diamond lattice of the figure).

In the uniform case, $J_A = J_B = J$, Eqs. (7) and (8) reduce to the simple recursion relation

$$x' = \frac{2x^2}{1 + x^4}\,, \tag{9}$$

with two trivial and stable fixed points, $x^* = 0$ and $x^* = 1$, and a non-trivial and unstable fixed point, $x^* = 0.543689\ldots$, which come from the polynomial equation

$$x^5 - 2x^2 + x = x[x^4 - 2x + 1] = 0\,. \tag{10}$$

In the aperiodic case ($J_A \neq J_B$) under consideration, the A components of the coordinates of the fixed points in the physical sectors of the x_A–x_B space ($0 \leqslant x_A, x_B \leqslant 1$) come from the solutions of the equation

$$x_A^9 + 2x_A^5 - 4x_A^3 + x_A = x_A[x_A^4 - 2x_A + 1][x_A^4 + 2x_A + 1] = 0\,. \tag{11}$$

Therefore, a comparison with Eq. (10) shows that the only fixed points are located along the $x_A = x_B$ direction, and given by the same values as in the uniform case, $x_A^* = x_B^* = 0$, $x_A^* = x_B^* = 1$, and $x_A^* = x_B^* = 0.543689\ldots$. The linearization about the non-trivial uniform fixed point yields the asymptotic equations

$$\begin{pmatrix} \Delta x_A' \\ \Delta x_B' \end{pmatrix} = C\mathbf{M}^T \begin{pmatrix} \Delta x_A \\ \Delta x_B \end{pmatrix}, \tag{12}$$

where $\mathbf{M}^T$ is the transpose of the substitution matrix, and the structure factor C is given by

$$C = \frac{1 - x_A^*}{x_A^*} = 0.839286\ldots . \tag{13}$$

The diagonalization of this linear form gives the eigenvalues

$$\Lambda_1 = C\lambda_1 = 2C = 1.678573\ldots , \tag{14}$$

and

$$\Lambda_2 = C\lambda_2 = -C = -0.839286\ldots , \tag{15}$$

where $\lambda_1 = 2$ and $\lambda_2 = -1$ are the eigenvalues of the substitution matrix. As $\Lambda_1 > 1$ and $|\Lambda_2| < 1$, the fixed point is a saddle node with a flipping approximation. Therefore, given the ratio $r = J_B/J_A$ between the exchange interactions, the critical temperature is defined by the flow into this uniform fixed point. From Eqs. (9) and (10), we see that the same eigenvalue Λ_1 characterizes the (unstable) fixed point of the uniform model. Thus, in this particular example, with the wandering exponent $\omega = 0$, the geometric fluctuations are unable to change the critical behavior with respect to the uniform system. We can draw a phase diagram, in terms of the ratio r and the temperature T, where the critical line displays the same (universal) exponents of the uniform case. Also, it is not difficult to check that the same sort of behavior (saddle point; largest eigenvalue associated with the uniform system) still holds for all finite values of the branching number q of the diamond ($b = 2$) structure.

To give an example with another value of the wandering exponent ω, where the geometric fluctuations become relevant, consider the (four-letter) Rudin–Shapiro sequence [8], $A \to AC$, $B \to DC$, $C \to AB$, and $D \to DB$. The substitution matrix is given by

$$\mathbf{M}_{RS} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \tag{16}$$

with eigenvalues $\lambda_1 = 2$, $\lambda_2 = -\lambda_3 = \sqrt{2}$, and $\lambda_4 = 0$, and the wandering exponent $\omega = 1/2$. For the Ising model on the diamond hierarchical lattice of the figure, it is easy to write the set of recursion relations

$$x_A' = \frac{2x_A x_C}{1 + x_A^2 x_C^2}, \qquad x_B' = \frac{2x_D x_C}{1 + x_D^2 x_C^2}, \tag{17}$$

$$x'_C = \frac{2x_A x_B}{1 + x_A^2 x_B^2}, \qquad x'_D = \frac{2x_D x_B}{1 + x_D^2 x_B^2} . \tag{18}$$

Again, there are two trivial fixed points, $x_A^* = x_B^* = x_C^* = x_D^* = 0$ and 1, and the non-trivial uniform fixed point, $x_A^* = x_B^* = x_C^* = x_D^* = 0.543689\ldots$, as in the uniform case. The linearization about this uniform fixed point gives a matrix relation of the same form as Eq. (12),

$$(\Delta \mathbf{x}') = C\mathbf{M}_{RS}^T(\Delta \mathbf{x}) , \tag{19}$$

with the same structure factor C, given by Eq. (13), and the eigenvalues

$$\Lambda_1 = C\lambda_1 = 2C = 1.678573\ldots , \tag{20}$$

$$\Lambda_{2,3} = C\lambda_{2,3} = \pm C\sqrt{2} = \pm 1.186930\ldots , \tag{21}$$

and

$$\Lambda_4 = C\lambda_4 = 0 . \tag{22}$$

Therefore, besides being unstable along the diagonal direction ($x_A = x_B = x_C = x_D$), this uniform fixed point is also unstable along two additional directions in the four-dimensional $x_A - x_B - x_C - x_D$ parameter space. Given the ratios between the exchange interactions, there is no temperature associated with any physical initial conditions in this parameter space so that we can reach the uniform fixed point. The critical behavior is of a much more subtle character as compared with the uniform case.

Now it is interesting to devise an analog of Luck's criterion to gauge the influence of the geometric fluctuations. As we have seen in the previous examples, the largest eigenvalue in the neighborhood of the uniform fixed point is given by

$$\Lambda_1 = \lambda_1 C = bC , \tag{23}$$

where it is important to remark that the calculations are always performed for substitutional sequences and MKH lattices such that $\lambda_1 = b$. The second largest eigenvalue is given by

$$\Lambda_2 = \lambda_2 C = \frac{\lambda_2}{b}\Lambda_1 . \tag{24}$$

Therefore, the fluctuations are relevant if

$$|\Lambda_2| = \frac{|\lambda_2|}{b}\Lambda_1 > 1 . \tag{25}$$

From the exact recursion relations between the free energies associated with successive generations of a uniform ferromagnetic Ising model on a MKH lattice [7], we can write

$$\Lambda_1 = b^{y_t} = b^{\frac{D}{2-\alpha}} , \tag{26}$$

where α is the critical exponent of the specific heat of the uniform model and D is the fractal dimension of the lattice. From the definition of the wandering exponent, given by Eq. (5), we can also write

$$|\lambda_2| = b^{\omega} . \tag{27}$$

Inserting these expressions into inequality (25), we show that the geometric fluctuations are relevant for

$$\omega > \omega_c = 1 - \frac{D}{2-\alpha} . \tag{28}$$

In the particular case of the diamond hierarchical lattice ($b=2$ and $q=2$) the fractal dimension is given by

$$D = \frac{\ln(qb)}{\ln b} = 2 , \tag{29}$$

so the criterion is reduced to the inequality

$$\omega > \omega_c = -\frac{\alpha}{2-\alpha} . \tag{30}$$

As $\Lambda_1 = 2C = 1.678573\ldots$, we have $\alpha = -0.676533\ldots$, and $\omega > 0.252764\ldots$, which explains the universal behavior of the first example ($\omega = 0$) and the relevance of the fluctuations in the case of the Rudin–Shapiro sequence ($\omega = 1/2$). This same criterion explains the change in the critical behavior of an aperiodic Potts model on the diamond hierarchical lattice above $4 + 2\sqrt{2}$ states, as recently shown by Magalhães, et al. [9].

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