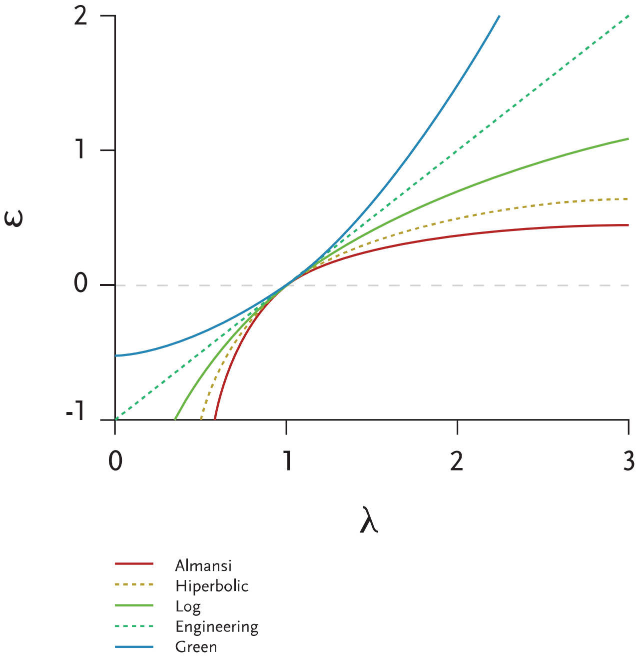


Different strain measures as a function of the stretching.

where σ_N is the nominal or engineering stress (see table above). The stress—strain conjugate pairs indicated in the table above form a family of stress—strain measures called consistent.

In the next chapter, we will introduce the concept of the gradient of deformation and use it to express strain and stress measures.

It is also interesting to note that the time derivative of the above equation gives the general stress rate,

$$\dot{\sigma}_m = \dot{\sigma}_N \lambda^{1-m} + (1-m)\lambda^{-m}\sigma_N \dot{\lambda},$$

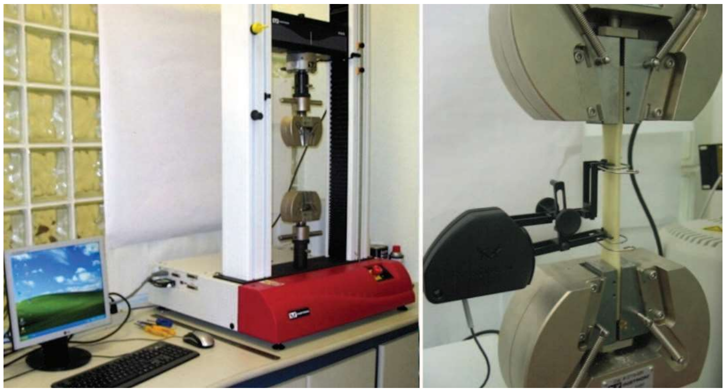
which does not converge to a common value, even for $\lambda \approx 1$.

7.2 Tensile tests

From the previous section, basic knowledge of a material mechanical behaviour can be gained by loading a rod and measuring the force and the stretching. Once these variables are known, one could express the mechanical behaviour of the material by the stress—strain curve. This curve plays a fundamental role in any structural analysis, as we have already experienced in previous chapters.

The tensile test machine, as the one depicted in the next figure, is the basic apparatus to measure the strength of a material and to obtain its stress—strain relation. The material sample is usually machined flat or on a cylindrical shape. There are standards describing the geometry and the test, although a given research may require alternative dimensions. The figure also shows a close–up of the tensile specimen fixed by the machine jaws. In the straight segment of the specimen, there is clip gauge to measure the displacement between the two knives of the gauge as the specimen is pulled by the machine. Also, on the top of the superior jaw, there is a load cell so it is possible to obtain the load and the stretching during the test.

We can use strain gauges to measure the strains in this test. One arrangement is two strain gauges glued in opposite sides of the specimen and connected in half—bridge configuration.



A tensile test machine and a composite material sample fixed in the jaws.

A remark is that tensile test machines have also an accurate sensor to measure the jaw displacement. However, this information needs to be handled with care since the machine transverse beam, which holds the jaws, deforms as the stretching load increases. This adds to the actual strain value experimented by the specimen. Hence, a local displacement measure, as the one yielded by strain or clip gauges, is far better to be used for the calculation of the strains.

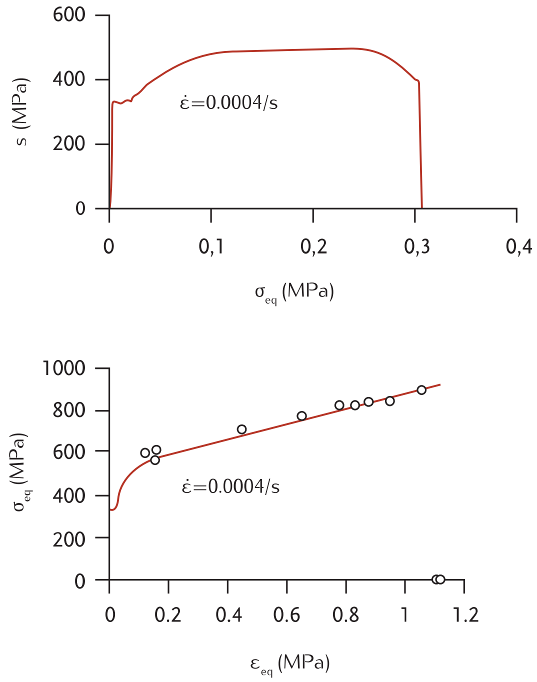
Currently, image processing techniques are being more and more used, allowing the mapping of the stretching of a sample only by tracking the motion of image pixels. We will explore this technique later in this chapter.

From the measured load, the initial specimen cross–section area and the jaw head displacement, we can obtain the nominal, or engineering, and the true stress–strain curves. From the previous sections, it is possible to change from the nominal to the true stress–strain definitions according to

$$\sigma = s(1+e) \qquad \text{and} \qquad \varepsilon = \ln(1+e),$$

when assuming that there is no volume change in the plastic regime, *ie* $AL = A_0 L_0$.

Note that the difference between engineering and true stress–strain values can be quite large for ductile materials and the next figure presents the case of a steel alloy. In this particular case, the true stress–strain curve is in the equivalent space, a concept that we will explore next.

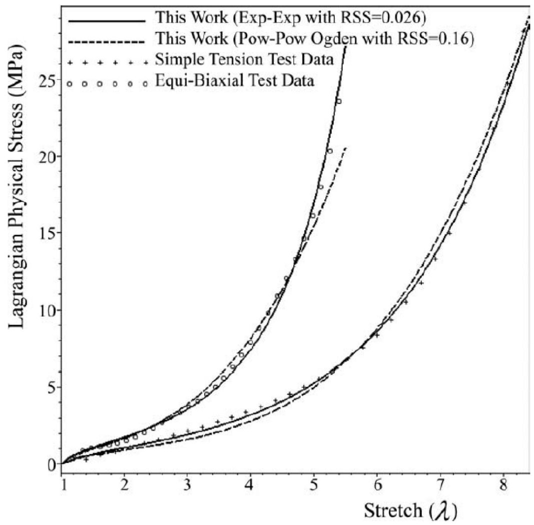


Stress–strain curves in the engineering and true equivalent spaces for the same ductile steel alloy.

We see from the results in the figure that the material in play presents a nearly linear behaviour for small strains. For many materials, when this linear stress—strain behaviour ceases, the material enters in the plastic regime, *ie* the removal of the load leaves the material permanently strained. This transition, not clear for some metals, is the yield stress, an important design parameter. There are many cases though, that even metals do not behave linearly in small strain regimes, like the cast iron seen in the opening of this chapter.

After yielding, a typical metal continues to increase its strength as it enters in the strain hardening regime. Observe that, as the strain grows, the discrepancy between the nominal and true stress curves become more evident. The true stress grows quicker because the sample cross–section area decreases.

It is important to realize that the fact that the material behaviour ceases to be linear does not mean that it entered the plastic regime. As shown in the next figure for a polymer, the behaviour is non–linear in an early stage but there is no guarantee that such a behaviour is plastic. The only macroscopic way to gauge plastic strains is by unloading the sample and measuring the left deformation. In other words, plastic strains are revealed only after load removal, given that what one measures during a test is the total strain. This is why the plastic strains are called an internal variable.



Stress—strain curve for a polymer showing non–linear behaviour at an early stage of loading (from H. Darijani, R. Naghdabadi and M.H. Kargarnovin, Constitutive modeling of rubberlike materials based on consistent strain energy density functions, Polymer Engineering and Science, p. 1058–1066, 2010).

7.2.1 Equivalent stress and strain

So far in this chapter, we have been dealing only with uniaxial stress. More generally, the stress in a point is represented by a set of stress components, according to the reference system used and to the assumptions made. Two important questions arise. First, how to represent this set of stress components by a single representative stress value. Second, to what value this single parameter should be compared to in order to allow us to access whether material plastic flow is established. As for the first question let us define the effective, or equivalent (true) stress, or von Mises stress as

Some lectures notes delivered by Richard von Mises at the School of Engineering, Harvard University, in 1946, were registered by M. Vargas and collect by R.T. Vargas in book form available at www.gmsie.usp.br.

$$\sigma_{eq}^2 = \frac{(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6(\sigma_{xy}^2 + \sigma_{yz}^2 + \sigma_{zx}^2)}{2},$$

which becomes simply $\sigma_{eq} = \sigma_z$ for the uni—dimensional stress case. For the case of pure shear, we have $\sigma_{eq} = \sqrt{3}\sigma_{xy}$. It will be shown later that this expression is proportional to one of the various invariants of the stress tensor. Clearly, this stress state quantification takes into account all components of the stress tensor.

We can define a similar expression for the equivalent (true) plastic strain. We expect that this variable should always increase, for both positive and negative values of the various strain components, so to go side by side with the equivalent stress, which is always positive. Accordingly, we define the equivalent plastic strain rate as

In finite element analysis, the material behaviour is usually informed by pairs of equivalent stress — equivalent strain.

$$(\dot{\varepsilon}^p_{eq})^2 = \frac{2}{9}\left\{(\dot{\varepsilon}^p_x - \dot{\varepsilon}^p_y)^2 + (\dot{\varepsilon}^p_y - \dot{\varepsilon}^p_z)^2 + (\dot{\varepsilon}^p_z - \dot{\varepsilon}^p_x)^2 + \right\}.$$

$$\cdot\frac{2}{9}\left\{+6\left[(\dot{\varepsilon}^p_{xy})^2 + (\dot{\varepsilon}^p_{yz})^2 + (\dot{\varepsilon}^p_{zx})^2\right]\right\}$$

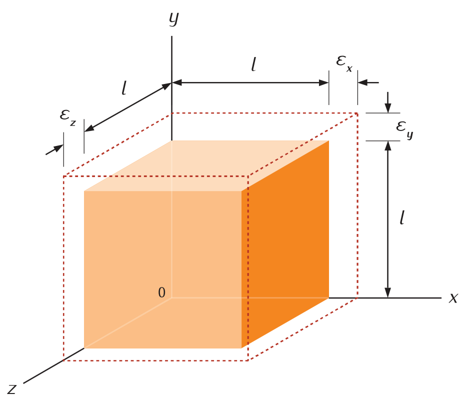
and from there the equivalent plastic strain

$$\varepsilon^p_{eq} = \int \dot{\varepsilon}^p_{eq}\mathrm{d}t.$$

For a circular bar being pulled in tension, before necking takes place, the equivalent stress reduces to the (true) unidimensional case, *ie*

$$\sigma_{eq} = \sigma_z = \frac{F}{A}.$$

To obtain an expression for the equivalent strain in such a bar, let us evaluate the variation of volume of the unity cube shown in the next figure.



A unity cube experimenting a change in its dimensions.

The variation in volume of the unity cube is $\Delta V = (1+\varepsilon_x)(1+\varepsilon_y)(1+\varepsilon_z) - 1$, which equals to $\Delta V = \varepsilon_x + \varepsilon_y + \varepsilon_z = \varepsilon_x(1-2\nu)$ when high order cross product are disregarded and use is made of the Poisson ratio definition, $\nu = -\varepsilon_y/\varepsilon_x = -\varepsilon_z/\varepsilon_x$. Now, various experiments in metals have shown that $\nu \approx 1/2$ as plastic flow grows, which indicates from the ΔV expression above that volume is conserved in the plastic regime.

For the cylinder under tension, we then write

$$\pi D_0^2 L_0/4 = \pi DL/4 \rightarrow 2\ln D/D_0 + \ln L/L_0 = 0 \rightarrow 2\varepsilon_\theta + \varepsilon_z = 0.$$

Using now the expression for the equivalent strain, we arrive at

$$\varepsilon_{eq} = \ln\frac{l}{l_0} = 2\ln\left(\frac{d_0}{d}\right),$$

which points to the fact that it is only necessary to trace the specimen diameter to obtain the equivalent strain (in the probing area). O course that we can measure the stretching of the bar along its length. The advantage of monitoring the diameter becomes more apparent when we seek for the equivalent stress—strain pair after localization, which ensues the phenomenon of necking.

7.2.2 Necking

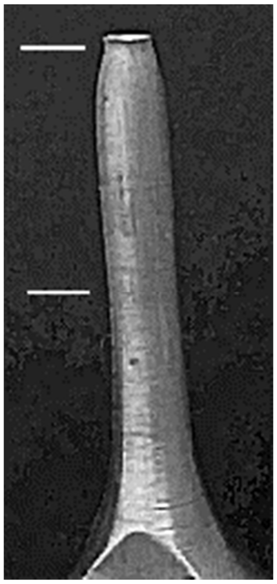
We have established that the true values of axial stress and natural axial strain that occur in a tensile test coincide with their equivalent counterparts. This is true up to a certain point in a test. If we continue to pull a tensile specimen, there will be a moment when material instability occurs and the deformation in the specimen becomes concentrated in a small necking zone, see next figure. The formation of the necking zone is roughly the point where the load reaches its maximum value, which is the ultimate engineering stress. Once necking starts, to measure the strain with the clip gauges and using $\varepsilon_{eq} = \ln(l/l_0)$ is meaningless since this measure will be a too rough value for the actual localized straining process in the neck.

The question now is the calculation of the equivalent stress and strain in the necking zone. This is important because it gives us information of the material behaviour at large strains, necessary when dealing, for instance, with crushing of a device. After necking, the stress field ceases to be unidimensional. An analysis of the stress field in the necking zone was developed by Bridgman. He found, and modern numerical models corroborate it , that the equivalent stress in this region is given by

See M. Alves and N. Jones, Influence of the hydrostatic stress on failure of axisymmetric notched specimens. Journal of the Mechanics and Physics of Solids, 47(3):643–667, 1999.

$$\sigma_{eq} = \frac{4F}{\pi d^2}\frac{1}{\left(1+\frac{4R}{d}\right)\ln\left(1+\frac{d}{4R}\right)}$$

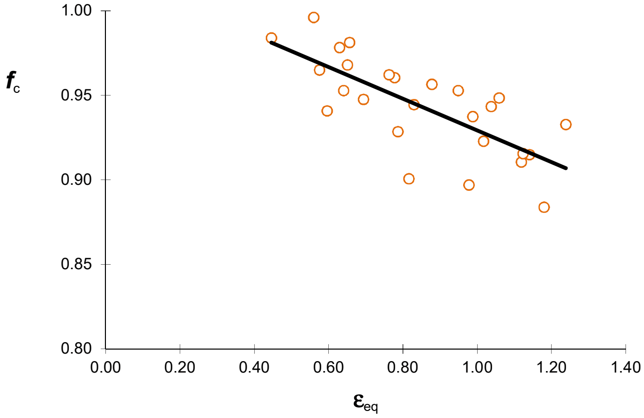
with the term $4F/\pi d^2$ being just the axial true stress and the remaining term a correction factor, f_c, which takes into account the three dimensionality of the stress field. Here, R is the necking radius, which needs



Necking zone in a tensile specimen and a sketch of the stress field.

to be measured as the test progresses, a somewhat easy task with image analysis systems.

The next figure plots the correction factor for a ductile steel alloy. We can see that the difference between the equivalent stress and the true unidimensional stress (F/A) can be significantly large and may lead to important errors in a non–linear finite element analysis if no distinction between true axial and true equivalent stresses are made.



The factor f_c for a ductile steel alloy.