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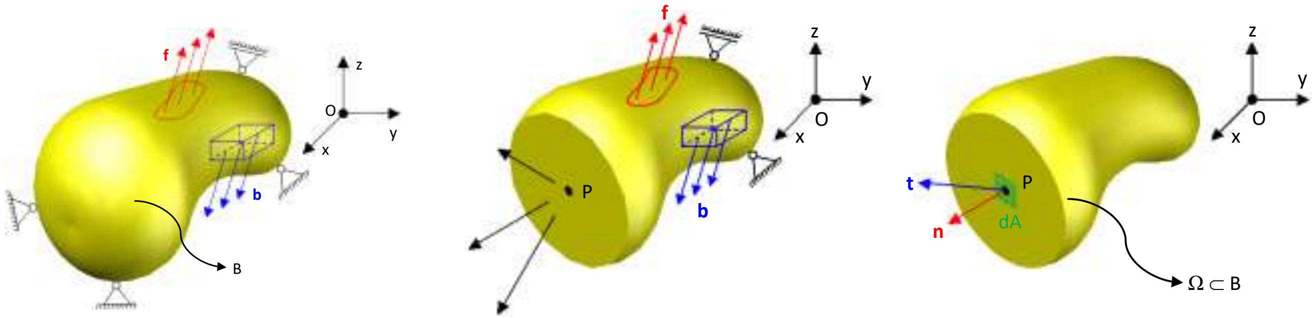
Equilibrium of Membranes

CAUCHY STRESSES
THEOREM OF VIRTUAL WORK
PARAMETRIC SURFACES
EQUILIBRIUM OF MEMBRANES
MINIMAL SURFACES

19/10/2018

Cauchy Stresses:



(a) A solid under displacement constraints and external loads;
(b/c) Internal stresses in a constrained and loaded solid

Cauchy Stress ("True Stress"): $\mathbf{t} = \mathbf{t}(P,\mathbf{n}) = \lim_{\Delta A \to 0} \dfrac{\Delta \mathbf{F}}{\Delta A} = -\mathbf{t}(P,-\mathbf{n})$

It can be shown that $\mathbf{t}(P,\mathbf{n}) = \mathbf{T}(P)\mathbf{n}$

where $\mathbf{T}$ is the Cauchy Tensor Field

In an implicit Cartesian basis $\langle \mathbf{e}_i \rangle \; i = 1,2,3$ $\quad [\mathbf{T}] = [\sigma_{ij}]$ where $\sigma_{ij} = \mathbf{e}_i \cdot \mathbf{t}(\mathbf{e}_j)$

Over the surface: $\mathbf{T}\mathbf{n} = \mathbf{f}$

Equilibrium

For any finite portion of the body $\Omega \subset B$ $\int_{\partial\Omega} \mathbf{f}\, dA + \int_{\Omega} \mathbf{b}\, dV = \mathbf{0}$

$$\boxed{\int_{\partial\Omega} \mathbf{T}\,\mathbf{n}\, dA + \int_{\Omega} \mathbf{b}\, dV = \mathbf{0}}$$

(Integral Equilibrium Equation)

Using Gauss Divergence Theorem: $\int_{\Omega} \operatorname{div}\Psi\, dV = \int_{\partial\Omega} \Psi\,\mathbf{n}\, dA$

where Ψ *is either a scalar, vector or tensor field*

we have $\int_{\Omega} \left(\operatorname{div}\mathbf{T}^T + \mathbf{b}\right) dV = \mathbf{0}, \quad \forall\Omega \subset B$

and since this must hold for $\forall\Omega$

$$\boxed{\begin{cases} \operatorname{div}\mathbf{T}^T + \mathbf{b} = \mathbf{0}, & \forall P \in B \\ \mathbf{T}\,\mathbf{n} = \mathbf{f}, & \forall P \in \partial B \end{cases}}$$

(Differential Equilibrium Equation)

where $\operatorname{div}\mathbf{T}^T = \sum_{i,j=1}^{3} \frac{\partial \sigma_{ij}}{\partial x_j}\mathbf{e}_i$

Equilibrium of moments around three coordinate axis provides $\boxed{\mathbf{T} = \mathbf{T}^T}$

So the transposition symbol can be omitted in the equilibrium equation, expressing both force and moment equilibrium!

Theorem of Virtual Work

A body is in equilibrium, if, and only f
$$\begin{cases} div\mathbf{T}+\mathbf{b}=\mathbf{0} & \forall P\in B \\ \mathbf{T}\mathbf{n}=\mathbf{f} & \forall P\in\partial B_{\mathbf{f}} \end{cases}$$

Thus, for any virtual displacement $\delta\mathbf{u}$ *compatible with the boundaries* $\left(\delta\mathbf{u}=\mathbf{0} \quad in \quad \partial B_{\mathbf{u}}\right)$

$$\delta\mathbf{u}\cdot\left(\mathrm{div}\mathbf{T}+\mathbf{b}\right)=0 \quad \forall P\in\partial B$$

$$\int_{\Omega}\delta\mathbf{u}\cdot\left(\mathrm{div}\mathbf{T}+\mathbf{b}\right)dV=0 \quad \forall\,\delta\mathbf{u}$$

Applying Green's divergence Theorem for the first term:

$$\int_{\Omega}\delta\mathbf{u}\cdot\left(\mathrm{div}\mathbf{T}^{T}\right)dV=\int_{\partial\Omega}\delta\mathbf{u}\cdot\left(\mathbf{T}\mathbf{n}\right)dA-\int_{\Omega}\mathbf{T}:\delta\mathbf{E}\,dV$$

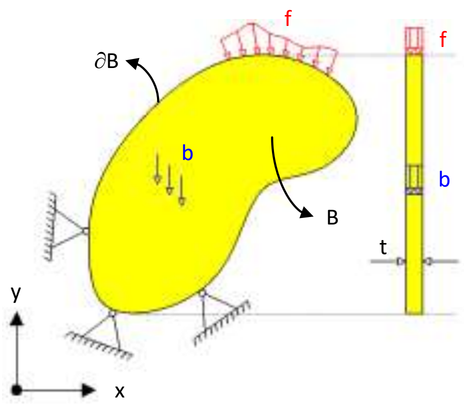
Thus equilibrium can be expressed by
$$\boxed{\int_{\Omega}\mathbf{T}:\delta\mathbf{E}\,dV=\int_{\partial\Omega}\delta\mathbf{u}\cdot\mathbf{t}\,dA+\int_{\Omega}\delta\mathbf{u}\cdot\mathbf{b}\,dV, \quad \forall\,\delta\mathbf{u}}$$

That is
$$\boxed{\delta W_{\mathrm{int}}=\delta W_{\mathrm{ext}}, \quad \forall\,\delta\mathbf{u}}$$

$$\delta\mathbf{E}=\frac{1}{2}\left[\delta\mathbf{F}+\delta\mathbf{F}^{T}\right]=\delta\mathbf{E}^{T}$$ *is the tensor of virtual displacements*

$$\delta\mathbf{F}=\nabla\delta\mathbf{u}=\left[\frac{\partial\left(\delta\mathbf{u}\right)}{\partial\mathbf{x}}\right]$$ *is the gradient of the virtual transformation*

Plane Stress State



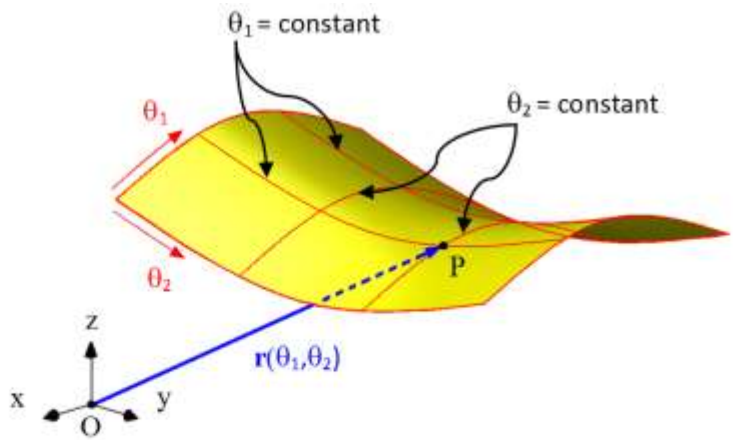
A solid under plane stress state

$$\begin{cases} \sigma_z = 0 \\ \tau_{xz} = \tau_{yz} = 0 \\ b_z = f_z = 0 \end{cases} \quad \Rightarrow \quad \mathbf{T} = \begin{bmatrix} \sigma_x & \tau_{xy} & 0 \\ \tau_{yx} & \sigma_y & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Equilibrium:

$$\forall P \in B, \quad \begin{cases} \dfrac{\partial \sigma_x}{\partial x} + \dfrac{\partial \tau_{xy}}{\partial y} + b_x = 0 \\ \dfrac{\partial \tau_{yx}}{\partial x} + \dfrac{\partial \sigma_y}{\partial y} + b_y = 0 \end{cases}$$

Parametric Surfaces



A parametric surface can be described by a position vector field $\mathbf{r} = \mathbf{r}\left(\theta_1, \theta_2\right)$

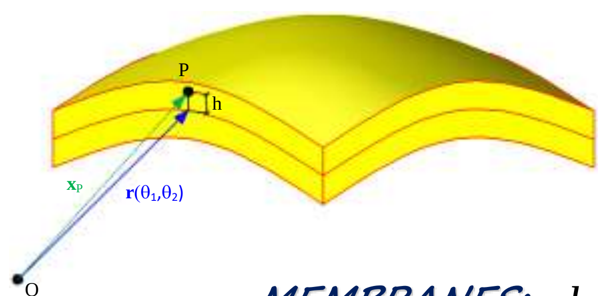
(we´ll refer to an implicit Cartesian coordinate system (x,y,z), so that no distinction will be done between covariant and contravariant base vectors, as it would be required for more general curvilinear coordinate systems…)

Some solids can be described by parametric surfaces, given $h = h\left(\theta_1, \theta_2\right)$

Such that: $\forall P \in B \quad : \quad \mathbf{x}_P = \mathbf{r}\left(\theta_1, \theta_2\right) + z\,\mathbf{e}_3\left(\theta_1, \theta_2\right)$

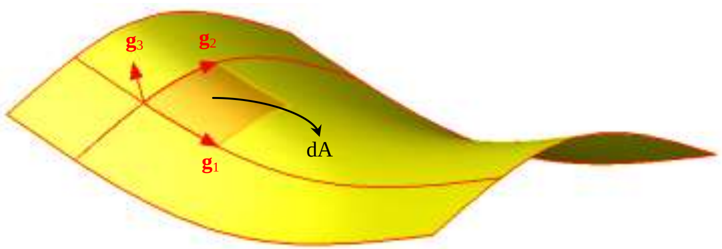
where: $z \in \left[-\frac{h}{2}, \frac{h}{2}\right]$

and: $0 < h < \rho_{\min}$



MEMBRANES: $h << 1$

Parametric Surfaces



Keeping $\theta_\beta = \text{constant}, \quad \beta = 2,1$

we define an in infinite set of coordinate curves, with associate tangent vectors

$$\mathbf{g}_\alpha = \frac{\partial \mathbf{r}}{\partial \theta_\alpha} \qquad \begin{cases} \theta_2 = \text{constant}, & \alpha = 1 \\ \theta_1 = \text{constant}, & \alpha = 2 \end{cases}$$

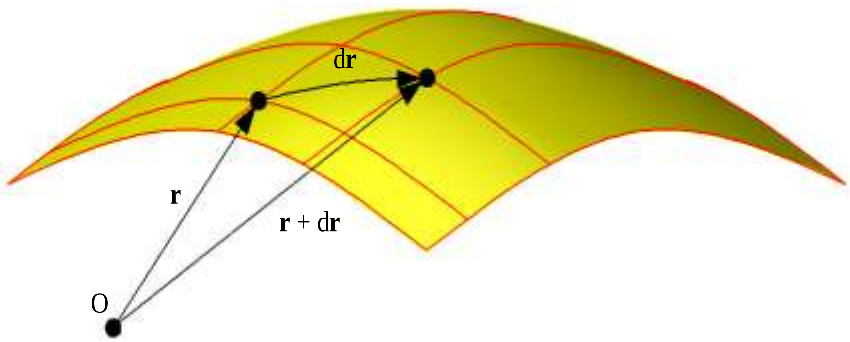
In general, $\|\mathbf{g}_\alpha\| \neq 1$

A unit vector field, always normal to te surface, is given by $\mathbf{g}_3 = \dfrac{\mathbf{g}_1 \times \mathbf{g}_2}{\|\mathbf{g}_1 \times \mathbf{g}_2\|}$

Differential area elements are given by $dA = \|\mathbf{g}_1 \times \mathbf{g}_2\| = (\mathbf{g}_1 \times \mathbf{g}_2) \cdot \mathbf{g}_3 = \det \mathbf{G}$

Where $\mathbf{G} = [a_{ij}], \quad a_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j, \quad i, j = 1,2,3$ *is the METRIC TENSOR*

Parametric Surfaces



A differential displacement over the surface is given by

$$d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial \theta_1} d\theta_1 + \frac{\partial \mathbf{r}}{\partial \theta_2} d\theta_2 = \sum_{\alpha=1}^{2} \mathbf{g}_\alpha d\theta_\alpha$$

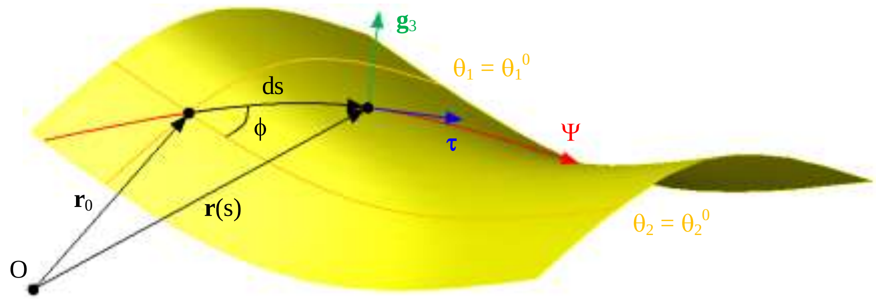
where $$\mathbf{g}_\alpha = \frac{\partial \mathbf{r}}{\partial \theta_\alpha}$$

(squared) length of an infinitesimal displacement: $$(ds)^2 = d\mathbf{r} \cdot d\mathbf{r} = \sum_{\alpha,\beta=1}^{2} \mathbf{g}_\alpha \cdot \mathbf{g}_\beta \, d\theta_\alpha d\theta_\beta$$

Denoting $a_{\alpha\beta} = \mathbf{g}_\alpha \cdot \mathbf{g}_\beta \quad \alpha, \beta = 1, 2$ *we have* $$(ds)^2 = \sum_{\alpha=1}^{2} \sum_{\beta=1}^{2} a_{\alpha\beta} \, d\theta_\alpha d\theta_\beta$$

'FIRST FUNDAMENTAL FORM
OF THE SURFACE'

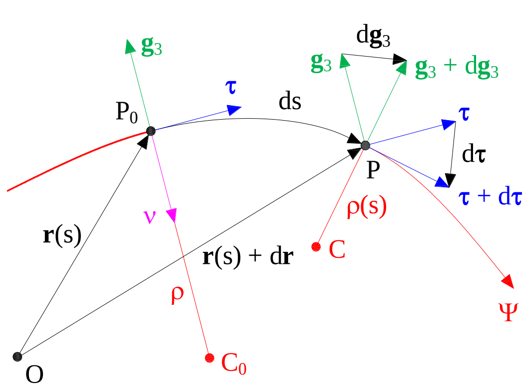
Normal Curvature

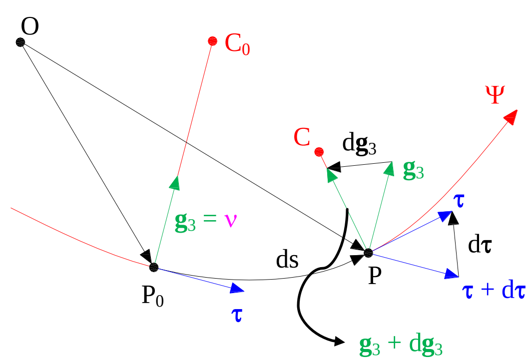


We consider a curve drawn onto the surface, parametrized by the arc-length:

$$\Psi \subset \Omega \quad : \quad \mathbf{r} = \mathbf{r}(s)$$

The curvature of Ψ *is given by* $\dot{\boldsymbol{\tau}} = \dfrac{d\boldsymbol{\tau}}{ds} = \dfrac{\boldsymbol{\nu}}{\rho} = \pm\dfrac{\mathbf{g}_3}{\rho}$





And since $\mathbf{g}_3 \perp \boldsymbol{\tau} \quad \forall s \quad \therefore \quad \dot{\mathbf{g}}_3 = \dfrac{d\mathbf{g}_3}{ds} = \pm\dfrac{\boldsymbol{\tau}}{\rho}$

So that, in any case $\dot{\mathbf{g}}_3 \cdot \boldsymbol{\tau} = \left(\dfrac{\boldsymbol{\tau}}{\rho}\cdot\right)\boldsymbol{\tau} = \dfrac{1}{\rho}$

Normal Curvature

Thus $\kappa = \frac{1}{\rho} = \dot{\mathbf{g}}_3 \cdot \boldsymbol{\tau}$ *may now be positive or negative, depending on the curve being concave or convex with respect to the surface orientation!*

$$\dot{\mathbf{g}}_3 = \frac{d\mathbf{g}_3}{ds} = \sum_{\beta=1}^{2} \frac{\partial \mathbf{g}_3}{\partial \theta_\beta} \frac{d\theta_\beta}{ds} \qquad \boldsymbol{\tau} = \frac{d\mathbf{r}}{ds} = \sum_{\alpha=1}^{2} \frac{\partial \mathbf{r}}{\partial \theta_\alpha} \frac{d\theta_\alpha}{ds} = \sum_{\alpha=1}^{2} \mathbf{g}_\alpha \frac{d\theta_\alpha}{ds}$$

$$\kappa = \frac{1}{\rho} = \dot{\mathbf{g}}_3 \cdot \boldsymbol{\tau} = \frac{1}{(ds)^2} \left[\sum_{\alpha=1}^{2} \sum_{\beta=1}^{2} \left(\mathbf{g}_\alpha \cdot \frac{\partial \mathbf{g}_3}{\partial \theta_\beta} \right) d\theta_\alpha d\theta_\beta \right]$$

Denoting $b_{\alpha\beta} = b_{\beta\alpha} = \mathbf{g}_\alpha \cdot \frac{\partial \mathbf{g}_3}{\partial \theta_\beta} \quad \alpha, \beta = 1, 2$

$$\boxed{\kappa = \frac{1}{\rho} = \frac{\sum_{\alpha=1}^{2} \sum_{\beta=1}^{2} b_{\alpha\beta}\, d\theta_\alpha d\theta_\beta}{\sum_{\alpha=1}^{2} \sum_{\beta=1}^{2} a_{\alpha\beta}\, d\theta_\alpha d\theta_\beta}}$$

"SECOND FUNDAMENTAL FORM OF THE SURFACE"

'FIRST FUNDAMENTAL FORM OF THE SURFACE'

Principal Curvatures

Expanding these forms $\kappa = \frac{1}{\rho} = \frac{b_{11}\, d\theta_1^{\,2} + 2b_{12}\, d\theta_1 d\theta_2 + b_{22}\, d\theta_2^{\,2}}{a_{11}\, d\theta_1^{\,2} + 2a_{12}\, d\theta_1 d\theta_2 + a_{22}\, d\theta_2^{\,2}} = \frac{B(d\theta_1, d\theta_2)}{A(d\theta_1, d\theta_2)}$

We arrive at $\boxed{A\kappa - B = 0}$, which <u>always holds</u>!

Therefore $\frac{\partial}{\partial d\theta_\alpha}(A\kappa - B) = \frac{\partial A}{\partial d\theta_\alpha}\kappa + A\frac{\partial \kappa}{\partial d\theta_\alpha} - \frac{\partial B}{\partial d\theta_\alpha} = 0 \qquad \alpha, \beta = 1, 2$

Furthermore, <u>in the directions for which k is extremum</u>: $\frac{\partial \kappa}{\partial d\theta_\alpha} = 0 \qquad \alpha, \beta = 1, 2$

That is, in the direction in which k is extremum: $\frac{\partial A}{\partial d\theta_\alpha}\kappa - \frac{\partial B}{\partial d\theta_\alpha} = 0 \qquad \alpha, \beta = 1, 2$

Proceeding with the derivations: $\begin{cases} (a_{11}\, d\theta_1 + a_{12}\, d\theta_2)\kappa - b_{11}\, d\theta_1 - b_{12}\, d\theta_2 = 0 \\ (a_{12}\, d\theta_1 + a_{22}\, d\theta_2)\kappa - b_{12}\, d\theta_1 + b_{22}\, d\theta_2 = 0 \end{cases}$

Principal Curvatures

Or, in matrix form:

$$\begin{bmatrix} \kappa a_{11} - b_{11} & \kappa a_{12} - b_{12} \\ \kappa a_{12} - b_{12} & \kappa a_{22} - b_{22} \end{bmatrix} \begin{bmatrix} d\theta_1 \\ d\theta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad (*)$$

This system has non-trivial solutions only if

$$\det\left(\begin{bmatrix} \kappa a_{11} - b_{11} & \kappa a_{12} - b_{12} \\ \kappa a_{12} - b_{12} & \kappa a_{22} - b_{22} \end{bmatrix} \right) = 0$$

'Characteristic equation':

$$\left(a_{11} a_{22} - a_{12}{}^2 \right) \kappa^2 - \left(a_{11} b_{22} - 2 a_{12} b_{12} + b_{11} a_{22} \right) k + \left(b_{11} b_{22} - b_{12}{}^2 \right) = 0$$

Roots are:

$$\kappa_{I,II} = K_M \pm \sqrt{K_M^2 - K_G}$$

where: $K_G = \kappa_I \kappa_{II} = \dfrac{b_{11} b_{22} - b_{12}^2}{a_{11} a_{22} - a_{12}^2}$ *Is the GAUSSIAN CURVATURE*

and where: $K_M = \dfrac{1}{2}\left(\kappa_I + \kappa_{II} \right) = \dfrac{1}{2} \dfrac{a_{11} b_{22} - 2 a_{12} b_{12} + b_{11} a_{22}}{a_{11} a_{22} - a_{12}^2}$ *Is the MEAN CURVATURE*

Principal Curvatures

$$K_G(P) > 0$$

$$\Downarrow$$

The point is ELLIPTIC

$$K_G > 0 \quad \forall P \in \Omega$$

$$\Downarrow$$

The surface is SINCLASTIC

$$K_G(P) = 0$$

$$\Downarrow$$

The point is PARABOLIC

$$K_G = 0 \quad \forall P \in \Omega$$

$$\Downarrow$$

The surface is SIMPLE or DEVELOPABLE

$$K_G(P) < 0$$

$$\Downarrow$$

The point is HIPERBOLIC

$$K_G < 0 \quad \forall P \in \Omega$$

$$\Downarrow$$

The surface is ANTICLASTIC

Relationships between Gaussian and Mean Curvatures:

$$K_M = \frac{\kappa_I + \kappa_{II}}{2} = \frac{\rho_I + \rho_{II}}{2\rho_I \rho_{II}} = \bar{\rho} K_G \quad ; \quad \bar{\rho} = \frac{\rho_I + \rho_{II}}{2}$$

Principal Directions & Curvature Lines

Substituting k_I and k_{II} in (*) we find the two mutually orthogonal direction (ϕ_I, ϕ_{II}), for which the curvature radiuses are extrema;

These directions are called PRINCIPAL DIRECTIONS, and curves that follow the principal directions are called <u>CURVATURE LINES</u>.

To show that indeed $\phi_I \perp \phi_{II}$, we consider that along the coordinate lines $d\theta_\alpha = 0$ (since $\theta_\alpha = \text{constant}$), so for curvature lines we have

$$\begin{cases} \left(a_{11}\, \overset{0}{d\theta_1} + a_{12}\, d\theta_2\right)\kappa_I - b_{11}\, \overset{0}{d\theta_1} - b_{12}\, d\theta_2 = 0 \\ \left(a_{12}\, d\theta_1 + a_{22}\, \underset{0}{d\theta_2}\right)\kappa_{II} - b_{12}\, d\theta_1 + b_{22}\, \underset{0}{d\theta_2} = 0 \end{cases}$$

That is

$$\begin{cases} a_{12}\kappa_I - b_{12} = 0 \\ a_{12}\kappa_{II} - b_{12} = 0 \end{cases}$$

$$a_{12}\left(\kappa_I - \kappa_{II}\right) = 0$$

Now, if $\kappa_I \neq \kappa_{II} \;\Rightarrow\; a_{12} = \mathbf{g}_1 \cdot \mathbf{g}_2 = 0 \;\Rightarrow\; \mathbf{g}_1 \perp \mathbf{g}_2$ *, that is, indeed the curvature lines are <u>mutually orthogonal!</u>*

There result simplified expressions for the principal curvatures, when the coordinate lines are curvature lines:

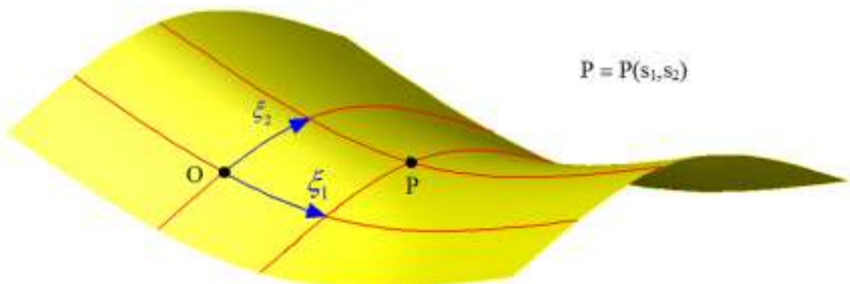
$$\kappa_I = \frac{b_{11}}{a_{11}} \quad ; \quad \kappa_{II} = \frac{b_{22}}{a_{22}}$$

If $\kappa_I = \kappa_{II} \neq 0$ *, the vicinity of the point is locally spherical, and all directions are principal ones.*

Curvature Lines

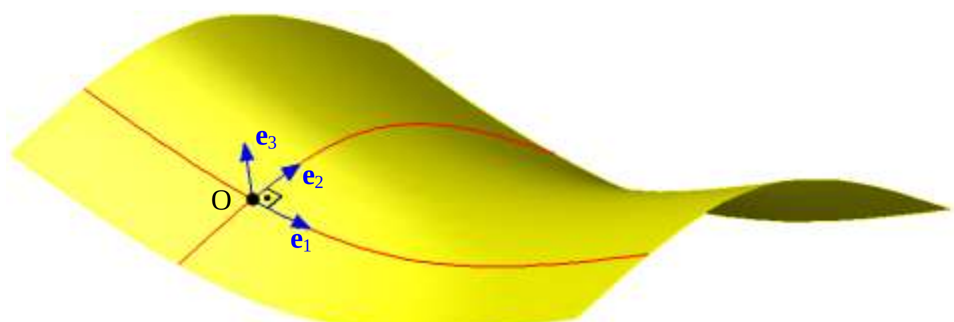
Determination of the Curvature Lines may be complicated, by assuming they are known, they may be taken as coordinate lines, respect to which the equilibrium of membranes is simpler to express;

We also keep the arc-lenght along the curvature lines as parameter, that is $\theta_\alpha = \xi_\alpha$



And we define, at each point, a local <u>Cartesian coordinate system</u> such that

$$\begin{cases} \mathbf{e}_\alpha = \dfrac{\partial \mathbf{r}}{\partial \xi_\alpha}, \quad \alpha = 1,2 \\ \mathbf{e}_3 = \mathbf{e}_1 \times \mathbf{e}_2 \end{cases}$$



Curvature Lines

- *It can be shown that necessary and sufficient conditions for coordinates lines be curvature lines are:*

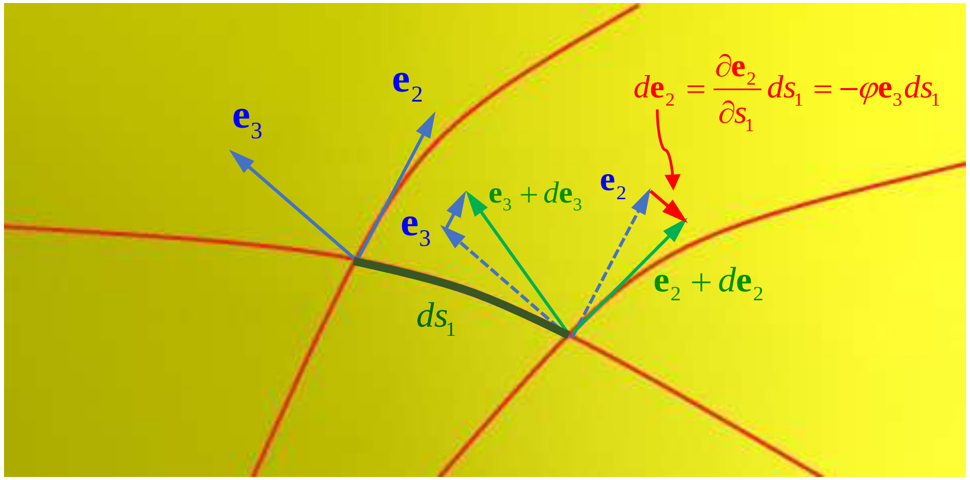
$$\begin{cases} \dfrac{\partial \mathbf{e}_3}{\partial \theta_\alpha} \cdot \mathbf{e}_\beta = 0 \\ \mathbf{e}_3 \cdot \dfrac{\partial \mathbf{e}_\alpha}{\partial \theta_\beta} = 0 \end{cases} \quad , \quad \alpha = 1,2 \; ; \; \beta = 2,1$$

- *Reciprocal conditions are:* $\dfrac{\partial \mathbf{e}_3}{\partial \theta_\alpha} \cdot \mathbf{e}_\beta = 0 \quad ; \quad \alpha = 1,2 \; ; \; \beta = 2,1$

Torsion

The TORSION of a curve is defined such that $\dfrac{\partial \mathbf{e}_\alpha}{\partial s_\beta} = \dfrac{\partial \mathbf{e}_\beta}{\partial s_\alpha} = -\varphi \mathbf{e}_3$

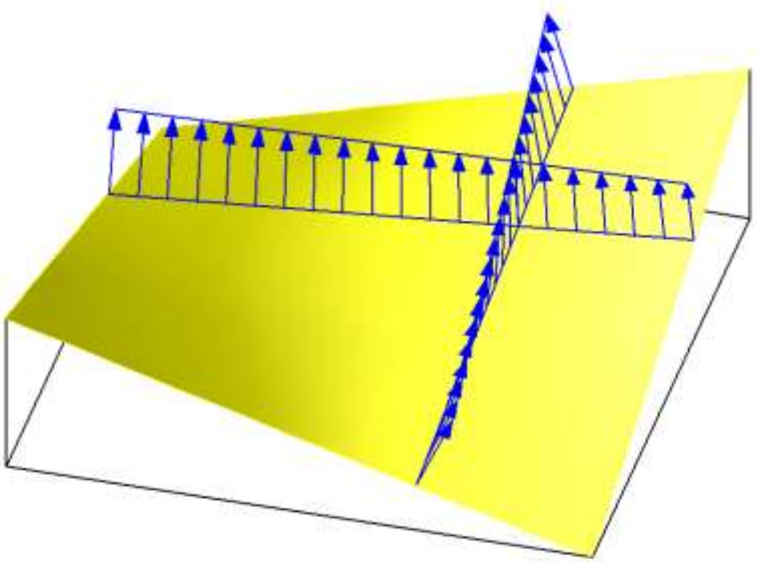
It expresses 'how fast' the osculating plane $(\mathbf{e}_1, \mathbf{e}_3)$ *rotates, in the vicinity of a point*



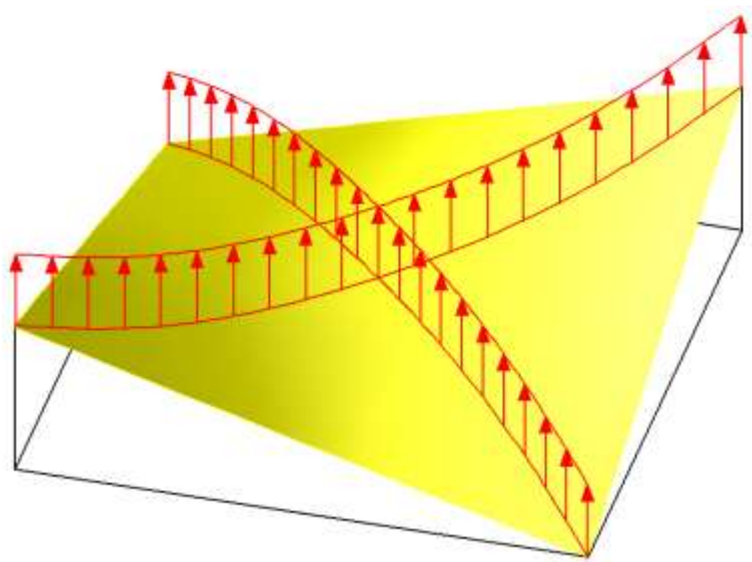
Torsion

<u>Curvature lines do not present any torsion</u>, since the previous conditions require that, for them:

$$\frac{\partial \mathbf{e}_\alpha}{\partial \xi_\beta} = \mathbf{0} \quad ; \quad \alpha = 1,2\ ;\ \beta = 2,1$$



(a) Lines with torsion



(b) curvature lines (without torsion)

Euler's Theorem & Curvature Tensor

The curvature of a line making an angle ϕ with a curvature line is

$$\boxed{\kappa = \kappa_I \cos^2\phi + \kappa_{II} \sin^2\phi}$$

Eulers Theorem highlights the tensorial nature of the curvatures around a point...

The components of the <u>Surface Curvature Tensor</u> on a point, in an intrinsic orthogonal coordinate system are given by:

$$\frac{\partial \mathbf{e}_\alpha}{\partial s_\beta} = K_{\alpha\beta}\mathbf{g}_3 \qquad (\alpha, \beta = 1, 2)$$

that is $$\mathbf{K} = \left[K_{\alpha\beta}\right] = \begin{bmatrix} \kappa_1 & -\varphi \\ -\varphi & \kappa_2 \end{bmatrix}$$

κ_I, κ_{II} are the eigenvalues of $\mathbf{K}$ and ϕ_I, ϕ_{II} its eigenvectors

DIVERGENCE OF A SURFACE TENSOR

If $\mathbf{T}$ *is resolved as a SURFACE TENSOR* $\mathbf{T}=\mathbf{T}(s_1,s_2)=\sum_{\alpha=1}^{2}\sum_{\beta=1}^{2}T_{\alpha\beta}\mathbf{e}_\alpha\otimes\mathbf{e}_\beta$

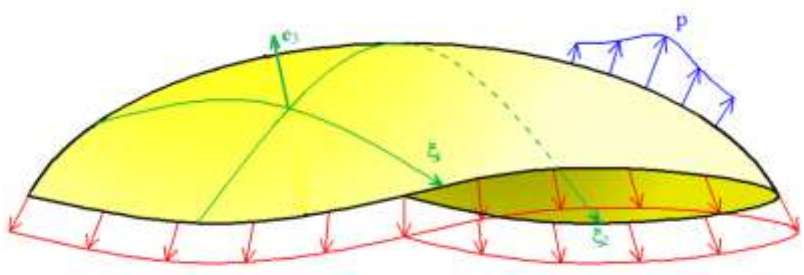
(where we consider orthogonal coordinates lines with arc-length parameters s_α
-- not necessarily curvature lines!)

Then its divergence vector is $\operatorname{div}\mathbf{T}=\sum_{\alpha=1}^{2}\left(\sum_{\beta=1}^{2}\frac{\partial T_{\alpha\beta}}{\partial s_\beta}\right)\mathbf{e}_\alpha+(\mathbf{K}:\mathbf{T})\mathbf{e}_3$

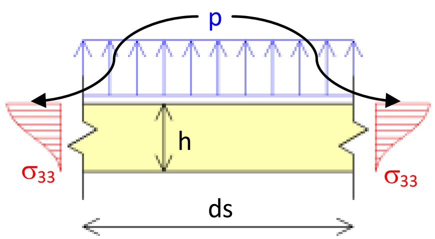
- *where* $\mathbf{K}$ *is the Surface Curvature Tensor*
- *and where the scalar product between tensors* $\mathbf{K}$ *and* $\mathbf{T}$ *:*

$$\mathbf{K}:\mathbf{T}=\operatorname{tr}\left(\mathbf{K}^T\mathbf{T}\right)=\sum_{\alpha=1}^{2}\sum_{\beta=1}^{2}K_{\alpha\beta}T_{\alpha\beta}$$

Equilibrium of Membranes – I



(a) External and boundary loads in a membrane, with h<<1 (constant);



(b) A transversal cut showing the variation of σ_{33} (usually disregarded!)

Equilibrium of Forces:

$$\boxed{\oint_{\partial\Omega} h\,\mathbf{Tn}\,ds = \int_{\Omega} \mathbf{b}A}$$

Applying the divergence theorem $$\left(\int_{\Omega} h\cdot \operatorname{div}\mathbf{T} + \mathbf{b}\right) dA = 0 \quad ,\forall\Omega$$

$$\boxed{\therefore \qquad h\cdot \operatorname{div}\mathbf{T} + \mathbf{b} = \mathbf{0} \quad \forall P \in \Omega}$$ (Equilibrium of Forces)

$$\boxed{\mathbf{T} = \mathbf{T}^T}$$ (Equilibrium of Momentum)

Equilibrium of Membranes - I

Remembering the expression of the divergence of a surface tensor equation
And denoting the external loads as

$$\mathbf{b} = -b_1\mathbf{e}_1 - b_2\mathbf{e}_2 - p\mathbf{e}_3$$

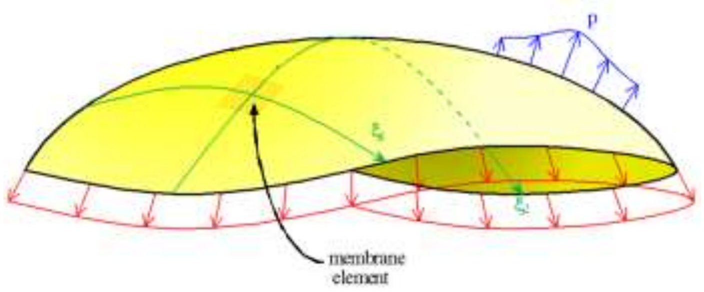
$$h\left(\left(\frac{\partial\sigma_{11}}{\partial s_1} + \frac{\partial\tau_{12}}{\partial s_2}\right)\mathbf{e}_1 + \left(\frac{\partial\tau_{12}}{\partial s_1} + \frac{\partial\sigma_{22}}{\partial s_2}\right)\mathbf{e}_2 + \left(\mathbf{K}:\mathbf{T}\right)_3 \mathbf{e}_3\right) - f_1\mathbf{e}_1 - f_2\mathbf{e}_2 - p\mathbf{e}_3 = \mathbf{0}$$

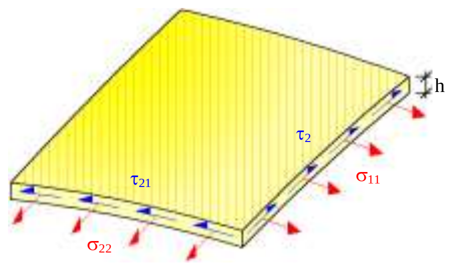
$$\begin{cases} \left.\begin{aligned} h\left(\frac{\partial\sigma_{11}}{\partial s_1} + \frac{\partial\tau_{12}}{\partial s_2}\right) = b_1 \\ h\left(\frac{\partial\tau_{12}}{\partial s_1} + \frac{\partial\sigma_{22}}{\partial s_2}\right) = b_2 \end{aligned}\right\} \text{Similar to a plane stress state} \\ h\left(\mathbf{K}:\mathbf{T}\right) = p \end{cases}$$

Surface stresses equilibrate transversal load by means of curvature!

Equilibrium of Membranes - II

The previous derivation might seems too much abstract, so we recast the membrane equilibrium in a more 'engineering' approach:





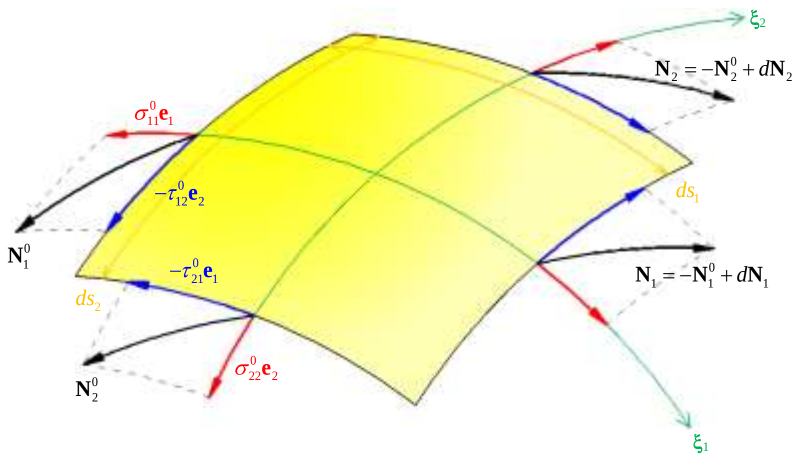
(a) External and boundary loads in a membrane;

(b) a membrane element under a surface stress field

We consider $\sigma_{\alpha\alpha}$ and $\tau_{\alpha\beta}$ constant through thickness h. and define the Surface Stresses [N/m]:

$$\boxed{N_{\alpha\alpha} = h\sigma_{\alpha\alpha} \quad ; \quad N_{\alpha\beta} = N_{\beta\alpha} = h\tau_{\alpha\beta}}$$

Equilibrium of Membranes – II



We again consider coordinates lines with length parameters s_α

(not necessarily curvature lines!)

We define: $\begin{cases} \mathbf{N}_1 = N_{11}\mathbf{e}_1 + N_{12}\mathbf{e}_2 \\ \mathbf{N}_2 = N_{21}\mathbf{e}_1 + N_{22}\mathbf{e}_2 \end{cases}$

And express Equilibrium:

Forces: $\boxed{d\mathbf{N}_1 ds_2 + d\mathbf{N}_2 ds_1 + \mathbf{b} ds_1 ds_2 = \mathbf{0}}$ (*)

Momentum: $\boxed{N_{12} = N_{21}}$

Dividing (*) by $ds_1 ds_2$: $\boxed{\dfrac{d\mathbf{N}_1}{ds_1} + \dfrac{d\mathbf{N}_2}{ds_2} + \mathbf{b} = \mathbf{0}}$ (**)

Equilibrium of Membranes – II

Now: $$\frac{d\mathbf{N}_1}{ds_1} = \frac{d}{ds_1}\left(N_{11}\mathbf{e}_1 + N_{12}\mathbf{e}_2\right) = \frac{\partial N_{11}}{\partial s_1}\mathbf{e}_1 + N_{11}\frac{\partial \mathbf{e}_1}{\partial s_1} + \frac{\partial N_{12}}{\partial s_1}\mathbf{e}_2 + N_{12}\frac{\partial \mathbf{e}_2}{\partial s_1}$$

But $\frac{\partial \mathbf{e}_1}{\partial s_1} = \kappa_1 \mathbf{e}_3$ *and* $\frac{\partial \mathbf{e}_2}{\partial s_1} = -\varphi \mathbf{e}_3$

Therefore $$\frac{\partial \mathbf{N}_1}{ds_1} = \frac{\partial N_{11}}{\partial s_1}\mathbf{e}_1 + \frac{\partial N_{12}}{ds_1}\mathbf{e}_2 + \left(k_1 N_{11} - \varphi N_{12}\right)\mathbf{e}_3$$

Likewise $$\frac{\partial \mathbf{N}_2}{ds_2} = \frac{\partial N_{21}}{\partial s_2}\mathbf{e}_1 + \frac{\partial N_{22}}{ds_2}\mathbf{e}_2 + \left(k_2 N_{22} - \varphi N_{21}\right)\mathbf{e}_3$$

Substituting in (**):

$$\boxed{\frac{\partial N_{11}}{\partial s_1}\mathbf{e}_1 + \frac{\partial N_{12}}{ds_1}\mathbf{e}_2 + \left(k_1 N_{11} - \varphi N_{12}\right)\mathbf{e}_3 + \frac{\partial N_{21}}{\partial s_2}\mathbf{e}_1 + \frac{\partial N_{22}}{ds_2}\mathbf{e}_2 + \left(k_2 N_{22} - \varphi N_{21}\right)\mathbf{e}_3 + \mathbf{b} = \mathbf{0}}$$

Differential Equilibrium in Membranes

Denoting $\mathbf{b} = -b_1\mathbf{e}_1 - b_2\mathbf{e}_2 - p\mathbf{e}_3$ *, considering that* $N_{12}=N_{21}$ *and collecting the terms according to the intrinsic basis:*

$$\left\{\begin{array}{l} \boxed{\begin{array}{l} \dfrac{\partial N_{11}}{\partial s_1} + \dfrac{\partial N_{12}}{\partial s_2} = b_1 \\ \dfrac{\partial N_{12}}{\partial s_1} + \dfrac{\partial N_{22}}{\partial s_2} = b_2 \end{array}} \\ \boxed{\dfrac{N_{11}}{\rho_1} + \dfrac{N_{22}}{\rho_2} - 2\varphi N_{12} = p} \end{array}\right.$$

Similar to a plane stress state!

"General Membrane Equation"

If ξ_α *are principal directions:* $\varphi = 0$

$$\left\{\begin{array}{l} \boxed{\begin{array}{l} \dfrac{\partial N_{11}}{\partial \xi_1} + \dfrac{\partial N_{12}}{\partial \xi_2} = b_1 \\ \dfrac{\partial N_{12}}{\partial \xi_1} + \dfrac{\partial N_{22}}{\partial \xi_2} = b_2 \end{array}} \\ \boxed{\dfrac{N_{11}}{\rho_I} + \dfrac{N_{22}}{\rho_{II}} = p} \end{array}\right.$$

Similar to a plane stress state!

"General Membrane Equation in Principal Directions"

Differential Equilibrium in Membranes

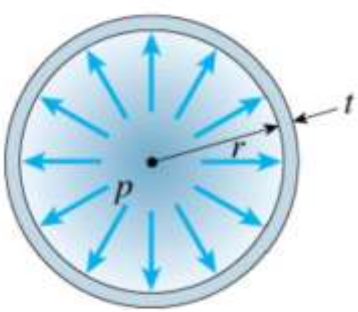
If the membrane is under a uniform and isotropic stress field

$$\begin{cases} N_{11} = N_{22} = \sigma_0 \\ N_{12} = N_{21} = 0 \end{cases} \Rightarrow \begin{cases} b_1 = b_2 = 0 \\ \boxed{h\sigma_0\left(\dfrac{1}{\rho_I} + \dfrac{1}{\rho_{II}}\right) = p} \end{cases}$$

Equation of Laplace-Young or "Soap Films Equation"

Rewriting: $$h\sigma_0\left(\frac{\rho_I + \rho_{II}}{\rho_I \rho_{II}}\right) = 2h\sigma_0 K_M = p$$

In particular, for a spherical membrane, of radius r and thickness t:



$$K_M = \frac{1}{r} \quad \Rightarrow \quad \sigma_0 = \frac{pr}{2t}$$

Minimum Surfaces (Plateu's Problem)

The Area of a Surface Ω with a prescribed closed boundary is given by

$$A = \int_{\Omega} dA = \int_{\Omega} \left\| \boldsymbol{g}_1 \times \boldsymbol{g}_2 \right\| d\theta_1 \, d\theta_2$$

We seek a surface Ω^* Spanned by a vector field $\boldsymbol{x}^*$ Such that A^* Is a minimum.

In other words, for every compatible perturbation $\delta \mathbf{u}$ around $\boldsymbol{x}^*$

$$\delta A^* = \left. \frac{\partial A}{\partial \mathbf{x}} \right|_{x^*} \delta \boldsymbol{u} = 0, \quad \forall \, \delta \boldsymbol{u}$$

Thus the 1st order condition for $\boldsymbol{x}^*$ To be a minimum is

$$\left. \frac{\partial A}{\partial \mathbf{x}} \right|_{x^*} = 0$$

With the equality constraint $\left(\boldsymbol{x}_P - \overline{\boldsymbol{x}}_P \right) = \mathbf{0}, \forall P \in \partial\Omega$

Where $\overline{\boldsymbol{x}}_P$ Spans the prescribe coordinates

Analytical solutions for this nonlinear equation maybe rather difficult!

Soap Film Analogy

We recast the problem of area minimization in a more nonlinear mechanics fashion:

$$\delta A = \int_{\Omega} (\delta J)\, dA = 0, \quad \forall \delta \boldsymbol{u} \quad \textit{around} \ \boldsymbol{x}^*$$

where $\delta J = \mathrm{tr}(\delta \mathbf{F})$ is the Jacobian of the deformation gradient $\delta \mathbf{F} = \mathbf{I} + \dfrac{\partial(\delta \boldsymbol{u})}{\partial \boldsymbol{x}^*}$ from $\boldsymbol{x}^*$ to $\boldsymbol{x}^* + \delta \boldsymbol{u}$

Measured with respect to $\boldsymbol{x}^*$

The configuration $\boldsymbol{x}^*$ That minimizes the functional A also fulfills the above equation $\forall \delta \boldsymbol{u}$

We can always consider coordinate systems for which $\boldsymbol{e}_3 = \boldsymbol{g}_3$, normal to Ω^*

So that $\delta F_{33} = 0$

And therefore

$$\boxed{\delta A = \int_{\Omega^*} \mathrm{tr}(\delta \mathbf{F})\, dA = 0 = \int_{\Omega^*} (\delta F_{11} + \delta F_{22})\, dA = 0 \quad \forall \delta \boldsymbol{u}}$$

Soap Film Analogy

We now consider the problem of finding the membrane geometry $\boldsymbol{x}^{\star}$

Compatible with a self-equilibrated Cauchy surface stress field $\mathbf{T}^{\star}$
(zero external loads, except reactions along the fixed boundaries)

Static equilibrium always requires $\delta W_{\text{int}} = \delta W_{\text{ext}}$

But since we assume that external loads arez ero (except at fixed boundaries): $\delta W_{\text{ext}} = 0$

Therefore, for self-equilibrated surfaces: $\delta W_{\text{int}} = 0, \quad \forall\, \delta \boldsymbol{u}$

When deformations are measured from the equilibrium configuration $\boldsymbol{x}^{\star}$

This condition can be recast as $\delta W_{\text{int}} = h \int_{\Omega^{\star}} \mathbf{T}^{\star} : \delta\mathbf{F}\, dA = 0, \quad \forall\, \delta \boldsymbol{u}$

Where h is the membrane thickness and $\delta\mathbf{F} = \mathbf{I} + \dfrac{\partial(\delta \boldsymbol{u})}{\partial \boldsymbol{x}^{\star}}$

Soap Film Analogy

We now particularize this condition for a homogeneous and isotropic Cauchy surface stress field, such that, in the chosen basis (with $\boldsymbol{e}_3 = \boldsymbol{g}_3$)

$$\mathbf{T}^{\star} = \sigma \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \sigma \tilde{\mathbf{1}}, \quad \forall P \in \Omega^{\star}$$

Even if the membrane presents non-zero deformations in the transversal direction, that is $\delta F_{33} \neq 0$, the virtual work developed by the stress field $\mathbf{T}^{\star}$ is given by

$$\boxed{\delta W_{\text{int}} = \sigma h \int_{\Omega^{\star}} \tilde{\mathbf{1}} : \delta \mathbf{F} \; dS = \sigma h \int_{\Omega^{\star}} \left(\delta F_{11} + \delta F_{22} \right) dA = 0, \quad \forall \delta \boldsymbol{u}}$$

<u>Comparing to the condition to minimum area we conclude that</u> $\boldsymbol{x}^{\star} = \boldsymbol{x}^{*}$

<u>That is, the geometry compatible to an auto-equilibrated, homogeneous and isotropic Cauchy surface stress field also complies to the condition of minimal area!</u>

Consequence for Minimal Surfaces

Considering the Membranes' Equation $$\frac{N_{11}}{\rho_I} + \frac{N_{22}}{\rho_{II}} = p$$

And taking into account that for minimum surfaces $$\begin{cases} p = 0 \\ N_{11} = N_{22} = N_0 \end{cases}$$

We have $$N_0 \left(\frac{1}{\rho_I} + \frac{1}{\rho_{II}} \right) = 0$$

Thus $$\rho_I = -\rho_{II}$$

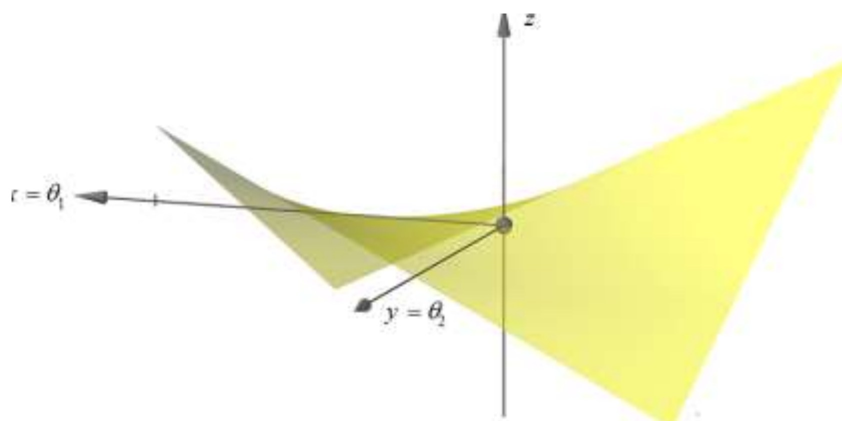
Or, in words, minimal surfaces have zero mean curvature K_M, with negative or zero Gaussian curvature K_G, that is, they are either anticlastic or flat!

- Soap films are minimal surfaces, when p=0.
- Strictly speaking, a soap bubble is not a minimal surface, although it is associated to a similar minimization problem ('minimize the area for a given volume').

Ex. 8 - Hyperbolic Paraboloid

Show that a hyperbolic paraboloid ("hypar")surface <u>IS NOT</u> a minimum surface.

$$z = axy$$

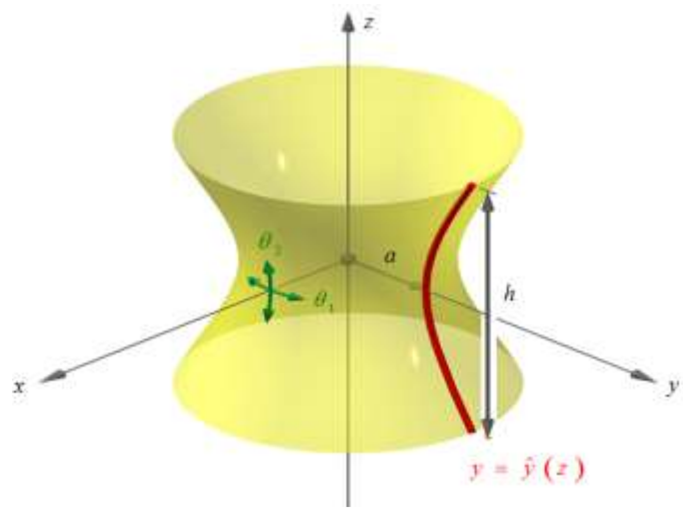


Hypars are usually mentioned as minimum surfaces in the literature about tension structures, and in fact they are practically indistinct to the minimum surface with the same boundary. The analytical expression of the true minimal surface was given by Scwarz in 1890, and is quite more complicated than the hypar expression!

H.A. Schwarz, *Gesammelte mathematische Abhandlungen*, 2 vols, Springer, Berlin, 1890.

Catenoid

The soap film analogy provides a shortcut to deduce the shape of the catenoid, that is, the shape of a minimal surface connecting two circular rings of radius R, with center on the save axis, and apart of each other by a distant h.



Symmetry requires that the solution is given by the rotation of a generatrix curve

$$y = y(z)$$

and that this curve is symmetric at mid-distance between the rings that is,

$$y'(0) = 0$$

Catenoid

We consider the vertical equilibrium of a slice of the surface bounded by to vertical radial planes, with and infinitesimal vertical strip spanned by horizontal angle $d\alpha$

We denote $y(0) = y_0$, so far an unknown constant value.

Using the soap analogy, we know that every cross section of the membrane is under a uniform surface stress $\sigma_0 t$, where t is the membrane thickness.

Thus vertical equilibrium of a vertical slice from z=0 to z(y) requires:

$$\boxed{\sigma_0 t\, y_0\, d\alpha = \left(\sigma_0 t\, y\, d\alpha\right)\cos\theta}$$

And since $\cos\theta = \dfrac{dz}{ds} = \sqrt{1+\left(y'\right)^2}$

We have $y_0 = \dfrac{y}{\sqrt{1+\left(y'\right)^2}}$

Catenoid

Rearranging: $$\frac{y'}{\sqrt{y^2 - y_0^2}} = \frac{1}{y_0}$$

Integrating along z: $$\int \left(\frac{y'}{\sqrt{y^2 - y_0^2}} \right) dz = \int \frac{1}{y_0} dz = \frac{z}{y_0} + C$$

From standard calculus $$\int \frac{du}{\sqrt{u^2 - a^2}} = \ln\left(u + \sqrt{u^2 - a^2} \right)$$

Thus, making $$y = u \quad \therefore \quad du = y'dz$$ $$a = y_0$$

Vertical equilibrium requires: $$\ln\left(y + \sqrt{y^2 - y_0^2} \right) = \frac{z}{y_0} + C$$

Catenoid

Denoting: $\eta = \dfrac{y}{y_0}$

We successively write:

$$\ln\left(y_0\left(\eta + \sqrt{\eta^2 - 1}\right)\right) = \frac{z}{y_0} + C$$

$$\ln y_0 + \ln\left(\eta + \sqrt{\eta^2 - 1}\right) = \frac{z}{y_0} + C$$

$$\ln\left(\eta + \sqrt{\eta^2 - 1}\right) = \frac{z}{y_0} + C - \ln y_0 = \frac{z}{y_0} + C^*$$

Remembering the inverse hyperbolic relationship:

$$\ln\left(\eta + \sqrt{\eta^2 - 1}\right) = \operatorname{arcosh}(\eta)$$

We have: $$\operatorname{arcosh}\left(\frac{y}{y_0}\right) = \frac{z}{y_0} + C^*$$

Catenoid

Also denoting: $\eta = \dfrac{y}{y_0}$

We successively write: $$\ln\left(y_0\left(\eta + \sqrt{\eta^2 - 1}\right)\right) = \frac{z}{y_0} + C$$

$$\ln\left(\eta + \sqrt{\eta^2 - 1}\right) = \frac{z}{y_0} + C - \ln y_0 = \frac{z}{y_0} + C^*$$

Remembering the inverse hyperbolic relationship:

$$\ln\left(\eta + \sqrt{\eta^2 - 1}\right) = \operatorname{arcosh}(\eta)$$

We have: $$\operatorname{arcosh}\left(\frac{y}{y_0}\right) = \frac{z}{y_0} + C^*$$

That is: $$y = y_0 \cosh\left(\frac{z}{y_0} + C^*\right)$$

Catenoid

Imposing the boundary condition

$$y(0) = y_0 = y_0 \cosh\left(C^*\right) \quad \therefore \quad \cosh\left(C^*\right) = 1 \quad \therefore \quad C^* = 0$$

$$\boxed{y = y_0 \cosh\left(\frac{z}{y_0}\right)}$$

That is, <u>the generatrix curve of the minimal surface connecting two circular rings is a catenary, and the surface itself is the catenoid</u>!

Defining the ratio $\lambda = \dfrac{h}{R}$ *we have* $y\left(\dfrac{h}{2}\right) = y_0 \cosh\left(\dfrac{\lambda R}{2 y_0}\right) = R$

Which provides y_0 *for a given ratio* λ

It can be shown that <u>the only surfaces of revolution which are also minimal surfaces are the catenoids</u> (Struik, 1950), as it is easy to conclude from the above development , since no particular restrictions have been set for $y = y(z)$

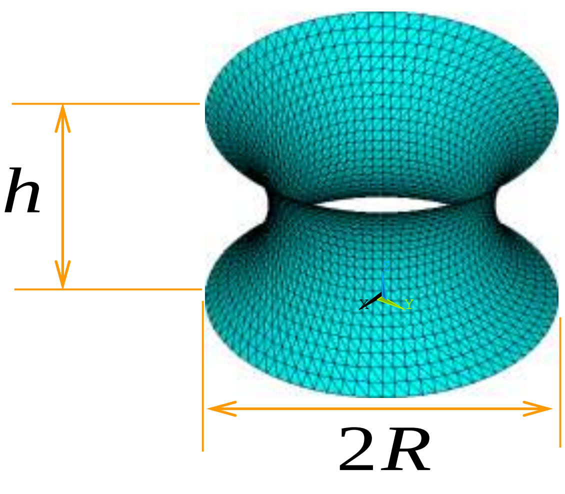
Goldschmidt limit

The maximum ring separation for a catenoid with equal lower and upper radiuses is

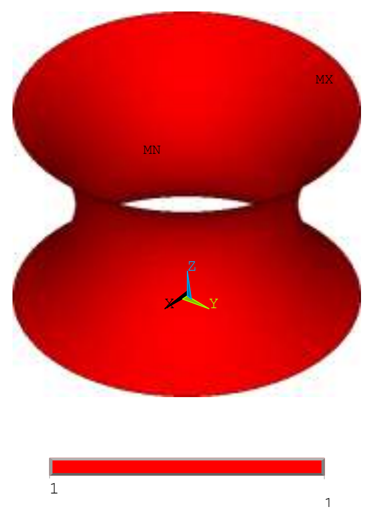
$$h \le h_{\lim} \cong 1.3254868R$$

This is known as Goldschmidt limit (1831).

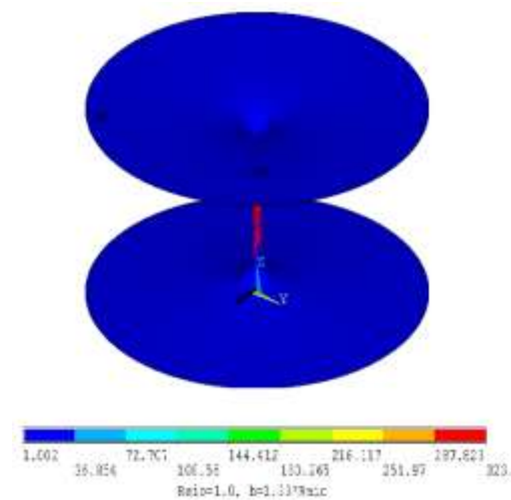
Carl Wolfgang Benjamin Goldschmidt, 'Determinatio superficiei minimae rotatione curvae data duo puncta jungentis circa datum axem ortae', Gottingae, MDCCCXXXI.



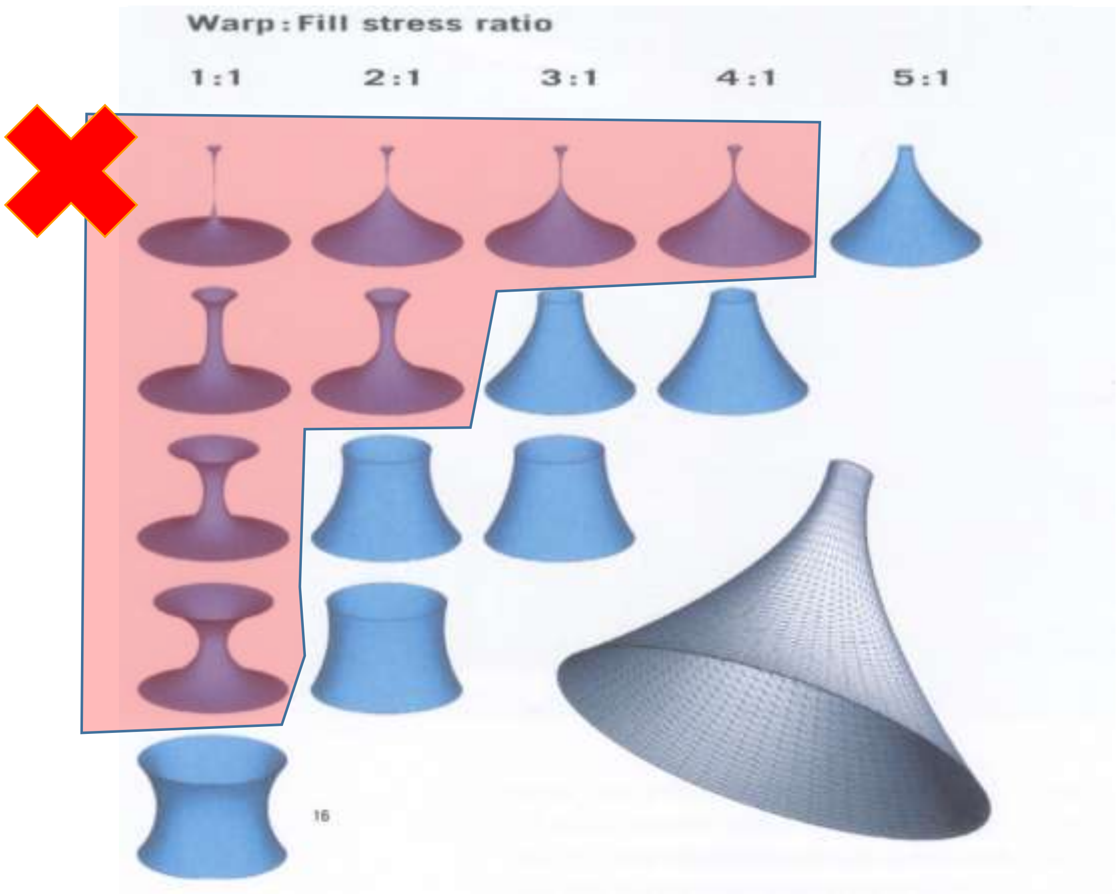
$$h = 1.32R$$



$$h = 1.33R$$



A family of conoids:



Ex. 9- Catenoid

Analytically determine the area of a catenoid bounded by two parallel and coaxial rings of radius R=1m, separated by a distance h=1m.Compare to numerical results obtained by direct area minimization and DRM.

Consider different degrees of discretization and check convergence of the numerical results to the analytical one.

Try to find the Goldschmidt limit, considering numerical models with separations just below and above h_{lim}=1.32R

Show that for distances h>1.056R, the area of the catenoid is actually greater than the area of two independent circular discs bounded by the rings (the solution for the minimum area problem with two separate discs is know as Goldschmidt discontinuous solution).