

# 25th-order high-temperature expansion results for three-dimensional Ising-like systems on the simple cubic lattice.

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## Abstract

25th-order high-temperature series are computed for a general nearest-neighbor three-dimensional Ising model with arbitrary potential on the simple cubic lattice. In particular, we consider three improved potentials characterized by suppressed leading scaling corrections. Critical exponents are extracted from high-temperature series specialized to improved potentials, obtaining  $\gamma = 1.2373(2)$ ,  $\nu = 0.63012(16)$ ,  $\alpha = 0.1096(5)$ ,  $\eta = 0.03639(15)$ ,  $\beta = 0.32653(10)$ ,  $\delta = 4.7893(8)$ . Moreover, biased analyses of the 25th-order series of the standard Ising model provide the estimate  $\Delta = 0.52(3)$  for the exponent associated with the leading scaling corrections.

By the same technique, we study the small-magnetization expansion of the Helmholtz free energy. The results are then applied to the construction of parametric representations of the critical equation of state, using a systematic approach based on a global stationarity condition. Accurate estimates of several universal amplitude ratios are also presented.

**Keywords:** Critical Phenomena, Ising Model, High-Temperature Expansion, Critical Exponents, Critical Equation of State, Universal Ratios of Amplitudes.

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# 1 Introduction

The Ising model is one of the most studied systems in the theory of phase transitions, not only because it is the simplest nontrivial model that has a critical behavior with nonclassical exponents, but also because it describes the critical behavior of many physical systems. Indeed, many systems characterized by short-range interactions and a scalar order parameter undergo a continuous phase transition belonging to the Ising universality class. We mention the liquid-vapor transition in simple fluids and the critical transitions in multicomponent fluid mixtures, in uniaxial antiferromagnetic materials, and in micellar systems. Continuous transitions belonging to the three-dimensional Ising universality class are also expected in high-energy physics, for instance in the electroweak theory at finite temperature and in the theory of strong interactions at finite temperature and finite baryon-number chemical potential. For a recent review, see, e.g., Ref. [1].

The high-temperature (HT) expansion is one of the most efficient approaches to the study of critical phenomena. Very precise results have been obtained by performing careful extrapolations to the critical point, by using several different methods, see, e.g., Ref. [2]. For moderately long series, such as those available for models in the three-dimensional Ising universality class, the nonanalytic confluent corrections are the main source of systematic errors. For instance, according to renormalization-group theory, the critical behavior of the magnetic susceptibility is given by the Wegner expansion

$$\chi = Ct^{-\gamma} \left( 1 + a_\chi t^\Delta + a_2 t^{2\Delta} + \dots + bt^{\Delta_2} + \dots + e_1 t + e_2 t^2 + \dots \right), \quad (1.1)$$

where  $t \equiv (T - T_c)/T_c$  is the reduced temperature and  $\Delta$  is a noninteger exponent,  $\Delta \approx 0.5$  in the Ising case. In the analysis of HT expansions these nonanalytic terms introduce large and dangerously undetectable systematic deviations in the results.

In order to obtain precise estimates of the critical parameters, the approximants of the HT series should properly allow for the confluent nonanalytic corrections [3–9]. However, the extensive numerical work that has been done shows that in practice, with the series of moderate length that are available today, no unbiased analysis is able to take effectively into account nonanalytic correction-to-scaling terms. In order to treat them properly, one should use biased methods in which the presence of the leading nonanalytic term with exponent  $\Delta$  is imposed (see, e.g., Refs. [10–17]). An alternative approach to this problem consists in considering models—we call them *improved*—that do not couple the leading irrelevant operator that gives rise to the confluent correction of order  $t^\Delta$ . Therefore, such correction does not appear in the expansion of *any* thermodynamic quantity near the critical point: for instance,  $a_\chi = 0$  in Eq. (1.1). In this case, we expect standard analysis techniques to be much more effective, since the main source of systematic error should have been eliminated. There are no methods that allow to determine exactly improved models, and one must therefore use numerical techniques. One may use HT expansions, but in this case the improved model is determined with a relatively large error [1, 6, 8, 9, 18, 19] so that the final results do not significantly improve the estimates obtained from standard analyses using biased approximants. Recently [19–27], it has been realized that Monte

Carlo (MC) simulations using finite-size scaling techniques are very effective for this purpose, obtaining accurate determinations of several improved models in the Ising, XY, and O(3) universality classes.

As shown in Refs. [19,25,27–29], analyses of the HT series for the improved models lead to a significant improvement in the estimates of the critical exponents and of other infinite-volume HT quantities. Our working hypothesis is that, with the series of current length, the systematic errors, i.e., the systematic deviations that are not taken into account in the analysis, are largely due to the leading confluent correction, so that improved models give results with smaller and, more importantly, reliable error estimates. This hypothesis can be checked by comparing the results obtained using different improved models: if correct, they should agree within error bars. In the following we shall report results that confirm our hypothesis. Indeed, the estimates obtained from three different improved Hamiltonians are perfectly consistent. Moreover, they are very stable with respect to the order of the series considered in the analysis, without showing dangerous trends, but only an apparent reduction of the error. The results obtained in Ref. [19] using 20th-order series are fully consistent with the 25th-order analysis that we present.

We consider a simple cubic lattice and scalar models with Hamiltonian

$$\mathcal{H} = -\beta \sum_{\langle i,j \rangle} \phi_i \phi_j + \sum_i V(\phi_i^2), \quad (1.2)$$

where  $\beta \equiv 1/T$ ,  $\langle i,j \rangle$  indicates nearest-neighbor sites,  $\phi_i$  are real variables, and  $V(\phi^2)$  is a generic potential satisfying appropriate stability constraints. These models are expected to have either a critical transition belonging to the Ising universality class or a first-order transition between a disordered and an ordered phase, apart from special cases that correspond to multicritical points. Using the linked-cluster expansion technique, we compute, for an arbitrary potential, the HT expansion of the two-point correlation function to 25th order on a simple cubic lattice. These results extend those of Ref. [19] that reported the two-point function to 20th order [30]. In particular, we consider three classes of models depending on an irrelevant parameter, which is fixed by requiring the absence of the leading scaling correction. The first one is the  $\phi^4$  lattice model with potential

$$V(\phi^2) = \phi^2 + \lambda_4(\phi^2 - 1)^2. \quad (1.3)$$

MC simulations using finite-size scaling techniques have shown that the model is improved for [31]

$$\lambda_4 = \lambda_4^* = 1.10(2). \quad (1.4)$$

A consistent but less precise estimate can be obtained from the HT expansion [19]. The second class of models is the  $\phi^6$  lattice model with potential

$$V(\phi^2) = \phi^2 + \lambda_4(\phi^2 - 1)^2 + \lambda_6(\phi^2 - 1)^3. \quad (1.5)$$

Fixing  $\lambda_6 = 1$ , the  $\phi^6$  Hamiltonian is improved for [19]

$$\lambda_4 = \lambda_4^* = 1.90(4). \quad (1.6)$$

Finally, we consider the spin-1 (or Blume-Capel) Hamiltonian

$$\mathcal{H} = -\beta \sum_{\langle i,j \rangle} s_i s_j + D \sum_i s_i^2, \quad (1.7)$$

where the variables  $s_i$  take the values  $0, \pm 1$ . An improved spin-1 model is obtained for [32]

$$D = D^* = 0.641(8). \quad (1.8)$$

The comparison of the results obtained using the above-mentioned improved Hamiltonians represents a strong check of the expected reduction of systematic errors in the HT results, and provides an estimate of the residual errors due to the subleading confluent corrections to scaling.

We also extend the HT expansion of the zero-momentum  $n$ -point correlation functions  $\chi_n$ . In particular, we compute  $\chi_4$ ,  $\chi_6$ , and  $\chi_8$  to 21st, 19th, and 17th order respectively. The analysis of such series provides information on the small-magnetization expansion of the Helmholtz free energy in the HT phase. These results are used to determine approximate representations of the equation of state that are valid in the critical regime in the whole  $(t, H)$  plane. For this purpose, following Ref. [19], we use a systematic approximation scheme based on polynomial parametric representations and on a global stationarity condition. This approach allows us to obtain an accurate determination of the critical equation of state in the whole critical region up to the coexistence curve.

In Table 1 we anticipate most of the results that we shall obtain in this paper. We report HT estimates of the critical exponents and of the coefficients parametrizing the small-magnetization expansion of the Helmholtz free energy: they are denoted by IHT, where the “I” stresses the fact that we are using improved models. Then, we report several amplitude ratios (definitions are given in Sec. 5). Those appearing in the column IHT-PR are obtained from an approximate representation of the equation of state that uses the HT results as inputs, those labelled by LT are obtained from the analysis of low-temperature expansions, while those reported under IHT-PR+LT are obtained combining the IHT-PR and LT results. The comparison with the corresponding Table XIII of Ref. [19] shows that the estimates obtained from the 25th-order series are essentially identical to those obtained by using the shorter 20th-order series. However, the longer series allow us to give error bars that are smaller by a factor of 1.5-2, depending on the observable. The estimates reported in Table 1 are in substantial agreement with, and substantially more precise than, the best theoretical and experimental results that have been previously obtained [10, 20–23, 35–50]. For a comprehensive recent review of theoretical and experimental results, see Ref. [1]. On the experimental side, we mention the planned experiments in microgravity environment described in Ref. [51], which may substantially improve the experimental determinations of the critical quantities and make the comparison with the theoretical computations more stringent.

After completion of this work, the study reported in Ref. [17] appeared, where analyses of 25th-order series for spin- $S$  models are reported. Results for the critical exponents are obtained by means of biased analyses, essentially by fixing  $\Delta$ . Comparing Ref. [17] with Refs. [10, 13], where 21st- and 23rd-order series are analyzed, a

trend appears towards better agreement with improved Hamiltonian results (Ref. [19] and present paper). The latest results of the authors of Ref. [17] are in full agreement with our estimates.

The paper is organized as follows.

In Sec. 2 we report the HT expansions. Section 3 reports the results of our analysis of the HT series for the critical exponents. In Sec. 4 we determine approximate representations of the critical equation of state. In Sec. 4.1 we give the definitions, in Sec. 4.2 we give estimates of the zero-momentum four-point coupling and of the first few coefficients of the small-magnetization expansion of the equation of state, in Sec. 4.3 we explain the method, and in Sec. 4.4 we give the final results. In Sec. 5 we present estimates of several universal amplitude ratios. In Sec. 6 we determine the low-momentum behavior of the two-point function in the HT phase.

## 2 High-temperature expansion

We considered a simple cubic lattice and computed the HT expansion of several quantities for a generic lattice model defined by the Hamiltonian (1.2), using the vertex- and edge-renormalized linked-cluster expansion technique, developed in Refs. [18, 52] and described in detail in Ref. [53]. Some technical points that allowed us to extend the computation of Ref. [53] will be reported in a forthcoming publication. We computed the 25th-order HT expansion of the two-point function

$$G(x) = \langle \phi(0)\phi(x) \rangle. \quad (2.1)$$

In the present context we consider its moments

$$m_{2j} = \sum_x |x|^{2j} G(x), \quad (2.2)$$

and therefore, the magnetic susceptibility  $\chi \equiv m_0$  and of the second-moment correlation length  $\xi^2 = m_2/(6\chi)$ .

We also calculated the HT expansion of the zero-momentum connected  $2j$ -point correlation functions  $\chi_{2j}$

$$\chi_{2j} = \sum_{x_2, \dots, x_{2j}} \langle \phi(0)\phi(x_2)\dots\phi(x_{2j-1})\phi(x_{2j}) \rangle_c \quad (2.3)$$

( $\chi = \chi_2$ ). More precisely, we computed  $\chi_4$  to 21st order,  $\chi_6$  to 19th order,  $\chi_8$  to 17th order. The correlation function  $\chi_{10}$  was computed to 15th order in Ref. [19].

It would be pointless to present here the full results for an arbitrary potential: the resulting expressions are only fit for further computer manipulation. They are available on request. In Table 2 we give the new coefficients only for the three improved models we have considered, i.e., for the  $\phi^4$  model at  $\lambda_4 = 1.10$ , for the  $\phi^6$  model at  $\lambda_6 = 1$  and  $\lambda_4 = 1.90$ , and for the spin-1 model at  $D = 0.641$ .

Table 1: Summary of the results obtained in this paper (unless a reference is explicitly cited) by our high-temperature calculations (IHT), by using the parametric representation of the equation of state (IHT-PR), by analyzing the low-temperature expansion (LT), and by combining the results of the two approaches (IHT-PR+LT). The estimates of critical exponents marked by an asterisk have been obtained using scaling and hyperscaling relations.

		IHT	IHT-PR	LT	IHT-PR+LT
critical exponents	$\gamma$	1.2373(2)			
	$\nu$	0.63012(16)			
	$\alpha$	0.110(2), *0.1096(5)			
	$\eta$	*0.03639(15)			
	$\beta$	*0.32653(10)			
	$\delta$	*4.7893(8)			
	$\Delta$	0.52(3)			
	$\omega$	0.83(5)			
	$\omega_{\text{NR}}$	2.0208(12) [19, 33]			
small-magnetization expansion of the free-energy in the HT phase	$g_4^+$	23.56(2)			
	$r_6$	2.056(5)			
	$r_8$	2.3(1)			
	$r_{10}$	-13(4) [19]	-10.6(1.8)		
universal amplitude ratios see Sec. 5 for notations	$U_0$		0.532(3)		
	$U_2$		4.76(2)		
	$U_4$		-9.0(2)		
	$R_c^+$		0.0567(3)		
	$R_c^-$		0.02242(12)		
	$R_4^+$		7.81(2)		
	$v_3$		6.050(13)		
	$R_4^-$		93.6(6)		
	$v_4$		16.17(10)		
	$R_\chi$		1.660(4)		
	$w^2$			4.75(4) [14]	
	$U_\xi$				1.956(7)
	$Q^+$		0.01880(8)		
	$R_\xi^+$		0.2659(4)		
	$Q^-$				0.00472(5)
	$Q_c$		0.3315(10)		
	$g_3^-$				13.19(6)
	$g_4^-$				76.8(8)
	$Q_\xi^+$	1.000200(3)			
	$Q_\xi^-$			1.032(4) [34]	
	$U_{\xi_{\text{gap}}}$				1.896(10)
	$Q_\xi^c$				1.024(4)
	$Q_2$				1.195(10)
	$P_m$		1.2498(6)		
	$P_c$		0.3933(7)		
	$R_p$		1.9665(10)		

Table 2: Coefficients of the HT series for the improved models. Lower-order coefficients appear in Ref. [19].

$n$	$\phi^4 : \lambda_4 = 1.10$	$\phi^6 : \lambda_6 = 1, \lambda_4 = 1.90$	spin-1: $D = 0.641$
$\chi$			
21	958465949.119795229380125	55356759.0258594943774739	521863527.549747127784405
22	2581828793.17418316658592	130996257.131383657648562	1367254366.70256684609648
23	6953921835.10625772660286	309956395.981892002096689	3581814299.63029965928082
24	18716342130.2600278822297	732873665.558914443007657	9376338630.49601545283933
25	50369768053.5367726030130	1732674465.68758001711514	24543094928.9205155990856
$m_2$			
21	32990320251.5660972216018	1900950559.23375555678011	17908950773.4801706544197
22	94071328367.8146359923071	4762044317.91673448231502	49684326561.5439542757331
23	267461898855.689392585599	11894571003.1970044574018	137433163639.457494472451
24	758423675496.642760823002	29631147101.2512233682029	379139772127.101469600055
25	2145329356955.42924803892	73634162230.2093808561076	1043350926215.22611634874
$m_4$			
20	541141652908.631074719231	35399348720.3598637148375	299758906549.791610350073
21	1643345014677.80358819408	94444621918.7858920241050	885976701269.736104292700
22	4961021084766.33884428748	250485298262.046958470064	2603026564263.78069815384
23	14895796670810.3387628037	660748522303.208118944668	7606210964865.32821158574
24	44504475774409.2126174407	1734347627024.93369651634	22115153167519.1984380502
25	132362288688779.709839376	4531641133142.45499870752	64005596692608.8036008995
$\chi_4$			
19	-141558376231.985023846408	-9210343000.40488445467068	-77210883309.3840433243811
20	-440895445559.088001425635	-25206881115.0765162521666	-234263398532.544236218037
21	-1363771989486.31756523825	-68511054288.5805997438372	-705801443484.646787710146
$\chi_6$			
18	25922773662329.4681285982	1657400403425.39611029038	13110582140461.8241625980
19	93214547843378.1420243052	5239283130720.37310719268	46080008679021.7095625364
$\chi_8$			
17	-3021378127745877.943411840	-188904527250502.5683919596	-1360671334948122.792253527

For the standard Ising model, we give below the coefficients of the terms that extend the expansions presented in Refs. [15, 53] for  $\chi$ ,  $m_2$ , and  $\chi_4$ :

$$\begin{aligned}
\chi &= \cdots + 18554916271112254v^{24} + 85923704942057238v^{25} + O(v^{26}) \\
m_2 &= \cdots + 977496788431483776v^{24} + 4767378698515169334v^{25} + O(v^{26}), \\
\chi_4 &= \cdots - 6306916133817628v^{18} - 34120335459595728v^{19} \\
&\quad - 183166058308506108v^{20} - 976373577976196368v^{21} + O(v^{22}), \quad (2.4)
\end{aligned}$$

where  $v \equiv \tanh \beta$ .

### 3 The critical exponents

In this section we shall report three different analyses for the critical exponents. In Sec. 3.1 we shall use integral approximants and derive the estimates reported in Table 1. In Secs. 3.2 and 3.3 we shall use two other methods that have been recently used in the literature [10, 16, 17] to confirm the integral-approximant results.

#### 3.1 Analysis using integral approximants

In order to estimate  $\gamma$  and  $\nu$ , we analyze the HT series of the magnetic susceptibility and of the second-moment correlation length respectively. We follow closely App. B of Ref. [19], to which the reader is referred for more details.

We use integral approximants (IA's) of first, second, and third order (see Ref. [2] for a review). Given an  $n$ th-order series  $f(\beta) = \sum_{i=0}^n c_i \beta^i$ , its  $k$ th-order integral approximant  $[m_k/m_{k-1}/\dots/m_0/l]$  IAK is a solution of the inhomogeneous  $k$ th-order linear differential equation

$$P_k(\beta)f^{(k)}(\beta) + P_{k-1}(\beta)f^{(k-1)}(\beta) + \dots + P_1(\beta)f^{(1)}(\beta) + P_0(\beta)f(\beta) + R(\beta) = 0, \quad (3.1)$$

where the functions  $P_i(\beta)$  and  $R(\beta)$  are polynomials of order  $m_i$  and  $l$  respectively, which are determined by the known  $n$ th-order small- $\beta$  expansion of  $f(\beta)$ . Following Fisher and Chen [9], we also consider integral approximants, FCIAK's, in which  $P_k(\beta)$  is a polynomial in  $\beta^2$ . FCIAK's allow for the presence of the antiferromagnetic singularity at  $\beta_c^{\text{af}} = -\beta_c$  [54]. In our analyses we consider diagonal or quasi-diagonal approximants, since they are expected to give the most accurate results. For each set of IAK's we determine the average of the values corresponding to all nondefective IAK's. The error bar from each class of IA's is essentially the spread of the results, and it is given by the standard deviation of the results obtained from all nondefective IA's. In most cases the nondefective IA's are more than 90%.

All IA's considered give perfectly consistent results. Moreover, the results turn out to be very stable with respect to the number of terms of the series, so that there is no need to perform problematic extrapolations in the number of terms in order to obtain the final estimates. In Fig. 1 we show the estimates of  $\gamma$  obtained by analyzing the series of  $\chi$  for the  $\phi^4$  model at  $\lambda_4 = 1.10$  by using IA1's, IA2's, IA3's, and FCIA2's, as a function of the order  $n$  of the series considered in the analysis. Perfect agreement is also found among the results for the three improved Hamiltonians. This is shown in Fig. 2, where the results of the IA2 analyses for the three improved Hamiltonians are reported versus  $n$ . In Fig. 2 we also show the results of the IA2 analysis applied to the series of  $\chi$  for the standard Ising spin-1/2 model. The corresponding results disagree with those obtained by using improved Hamiltonians: clearly, there is a large error that is not taken into account by the spread of the approximants. The results for the Ising model improve if one biases the analysis by using the very accurate MC estimate of  $\beta_c$  [47]:  $\beta_c = 0.22165459(10)$ . Indeed,  $\gamma$  drops from 1.245 to  $\gamma = 1.2400(5)$ . However, the error obtained from the spread of the approximants is still incorrect. Results that are closer to those obtained by using the improved Hamiltonians (and



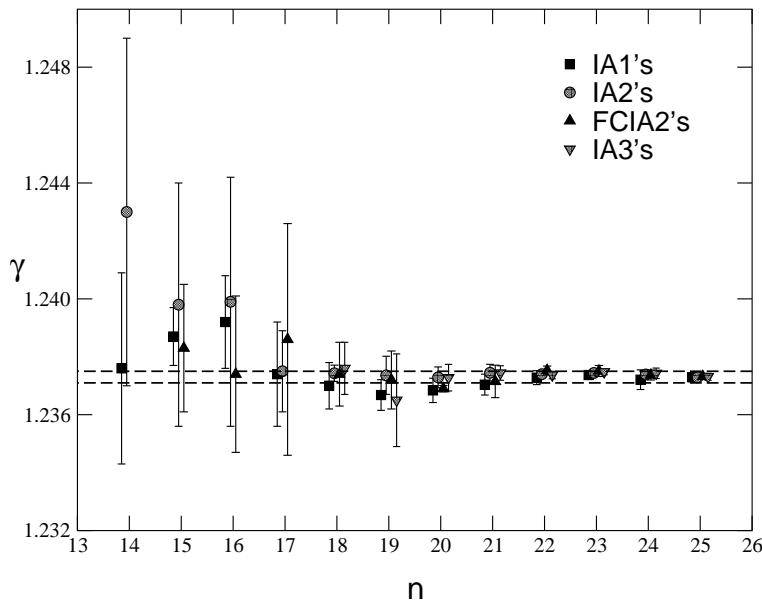


Figure 1: Estimates of  $\gamma$  as obtained by analyzing the HT series of  $\chi$  for the  $\phi^4$  model at  $\lambda_4 = 1.10$  versus the order  $n$  of the series considered in the analysis. Several approximants (defined in the text) are considered.

substantially compatible with them) are only obtained by additionally biasing the series, allowing for  $O(t^\Delta)$  confluent corrections, see, e.g., Ref. [10].

For the  $\phi^4$  lattice model we obtained

$$\beta_c(\lambda_4 = 1.10) = 0.3750975(5), \quad (3.2)$$

$$\gamma_e(\lambda_4) = 1.23732(10) + 0.006(\lambda_4 - 1.10), \quad (3.3)$$

where  $\gamma_e(\lambda_4)$  is the effective critical exponent obtained in the IA analysis, which has a small but nonvanishing dependence on  $\lambda_4$  around the favorite value  $\lambda_4 = 1.10$ . (Here and in the following, we report explicitly the dependance on  $\lambda_4$  and equivalent couplings: should a better estimate of  $\lambda_4^*$  become available, it can be immediately used to improve our results.) The number between parentheses is basically the spread of the approximants at  $\lambda_4 = 1.10$ . The  $\lambda_4$ -dependence is estimated by determining the variation of the results when changing  $\lambda_4$  around  $\lambda_4 = 1.10$ . The best estimate of  $\gamma$  should be obtained at  $\lambda_4 = \lambda_4^*$ . Thus, using the MC estimate of  $\lambda_4^*$ , i.e.,  $\lambda_4^* = 1.10(2)$ , and taking into account its uncertainty, we obtain the estimate  $\gamma = 1.23732(10)$ [12] (which is also reported in Table 3), where the error in brackets is related to the uncertainty on  $\lambda_4^*$ . As final error we consider, prudentially, the sum of these two numbers. The estimate (3.2) is in substantial agreement with the MC estimate of  $\beta_c$  [23] obtained using finite-size scaling techniques,  $\beta_c(\lambda_4 = 1.10) = 0.3750966(4)$ .

Similarly, for the  $\phi^6$  lattice model we obtain

$$\beta_c(\lambda_4 = 1.90, \lambda_6 = 1) = 0.4269791(5), \quad (3.4)$$

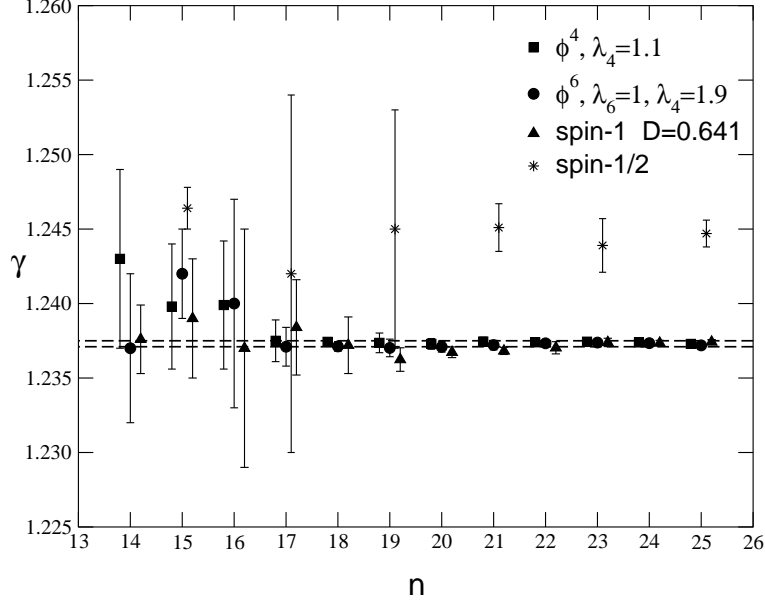


Figure 2: Estimates of  $\gamma$  as obtained by analyzing the HT series of  $\chi$  for the improved models and for the spin-1/2 model, versus the order  $n$  of the series considered in the analysis. IA2's are considered.

Table 3: Critical exponents obtained from the HT analysis. In parentheses we report the approximant error at  $\lambda^*$  or  $D^*$ , in brackets the uncertainty due to the error on  $\lambda^*$  or  $D^*$ , in braces the uncertainty due to the error on  $\beta_c$ .

		$\phi^4$	$\phi^6$	spin-1
$\gamma$	$\chi$ -series	1.23732(10)[12]	1.23726(10)[22]	1.23725(20)[10]
$\alpha$	$\chi$ -series (af)	0.110(2)	0.110(2)	0.112(5)
$\nu$	$\xi^2$ -series	0.6302(2)[1]	0.6301(3)[3]	0.6300(2)[1]
$\nu$	$\xi^2$ -series ( $\beta_c$ -biased)	0.63014(1){6}[9]	0.63009(1){16}[16]	0.63010(1){10}[9]
$\eta\nu$	$\chi, \xi^2$ -series (CPRM)	0.02294(3)[6]	0.02291(2)[10]	0.02294(8)[4]

$$\gamma_e(\lambda_4, \lambda_6 = 1) = 1.23726(10) + 0.0055(\lambda_4 - 1.90), \quad (3.5)$$

and, using the MC result  $\lambda_4^* = 1.90(4)$ , the estimate  $\gamma = 1.23726(10)[22]$ ; for the spin-1 model

$$\beta_c(D = 0.641) = 0.3856717(10), \quad (3.6)$$

$$\gamma_e(D) = 1.23725(20) - 0.012(D - 0.641) \quad (3.7)$$

and therefore, using  $D^* = 0.641(8)$ ,  $\gamma = 1.23725(20)[10]$ .

Our final estimate of  $\gamma$  is obtained by combining the results of the three improved Hamiltonians: as estimate we take the weighted average of the three results, and as estimate of the uncertainty the smallest of the three errors. According to this rather subjective but reasonable procedure we obtain

$$\gamma = 1.2373(2). \quad (3.8)$$

A direct estimate of the specific-heat exponent  $\alpha$  is obtained from the singular behavior of  $\chi$  at the antiferromagnetic critical point  $\beta_c^{\text{af}} = -\beta_c$ , since [54]

$$\chi = c_0 + c_1 (\beta - \beta_c^{\text{af}})^{\theta_{\text{af}}} + \dots \quad (3.9)$$

where

$$\theta_{\text{af}} = 1 - \alpha. \quad (3.10)$$

FCIAK's provide rather precise estimates of  $\theta_{\text{af}}$ . The corresponding results for  $\alpha$  are reported in the second line of Table 3. No error in brackets is reported since the dependence on  $\lambda_4, D$  is negligible. As final estimate we give

$$\alpha = 0.110(2). \quad (3.11)$$

The exponent  $\nu$  is obtained from the series of the second-moment correlation length  $\xi$ , since  $\xi^2 \sim (\beta_c - \beta)^{-2\nu}$ . Unbiased analyses of the 24th-order series of  $\xi^2/\beta$  provide the results reported in the third line of Table 3. The corresponding estimates of  $\beta_c$  are consistent with those derived from  $\chi$ , although less precise. For instance, for the  $\phi^4$  model at  $\lambda_4 = 1.10$  we found  $\beta_c = 0.375098(2)$ .

In order to get a more precise estimate of  $\nu$ , we follow the procedure suggested in Ref. [2], i.e., we use the estimate of  $\beta_c$  obtained from  $\chi$  to bias the analysis of  $\xi^2$ . For this purpose we use IA's that have a singularity at a fixed value of  $\beta_c$ , or, in order to take into account the antiferromagnetic singularity, a pair of singularities at  $\pm\beta_c$ ; the two choices give equivalent results. This analysis provides the following effective exponents for the three classes of models. For  $\lambda_4 \approx \lambda_4^*$

$$\nu_e(\lambda_4) = 0.63014(1)\{6\} + 0.0045(\lambda_4 - 1.10) \quad (3.12)$$

for the  $\phi^4$  model, where the number in braces gives the variation of the estimate when  $\beta_c$  varies within one error bar;

$$\nu_e(\lambda_4, \lambda_6 = 1) = 0.63009(1)\{16\} + 0.004(\lambda_4 - 1.90) \quad (3.13)$$

for the  $\phi^6$  model;

$$\nu_e(D) = 0.63010(1)\{10\} - 0.011(D - 0.641) \quad (3.14)$$

for the spin-1 model. Then, using the MC estimates of  $\lambda_4^*, D^*$ , one obtains the results reported in Table 3, where the error due to the uncertainty on  $\lambda_4^*$  and  $D^*$  is reported between brackets. They are perfect consistent with the results of the unbiased analysis, but more precise. Combining the results of Table 3 as we did for  $\gamma$ , we obtain

$$\nu = 0.63012(16). \quad (3.15)$$

Using the hyperscaling relation  $\alpha = 2 - 3\nu$ , we derive

$$\alpha = 0.1096(5), \quad (3.16)$$

which is fully consistent with, but more precise than, the direct estimate (3.11).

Using the above-reported results for  $\gamma$  and  $\nu$  and the scaling relation  $\gamma = (2 - \eta)\nu$ , we obtain  $\eta = 0.0364(6)$ , where the error is estimated by considering the errors on  $\gamma$  and  $\nu$  as independent, which is of course not true. We can obtain an estimate of  $\eta$  with a smaller, yet reliable, error by applying the so-called critical-point renormalization method [55] to the series of  $\chi$  and  $\xi^2$ . This method provides an estimate for the combination  $\eta\nu$ . Proceeding as before, we obtain

$$[\eta\nu]_e(\lambda_4) = 0.02294(3) + 0.003(\lambda_4 - 1.10) \quad (3.17)$$

for the  $\phi^4$  model,

$$[\eta\nu]_e(\lambda_4, \lambda_6 = 1) = 0.02291(2) + 0.0025(\lambda_4 - 1.90) \quad (3.18)$$

for the  $\phi^6$  model, and

$$[\eta\nu]_e(D) = 0.02294(8) - 0.005(D - 0.641) \quad (3.19)$$

for the spin-1 model. We then obtain the results reported in Table 3, which lead to an estimate of  $\eta$  with a considerably smaller error:

$$\eta = 0.03639(15). \quad (3.20)$$

Then, by using the scaling relations we obtain

$$\delta = \frac{5 - \eta}{1 + \eta} = 4.7893(8), \quad (3.21)$$

$$\beta = \frac{\nu}{2}(1 + \eta) = 0.32653(10), \quad (3.22)$$

where the error on  $\beta$  has been estimated by considering the errors of  $\nu$  and  $\eta$  as independent.

Finally, we estimate the exponent  $\Delta$ . For this purpose, we analyze the HT expansion of  $t^\gamma\chi$  that behaves like

$$t^\gamma\chi = C^+ (1 + a_\chi t^\Delta + \dots), \quad (3.23)$$

for  $t \equiv 1 - \beta/\beta_c \rightarrow 0$ . We consider the spin-1/2 model—here improved models are not useful since  $a_\chi \approx 0$ —fix the exponent  $\gamma$  to our best estimate,  $\gamma = 1.2373$ , and use biased IA's that are singular at  $\beta_c = 0.22165459(10)$ , which is the most precise MC estimate of the critical point [47]. We obtain

$$\Delta = 0.52(3), \quad (3.24)$$

where the error takes into account the uncertainty on  $\beta_c$  and  $\gamma$ . Correspondingly, we obtain  $\omega = \Delta/\nu = 0.83(5)$ . Consistent results are obtained from the analysis of the series of  $t^{2\nu}\xi^2$ , fixing  $\nu$  and  $\beta_c$ .

### 3.2 The ratio method

In order to check the above-reported results, we consider the ratio method proposed by Zinn-Justin in Ref. [3] (see also Ref. [2]). Such a method has been recently employed in Refs. [10, 17] to analyze the 25th-order HT expansions of spin- $S$  models on the simple cubic and on the body-centered cubic lattice.

According to this method, given a quantity

$$S = \sum_n c_n \beta^n \approx A_S (\beta_c - \beta)^{-\zeta} [1 + a_S (\beta_c - \beta)^\epsilon + \dots], \quad (3.25)$$

one considers the sequences

$$\beta_c^{(n)} = \left( \frac{c_{n-2} c_{n-3}}{c_n c_{n-1}} \right)^{1/4} \exp \left[ \frac{s_n + s_{n-2}}{2 s_n (s_n - s_{n-2})} \right], \quad (3.26)$$

$$\zeta^{(n)} = 1 + 2 \frac{s_n + s_{n-2}}{(s_n - s_{n-2})^2}, \quad (3.27)$$

where

$$s_n = -\frac{1}{2} \left[ \frac{1}{\ln(c_n c_{n-4}/c_{n-2}^2)} + \frac{1}{\ln(c_{n-1} c_{n-5}/c_{n-3}^2)} \right]. \quad (3.28)$$

Asymptotically, the two sequences  $\beta_c^{(n)}$  and  $\zeta^{(n)}$  approach  $\beta_c$  and  $\zeta$ , with corrections of  $O(1/n^{1+\epsilon})$  and  $O(1/n^\epsilon)$  respectively. More precisely, if

$$c_n \approx \beta_c^{-n} n^{\zeta-1} (A_0 + A_\epsilon n^{-\epsilon}) \quad (3.29)$$

for  $n \rightarrow \infty$ , then

$$\beta_c^{(n)} \approx \beta_c \left[ 1 + \frac{A_\epsilon}{2A_0} \epsilon^2 (\epsilon - 1) \frac{1}{n^{1+\epsilon}} \right] \quad (3.30)$$

$$\zeta^{(n)} \approx \zeta \left[ 1 + \frac{A_\epsilon}{\zeta A_0} \epsilon (\epsilon^2 - 1) \frac{1}{n^\epsilon} \right]. \quad (3.31)$$

Note that, if only analytic corrections are present, i.e.,  $\epsilon = 1$ , the convergence is faster with corrections of order  $n^{-3}$  and  $n^{-2}$  for  $\beta_c$  and  $\zeta$ :

$$\beta_c^{(n)} \approx \beta_c \left[ 1 - \left( \frac{A_1^2 \zeta - 2}{A_0^2 \zeta - 1} + \frac{7}{12} (\zeta - 1) \right) \frac{1}{n^3} \right], \quad (3.32)$$

$$\zeta^{(n)} \approx \zeta \left[ 1 - \left( 3 \frac{A_1^2 \zeta - 2}{A_0^2 \zeta - 1} + \frac{3}{4} (\zeta - 1) \right) \frac{1}{\zeta n^2} \right]. \quad (3.33)$$

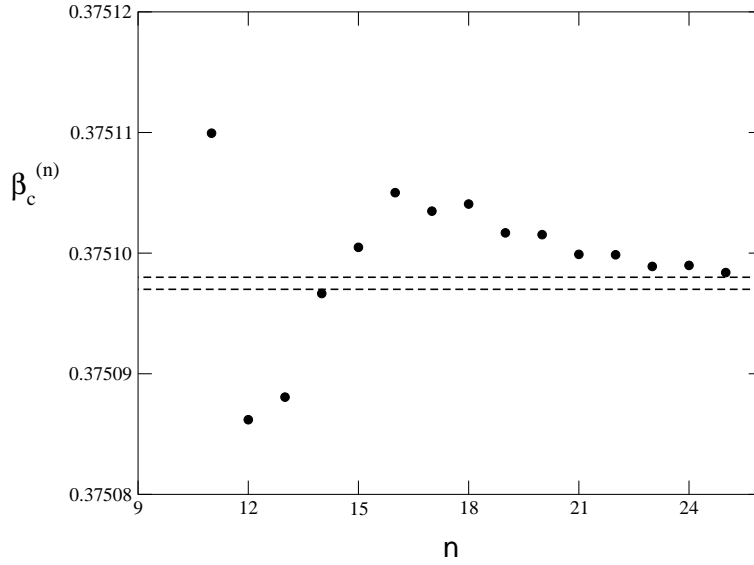


Figure 3: Sequence  $\beta_c^{(n)}$  for the  $\phi^4$  model at  $\lambda_4 = 1.10$  using the series for  $\chi$ . The dashed lines indicate the IA estimate of  $\beta_c$ .

In Figs. 3 and 4 we show, for the  $\phi^4$  model at  $\lambda_4 = 1.10$  and for the spin-1 model at  $D = 0.641$  respectively, the sequence  $\beta_c^{(n)}$  obtained using  $S = \chi$ . The sequence clearly approaches the IA estimate. For the  $\phi^4$  model the agreement is quite good and indeed  $\beta_c^{(n)}$  differs from the IA estimate (3.2) by  $15 \times 10^{-7}$  and  $9 \times 10^{-7}$  for  $n = 24, 25$  (note that the error on the IA estimate of  $\beta_c$  is  $5 \times 10^{-7}$ ). In principle, one could try to extrapolate the sequence  $\beta_c^{(n)}$  to get a better estimate of  $\beta_c$ . For this purpose, we have tried to fit  $\beta_c^{(n)}$  assuming a behavior of the form

$$\beta_c^{(n)} = a + bn^{-\sigma}, \quad (3.34)$$

where  $a$ ,  $b$ , and  $\sigma$  are free parameters. If we interpolate  $\beta_c^{(n)}$  for  $n = 21, 23, 25$  with Eq. (3.34), we obtain

$$\beta_c^{(n)} = 0.3750977(-5) + 3.0(+4) \times 10^{-6} \left(\frac{n}{20}\right)^{-6.6(-1.5)}, \quad (3.35)$$

where the “errors” show the variation of the parameters between the interpolation with  $n = 21, 23, 25$  and  $n = 19, 21, 23$ . Analogously, the even sequence  $n = 20, 22, 24$  gives

$$\beta_c^{(n)} = 0.3750968(-25) + 4(+2) \times 10^{-6} \left(\frac{n}{20}\right)^{-6(-2)}. \quad (3.36)$$

The extrapolated values are in perfect agreement with Eq. (3.2), but it is quite difficult to interpret the results for  $\sigma$ . Indeed, in an improved model the leading corrections in the coefficients  $c_n$  are of order  $n^{-\Delta_2}$ ,  $n^{-1}$ , with [56]  $\Delta_2 \approx 1$ . The analytic term

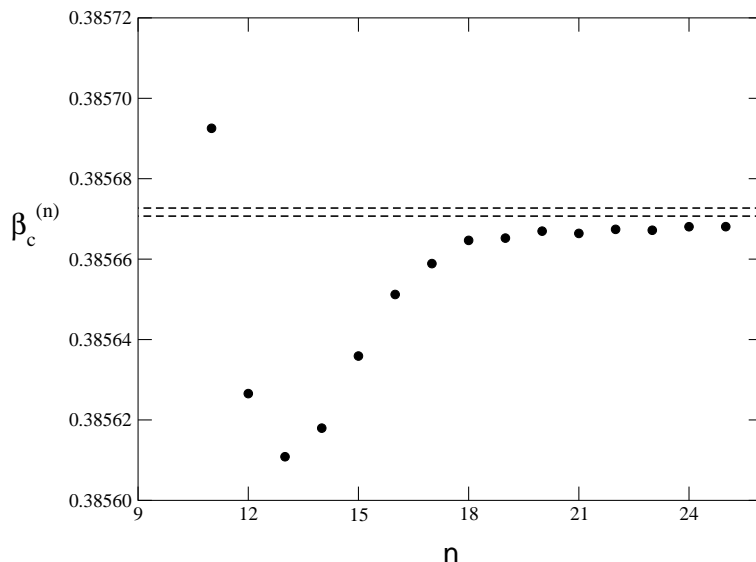


Figure 4: Sequence  $\beta_c^{(n)}$  for the spin-1 model at  $D = 0.641$  using the series for  $\chi$ . The dashed lines indicate the IA estimate of  $\beta_c$ .

gives a contribution of order  $n^{-3}$ , while the nonanalytic one gives a correction of order  $n^{-\Delta_2-1}$ . However, its amplitude is of order  $\Delta_2 - 1$ , and thus, since  $\Delta_2 \approx 1$ , it could be very small. The next correction terms are of order  $n^{-\Delta_3}$ ,  $n^{-1-\Delta}$ , and give rise to corrections of order  $n^{-1-\Delta_3}$ ,  $n^{-2-\Delta}$ . Inclusion of corrections with  $2 < \sigma \lesssim 5/2$  does not improve the fit [57]. Clearly, we are not yet sufficiently asymptotic to be able to extrapolate using the leading asymptotic behavior. At the values of  $n$  we are considering, several corrections are still important and apparently conspire to give a uniformly small correction.

The same behavior is observed in the  $\phi^6$  model, where both odd and even points extrapolate to 0.4269787, with effective exponent  $\sigma \approx 12, 8$ . The agreement with the IA estimate (3.4) is quite good. We finally analyze the spin-1 results. Even points show again a very fast convergence with  $\sigma \approx 9$  and extrapolate to 0.3856662. Odd points instead are well fitted by assuming corrections of order  $n^{-2}$  or  $n^{-5/2}$ . Fixing  $\sigma = -2$ , we obtain 0.3856730, while for  $\sigma = -5/2$  we have 0.3856719. Again, the IA result (3.6) is very well confirmed.

For comparison, in Fig. 5 we plot the sequence  $\beta_c^{(n)}$  for the spin-1/2 model versus  $1/n^{3/2}$  which should be approximately the leading correction. The higher- $n$  results have apparently the predicted  $O(n^{-3/2})$  behavior, and indeed an extrapolation with Eq. (3.34) and  $\sigma = 3/2$  gives results that are close to the MC estimate of  $\beta_c$ . The odd (resp. even) points extrapolate to 0.22165686 (resp. 0.22165717): they are close to the MC estimate [47] 0.22165459(10). However, it is hard to go beyond a relative precision of  $10^{-5}$ .

In Fig. 6 we show the sequence  $\gamma^{(n)}$  as obtained from the series of  $\chi$  for the three

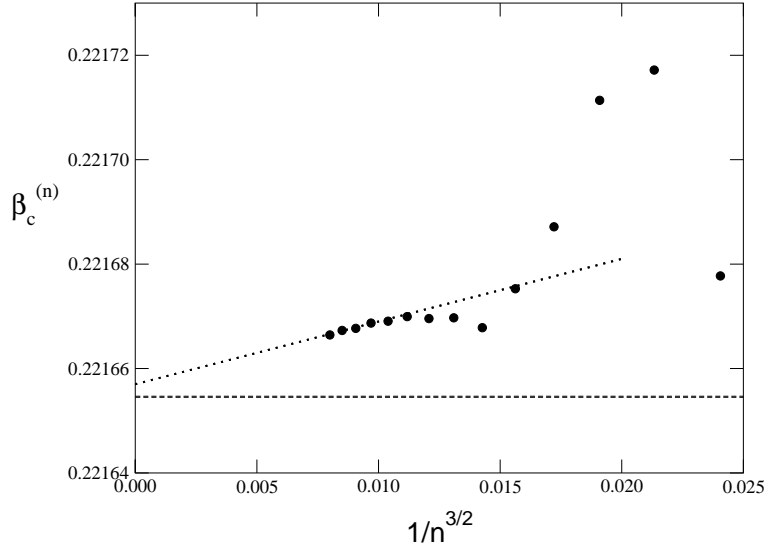


Figure 5: Sequence  $\beta_c^{(n)}$  for the spin-1/2 model using the series for  $\chi$ . The dashed lines indicate the MC estimate of  $\beta_c$ , while the dotted line corresponds to a  $n^{-3/2}$  extrapolation of the four points with  $n = 22, 23, 24, 25$ .

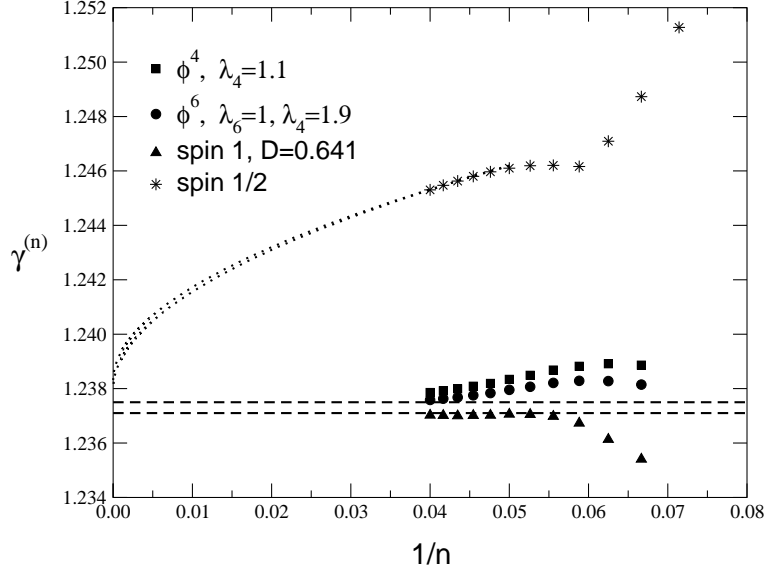


Figure 6: Sequences  $\gamma^{(n)}$  for the  $\phi^4$  model at  $\lambda_4 = 1.10$ , the  $\phi^6$  model at  $\lambda_4 = 1.90$ , the spin-1 model at  $D = 0.641$ , and the standard Ising model. The dashed lines indicate the IA estimate of  $\gamma$ .



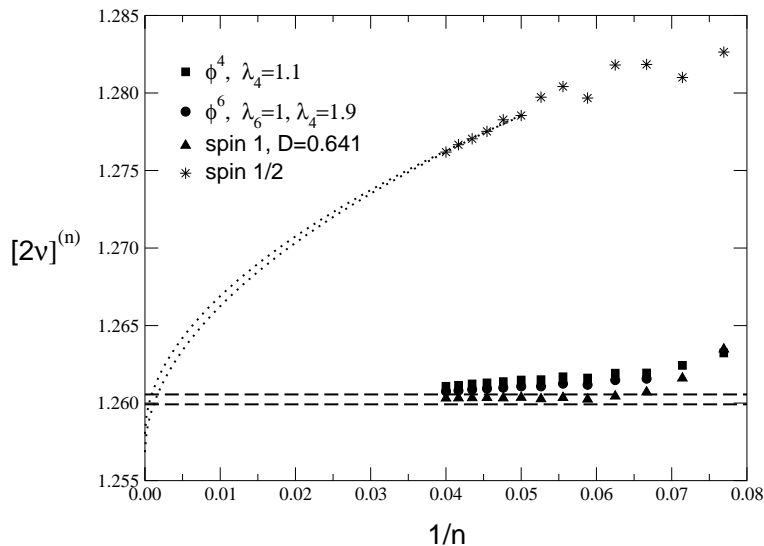


Figure 7: Sequences  $[2\nu]^{(n)}$  for the  $\phi^4$  model at  $\lambda_4 = 1.10$ , the  $\phi^6$  model at  $\lambda_4 = 1.90$ , the spin-1 model at  $D = 0.641$ , and the standard Ising model. The dashed lines indicate the IA estimate of  $2\nu$ .

improved models and for the standard Ising model. The improved results clearly approach our best estimate  $\gamma = 1.2373(2)$ , the  $\phi^4$  and  $\phi^6$  models from above and the spin-1 model from below. Note that the results are extremely flat and no extrapolation is needed. We also report the sequence  $\gamma^{(n)}$  for the Ising model. If we extrapolate the results assuming a behavior of the form  $a + bn^{-\Delta}$ , with  $\Delta = 0.52$ , we obtain  $\gamma = 1.23857, 1.23832, 1.23801$  using pairs  $n = (21, 23), (22, 24)$ , and  $(23, 25)$ . Clearly, the estimates converge towards the IA estimate  $\gamma = 1.2373(2)$ .

In Fig. 7 we show the sequence  $[2\nu]^{(n)}$  obtained from the series of  $\xi^2$ . Again, the improved models show a very good convergence to the IA estimate, in spite of the fact that the analysis is unbiased—the value of  $\beta_c$  is not fixed. The Ising results are sensibly higher and steadily decreasing, reaching  $\nu \approx 0.638$  for  $n = 25$ . Results that are closer to the IA estimate are obtained by an extrapolation. Assuming a behavior of the form  $a + bn^{-\Delta}$ , we obtain  $\nu = 0.6290$  and  $0.6284$  from even and odd sequences respectively. Again, the agreement is satisfactory.

In conclusion, this analysis based on the variant of the ratio method proposed by Zinn-Justin [3] supports the IA estimate obtained in Sec. 3.1.

### 3.3 Matching the coefficients with their asymptotic form

In the preceding section we have determined the critical exponents and  $\beta_c$  by generating sequences that converge to the asymptotic value for  $n \rightarrow \infty$ . In this Section, following Ref. [16], we wish to perform a more straightforward analysis, both conceptually and practically. The idea is to generate sequences of estimates by fitting

the expansion coefficients with their asymptotic form. By adding a sufficiently large number of terms we can make the convergence as fast as possible, although of course the procedure becomes unstable if the number of terms included is too large compared to the number of available terms. In practice, one should include those terms that give rise to the maximal stability of the results. In some sense, the variant ratio method of the previous section corresponds to considering the leading singular behavior and the first analytic correction—and also the leading nonanalytic term if we further extrapolate the sequence.

On the cubic lattice, the large-order behavior is dictated by the singularities at  $\pm\beta_c$ . Indeed, given an observable  $S$  with expansion  $S = \sum_n c_n \beta^n$ , for  $n \rightarrow \infty$  the expansion coefficients behave like

$$\begin{aligned} \beta_c^n c_n = & n^{\zeta-1} \left( A_0 + \frac{A_1}{n^\Delta} + \frac{A_2}{n} + \frac{A_3}{n^{1+\Delta}} + \frac{A_4}{n^2} + \dots \right) \\ & + (-1)^n n^{-(\theta_{\text{af}}+1)} \left( B_0 + \frac{B_1}{n} + \frac{B_2}{n^2} + \frac{B_{\Delta_{\text{af}}}}{n^{\Delta_{\text{af}}}} + \dots \right). \end{aligned} \quad (3.37)$$

Here, we have neglected all subleading exponents except the first one  $\Delta$ , and in particular, the first subdominant  $\Delta_2$ . However, since  $\Delta_2 \approx 1$  [56], for all practical purposes a term  $n^{-\Delta_2}$  cannot be distinguished from a purely analytic correction. Also, we do not write terms of order  $n^{-k\Delta}$  since they cannot be distinguished from the analytic terms and corrections of order  $n^{-m-\Delta}$ . Note also the presence of the parity-dependent corrections with exponent  $\theta_{\text{af}}$  and the subleading corrections with exponent  $\Delta_{\text{af}}$ . For the susceptibility  $\chi$ , it is known [54] that  $\theta_{\text{af}} = 1 - \alpha$ . The argument can be generalized to all moments  $m_{2k}$  and thus in all cases we predict  $\theta_{\text{af}} = 1 - \alpha$ . We have tested this prediction for  $\chi$ , cf. Sec. 3.1,  $m_2$ , and  $m_4$ . By analyzing the expansion of  $m_2$  with biased IAK's that have a pair of singularities in  $\pm\beta_c$ , we obtain  $\theta_{\text{af}} = 0.884(12)$ , while from the expansion of  $m_4$  we obtain  $\theta_{\text{af}} = 0.90(9)$ . These results are clearly compatible with the prediction  $\theta_{\text{af}} = 1 - \alpha = 0.8904(5)$ . For the exponent  $\Delta_{\text{af}}$  nothing is known. We have analyzed the expansion using  $\Delta_{\text{af}} = 1/2$  and  $\Delta_{\text{af}} = 1$ . The results appear to be quite insensitive on either choice. For this reason, in the following we only report the results corresponding to purely analytic corrections, i.e., we set  $B_{\Delta_{\text{af}}} = 0$ .

Note that this method allows to determine the nonuniversal amplitudes  $A_0, A_\Delta, \dots$ , and consequently the amplitudes  $a_i$  appearing in the expansion of  $S$  for  $\beta \rightarrow \beta_c$ . If

$$S = A_S (\beta_c - \beta)^{-\zeta} \left[ 1 + a_S (\beta_c - \beta)^\Delta \right] \quad (3.38)$$

then

$$A_S = \Gamma(\zeta) A_0, \quad (3.39)$$

$$a_S = \frac{\Gamma(\zeta - \Delta) A_1}{\Gamma(\zeta) A_0}. \quad (3.40)$$

In the following, we shall perform two different analyses: (essentially) unbiased analyses in order to determine the exponents  $\zeta$  and  $\beta_c$  and biased analyses in which  $\zeta$

and  $\beta_c$  are fixed. In all cases we fix the value of  $\Delta$ ,  $\Delta = 0.52$ , and the exponent of the antiferromagnetic singularity. In the unbiased analyses, in order to have a linear problem, we consider  $\ln c_n$  that behaves

$$\begin{aligned} \ln c_n = & -\ln(\beta_c) n + (\zeta - 1)\ln n + b_0 + \frac{b_1}{n^\Delta} + \frac{b_2}{n} + \frac{b_3}{n^{1+\Delta}} + \frac{b_4}{n^2} + \dots \quad (3.41) \\ & + (-1)^n n^{-(\zeta+\theta_{\text{af}})} \left( d_0 + \frac{d_1}{n^\Delta} + \frac{d_2}{n} + \frac{d_3}{n^{1+\Delta}} + \frac{d_4}{n^2} \dots \right). \end{aligned}$$

As before, we have neglected terms that have exponents similar to those already present: for instance, terms  $O(n^{-k\Delta-m})$  or  $O(n^{-k\Delta_2-h\Delta-m})$ . In the expansion of the antiferromagnetic part we have assumed  $\Delta_{\text{af}} = \Delta$ , or  $\Delta_{\text{af}} = 1$ . Note that if only analytic terms are present in Eq. (3.37), i.e.,  $B_{\text{af}} = 0$ , then  $d_1$  is proportional to  $A_1$  and therefore it vanishes in improved models.

We first analyze improved models and we verify that  $A_1 \approx 0$ . For this purpose, we consider the susceptibility  $\chi$  and, for each improved Hamiltonian, we generate two sequences of amplitudes in the following way:

- (a) We choose two integers  $h, k$  and consider Eq. (3.41) keeping only  $b_0, \dots, b_{h-1}$  in the ferromagnetic part and  $d_0, \dots, d_{k-1}$  in the antiferromagnetic one. Then, we generate sequences  $\beta_c^{(n)}, \gamma^{(n)}, b_0^{(n)}, \dots, b_{h-1}^{(n)}, d_0^{(n)}, \dots, d_{k-1}^{(n)}$ , by solving the  $(h+k+2)$  equations  $\ln c_{n-m} = R_{n-m}$ ,  $m = 0, \dots, h+k+1$ , where  $R_n$  is the right-hand side of Eq. (3.41). We use  $\Delta = 0.52$ ,  $\gamma + \theta_{\text{af}} = 2.1277$ .
- (b) We choose two integers  $h, k$  and consider Eq. (3.37) keeping only  $A_0, \dots, A_{h-1}$  in the ferromagnetic part and  $B_0, \dots, B_{k-1}$  in the antiferromagnetic one. We use  $\Delta = 0.52$ ,  $\gamma = 1.2373$ ,  $\theta_{\text{af}} = 0.8904$ , the IA estimate of  $\beta_c$ , and  $B_{\text{af}} = 0$ . Then, we generate sequences  $A_0^{(n)}, \dots, A_{h-1}^{(n)}, B_0^{(n)}, \dots, B_{k-1}^{(n)}$ , by solving the  $(h+k)$  equations  $c_{n-m} = R_{n-m}$ ,  $m = 0, \dots, h+k-1$ , where  $R_n$  is the right-hand side of Eq. (3.37).

In both cases we vary  $h$  and  $k$ , trying to find the values that give the best stability of the exponents or of the leading amplitudes. In the unbiased analysis (a), the preferred choice is  $(h, k) = (4, 4)$ , while for analysis (b) we use  $(h, k) = (3, 2)$ . For these choices of the parameters, in Fig. 8 we report the corresponding sequence of  $a_\chi^{(n)} \equiv a_1^{(n)}$ , obtained using Eq. (3.40). In the unbiased analysis (a),  $a_\chi^{(n)}$  clearly converges to zero for the improved Hamiltonians  $\phi^4$  and  $\phi^6$ , as expected. For the spin-1 model, it is not that clear, and presumably more orders are needed to observe convincingly  $a_\chi = 0$ . In the case of the biased analysis,  $a_\chi^{(n)}$  is very stable and small already for  $n \gtrsim 15$ . For all Hamiltonians we observe  $|a_\chi| \lesssim 10^{-3}$ .

As a second check of consistency we have verified that our estimates of  $a_\chi$  are compatible with the quoted error bars on  $\lambda_4^*$  and  $D^*$ . For this purpose, using the analysis of type (b) reported above, we have computed  $a_\chi$  for  $\lambda_4^* \pm \Delta\lambda_4$ , where  $\Delta\lambda_4$  is the quoted error bar—for the spin-1 model we are referring to  $D^* \pm \Delta D$ . In all cases, we find  $|a_\chi(\lambda_4^* \pm \Delta\lambda_4)| > |a_\chi(\lambda_4^*)|$  and that  $a_\chi(\lambda_4^* + \Delta\lambda_4)$  and  $a_\chi(\lambda_4^* - \Delta\lambda_4)$  have opposite sign. This confirms the correctness of our estimates of  $\lambda_4^*$  and  $D^*$ . Of

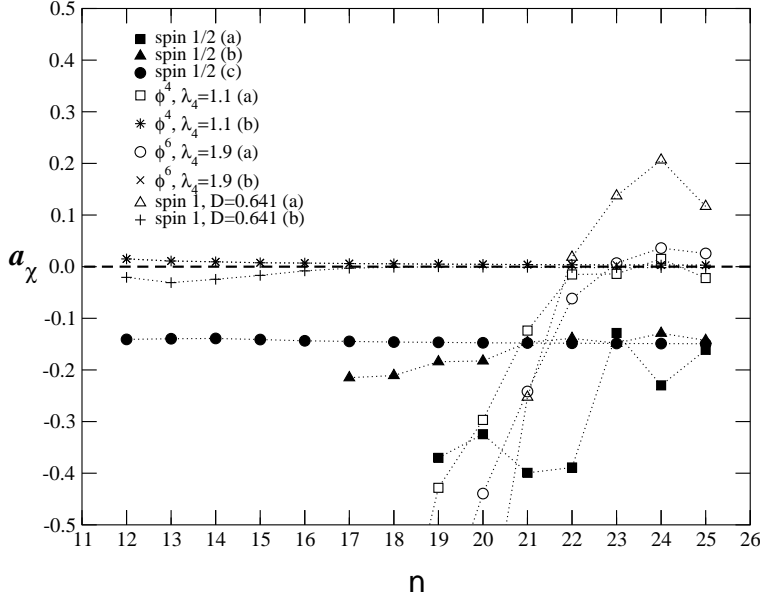


Figure 8: Amplitude  $a_\chi$  of the leading scaling correction as obtained from several different analyses of  $\chi$  for the standard spin-1/2 Ising model and for the three improved models. Details are explained in the text.

course, since we use  $\beta_c$  and  $\gamma$  obtained in the IA analysis, the above results represent only a check of consistency. Indeed: (i) we determine  $\beta_c$  and  $\gamma$  by performing a IA analysis whose results should be reliable only if the models are improved (in some sense we assume weakly here  $a_\chi \approx 0$ ); (ii) using such values of  $\beta_c$  and  $\gamma$ , we estimate  $a_\chi$  and find  $a_\chi \approx 0$ .

Once we have verified that  $A_1$  is very small and compatible with zero within the precision of the analysis, we have performed several analyses fixing  $A_1 = 0$  and  $b_1 = 0$ . At the same time, we have set  $d_1 = 0$ , which corresponds to assuming  $\Delta_{af} = 1$ . We have determined the exponents by performing the analysis (a) reported above. In the case of the  $\phi^4$  model for  $\lambda_4 = 1.10$ , this analysis gives  $\gamma \approx 1.2374$ . Similarly, we obtain  $\gamma \approx 1.2375$  for the  $\phi^6$  model at  $\lambda_4 = 1.90$  and for the spin-1 model at  $D = 0.641$ . In Fig. 9 we show the sequence  $\gamma^{(n)}$  for  $(h, k) = (5, 5)$  (since two coefficients vanish, we are considering four amplitudes in the ferromagnetic and antiferromagnetic expansion). We observe a very good agreement with the IA estimate  $\gamma = 1.2373(2)$ . It is difficult to estimate the uncertainty, since the results do not show a sufficiently robust stability with respect to the number  $(h, k)$  of coefficients used in the analysis.

Finally, we report the estimates of the amplitudes obtained in the analysis of type (b) for the magnetic susceptibility:

$$\phi^4 : A_0^{(\chi)} \approx 0.5246, A_2^{(\chi)} \approx 0.13, B_0^{(\chi)} \approx -0.0351;$$

$$\phi^6 : A_0^{(\chi)} \approx 0.4601, A_2^{(\chi)} \approx 0.11, B_0^{(\chi)} \approx -0.0311;$$

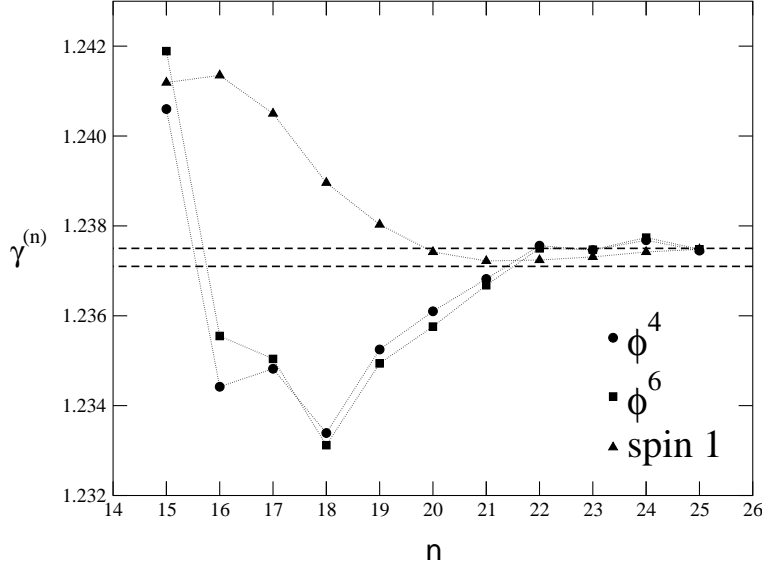


Figure 9: Exponent  $\gamma$  as obtained from the analysis (a) of  $\chi$  for the three improved models. Details are explained in the text.

spin-1 :  $A_0^{(\chi)} \approx 0.5126$ ,  $A_2^{(\chi)} \approx 0.12$ ,  $B_0^{(\chi)} \approx -0.0359$ .

Moreover,  $|A_3^{(\chi)}| \lesssim 10^{-2}$  for the  $\phi^4$  and  $\phi^6$  models, while  $A_3^{(\chi)} \approx -0.02$  for the spin-1 model. Errors should be  $\pm 1$  on the last reported digit and include the uncertainty on the  $n \rightarrow \infty$  extrapolation of the sequences and the variation of the estimates for  $h$  and  $k$  in the range  $h = 3 - 5$  and  $k = 2 - 4$ . They do not take into account instead the variation of the estimates with  $\gamma$  and  $\beta_c$ . Note that the estimate of  $A_2$  is purely phenomenological and in practice it should correspond to the sum of the amplitude of  $n^{-1}$  and of  $n^{-\Delta_2}$  (note that in improved models the amplitude of  $n^{-2\Delta}$  vanishes).

We have performed similar analyses for the spin-1/2 Ising model, in order to compute the nonuniversal amplitudes. We have performed: (a) an analysis of type (a) using  $(h, k) = (4, 4)$ ; (b) an analysis of type (a) in which we have fixed  $\beta_c$  to its MC value using  $(h, k) = (4, 3)$ ; (c) an analysis of type (b) using  $(h, k) = (3, 2)$ . The results for  $a_\chi^{(n)}$  are reported in Fig. 8. These analyses give perfectly consistent results and allow us to determine the amplitudes:

$$A_0^{(\chi)} = 1.233 - 10(\gamma - 1.2373) - 0.013(\Delta - 0.52), \quad (3.42)$$

$$A_1^{(\chi)} = -0.13 - 0.7(\Delta - 0.52) + 50(\gamma - 1.2373), \quad (3.43)$$

$$B_0^{(\chi)} = -0.073, \quad (3.44)$$

where we have explicitly written the dependence on the input parameters (when it turns out to be relevant). We have repeated the same analysis for the second moment  $m_2$ . We obtain

$$A_0^{(m_2)} = 1.301 - 10(\gamma + 2\nu - 2.49754) - 0.07(\Delta - 0.52), \quad (3.45)$$

$$A_1^{(m_2)} = -0.73 - 4(\Delta - 0.52) + 55(\gamma + 2\nu - 2.49754), \quad (3.46)$$

$$B_0^{(m_2)} = 0.06. \quad (3.47)$$

Using the above results and Eq. (3.40), one can determine the amplitudes  $a_\chi$  and  $a_\xi$ , associated with the  $O(t^\Delta)$  of the scaling corrections in the Wegner expansion of  $\chi$  and  $\xi$  respectively, and evaluate their universal ratio. We obtain  $a_\xi/a_\chi = 0.9(1)$ , where the error takes also into account the uncertainty on the input parameters of the biased analysis. For comparison we mention the recent HT result  $a_\xi/a_\chi = 0.76(6)$  [17], and the field theoretical estimate  $a_\xi/a_\chi = 0.68(2)$  [35].

## 4 The critical equation of state

### 4.1 Definitions

The equation of state relates the magnetization  $M$ , the magnetic field  $H$ , and the reduced temperature  $t \equiv (T - T_c)/T_c$ . In the neighborhood of the critical point  $t = 0$ ,  $H = 0$ , it can be written in the scaling form

$$H = B_c^{-\delta} M^\delta f(x), \quad (4.1)$$

$$x \equiv t(M/B)^{-1/\beta}, \quad (4.2)$$

where  $B_c$  and  $B$  are the amplitudes of the magnetization on the critical isotherm and on the coexistence curve,

$$M = B_c H^{1/\delta} \quad t = 0, \quad (4.3)$$

$$M = B(-t)^\beta \quad H = 0, \quad t < 0. \quad (4.4)$$

Using these normalizations the coexistence curve corresponds to  $x = -1$ , and the universal function  $f(x)$  satisfies  $f(-1) = 0$ ,  $f(0) = 1$ . Griffiths' analyticity implies that  $f(x)$  is regular everywhere for  $x > -1$ . It has a regular expansion in powers of  $x$ ,

$$f(x) = 1 + \sum_{n=1}^{\infty} f_n^0 x^n, \quad (4.5)$$

and a large- $x$  expansion of the form

$$f(x) = x^\gamma \sum_{n=0}^{\infty} f_n^\infty x^{-2n/\beta}. \quad (4.6)$$

At the coexistence curve, i.e., for  $x \rightarrow -1$ ,  $f(x)$  has at most an essential singularity [58]. It can be asymptotically expanded as

$$f(x) \approx \sum_{n=1}^{\infty} f_n^{\text{coex}} (1+x)^n. \quad (4.7)$$

It is useful to rewrite the equation of state in terms of a variable proportional to  $Mt^{-\beta}$ , although in this case we must distinguish between  $t > 0$  and  $t < 0$ . For  $t > 0$  we define

$$\begin{aligned} H &= \left( \frac{C^+}{C_4^+} \right)^{1/2} t^{\beta\delta} F(z), \\ z &\equiv \left[ -\frac{C_4^+}{(C^+)^3} \right]^{1/2} Mt^{-\beta}, \end{aligned} \quad (4.8)$$

while for  $t < 0$  we set

$$\begin{aligned} H &= \frac{B}{C^-} (-t)^{\beta\delta} \Phi(u), \\ u &\equiv \frac{M}{B} (-t)^{-\beta}. \end{aligned} \quad (4.9)$$

The constants  $C^\pm$  and  $C_4^+$  are the amplitudes appearing in the critical behavior of the zero-momentum connected  $n$ -point correlation functions  $\chi_n$ :

$$\chi_n = C_n^\pm |t|^{-\gamma-(n-2)\beta\delta}. \quad (4.10)$$

The susceptibility  $\chi$  corresponds to  $\chi_2$  and we simply write  $C^\pm = C_2^\pm$ .

With the chosen normalizations [41, 46, 49]

$$F(z) = z + \frac{1}{6}z^3 + \sum_{j=3}^{\infty} \frac{1}{(2j-1)!} r_{2j} z^{2j-1}, \quad (4.11)$$

$$\Phi(u) = (u-1) + \sum_{j=3}^{\infty} \frac{1}{(j-1)!} v_j (u-1)^{j-1}. \quad (4.12)$$

The functions  $F(z)$  and  $\Phi(u)$  are related to  $f(x)$ . Indeed,

$$z^{-\delta} F(z) = F_0^\infty f(x), \quad z = z_0 x^{-\beta}, \quad (4.13)$$

and

$$u^{-\delta} \Phi(u) = \frac{C^- B^{\delta-1}}{B_c^\delta} f(x), \quad u = (-x)^{-\beta}. \quad (4.14)$$

The constant  $F_0^\infty$  is defined by the large- $z$  behavior of  $F(z)$ ,

$$F(z) = z^\delta \sum_{k=0}^{\infty} F_k^\infty z^{-k/\beta}, \quad (4.15)$$

while

$$z_0 = \left[ -\frac{C_4^+}{(C^+)^3} \right]^{1/2} B. \quad (4.16)$$

To compare with experimental data, it is useful to determine the magnetization as a function of  $tH^{-1/\beta\delta}$ . Therefore, we define

$$E(y) \equiv B_c^{-1} M H^{-1/\delta} = f(x)^{-1/\delta}, \quad (4.17)$$

$$y \equiv (B/B_c)^{1/\beta} t H^{-1/(\beta\delta)} = x f(x)^{-1/(\beta\delta)}. \quad (4.18)$$

Finally, we shall also determine the scaling behavior of the susceptibility, by defining

$$D(y) \equiv B_c^{-1} H^{1-1/\delta} \chi = \frac{f(x)^{1-1/\delta}}{\delta f(x) - \frac{1}{\beta} x f'(x)}. \quad (4.19)$$

## 4.2 Small-magnetization behavior

In this section we determine the first few coefficients  $r_{2j}$  appearing in the expansion of the scaling function  $F(z)$ , cf. Eq. (4.8). We shall also compute the four-point renormalized coupling constant  $g_4$ , which, although not related to the equation of state, is relevant for the field-theoretical approach and will be used to determine amplitude ratios involving the second-moment correlation length.

In order to estimate the critical limit of  $g_4$  and of  $r_{2j}$  we first determine their HT expansions using the corresponding results for  $\chi_{2j}$  and  $m_2$

$$g_4 = -\frac{\chi_4}{\chi^2 \xi^3}, \quad (4.20)$$

$$r_6 = 10 - \frac{\chi_6 \chi_2}{\chi_4^2}, \quad (4.21)$$

$$r_8 = 280 - 56 \frac{\chi_6 \chi_2}{\chi_4^2} + \frac{\chi_8 \chi_2^2}{\chi_4^3}. \quad (4.22)$$

The corresponding series [59] have been analyzed following closely the procedure presented in App. B.3 of Ref. [25]. We use biased IA1's with a singularity at  $\beta_c$  or a pair of singularities at  $\pm\beta_c$ , where  $\beta_c$  is obtained from the analysis of the susceptibility. Around  $\beta_c$ , IA1's behave like [60]

$$\text{IA1} \approx f(\beta) (1 - \beta/\beta_c)^\zeta + g(\beta), \quad (4.23)$$

where  $f(\beta)$  and  $g(\beta)$  are regular at  $\beta_c$ , provided  $\zeta$  is not a negative integer. In particular

$$\zeta = \frac{P_0(\beta_c)}{P_1'(\beta_c)}, \quad g(\beta_c) = -\frac{R(\beta_c)}{P_0(\beta_c)} \quad (4.24)$$

(see Eq. (3.1) for the definition of the above quantities). In the case we are considering,  $\zeta$  is positive and, therefore,  $g(\beta_c)$  provides the desired estimate.

In Table 4 (first line) we report the estimates of  $g_4$  obtained for the three improved Hamiltonians. The error in parentheses is related to the spread of the approximants and the second one in brackets to the uncertainty on  $\lambda_4^*$ ,  $D^*$ . The error induced by the uncertainty on  $\beta_c$  is negligible. The results are perfectly consistent. Our final estimate is

$$g_4 = 23.56(2). \quad (4.25)$$

The result for the exponent  $\zeta$  in Eq. (4.23) is  $\zeta = 1.3(3)$ , which is consistent with our expectation for improved models, i.e.,  $\zeta = \Delta_2 \approx 2\Delta$  and  $\Delta \approx 0.5$ . For comparison, the same analysis applied to the standard Ising model gives  $g_4 = 23.5(5)$  and  $\zeta = 0.6(3)$ , in agreement with the fact that in this case  $\zeta = \Delta$ . Notice that



Table 4: Results for  $g_4$ ,  $r_6$  and  $r_8$ .

	$\phi^4$	$\phi^6$	spin-1	final estimates
$g_4$	23.559(8)[11]	23.554(8)[20]	23.560(20)[5]	23.56(2)
$r_6$	2.057(4)[1]	2.056(4)[2]	2.052(8)[2]	2.056(5)
$r_8$	2.29(9)[3]	2.31(5)[5]	2.37(7)[3]	2.3(1)

the small difference with the estimate of  $g_4$  reported in Ref. [19] is essentially due to the different analysis employed here, which is better justified due to the nonanalytic behavior at  $\beta_c$  predicted by renormalization group [61]. With respect to standard Padé approximants, biased IA1's require more terms of the series to give reasonable results, but they are less subject to systematic errors since they allow for confluent nonanalytic corrections at  $\beta_c$ . Biased IA1's give [1]  $g_4 = 23.54(4)$  when applied to the 17th-order series of Ref. [19].

Results for  $r_6, r_8$  are obtained using the same method and are reported in Table 4. We finally recall that a rough estimate of  $r_{10}$  was obtained in Ref. [19] from the analysis of its 15th-order series, obtaining  $r_{10} = -13(4)$ . A review of the available results for these quantities can be found in Ref. [1].

### 4.3 Parametric representations of the equation of state

In this section we shall determine the equation of state using parametric representations, improving the results of Refs. [19, 41]. This method has also been applied in two dimensions [62], and to the three-dimensional  $XY$  [25, 29] and Heisenberg [27] universality classes.

In order to obtain approximate expressions for the equation of state, we parametrize the thermodynamic variables in terms of two parameters  $R$  and  $\theta$ , implementing all expected scaling and analytic properties. Explicitly, we write [63–65]

$$\begin{aligned} M &= m_0 R^\beta \theta, \\ t &= R(1 - \theta^2), \\ H &= h_0 R^{\beta\delta} h(\theta), \end{aligned} \tag{4.26}$$

where  $h_0$  and  $m_0$  are normalization constants. The function  $h(\theta)$  is odd and normalized so that  $h(\theta) = \theta + O(\theta^3)$ . The smallest positive zero of  $h(\theta)$ , which should satisfy  $\theta_0 > 1$ , corresponds to the coexistence curve, i.e., to  $T < T_c$  and  $H \rightarrow 0$ . We mention that alternative versions of the parametric representations have been considered in Ref. [66].

It is easy to express the scaling functions introduced in Sec. 4.1 in terms of  $\theta$ . The scaling function  $f(x)$  is obtained from

$$x = \frac{1 - \theta^2}{\theta_0^2 - 1} \left( \frac{\theta_0}{\theta} \right)^{1/\beta},$$

$$f(x) = \theta^{-\delta} \frac{h(\theta)}{h(1)}, \quad (4.27)$$

while  $F(z)$  is obtained by

$$\begin{aligned} z &= \rho \theta (1 - \theta^2)^{-\beta}, \\ F(z(\theta)) &= \rho (1 - \theta^2)^{-\beta\delta} h(\theta), \end{aligned} \quad (4.28)$$

where  $\rho$  can be related to  $m_0$ ,  $h_0$ ,  $C^+$  and  $C_4^+$  using Eqs. (4.8) and (4.26).

It is important to note that Eq. (4.26) and the normalization condition  $h(\theta) \approx \theta$  for  $\theta \rightarrow 0$  do not completely fix the function  $h(\theta)$ . Indeed, one can rewrite the relation between  $x$  and  $\theta$  in the form

$$x^\gamma = h(1) f_0^\infty (1 - \theta^2)^\gamma \theta^{1-\delta}. \quad (4.29)$$

Thus, given  $f(x)$ , the value of  $h(1)$  can be arbitrarily chosen to completely fix  $h(\theta)$ . In the expression (4.28) we can fix this arbitrariness by choosing arbitrarily the parameter  $\rho$  [1, 19, 29, 41].

As suggested by arguments based on the  $\epsilon$ -expansion [19, 41], we approximate  $h(\theta)$  with polynomials, i.e., we set

$$h(\theta) = \theta + \sum_{n=1}^k h_{2n+1} \theta^{2n+1}. \quad (4.30)$$

This choice is further supported by the effectiveness of its simplest version with  $k = 1$ , which is the so-called linear model. If we require the approximate parametric representation to give the correct  $(k - 1)$  universal ratios  $r_6$ ,  $r_8$ ,  $\dots$ ,  $r_{2k+2}$ , we obtain

$$h_{2n+1} = \sum_{m=0}^n c_{nm} 6^m (h_3 + \gamma)^m \frac{r_{2m+2}}{(2m+1)!}, \quad (4.31)$$

where

$$c_{nm} = \frac{1}{(n-m)!} \prod_{k=1}^{n-m} (2\beta m - \gamma + k - 1), \quad (4.32)$$

and we have set  $r_2 = r_4 = 1$ . Moreover, by requiring that  $F(z) = z + \frac{1}{6}z^3 + \dots$ , we obtain the relation

$$\rho^2 = 6(h_3 + \gamma). \quad (4.33)$$

In the exact parametric representation, the coefficient  $h_3$  can be chosen arbitrarily. Of course, this is no longer true when we use our truncated function  $h(\theta)$ , and the related approximate function  $f_{\text{approx}}^{(k)}(x, h_3)$  depends on  $h_3$ . We must thus fix a particular value for this parameter. Here we use a variational approach, requiring the approximate function  $f_{\text{approx}}^{(k)}(x, h_3)$  to have the smallest possible dependence on  $h_3$ . Thus, we set  $h_3 = h_{3,k}$ , where  $h_{3,k}$  is a solution of the global stationarity condition

$$\left. \frac{\partial f_{\text{approx}}^{(k)}(x, h_3)}{\partial h_3} \right|_{h_3=h_{3,k}} = 0 \quad (4.34)$$

Table 5: Polynomial approximations of  $h(\theta)$  using the global stationarity condition for various values of the parameter  $k$ . The reported expressions are obtained by using the central values of the input parameters. The last column shows the corrections to the simple linear model  $h_{\text{lin}}(\theta, \theta_0) \equiv \theta(1 - \theta^2/\theta_0^2)$ .

$k$	$h(\theta)/\theta$	$\theta_0^2$	$h(\theta)/h_{\text{lin}}(\theta, \theta_0)$
1	$1 - 0.734732\theta^2$	1.36104	1
2	$1 - 0.731630\theta^2 + 0.009090\theta^4$	1.39085	$1 - 0.0126429\theta^2$
3	$1 - 0.736743\theta^2 + 0.008904\theta^4 - 0.000472\theta^6$	1.37861	$1 - 0.0113775\theta^2 + 0.0006511\theta^4$

for all  $x$ . Equivalently one may require that, for *any* universal ratio  $R$  that can be obtained from the equation of state, its approximate expression  $R_{\text{approx}}^{(k)}$  obtained by using the parametric representation satisfies

$$\left. \frac{dR_{\text{approx}}^{(k)}(h_3)}{dh_3} \right|_{h_3=h_{3,k}} = 0. \quad (4.35)$$

The existence of such a value of  $h_3$  is a nontrivial mathematical fact. The stationary value of  $h_3$  is the solution of the algebraic equation [19]

$$\left[ 2(2\beta - 1)(h_3 + \gamma) \frac{\partial}{\partial h_3} - 2\gamma + 2k \right] h_{2k+1} = 0. \quad (4.36)$$

For  $k = 1$ , the so-called linear model, Eq. (4.36) gives

$$h_3 = \frac{\gamma(1 - 2\beta)}{\gamma - 2\beta}, \quad (4.37)$$

which is the optimal value of  $h_3$  considered in Ref. [64]. Thus, the optimal (sometimes called restricted) linear model represents the first approximation of our scheme.

## 4.4 Results

Following Ref. [19], we apply the variational method by using the HT results for  $\gamma = 1.2373(2)$ ,  $\nu = 0.63012(16)$ ,  $r_6 = 2.056(5)$ ,  $r_8 = 2.3(1)$ , and  $r_{10} = -13(4)$  as input parameters of the approximation scheme. This provides different approximations with  $k = 1, 2, 3, 4$ . In Table 5 we report the polynomials  $h(\theta)$  for  $k = 1, 2, 3$ , that are obtained in the variational approach for the central values of the input parameters. The fast decrease of the coefficients of the higher-order terms in  $h(\theta)$  gives further support to the effectiveness of the approximation scheme. We do not report  $h(\theta)$  for  $k = 4$ , since it requires  $r_{10}$  and its available estimate is rather imprecise. Using the results reported in Table 5 and Eqs. (4.27), (4.28), and (4.14), one may easily

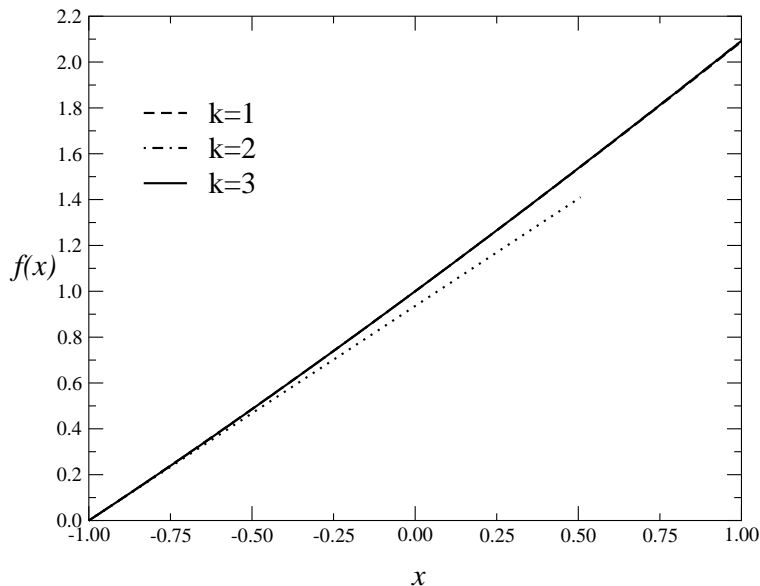


Figure 10: The scaling function  $f(x)$ . We also plot the asymptotic behavior of  $f(x)$  at the coexistence curve (dotted line), i.e.,  $f(x) \approx f_0^{\text{coex}}(1+x)$  for  $x \rightarrow -1$ .

compute the corresponding approximations for the scaling functions  $f(x)$ ,  $F(z)$ , and  $\Phi(u)$ . The results show a good convergence with increasing  $k$ . Actually, the results for  $k = 2, 3, 4$  are already consistent within the errors induced by the uncertainty on the input parameters, indicating that the systematic error due to the truncation is at most of the same order of the error induced by the input data. In Figs. 10, 11, and 12 we show respectively the scaling functions as obtained from  $h(\theta)$  for  $k = 1, 2, 3$ .

In Table 6 we report results concerning the behavior of the scaling function  $f(x)$ ,  $F(z)$  and  $\Phi(u)$  for  $H = 0$  and on the critical isotherm, cf. Eqs. (4.5), (4.6), (4.7), (4.11), (4.12), (4.15). Note that the results for  $k = 1, 2, 3$  oscillate and that the uncertainty due to the input parameters on the  $k = 3$  results is approximately the same as the difference between the estimates with  $k = 2$  and  $k = 3$ . Therefore, it is reasonable to consider the  $k = 3$  truncation as the best approximation of the method using the available input parameters and to use the corresponding errors as final uncertainties.

In Fig. 13 we give the behavior of the magnetization as a function of  $t$  and  $H$ , reporting the scaling function  $E(y)$ . The behavior of the susceptibility can be obtained from the scaling function  $D(y)$ . The function  $D(y)$  has a maximum for  $y_{\text{max}} = 1.980(4)$ , corresponding to the so-called crossover or pseudocritical line (see Sec. 5). In order to simplify possible comparisons, it may be convenient to consider the rescaled function

$$C(y_R) = \frac{D(y)}{D(y_{\text{max}})},$$

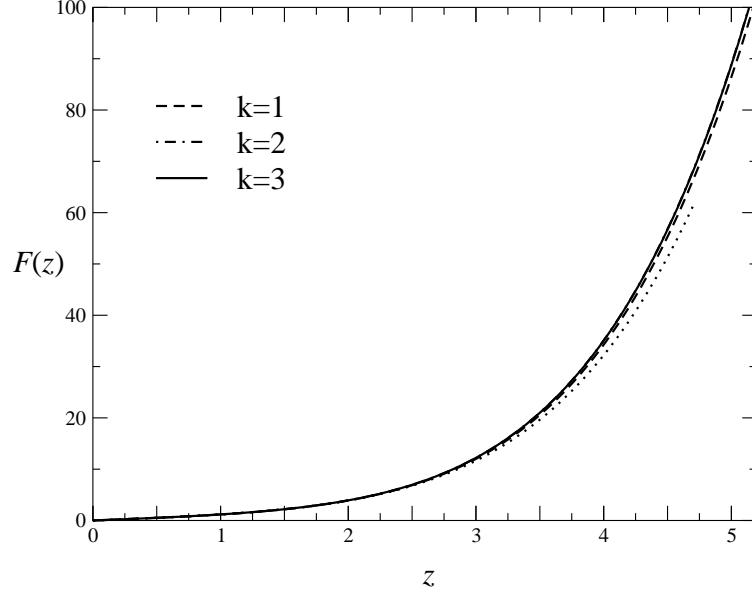


Figure 11: The scaling function  $F(z)$ . We also show the plot of the small- $z$  expansion (dotted line), i.e.,  $F(z) \approx z + \frac{1}{6}z^3 + \frac{1}{120}r_6z^5$  for  $z \rightarrow 0$ .

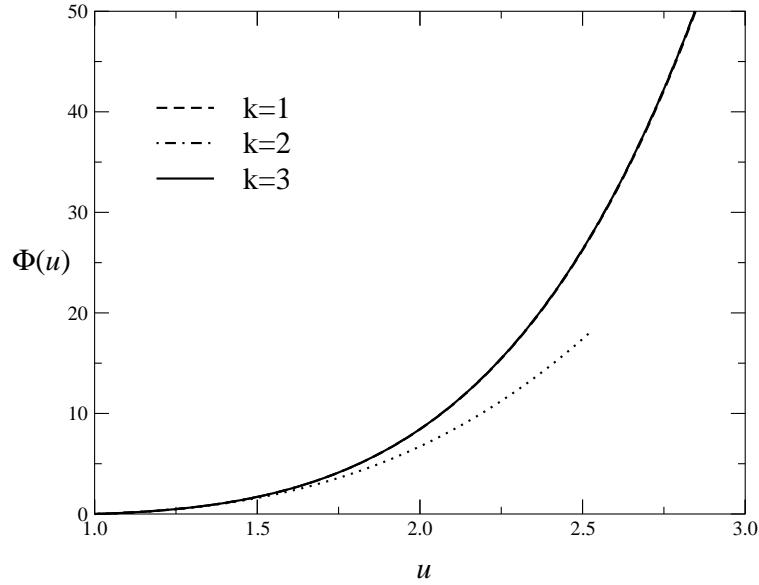


Figure 12: The scaling function  $\Phi(u)$ . We plot also the asymptotic behavior of  $\Phi(u)$  at the coexistence curve (dotted line), i.e.,  $\Phi(u) \approx (u-1) + \frac{1}{2}v_3(u-1)^2 + \frac{1}{6}v_4(u-1)^3$  for  $u \rightarrow 1$ .

Table 6: Expansion coefficients for the scaling equation of state obtained by the variational approach. See text for definitions. Numbers marked with an asterisk are inputs, not predictions.

	$k = 1$	$k = 2$	$k = 3$	$k = 4$
$\theta_0^2$	1.3610(8)	1.390(2)	1.38(2)	1.34(6)
$\rho$	1.7365(8)	1.741(1)	1.733(10)	1.69(6)
$r_6$	1.938(3)	*2.056(5)	*2.056(5)	*2.056(5)
$r_8$	2.50(2)	2.39(3)	*2.3(1)	*2.3(1)
$r_{10}$	-12.59(2)	-12.08(5)	-10.6(1.8)	*-13(4)
$F_0^\infty$	0.03277(8)	0.03388(11)	0.03382(15)	0.0338(2)
$z_0$	2.8254(7)	2.792(2)	2.794(3)	2.798(8)
$f_1^0$	1.05041(7)	1.0532(2)	1.0527(7)	1.051(2)
$f_2^0$	0.04298(6)	0.04494(13)	0.0446(4)	0.0439(13)
$f_3^0$	-0.02474(4)	-0.02595(8)	-0.0254(7)	-0.023(4)
$f_0^\infty$	0.5960(4)	0.6031(7)	0.6024(15)	0.601(4)
$f_1^{\text{coex}}$	0.93912(9)	0.9347(3)	0.9357(11)	0.938(4)
$v_3$	6.013(4)	6.062(4)	6.050(13)	6.02(5)
$v_4$	16.32(3)	16.10(4)	16.17(10)	16.4(3)

$$y_R = \frac{y}{y_{\max}}, \quad (4.38)$$

which is such that the maximum corresponds to  $y_R = 1$  and satisfies  $C(1) = 1$ . In Fig. 14 we plot the scaling function  $C(y_R)$ , as obtained from the  $k = 1, 2, 3$  approximate parametric representations.

In experimental work on magnetic systems, it is customary to report [67,68]  $h/m \equiv H|t|^{-\gamma}/M$  versus  $m^2 = M^2|t|^{-2\beta}$ . Such a function can be easily obtained from our approximations for  $f(x)$ , since  $m^2 = B^2|x|^{-2\beta}$  and

$$\frac{h}{m} = k|x|^{-\gamma}f(x), \quad (4.39)$$

where the constant  $k$  can be written as

$$k = B_c^{-\delta} B^{\gamma/\beta} = \frac{R_\chi}{C^+}, \quad (4.40)$$

where  $R_\chi \equiv C^+ B^{\delta-1}/B_c^\delta$  is a universal constant, see Sec. 5. A plot of  $m^2/B^2$  versus  $C^+ h/m$  for the two cases  $t > 0$  and  $t < 0$  is reported in Fig. 15.

It is interesting to observe that in a neighborhood of the critical isotherm the equation of state can be written in the Arrott-Noakes form [69]

$$\left(\frac{H}{M}\right)^{1/\gamma} = at + bM^{1/\beta}, \quad (4.41)$$

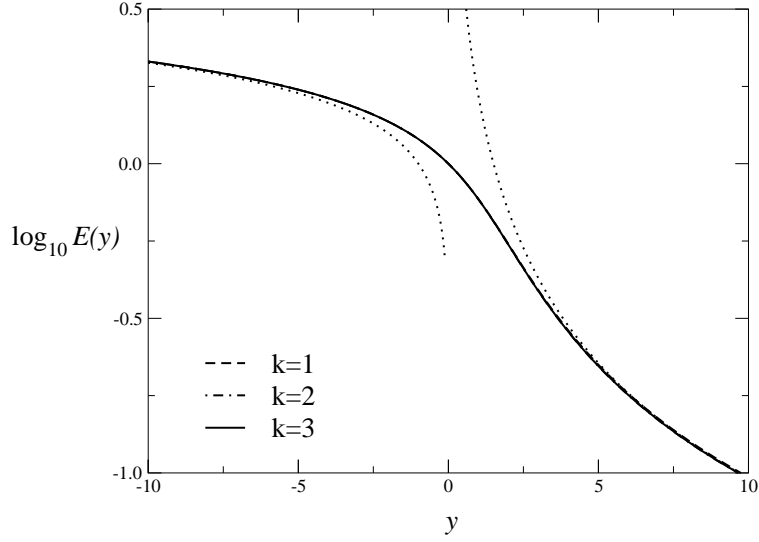


Figure 13: The scaling function  $E(y)$ . We also report its asymptotic behaviors (dotted lines):  $E(y) \approx R_\chi y^{-\gamma}$  for  $y \rightarrow +\infty$ , and  $E(y) \approx -y^\beta$  for  $y \rightarrow -\infty$ .

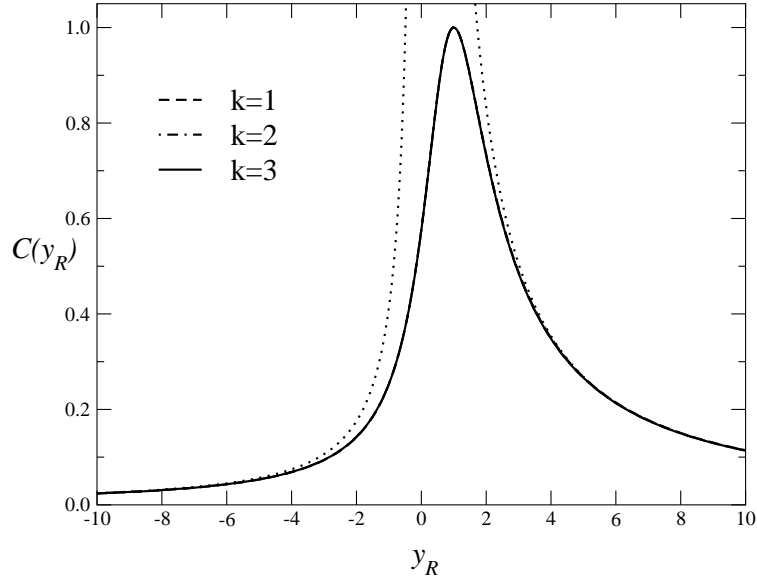


Figure 14: The scaling function  $C(y_R)$ . We also report its asymptotic behaviors (dotted lines):  $C(y_R) \approx R_\chi y_{\max}^{-\gamma} D(y_{\max})^{-1} y_R^{-\gamma} \approx 1.97 y_R^{-\gamma}$  for  $y_R \rightarrow +\infty$ , and  $C(y_R) \approx \beta(f_1^{\text{coex}})^{-1} y_{\max}^{-\gamma} D(y_{\max})^{-1} (-y_R)^{-\gamma} \approx 0.413 (-y_R)^{-\gamma}$  for  $y_R \rightarrow -\infty$ .

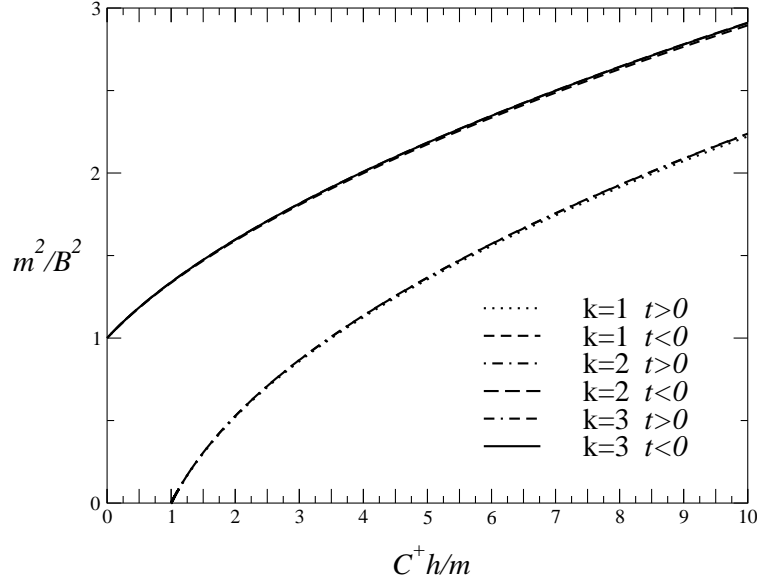


Figure 15: Plot of  $m^2/B^2$  versus  $C^+h/m$ .

where  $a$  and  $b$  are numerical constants. Indeed, using the results of Table 6 for  $k = 3$ , we obtain

$$\left(\frac{H}{M}\right)^{1/\gamma} k^{-1/\gamma} = \left(\frac{M}{B}\right)^{1/\beta} + 0.851 t - 0.050 t^2 \left(\frac{M}{B}\right)^{-1/\beta} - 0.008 t^3 \left(\frac{M}{B}\right)^{-2\beta} \dots \quad (4.42)$$

Thus, corrections to Eq. (4.41) are small and thus this expression has a quite wide range of validity.

## 5 Universal amplitude ratios

From the critical equation of state one may derive estimates of several universal amplitude ratios. They are expressed in terms of the amplitudes of the magnetization, cf. Eqs. (4.3) and (4.4), of the magnetic susceptibility and  $n$ -point correlation functions, cf. Eq. (4.10), of the specific heat

$$C_H = A^\pm |t|^{-\alpha}, \quad (5.1)$$

of the second-moment correlation length

$$\xi = f^\pm |t|^{-\nu}, \quad (5.2)$$

and of the true (on-shell) correlation length, describing the large distance behavior of the two-point function,

$$\xi_{\text{gap}} = f_{\text{gap}}^\pm |t|^{-\nu}. \quad (5.3)$$



$U_0 \equiv A^+/A^-$	$U_2 \equiv C^+/C^-$
$U_4 \equiv C_4^+/C_4^-$	$R_4^+ \equiv -C_4^+ B^2/(C^+)^3$
$R_c^+ \equiv \alpha A^+ C^+/B^2$	$R_c^- \equiv \alpha A^- C^-/B^2$
$R_4^- \equiv C_4^- B^2/(C^-)^3$	$R_\chi \equiv C^+ B^{\delta-1}/(B_c)^\delta$
$v_3 \equiv -C_3^- B/(C^-)^2$	$v_4 \equiv -C_4^- B^2/(C^-)^3 + 3v_3^2$
$g_4^+ \equiv -C_4^+ /[(C^+)^2 (f^+)^3]$	$w^2 \equiv C^-/[B^2 (f^-)^d]$
$U_\xi \equiv f^+/f^-$	$U_{\xi_{\text{gap}}} \equiv f_{\text{gap}}^+/f_{\text{gap}}^-$
$Q^+ \equiv \alpha A^+ (f^+)^3$	$Q^- \equiv \alpha A^- (f^-)^d$
$R_\xi^+ \equiv (Q^+)^{1/3}$	$Q_\xi^+ \equiv f_{\text{gap}}^+/f^+$
$Q_\xi^- \equiv f_{\text{gap}}^-/f^-$	$Q_\xi^c \equiv f_{\text{gap}}^c/f^c$
$Q_c \equiv B^2 (f^+)^3/C^+$	$Q_2 \equiv (f^c/f^+)^{2-\eta} C^+/C_c$
$P_m \equiv T_p^\beta B/B_c$	$P_c \equiv -T_p^{2\beta\delta} C^+/C_4^+$
$R_p \equiv C^+/C_p$	

Table 7: Amplitude-ratio definitions.

One can also define amplitudes along the critical isotherm, e.g.

$$\chi = C^c |H|^{-\gamma/\beta\delta}, \quad (5.4)$$

$$\xi = f^c |H|^{-\nu/\beta\delta}, \quad (5.5)$$

$$\xi_{\text{gap}} = f_{\text{gap}}^c |H|^{-\nu/\beta\delta}. \quad (5.6)$$

We also consider the crossover (or pseudocritical) line  $t_{\text{max}}(H)$ , that is defined as the reduced temperature for which the magnetic susceptibility has a maximum at  $H$  fixed. Renormalization group predicts

$$t_{\text{max}}(H) = T_p H^{1/(\gamma+\beta)}, \quad (5.7)$$

$$\chi(t_{\text{max}}, H) = C_p t_{\text{max}}^{-\gamma}. \quad (5.8)$$

We consider several universal amplitude ratios. They are defined in Table 7.

In Table 8 we report the universal amplitude ratios, as derived by the approximate polynomial representations of the equation of state for  $k = 1, 2, 3, 4$ . The reported errors are only due to the uncertainty of the input parameters and do not include the systematic error of the procedure, which may be determined by comparing the results of the various approximations. In Table 8 we also show results for  $z_{\text{max}}$ ,  $x_{\text{max}}$  and  $y_{\text{max}}$  which are the values of the scaling variable  $z$ ,  $x$  and  $y$  ( $y$  was defined in

Table 8: Universal amplitude ratios obtained by taking different approximations of the parametric function  $h(\theta)$ .

	$k = 1$	$k = 2$	$k = 3$	$k = 4$
$U_0$	0.5231(11)	0.533(2)	0.5319(25)	0.529(6)
$U_2$	4.826(7)	4.745(10)	4.758(19)	4.78(5)
$U_4$	-9.73(3)	-8.85(6)	-9.0(2)	-9.3(5)
$R_c^+$	0.05545(7)	0.0570(1)	0.0567(3)	0.0562(11)
$R_c^-$	0.021967(11)	0.02253(3)	0.02242(12)	0.0222(4)
$R_4^+$	7.983(4)	7.794(8)	7.81(2)	7.83(4)
$R_4^-$	92.15(13)	94.13(13)	93.6(6)	92(2)
$R_\chi$	1.6779(11)	1.658(2)	1.660(4)	1.665(10)
$U_2 R_4^+$	38.52(5)	36.98(10)	37.1(2)	37.4(6)
$R_4^+ R_c^+$	0.4427(7)	0.4444(7)	0.443(2)	0.440(6)
$P_m$	1.25203(6)	1.2493(2)	1.2498(6)	1.251(2)
$P_c$	0.3831(3)	0.3938(5)	0.3933(7)	0.3930(11)
$R_p$	1.9789(3)	1.9658(6)	1.9665(10)	1.9671(16)
$z_{\max}$	1.2443(4)	1.2317(5)	1.2322(8)	1.2326(12)
$x_{\max}$	12.32(3)	12.26(3)	12.27(4)	12.31(8)
$w_{\max}$	1.990(1)	1.977(2)	1.980(4)	1.984(9)
$D(w_{\max})$	0.36179(4)	0.36277(7)	0.36268(14)	0.3626(3)

Eq. (4.18)) associated with the crossover line. As already mentioned in Sec. 4.4, we consider the  $k = 3$  results as our best estimates, and report them in Table 1.

Estimates of universal ratios of amplitudes involving correlation-length amplitudes, such as  $Q^+$ ,  $R_\xi^+$ , and  $Q_c$ , can be obtained using the HT estimate of  $g_4$ . For instance  $Q^+ = R_4^+ R_c^+ / g_4$ . Other universal ratios can be obtained by supplementing the above results with the estimates of  $w^2$  and  $Q_\xi^-$  (see Table 7), obtained by an analysis of the corresponding low-temperature expansions [14, 34], and the HT estimate of  $Q_\xi^+$  (see Sec. 6). Moreover, using approximate parametric representations of the correlation length, see Refs. [1, 19] for details, one may also estimate the universal ratios  $Q_\xi^c$  and  $Q_2$  defined in Table 7.

## 6 Low-momentum behavior of the structure factor

In this section we update the determination of the first few coefficients that parametrize the low-momentum expansion of the scaling two-point function in the HT phase [19, 34, 70]

$$g(y) \equiv \frac{\chi}{\widetilde{G}(k)} = 1 + y + \sum_{i=2}^{\infty} c_i y^i, \quad (6.1)$$

where  $y = k^2 \xi^2$ .

The coefficients  $c_i$  can be related to the critical limit of appropriate dimensionless

Table 9: Estimates of the coefficients  $c_i$ ,  $i = 2, 3, 4$ , of the low-momentum expansion of the structure factor.

	$\phi^4$	$\phi^6$	spin-1	final estimates
$c_2$	$-0.390(7) \times 10^{-3}$	$-0.390(6) \times 10^{-3}$	$-0.389(12) \times 10^{-3}$	$-0.390(6) \times 10^{-3}$
$c_3$	$0.882(8) \times 10^{-5}$	$0.882(6) \times 10^{-5}$	$0.88(4) \times 10^{-5}$	$0.882(6) \times 10^{-5}$
$c_4$	$-0.4(1) \times 10^{-6}$	$-0.4(1) \times 10^{-6}$	$-0.4(1) \times 10^{-6}$	$-0.4(1) \times 10^{-6}$

ratios of spherical moments  $m_{2j}$ . See Ref. [34] for details. We have estimated the first few coefficients  $c_i$  from the corresponding series derived from the 25th-order expansions of  $m_{2j}$ , using the analysis described in Sec. 4.2. The results for the three improved models and our final estimates are reported in Table 9. Other interesting quantities are

$$S_M \equiv M_{\text{gap}}^2/M^2, \quad (6.2)$$

$$S_Z \equiv \chi M^2/Z_{\text{gap}}, \quad (6.3)$$

where  $M_{\text{gap}}$  (the mass gap of the theory) and  $Z_{\text{gap}}$  determine the long-distance behavior of the two-point function:

$$G(x) \approx \frac{Z_{\text{gap}}}{4\pi|x|} e^{-M_{\text{gap}}|x|}. \quad (6.4)$$

As discussed in Refs. [19,34], one may estimate  $S_M$  and  $S_Z$  from  $c_2$ ,  $c_3$ , and  $c_4$ . Indeed, we have

$$S_M = 1 + c_2 - c_3 + c_4 + 2c_2^2 + \dots \quad (6.5)$$

$$S_Z = 1 - 2c_2 + 3c_3 - 4c_4 - 2c_2^2 + \dots \quad (6.6)$$

where the ellipses indicate contributions that are negligible with respect to  $c_4$ . Therefore, one finds  $S_M = 0.999601(6)$  and  $S_Z = 1.000810(13)$ . From the result for  $S_M$ , one obtains  $Q_\xi^+ \equiv f_{\text{gap}}^+/f^+ = 1.000200(3)$ .

A more detailed analysis of the behavior of the structure factor for all momenta can be found in Ref. [71].

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