FIFTEENTH PAN-AMERICAN CONGRESS OF APPLIED MECHANICS

PACAM XV

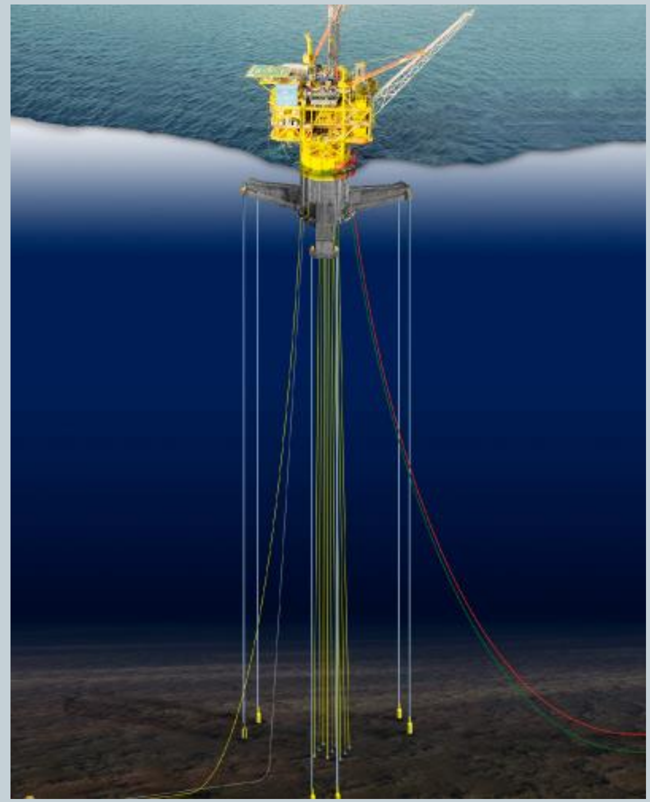
Parametric instability analysis of straight risers via a ROM based on non-linear Bessel-like modes

Carlos Mazzilli and Thiago Dias

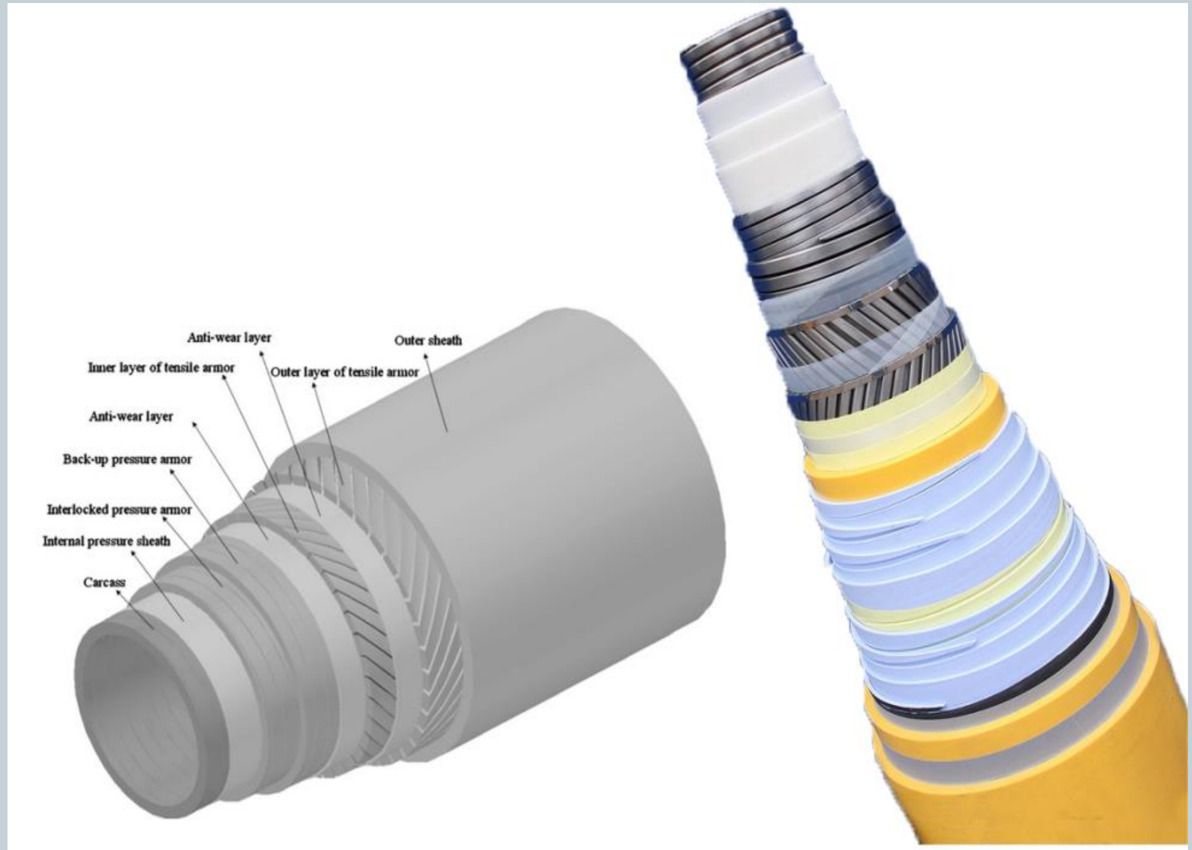
Escola Politécnica

Universidade de São Paulo

Vertical risers



rigid riser

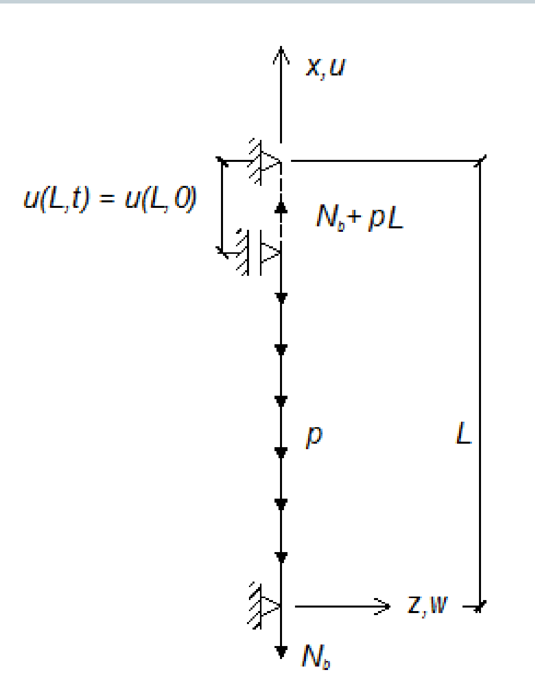


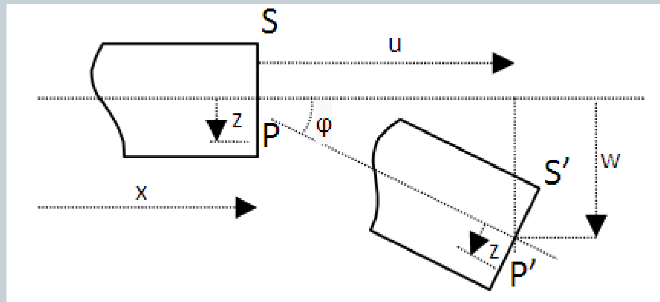
flexible riser

PART I

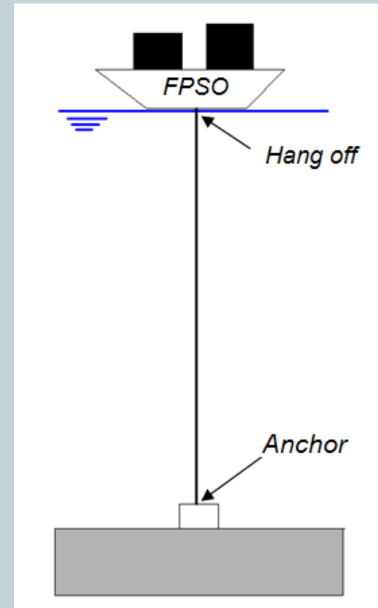
Free (undamped!) vibrations

Free vibrations





$$\begin{cases} m\ddot{u} - \left[EA\left(u' + \frac{1}{2}w'^2\right)\right]' + p = 0 \\ m\ddot{w} + EIw^{IV} - \left[EA\left(u' + \frac{1}{2}w'^2\right)w'\right]' = 0 \end{cases}$$



Approximation:

$$m\ddot{u} \cong 0 \Rightarrow N(x,t) = N_b(t) + px = N_t(t) - p(L - x)$$

$$N(x,t) = \left[EA\left(u' + \frac{1}{2}w'^2\right)\right];\; N_b(t) = N(L,t);\;\; N_t(t) = N(0,t)$$

Free vibrations

$$N(x,t) = N_b(t) + px = EA\left[u' + \frac{1}{2}w'^2\right] \Rightarrow N_b(t) = \frac{EA}{L}u(L,0) - \frac{pL}{2} + \frac{EA}{2L}\int_0^L [w'(x,t)]^2 dx$$

$$\Downarrow$$

$$N(x,t) = \underbrace{\frac{EA}{L}u(L,0) - \frac{pL}{2} + px}_{N(x,0)} + \frac{EA}{2L}\int_0^L [w'(x,t)]^2 dx = N(x,0) + \frac{EA}{2L}\int_0^L [w'(x,t)]^2 dx$$

$$\Downarrow$$

$$m\ddot{w} + EIw^{IV} - \left[\underbrace{EA\left(u' + \frac{1}{2}w'^2\right)}_{N(x,t)} w'\right]' = 0 \Rightarrow m\ddot{w} + EIw^{IV} - \left[N(x,0)w' + \frac{EA}{2L}w'\int_0^L [w'(x,t)]^2 dx\right]' = 0$$

$$\Downarrow$$

$$\boxed{EIw^{IV} - N(x,0)w'' - pw' - \frac{EA}{2L}w''\int_0^L w'^2 dx + m\ddot{w} = 0}$$

Further approximations...

- Single-mode dynamics

$$w(x,t) = W(x)\sin\omega t \quad \Rightarrow \quad EIW^{IV} - N(x,0)W'' - pW' - \frac{EA}{2L}W''\sin^2\omega t\int_0^L W'^2dx - m\omega^2 W = 0$$

- Temporal Galerkin projection

$$EIW^{IV} - N(x,0)W'' - pW' - \frac{3EA}{8L}W''\int_0^L W'^2dx - m\omega^2 W = 0$$

- Definition of fictitious normal force $\Delta N(x)$ such that

$$EIW^{IV} - \frac{3EA}{8L}W''\int_0^L W'^2dx = -\Delta N(x)W''$$

Senjanovic et al (2006)

...lead to "equivalent" cable equation...

$$\boxed{[N(x) + \Delta N(x)]W'' + pW' + m\omega^2 W = 0}$$

for tension $N(x) + \Delta N(x)$

Still another approximation...

non-linear correction

$$\Delta N(x) \cong \Delta N_{0n} = \left(\frac{n\pi}{L}\right)^2 EI_{eq} \quad \text{with} \quad EI_{eq} = EI\left(1 + \frac{3\eta_n^2}{16}\right) \qquad \eta_n = \frac{W_{0n}}{r} \qquad r = \sqrt{\frac{I}{A}}$$

"Fictitious normal force" for linear mode $W(x) = W_{0n} sin\left(\frac{n\pi x}{L}\right)$

then the "equivalent cable" equation becomes...

$$a\left[x + \frac{N_{bn}}{p}\right] W_n'' + aW_n' + W_n = 0 \qquad \text{with} \quad a = \frac{p}{m{\omega_n}^2} \qquad N_{bn} = N_b(0) + \Delta N_{0n}$$

or $\dfrac{d^2W_n}{dz^2} + \dfrac{1}{z}\dfrac{dW_n}{dz} + W_n = 0$ after variable transformation

$$x = \frac{az^2}{4} - \frac{N_{bn}}{p}$$

$$z = 2\frac{\sqrt{m(N_{bn} + px)}}{p}\omega_n$$

Non-linear mode in *lato-sensu*...

$$W_n(z) = A_n\, J_0(z) + B_n\, Y_0(z)$$

where $J_0(z)$ and $Y_0(z)$ are the Bessel functions of first and second kind and zero order

A_n and B_n are integration constants, which depend on the boundary conditions

The frequency ω_n of the non-linear mode is determined when the condition for a non-trivial solution for A_n and B_n is imposed

Alternative determination of non-linear mode in *lato-sensu*...

Still a variable change... $W_n = \frac{1}{\sqrt{z}} X_n \Rightarrow \frac{d^2 X_n}{dz^2} + \left(1 + \frac{1}{4z^2}\right) X_n = 0$

with $\frac{1}{4z^2} \ll 1$ and also its average value $2\varepsilon = \frac{1}{z_t - z_b} \int_{z_b}^{z_t} \frac{dz}{4z^2} = \frac{1}{4 z_b z_t} \ll 1$

$$\frac{d^2 X_n}{dz^2} + (1 + 2\varepsilon) X_n = 0$$

$X_n(z) \cong C_n \sin \beta_n z + D_n \cos \beta_n z$ with $\beta_n = \sqrt{1 + 2\varepsilon} \cong 1 + \varepsilon \cong 1$

$$W_n = \frac{1}{\sqrt{z}} [C_n \sin z + D_n \cos z]$$

Bessel-like modes

From the boundary conditions...

$$W_n(z_b) = W_n(z_t) = 0$$

$$\begin{bmatrix} \dfrac{\sin z_b}{\sqrt{z_b}} + \dfrac{\cos z_b}{\sqrt{z_b}} \\ \dfrac{\sin z_t}{\sqrt{z_t}} + \dfrac{\cos z_t}{\sqrt{z_t}} \end{bmatrix} \begin{Bmatrix} C_n \\ D_n \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

and non-trivial solutions... it comes out $\dfrac{1}{z_b z_t}\sin(z_t - z_b) = 0 \Rightarrow z_t - z_b = n\pi$

recall... $N_{bn} = N_b(0) + \Delta N_{0n}$ and $N_{tn} = N_t(0) + \Delta N_{0n}$

define... $C_n = \sqrt{z_b}\, cos(z_b)$

$$\omega_n = \frac{n\pi p}{2\sqrt{m}\left(\sqrt{N_{tn}} - \sqrt{N_{bn}}\right)} = \frac{n\pi}{2L\sqrt{m}}\left(\sqrt{N_{tn}} + \sqrt{N_{bn}}\right)$$

$$W_n(z) = \sqrt{\frac{z_b}{z}}\sin(z - z_b)$$

$$z = \frac{\sqrt{N_{bn} + px}}{\sqrt{N_{tn}} - \sqrt{N_{bn}}} n\pi \qquad z_b = \frac{\sqrt{N_{bn}}}{\sqrt{N_{tn}} - \sqrt{N_{bn}}} n\pi$$

Bessel-like modes

Previously… $W_n(z) = \sqrt{\frac{z_b}{z}}\sin(z - z_b)$ $\quad z = \frac{\sqrt{N_{bn} + px}}{\sqrt{N_{tn}} - \sqrt{N_{bn}}}n\pi$ $\quad z_b = \frac{\sqrt{N_{bn}}}{\sqrt{N_{tn}} - \sqrt{N_{bn}}}n\pi$

setting $\xi = \frac{x}{L}$ and $a_n = \frac{p}{N_{bn}}$

recalling $N_{bn} = N_{bo} + \left(\frac{n\pi}{L}\right)^2 EI\left(1 + \frac{3}{16}{\eta_n}^2\right)$ $\quad \eta_n = \frac{W_{0n}}{r}$ $\quad r = \sqrt{\frac{I}{A}}$ $\quad b_n = \frac{n\pi}{\sqrt{1+a_n L} - 1}$

$$\boxed{\psi_n(\xi, \eta_n) = (1 + a_n L\xi)^{-\frac{1}{4}}\ \sin\left[b_n\left(\sqrt{1 + a_n L\xi} - 1\right)\right]}$$

W_{0n} is the dimensional modal amplitude (defined, for convenience, where the linear- theory sinusoidal mode has a maximum)

r is the gyration radius of the riser cross section

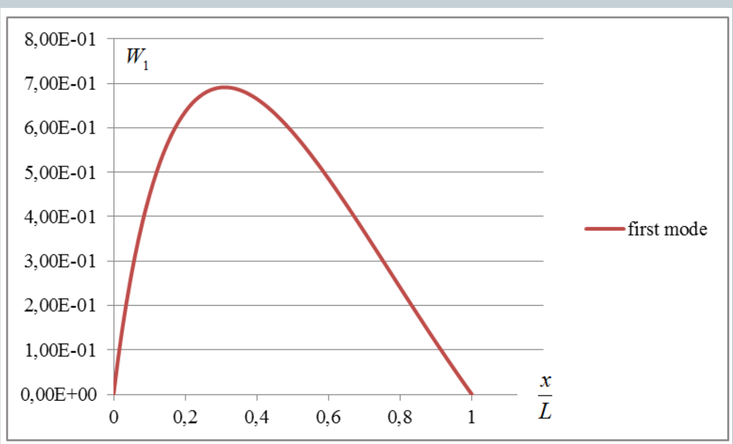
η_n is the non-dimensional amplitude

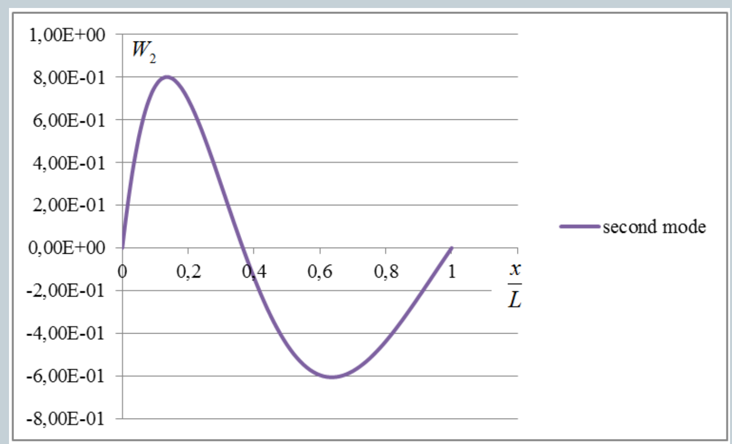
Example 1: cable model

$$EI=0;\ L=2000m;\ p=3433.2Nm^{-1};\ N_b=6.87E+05N;\ N_t=7.55E+06N,\ m=1200kgm^{-1}$$

Mode n	$\omega_n\left(rad\,s^{-1}\right)$ [1]	$\omega_n\left(rad\,s^{-1}\right)$ [2]	$\omega_n\left(rad\,s^{-1}\right)$ [3]
1	8.11E-02	8.11E-02	7.97E-02
2	1.62E-01	1.62E-01	1.61E-01
3	2.43E-01	2.43E-01	2.43E-01
4	3.24E-01	3.24E-01	3.24E-01
5	4.05E-01	4.05E-01	4.05E-01
10	8.11E-01	8.11E-01	8.11E-01

approximate solution[1] asymptotic solution[2] [Senjanovic et al] exact solution[3] [Senjanovic et al]



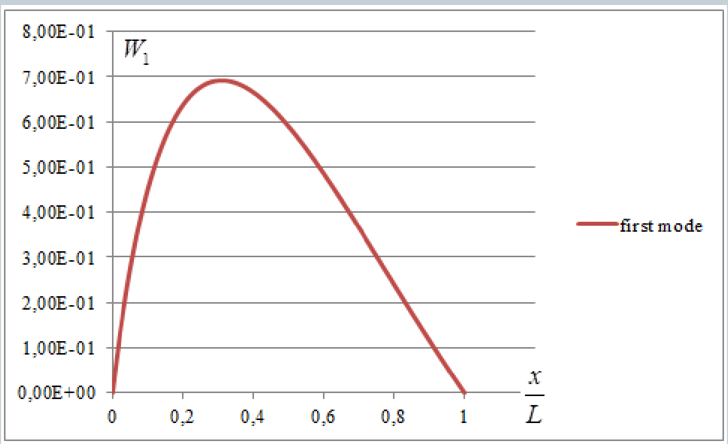


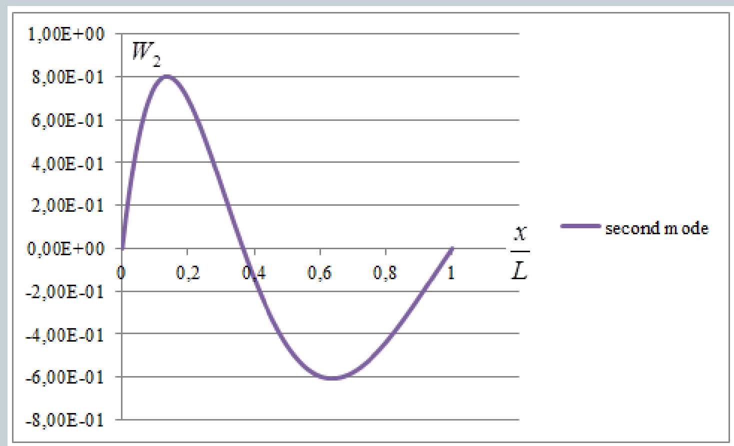
Example 2: beam model (linear $W_{0n} = 0$)

$$EI = 3.19E+08Nm^{-2};\ L = 2000m;\ p = 3433.2Nm^{-1};\ N_b = 6.87E+05N;\ N_t = 7.55E+06N;\ m = 1200kgm^{-1}$$

Mode n	$\omega_n\left(rad\ s^{-1}\right)$ [1]	$\omega_n\left(rad\ s^{-1}\right)$ [2]	$\omega_n\left(rad\ s^{-1}\right)$ [3]
1	8.11E-02	8.12E-02	7.98E-02
2	1.62E-01	1.62E-01	1.62E-01
3	2.44E-01	2.44E-01	2.44E-01
4	3.25E-01	3.26E-01	3.26E-01
5	4.07E-01	4.09E-01	4.09E-01
10	8.25E-01	8.36E-01	8.36E-01

approximate solution[1] asymptotic solution[2] [Senjanovic et al] finite elements[3] [Senjanovic et al]





Example 2: beam model (non-linear)

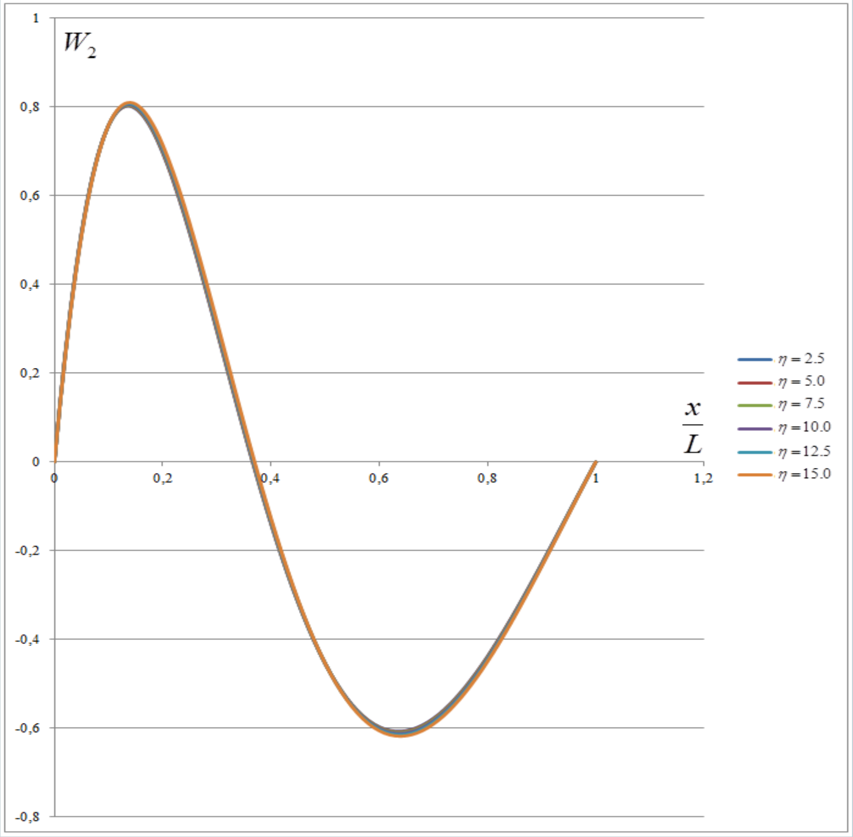
$$EI = 3.19E+08Nm^{-2};\ L = 2000m;\ p = 3433.2Nm^{-1};\ N_b = 6.87E+05N;\ N_t = 7.55E+06N;\ m = 1200kgm^{-1}$$

Mode n	$\omega_n\ (rad\ s^{-1})$ $\eta_n = 0.0$	$\omega_n\ (rad\ s^{-1})$ $\eta_n = 2.5$	$\omega_n\ (rad\ s^{-1})$ $\eta_n = 5.0$	$\omega_n\ (rad\ s^{-1})$ $\eta_n = 7.5$	$\omega_n\ (rad\ s^{-1})$ $\eta_n = 10.0$	$\omega_n\ (rad\ s^{-1})$ $\eta_n = 12.5$	$\omega_n\ (rad\ s^{-1})$ $\eta_n = 15.0$
1	8.11e-02	8.11e-02	8.12e-02	8.13e-02	8.14e-02	8.15e-02	8.17e-02
2	1.62e-01	1.62e-01	1.63e-01	1.63e-01	1.64e-01	1.66e-01	1.67e-01
3	2.44e-01	2.44e-01	2.45e-01	2.48e-01	2.50e-01	2.54e-01	2.58e-01
4	3.25e-01	3.26e-01	3.29e-01	3.34e-01	3.41e-01	3.49e-01	3.59e-01
5	4.07e-01	4.09e-01	4.15e-01	4.25e-01	4.37e-01	4.52e-01	4.69e-01
10	8.25e-01	8.40e-01	8.83e-01	9.46e-01	1.02e-00	1.11e-00	1.21e-00

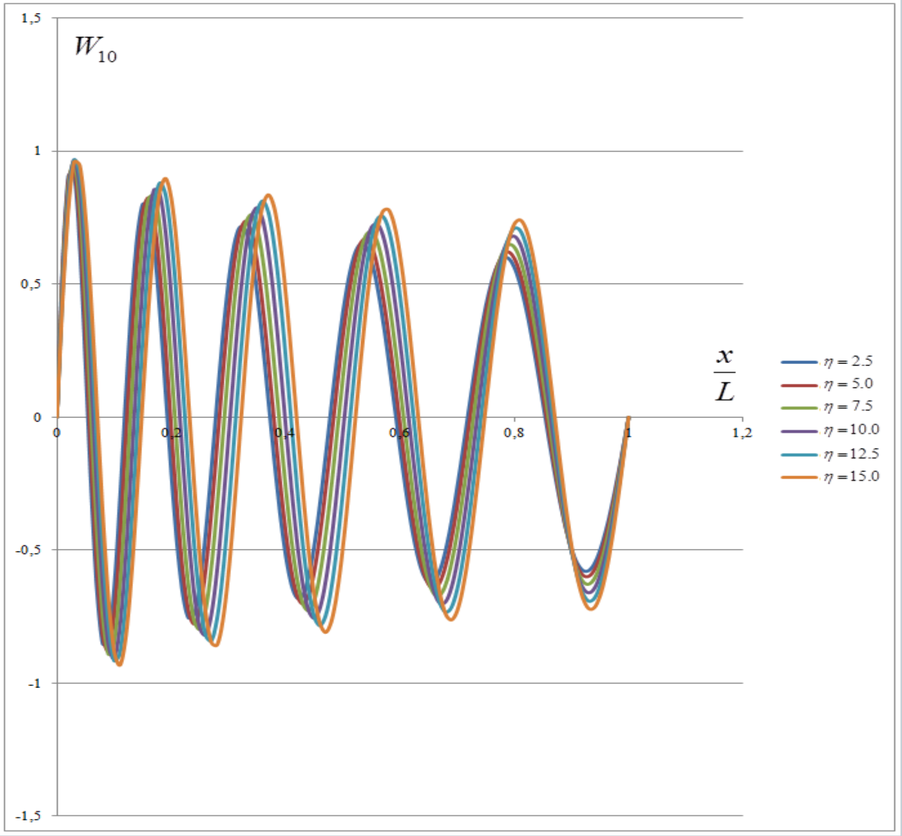
FIFTEENTH PAN-AMERICAN CONGRESS OF APPLIED MECHANICS

PACAM XV

Example 2: beam model (non-linear)



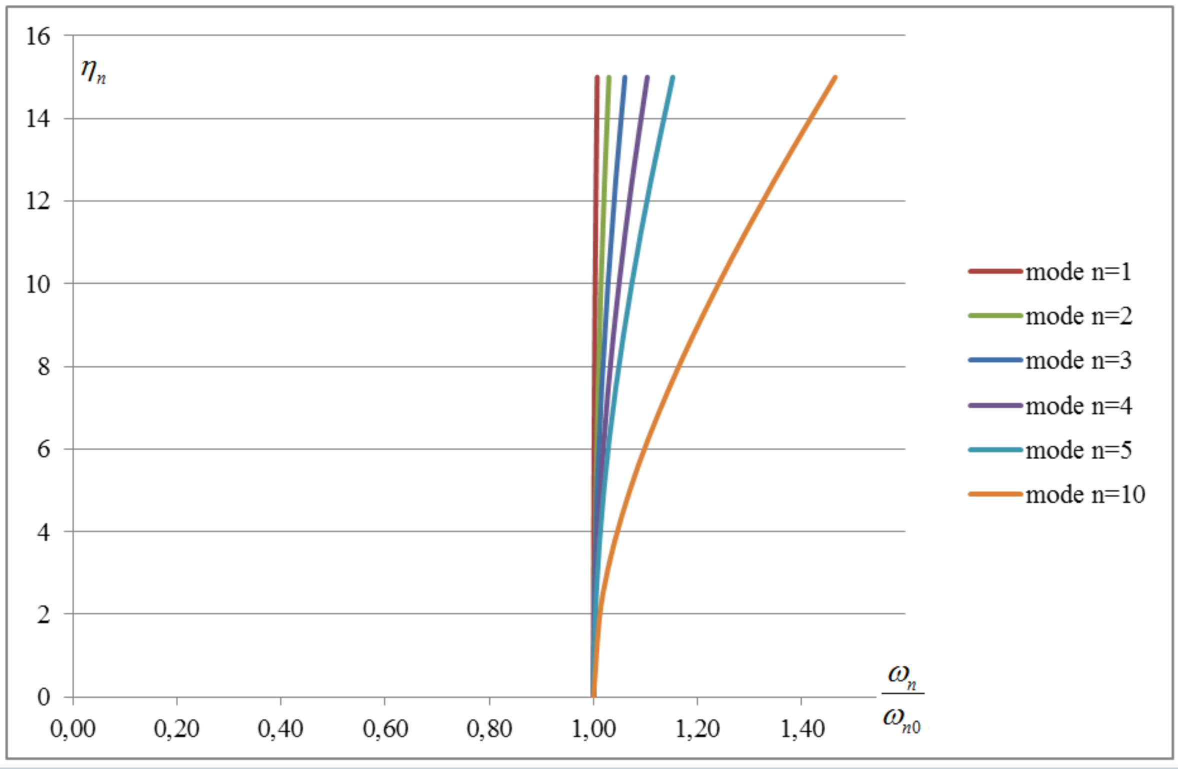
second mode



tenth mode

Example 2: beam model (non-linear)

$$EI = 3.19E+08Nm^{-2};\ L = 2000m;\ p = 3433.2Nm^{-1};\ N_b = 6.87E+05N;\ N_t = 7.55E+06N;\ m = 1200kgm^{-1}$$



backbone curve

Remarks on Bessel-like modes

- Comparison between the proposed simplified analysis and ‘exact’ or finite element solutions, whenever available, indicates minor differences (less than 1.5%), as far as the riser “linear” natural frequencies up to the tenth mode are considered.

- As for the non-linear analysis, the approximate solution is still expected to supply accurate information regarding modal frequencies and shapes to be used, for example, as projection functions within non-linear Galerkin procedures to obtain reduced-order models (ROM’s).

PART II

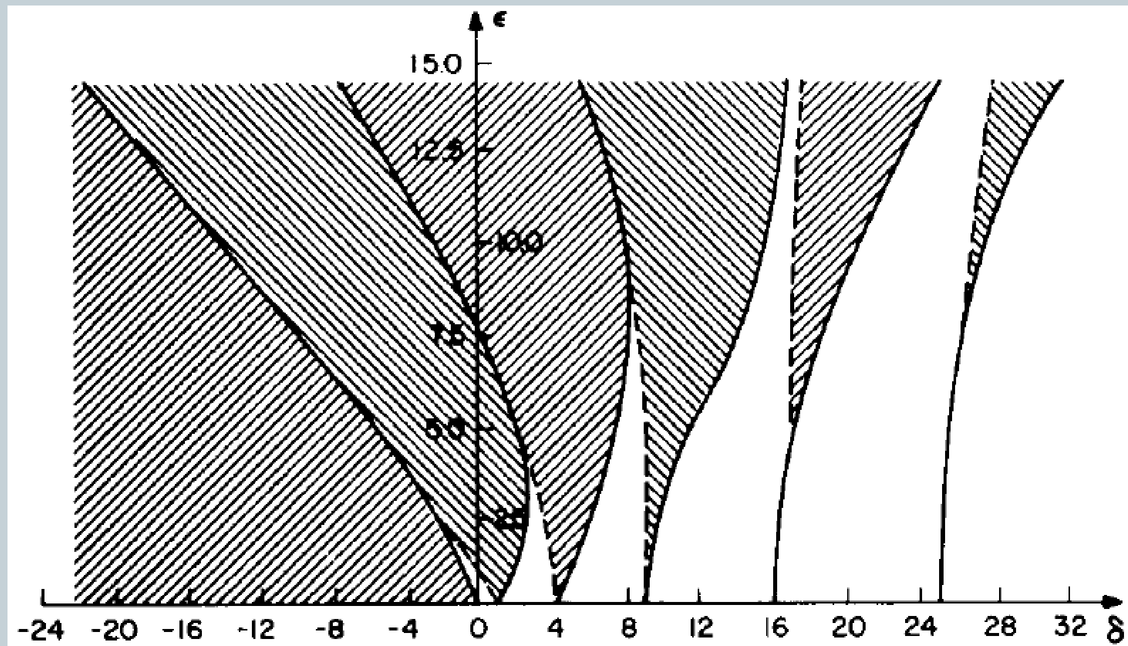
Reduced-Order Model (ROM) for heave-imposed vibrations

Underlying problem: parametric instability!

Mathieu Equation $\ddot{x} + (\delta + 2\epsilon \cos 2t)x = 0$

where $\delta = \left(\frac{2\omega}{\Omega}\right)^2$; ϵ is forcing amplitude

Hatched areas denote instability of trivial solution (amplitude grows without bound)

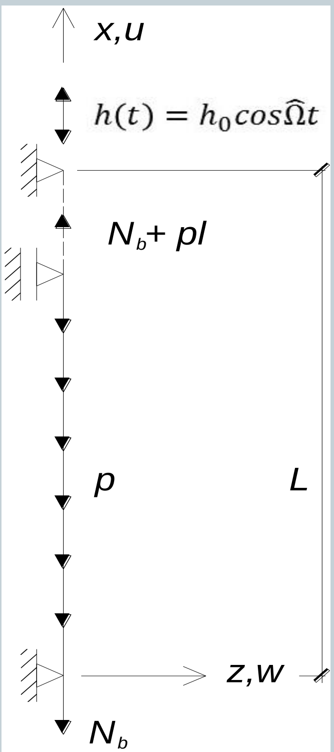


Strutt's diagram (from Nayfeh and Mook)

Heave-imposed vibrations

$$m\frac{\partial^2 w}{\partial t^2}+EI\frac{\partial^4 w}{\partial x^4}-\frac{\partial}{\partial x}\left[\left(N_b(0)+px+\frac{EAh(t)}{L}+\frac{EA}{2L}\int_0^L\left(\frac{\partial w}{\partial x}\right)^2 dx\right)\frac{\partial w}{\partial x}\right]$$
$$=-C_a\rho_w\frac{\pi D^2}{4}\frac{\partial^2 w}{\partial t^2}-\frac{1}{2}C_d\rho_w D\left|\frac{\partial w}{\partial t}\right|\frac{\partial w}{\partial t}$$

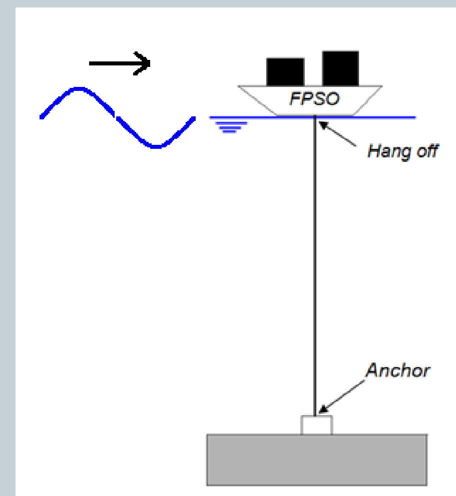
Morison force



$$\frac{\partial^2 w}{\partial t^2}+\alpha\frac{\partial^4 w}{\partial x^4}-[\beta+\beta_h(t)]\frac{\partial^2 w}{\partial x^2}+\gamma\left(\frac{L}{2}-x\right)\frac{\partial^2 w}{\partial x^2}$$
$$-\gamma\frac{\partial w}{\partial x}-\mu\frac{\partial^2 w}{\partial x^2}\int_0^L\left(\frac{\partial w}{\partial x}\right)^2 dx=\vartheta_w(t)$$

$$\alpha=\frac{EI}{m};\ \beta=\frac{\bar{N}}{m};\ \beta_h(t)=\frac{EAh(t)}{mL};\ \gamma=\frac{p}{m}$$

$$\mu=\frac{EA}{ml};\ \vartheta(x,t)=-\frac{C_a}{m}\rho_w\frac{\pi D^2}{4}\frac{\partial^2 w}{\partial t^2}-\frac{1}{2}\frac{C_d}{m}\rho_w D\left|\frac{\partial w}{\partial t}\right|\frac{\partial w}{\partial t}$$



Normalization

$$v = \frac{w}{D} \qquad \xi = \frac{x}{L} \qquad \tau = \widehat{\omega}_1 t \qquad \widehat{\omega}_1 = \frac{\omega_1}{\sqrt{1 + C_a^*}} \qquad \omega_1 = \frac{\pi p}{2\sqrt{m}\left(\sqrt{N_{tn}} - \sqrt{N_{bn}}\right)} \qquad C_a^* = \frac{\rho_w \frac{\pi D^2}{4}}{m} C_a$$

$$\frac{\partial^2 v}{\partial \tau^2} + \alpha_1 \left|\frac{\partial v}{\partial \tau}\right| \frac{\partial v}{\partial \tau} + \alpha_2 \frac{\partial^4 v}{\partial \xi^4} - \left[\alpha_3 + \alpha_h \cos\left(\frac{\widehat{\Omega}}{\widehat{\omega}_1}\tau\right)\right] \frac{\partial^2 v}{\partial \xi^2} + \alpha_4 \left(\frac{1}{2} - \xi\right) \frac{\partial^2 v}{\partial \xi^2} - \alpha_5 \frac{\partial v}{\partial \xi} - \alpha_6 \frac{\partial^2 v}{\partial \xi^2} \int_0^1 \left(\frac{\partial v}{\partial \xi}\right)^2 d\xi = 0$$

with

$$\alpha_1 = \frac{C_d \rho_w D}{2m(1 + C_a^*)} \qquad \alpha_2 = \frac{\alpha}{\widehat{\omega}_1^2 L^4 (1 + C_a^*)} \qquad \alpha_3 = \frac{\beta}{\widehat{\omega}_1^2 L^2 (1 + C_a^*)} \qquad \alpha_h = \frac{E A h_0}{m \widehat{\omega}_1^2 L^3 (1 + C_a^*)}$$

$$\alpha_4 = \alpha_5 = \frac{\gamma}{\widehat{\omega}_1^2 L (1 + C_a^*)} \qquad \alpha_6 = \frac{\mu r^2}{\widehat{\omega}_1^2 L^3 (1 + C_a^*)}$$

Galerkin projection onto Bessel-like modes

$$\ddot{\eta} I_0 + \dot{\eta}|\dot{\eta}|\alpha_1 I_1 + \eta\left\{\alpha_2 I_2 - \left[\alpha_3 + \alpha_h \cos\left(\frac{\hat{\Omega}}{\hat{\omega}_1}\tau\right)\right] I_3 + \alpha_4 I_4 - \alpha_5 I_5\right\} - \eta^3 \alpha_6 I_6 = 0$$

with

$$I_0 = \int_0^1 \phi^2 d\xi;\ \ I_1 = \int_0^1 |\phi|\phi^2 d\xi;\ \ I_2 = \int_0^1 \phi^{IV}\phi\, d\xi;\ \ I_3 = \int_0^1 \phi^{II}\phi\, d\xi$$

$$I_4 = \int_0^1 \left(\frac{1}{2} - \xi\right)\phi^{II}\phi d\xi;\ \ I_5 = \int_0^1 \phi^{I}\phi d\xi;\ \ I_6 = \int_0^1 \phi^{II}\phi\left(\int_0^1 (\phi^{I})^2\, d\xi\right) d\xi$$

Normalization for mode n: $\phi(\xi, \eta_n) = \dfrac{\psi_n(\xi, \eta)}{\psi_n(\bar{\xi}, \eta)}$ $\quad \bar{\xi} = (2n)^{-1}$ $\quad \sin(n\pi\bar{\xi}) = 1$

Two Strategies

Fictitious normal force $\Delta N_{0n} = \left(\frac{n\pi}{L}\right)^2 EI \left(1 + \frac{3\eta_n^2}{16}\right)$

Strategy 1
Neglect amplitude correction in ΔN

In this strategy, the modal-amplitude correction in ΔN is considered null throughout the process

$$\eta_0 = 0$$

Strategy 2
Consider amplitude correction in ΔN

In this strategy, the modal amplitude is updated (and so it is ΔN) through an iterative procedure until convergence is achieved

$$\eta_0 = 0 \rightarrow \eta_1 \rightarrow \eta_2 \ldots \rightarrow \eta_n \rightarrow \eta_n$$

Strategy 2
Considering amplitude correction in ΔN

In this strategy, the modal amplitude is corrected through an iterative procedure until convergence is achieved. It starts with null amplitude and obtains a new value. This value serves as input for the next iteration, and the process is repeated until convergence is achieved. Intermediate values for η are shown:

Iteration	Input	Output
1ª	0.0000	2.2162
2ª	2.2162	2.2168
3ª	2.2168	2.2168

Case study

System parameters

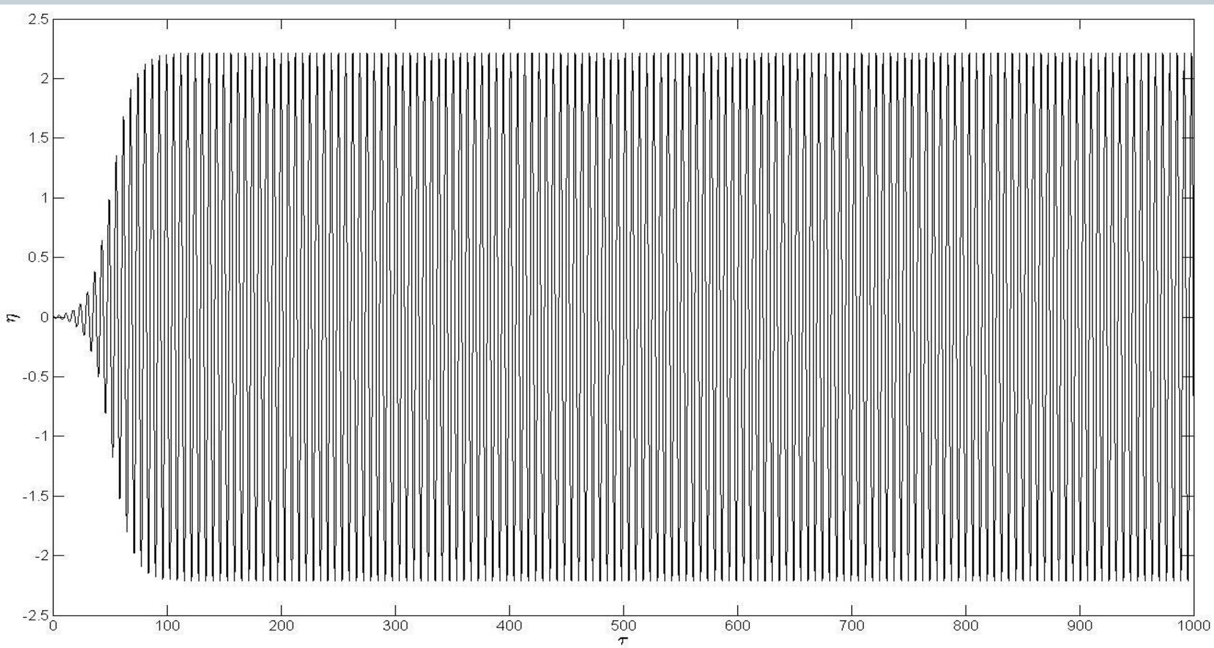
Unstretched length	L	2552 mm	Fluid density	ρ_w	1000kgm^{-3}
External diameter	D	22.2mm	Mass ratio	C_a^*	0.3252
Gyration radius	r	6.8 mm	Added mass coefficient	C_a	0.998
Riser unit mass	m	1.190kgm^{-1}	Drag coefficient	C_d	2.84
Immersed weight	p	7.869Nm^{-1}	Natural frequency in air	ω_1	5.773rads^{-1}
Axial stiffness	EA	1207N	Natural frequency in still water	$\hat{\omega}_1$	5.015rads^{-1}
Bending stiffness	EI	0.056Nm^2	Heave amplitude	h_0	0.025m
Average normal force	$\bar{N}$	23.65N	Heave frequency	$\hat{\Omega}$	10.030rads^{-1}

Data of experimental model tested at IPT laboratory in São Paulo

FIFTEENTH PAN-AMERICAN CONGRESS OF APPLIED MECHANICS

PACAM XV

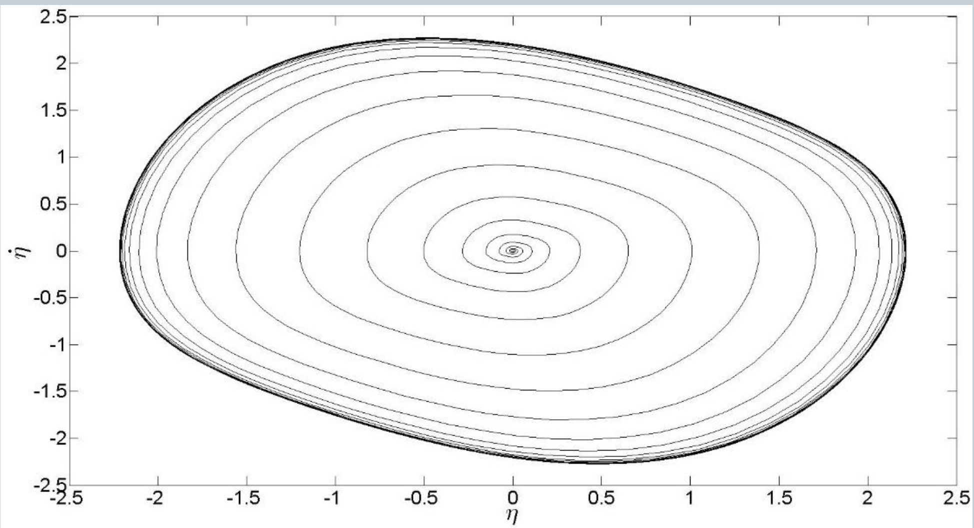
Case study (for mode n=1)



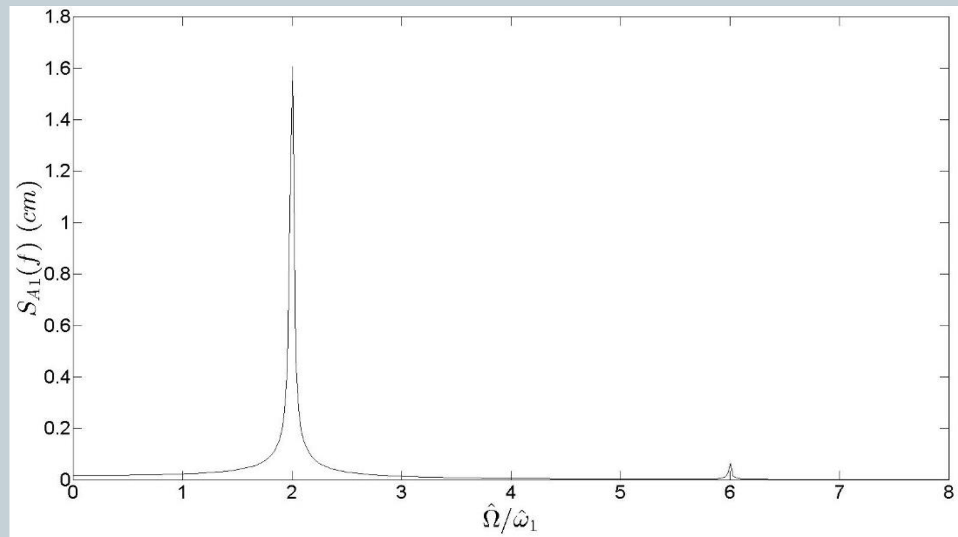
Maximum normalized amplitude $\eta_{max} = 2.2168$.

Maximum dimensional amplitude (after multiplication by the gyration radius $r = 0.0068m$)
$W_{01} = 0.0151m$.

Case study (for mode n=1)

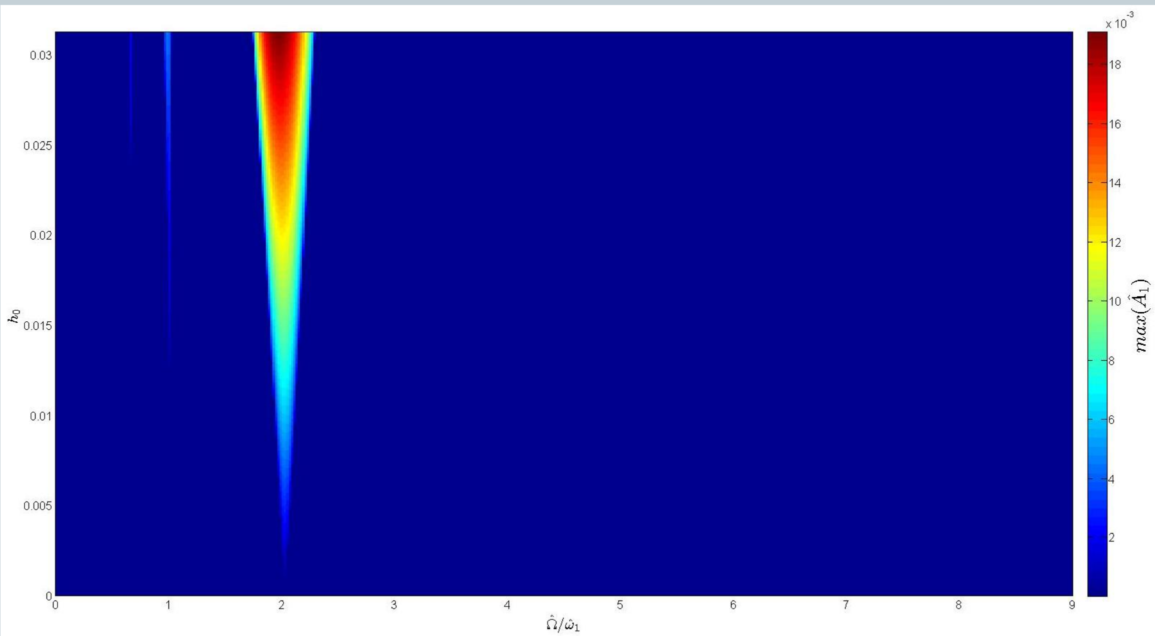


phase space



amplitude spectrum

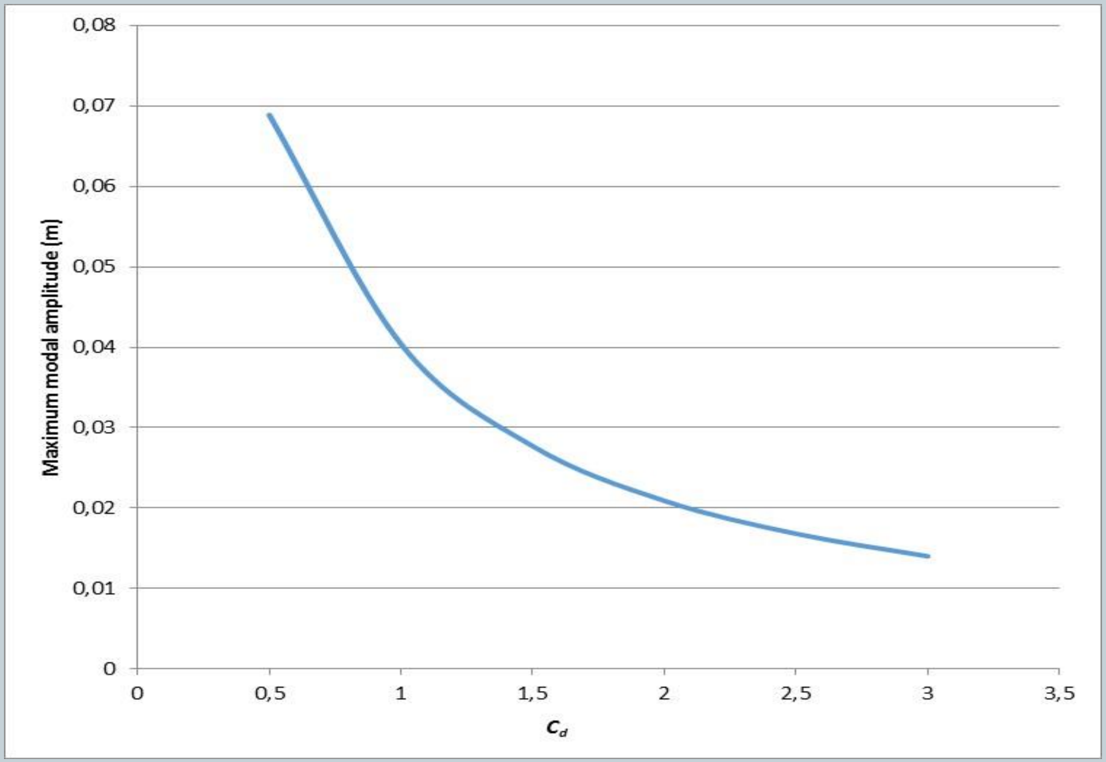
Case study (for mode n=1)



Response-amplitude mapping in the control-parameter plane $\left(\frac{\widehat{\Omega}}{\widehat{\omega}_1}, h_0\right)$, for post-critical steady states, which recalls the classic Strutt's diagram (for linear systems)

Diagram Franzini-Sakamoto

Case study



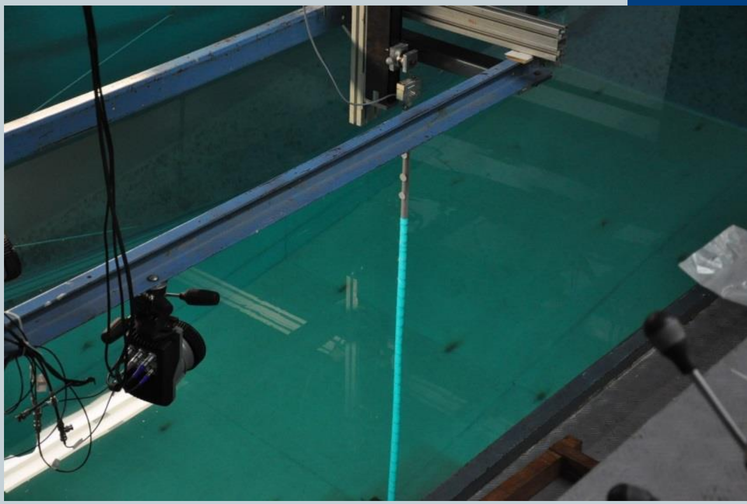
Post-critical amplitude as a function of the drag coefficient

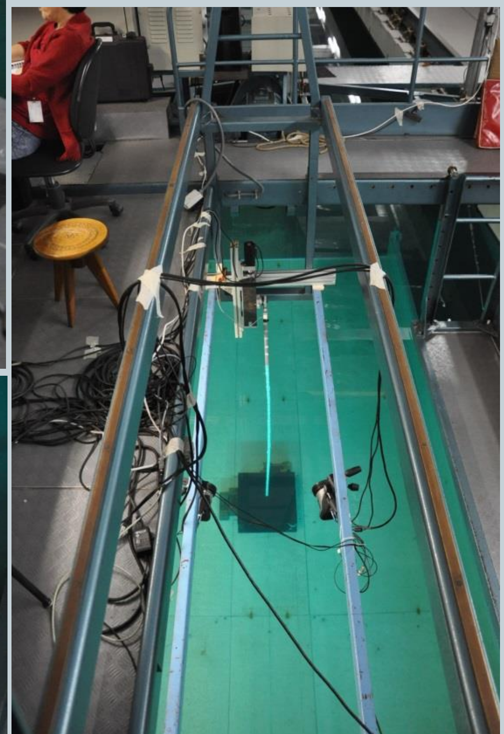
PART III

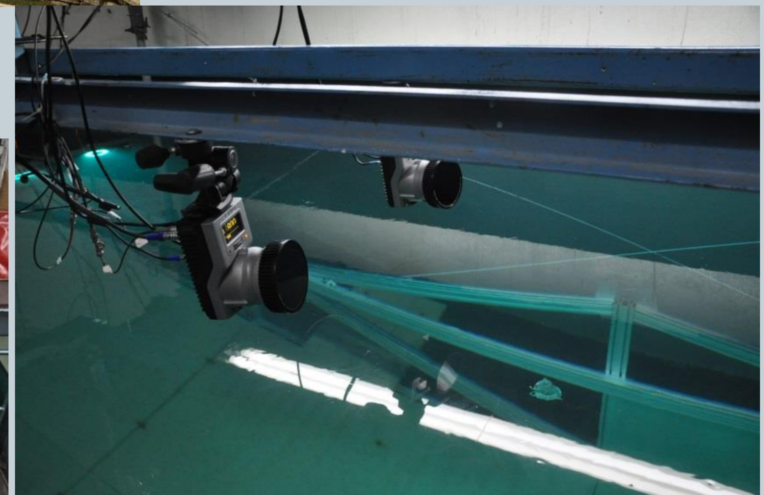
Experimental research for n=1

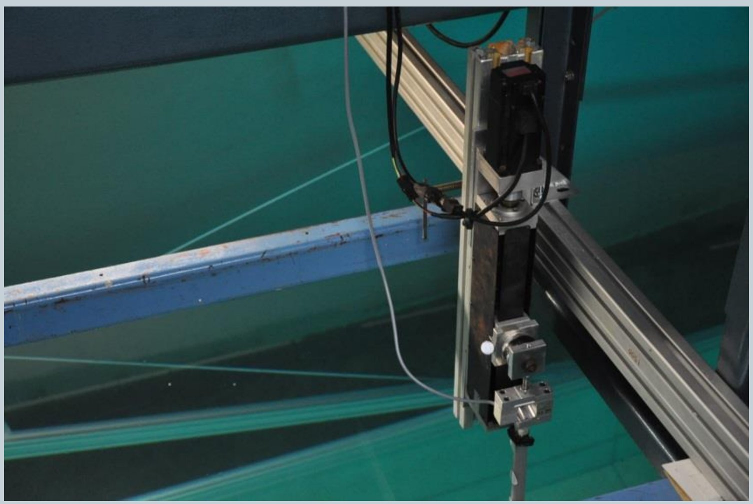
FIFTEENTH PAN-AMERICAN CONGRESS OF APPLIED MECHANICS

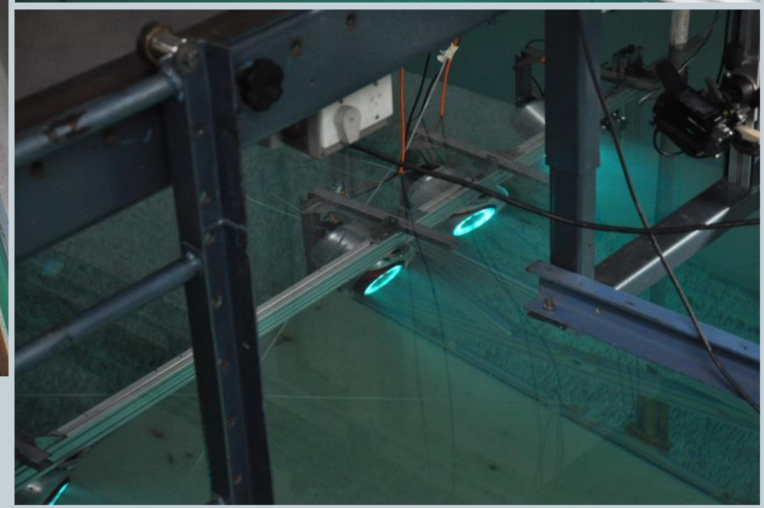
PACAM XV







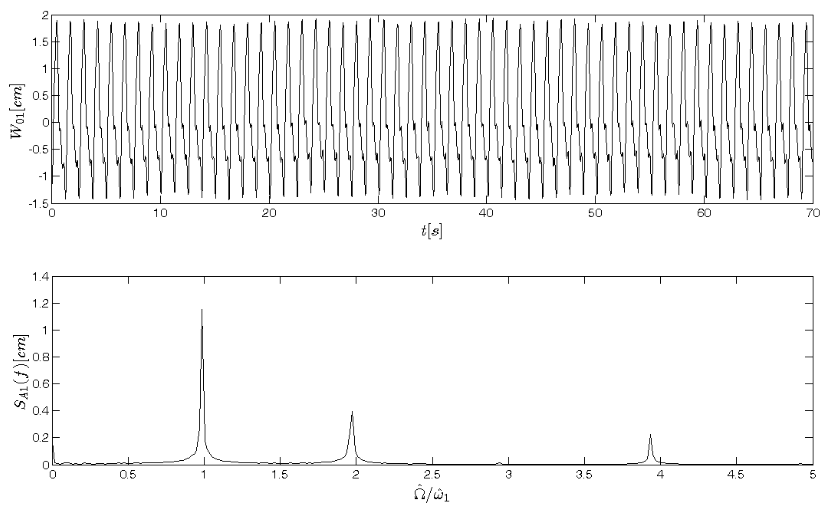




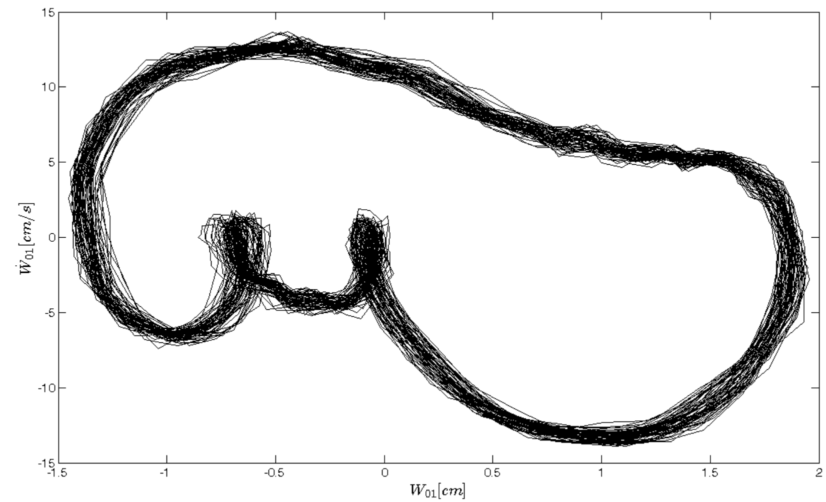
FIFTEENTH PAN-AMERICAN CONGRESS OF APPLIED MECHANICS

PACAM XV

Experimental Results



time response and amplitude spectrum



phase space

Experimental Results

Model	$h_0\ (cm)$	$\hat{\Omega}\ (\frac{rad}{s})$	$W_{01}\ (cm)$
Experimental	2.50	10.03	1.50
Orcaflex	2.50	10.03	3.80
ROM trigonometric	2.50	10.03	1.23
ROM Bessel-like 1	2.50	10.03	1.51
ROM Bessel-like 2	2.50	10.03	1.51

ROM's were calibrated to have the same first natural frequency $\hat{\omega}_1 = 5.015\,\frac{rad}{s}$, which required an added mass coefficient $C_a = 0.998$.

The drag coefficient in all ROM's was $C_d = 2.84$

.

Remarks on the comparison with experimental work for n=1

- Both ROM models based on the Bessel-like modes presented a very accurate correlation with the experimental results, as far as the maximum amplitude values and positioning were concerned, provided adequate calibration of added-mass and drag coefficients were carried out.

- ROM based on the classic trigonometric mode with the same calibration underestimated the maximum amplitude.

- High-hierarchy FEM model didn’t supply a good estimate for the maximum amplitude.

PART IV

Concluding Remarks

- ROM’s for vertical risers based on Bessel-like modes are useful for the analysis of vibrations induced by imposed motions at the riser top.

- Ongoing research also shows that ROM’s for vertical risers based on Bessel-like modes coupled to phenomenological fluid-oscillator models are useful for the analysis of vortex-induced vibrations (VIV)

Acknowledgements

The support of CNPq (Brazilian Science Research Council) under Grant 302757/2013-9 is acknowledged.

Thanks are also expressed to Fabio Rizza (MSc student from Politecnico di Milano)